Isotropic Flow of Homeomorphisms on $\mathbb{S}^d with Respect to the Metric H^{(d+2)/2}$

Fang, Shizan and Zhang, Tusheng

2006

MIMS EPrint: **2006.53**

ISSN 1749-9097

Isotropic Flow of Homeomorphisms on S^d with Respect to the Metric $H^{d+2/2}$

Shizan Fang & Tusheng Zhang

First version: 27 September 2005

Research Report No. 13, 2005, Probability and Statistics Group
School of Mathematics, The University of Manchester

Isotropic flow of homeomorphisms on S^d
with respect to the metric $H^{d+2/2}$

Shizan FANG Tusheng ZHANG

S.F.: I.M.B, UFR Sciences et techniques, Université de Bourgogne, 9 avenue Alain Savary, B.P. 47870, 21078 Dijon, France.

T.Z.: Department of Mathematics, University of Manchester, Oxford road, Manchester, M13 9PL, England.

Abstract . In this work, we shall deal with the critical Sobolev isotropic Brownian flows on the sphere S^d. Based on previous works by O. Raimond and LeJan-Raimond (see *Ann. Inst. H. Poincaré*, **35** (1999), p. 313-354 and *Ann. of Prob.*, **30** (2002), p. 826-873), we prove that the associated flow is a flow of homeomorphisms.

1. Introduction

Let Δ be the Laplace operator on S^d, acting on vector fields. The spectrum of Δ is given by spectrum$(\Delta) = \{-c_{\ell,d};\ \ell \geq 1\} \cup \{-c_{\ell,\delta};\ \ell \geq 1\}$, where $c_{\ell,d} = \ell(\ell+d-1)$, $c_{\ell,\delta} = (\ell+1)(\ell+d-2)$. Let $\mathcal{G}_\ell$ be the eigenspace associated to $c_{\ell,d}$ and $\mathcal{D}_\ell$ the eigenspace associated to $c_{\ell,\delta}$. Their dimension will be denoted by $D_{\ell,1} = \dim \mathcal{G}_\ell$, $\quad D_{\ell,2} = \dim \mathcal{D}_\ell$. It is known (see [6]) that

$$D_{\ell,1} = O(\ell^{d-1}), \quad D_{\ell,2} = O(\ell^{d-1}) \quad \text{as } \ell \to +\infty. \tag{1.1}$$

Denote by $\{A^i_{\ell,k};\ k = 1, \cdots, D_{\ell,i},\ \ell \geq 1\}$ for $i = 1, 2$ the orthonormal basis of $\mathcal{G}_\ell$ or $\mathcal{D}_\ell$ in L^2:

$$\int_{S^d} \langle A^i_{\ell,k}(x), A^j_{\alpha,\beta}(x) \rangle \, dx = \delta_{ij} \delta_{\ell\alpha} \delta_{k\beta}$$

where δ_{ij} is the Kronecker symbol and dx is the normalized Riemannian measure on S^d, which is the unique one invariant by actions of $g \in \mathrm{SO}(d+1)$. By Weyl theorem, the vector fields $\{A^i_{\ell,k}\}$ are smooth.

Let $s > 0$ and $H^s(S^d)$ be the Sobolev space of vector fields on S^d, which is the completion of smooth vector fields with respect to the norm

$$||V||^2_{H^s} = \int_{S^d} \langle (-\Delta + 1)^s V, V \rangle \, dx. \tag{1.2}$$

Then $\Big\{ A^1_{\ell,k}/(1 + c_{\ell,d})^{s/2}, A^2_{\ell,\beta}/(1 + c_{\ell,\delta})^{s/2};\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,1}\,, 1 \leq \beta \leq D_{\ell,2} \Big\}$ is an orthonormal basis of H^s. If we consider

$$a_\ell = \frac{a}{(\ell-1)^{1+\alpha}},\ b_\ell = \frac{b}{(\ell-1)^{1+\alpha}}, \quad \alpha > 0,\ a, b > 0,\ \ell \geq 2 \tag{1.3}$$

then

$$\sqrt{\frac{a_\ell}{D_{\ell,1}}} = O(\frac{1}{\ell^{(\alpha+d)/2}}), \quad \sqrt{\frac{b_\ell}{D_{\ell,2}}} = O(\frac{1}{\ell^{(\alpha+d)/2}}). \tag{1.4}$$

Let $\{B^i_{\ell,k}(t);\ \ell\geq 1,\ 1\leq k\leq D_{\ell,i}\}$ for $i=1,2$ be two family of independent standard Brownian motions defined on a probability space $(\Omega,\mathcal{F},P)$. Consider the series

$$W_t(\omega)=\sum_{\ell\geq 1}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}B^1_{\ell,k}(t)A^1_{\ell,k}\ +\ \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}B^2_{\ell,k}(t)A^2_{\ell,k}\Big\} \tag{1.5}$$

which converges in L^2, but uniformly with respect to t in any compact subset of $[0,+\infty[$. According to (1.4), $(W_t)_{t\geq 0}$ is a *cylinder* Brownian motion in the Sobolev space $H^{(\alpha+d)/2}$. Moreover, W_t takes values in the space $H^s(S^d)$ for any $0<s<\alpha/2$. By Sobolev embedding theorem, in order to ensure that W_t takes values in the space of C^2 vector fields, α must be *large* than $d+2$. In this last case, the classical Kunita's framework ([5]) can be applied to integrate the vector field W_t so that we obtain a flow of diffeomorphisms. For the case of small α, the notion of statistical solutions was introduced in [6] and the phenomenon of phase transition appears when $0<\alpha<2$. It was also shown in [6] that the statistical solutions give rise to a flow of maps if $\alpha>2$, but the regularity was not discussed. The main objective of this work is to deal with the critical case $\alpha=2$. Instead of introducing $(W_t)_{t\geq 0}$ as in (1.5), we consider first the stochastic differential equations on S^d

$$dx^n_t=\sum_{\ell=1}^{2^n}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(x^n_t)\circ dB^1_{\ell,k}(t)\ +\ \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(x^n_t)\circ dB^2_{\ell,k}(t)\Big\},\quad x^n_0=x. \tag{1.6}$$

Using the specific properties of eigen vector fields, we prove that $x^n_t(x)$ converges uniformly in $(t,x)\in[0,T]\times S^d$, so that we obtain the following main result of this work.

Theorem A. *Let $\alpha=2$ in definition (1.3). Then the stochastic differential equation on S^d:*

$$dx_t=\sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(x_t)\circ dB^1_{\ell,k}(t)\ +\ \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(x_t)\circ dB^2_{\ell,k}(t)\Big\},\quad x_0=x \tag{1.7}$$

has one unique strong solution $(x_t(x))_{t\geq 0}$, which gives rise to a flow of homeomorphisms.

In the case of the circle S^1, this property of flow of homeomorphisms was discovered in [7].

The main feature of this work is to handle the non-Lipschitzian stochastic differential equations: it complements our work [4].

2. Approximating flows

In this section, we discuss the approximating flows and establish some necessary estimates. Let $n\geq 1$. Consider the Stratanovich stochastic differential equation on S^d:

$$dx^n_t=\sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(x^n_t)\circ dB^1_{\ell,k}(t)+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(x^n_t)\circ dB^2_{\ell,k}(t)\Big] \tag{2.1}$$

with $x^n_0=x\in S^d$ given. Since $A^i_{\ell,k}$ are smooth, it is known (see [2], [5], [8]) that the stochastic differential equation (2.1) defines a flow of diffeomorphisms $\varphi^n_t(x)$ on S^d.

For $x, y \in S^d$, consider the Riemannian distance $d(x,y)$, which satisfies the formula

$$\cos d(x,y) = \langle x, y\rangle \tag{2.2}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbf{R}^{d+1}$, with the Euclidean distance $|\cdot|$. We have the relation

$$|x-y| \le d(x,y) \le \frac{\pi}{2}|x-y|. \tag{2.3}$$

In what follows, we shall compute the term $\langle x_t^n, x_t^{n+1}\rangle$. By Itô formula,

$$\begin{aligned} d\langle x_t^n, x_t^{n+1}\rangle &= \langle \circ dx_t^n, x_t^{n+1}\rangle + \langle x_t^n, \circ dx_t^{n+1}\rangle \\ &= \sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\langle x_t^{n+1}, A_{\ell,k}^1(x_t^n)\circ dB_{\ell,k}^1(t)\rangle \\ &\qquad + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\langle x_t^{n+1}, A_{\ell,k}^2(x_t^n)\circ dB_{\ell,k}^2(t)\rangle\Big] \\ &\quad + \sum_{\ell=1}^{2^{n+1}}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\langle x_t^n, A_{\ell,k}^1(x_t^{n+1})\circ dB_{\ell,k}^1(t)\rangle \\ &\qquad + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\langle x_t^n, A_{\ell,k}^2(x_t^{n+1})\circ dB_{\ell,k}^2(t)\rangle\Big]. \end{aligned} \tag{2.4}$$

Let $x \in S^d$. Denote by T_xS^d the tangent space at the point x. Consider the orthogonal projection $Q_x : \mathbf{R}^{d+1} \to T_xS^d$. We have for $i = 1, 2$,

$$\langle A_{\ell,k}^i(x_t^n)\circ dB_{\ell,k}^i,\ x_t^{n+1}\rangle = \langle A_{\ell,k}^i(x_t^n),\ Q_{x_t^n}x_t^{n+1}\rangle_{T_{x_t^n}S^d}\circ dB_{\ell,k}^i.$$

Set $\Lambda_t = Q_{x_t^n}x_t^{n+1}$. Then Λ_t has the expression

$$\Lambda_t = x_t^{n+1} - \langle x_t^n, x_t^{n+1}\rangle x_t^n \in T_{x_t^n}S^d.$$

Viewing Λ_t as a process in $\mathbf{R}^{d+1}$, we have

$$d\Lambda_t = dx_t^{n+1} - \langle \circ dx_t^n, x_t^{n+1}\rangle x_t^n - \langle x_t^n, \circ dx_t^{n+1}\rangle x_t^n - \langle x_t^n, x_t^{n+1}\rangle\circ dx_t^n.$$

Denote by $\frac{D}{dt}$ the covariant derivative along x_t^n. Then

$$\frac{D}{dt}\Lambda_t = Q_{x_t^n}\circ d\Lambda_t = Q_{x_t^n}\circ dx_t^{n+1} - \langle x_t^{n+1}, x_t^n\rangle\circ dx_t^n$$

which is equal to

$$\begin{aligned} &\sum_{\ell=1}^{2^{n+1}}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}Q_{x_t^n}(A_{\ell,k}^1(x_t^{n+1}))\circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}Q_{x_t^n}(A_{\ell,k}^2(x_t^{n+1}))\circ dB_{\ell,k}^2(t)\Big] \\ &- \langle x_t^n, x_t^{n+1}\rangle\sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A_{\ell,k}^1(x_t^n)\circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A_{\ell,k}^2(x_t^n)\circ dB_{\ell,k}^2(t)\Big]. \end{aligned}$$

It follows that the Itô contraction $\left\langle \frac{D}{dt}\Lambda_t, A^i_{\ell k}(x^n_t)\right\rangle \cdot dB^i_{\ell k}(t)$ is given by

$$\sqrt{\frac{da_\ell}{D_{\ell,1}}}\Big\{\big\langle Q_{x^n_t}A^i_{\ell k}(x^{n+1}_t), A^i_{\ell k}(x^n_t)\big\rangle - \big\langle x^n_t, x^{n+1}_t\big\rangle\big\langle A^i_{\ell k}(x^n_t), A^i_{\ell k}(x^n_t)\big\rangle\Big\}\,dt. \tag{2.5}$$

On other hand, the Itô contraction $\sqrt{\frac{da_\ell}{D_{\ell,1}}}\big\langle \frac{D}{dt}A^1_{\ell k}, \Lambda_t\big\rangle \cdot dB^1_{\ell k}$ is equal to

$$\big\langle (\nabla_{A^1_{\ell k}}A^1_{\ell k})(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle\,dt. \tag{2.6}$$

Now passing to Itô integrals, we get

$$\begin{aligned}\big\langle A^1_{\ell,k}(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle \circ dB^1_{\ell,k} &= \big\langle A^1_{\ell,k}(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle\,dB^1_{\ell,k}\\ &+\frac{1}{2}\sqrt{\frac{da_\ell}{D_{\ell,1}}}\Big\{\big\langle (\nabla_{A^1_{\ell,k}}A^1_{\ell,k})(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle\\ + \big\langle A^1_{\ell,k}(x^n_t),\ Q_{x^n_t}A^1_{\ell,k}(x^{n+1}_t)\big\rangle &- \big\langle x^n_t, x^{n+1}_t\big\rangle\big\langle A^1_{\ell,k}(x^n_t), A^1_{\ell,k}(x^n_t)\big\rangle\Big\}\,dt.\end{aligned}$$

Using $(A.6)$, we get

$$\begin{aligned}\sum_{k=1}^{D_{\ell,1}}\big\langle A^1_{\ell,k}(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle \circ dB^1_{\ell,k} &= \sum_{k=1}^{D_{\ell,1}}\big\langle A^1_{\ell,k}(x^n_t), Q_{x^n_t}x^{n+1}_t\big\rangle\,dB^1_{\ell,k}\\ +\frac{1}{2}\sum_{k=1}^{D_{\ell,1}}\sqrt{\frac{da_\ell}{D_{\ell,1}}}\Big\{\big\langle A^1_{\ell,k}(x^n_t),\ A^1_{\ell,k}(x^{n+1}_t)\big\rangle &- \big\langle x^n_t, x^{n+1}_t\big\rangle |A^1_{\ell,k}(x^n_t)|^2\Big\}\,dt.\end{aligned}$$

The same kind of calculations hold for vector fields $\{A^2_{\ell k}\}$. Let M^n_t be the martingale part of $\big\langle x^n_t, x^{n+1}_t\big\rangle$ and V^n_t the drift part:

$$d\big\langle x^n_t, x^{n+1}_t\big\rangle = dM^n_t + V^n_t\,dt.$$

Then by above calculations, we have

$$\begin{aligned}dM^n_t &= \sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\langle A^1_{\ell k}(x^n_t),\ x^{n+1}_t\rangle dB^1_{\ell k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\langle A^2_{\ell k}(x^n_t),\ x^{n+1}_t\rangle dB^2_{\ell k}(t)\Big]\\ &+\sum_{\ell=1}^{2^{n+1}}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\langle A^1_{\ell k}(x^{n+1}_t),\ x^n_t\rangle dB^1_{\ell k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\langle A^2_{\ell k}(x^{n+1}_t),\ x^n_t\rangle dB^2_{\ell k}(t)\Big].\end{aligned} \tag{2.7}$$

By $(A.4)$, we have

$$\sum_{k=1}^{D_{\ell,1}}|A^1_{\ell,k}(x)|^2 = \frac{D_{\ell,1}}{d}\sum_{i=1}^{d}\sum_{k=1}^{D_{\ell,1}}|T^\ell_{ki}|^2(g) = D_{\ell,1},$$

where $g \in \mathrm{SO}(d{+}1)$ such that $x = gP_0$. So V^n_t has the following expression

$$
\begin{aligned}
V_t^n = &\sum_{\ell=1}^{2^n}\Big[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big\langle A^1_{\ell k}(x_t^n), A^1_{\ell k}(x_t^{n+1})\big\rangle + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big\langle A^2_{\ell k}(x_t^n), A^2_{\ell k}(x_t^{n+1})\big\rangle\Big] \\
&- \sum_{\ell=1}^{2^n} d(a_\ell+b_\ell)\big\langle x_t^n, x_t^{n+1}\big\rangle \\
&+ \frac{1}{2}\sum_{\ell=2^n+1}^{2^{n+1}}\Big[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big\langle A^1_{\ell k}(x_t^n), A^1_{\ell k}(x_t^{n+1})\big\rangle + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big\langle A^2_{\ell k}(x_t^n), A^2_{\ell k}(x_t^{n+1})\big\rangle\Big] \\
&- \frac{1}{2}\sum_{\ell=2^n+1}^{2^{n+1}} d(a_\ell+b_\ell)\big\langle x_t^n, x_t^{n+1}\big\rangle.
\end{aligned}
\tag{2.8}
$$

Define

$$
\begin{aligned}
f_\ell(\theta) = &\, a_\ell\Big[(d-1+\cos^2\theta)\,\gamma_\ell(\cos\theta) - \cos\theta\sin^2\theta\,\gamma_\ell'(\cos\theta)\Big] \\
&+ b_\ell\Big[d\cos\theta\,\gamma_\ell(\cos\theta) - \sin^2\theta\,\gamma_\ell'(\cos\theta)\Big] - d(a_\ell+b_\ell)\,\cos\theta.
\end{aligned}
\tag{2.9}
$$

Let $\theta_t^n = d(x_t^n, x_t^{n+1})$. Then, $\big\langle x_t^n, x_t^{n+1}\big\rangle = \cos\theta_t^n$. Then according to proposition A.4 and (2.8), V_t^n has the expression

$$
V_t^n = \sum_{\ell=1}^{2^n} f_\ell(\theta_t^n) + \frac{1}{2}\sum_{\ell=2^n+1}^{2^{n+1}} f_\ell(\theta_t^n).
\tag{2.10}
$$

Now by expression (2.7), the quadratic variation $d\Theta_t^n = dM_t^n \cdot dM_t^n$ of M_t^n is given by

$$
\begin{aligned}
d\Theta_t^n = &\sum_{\ell=1}^{2^n}\Big\{\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\Big(\big\langle A^1_{\ell k}(x_t^n),\ x_t^{n+1}\big\rangle + \big\langle A^1_{\ell k}(x_t^{n+1}),\ x_t^n\big\rangle\Big)^2 \\
&+ \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\Big(\big\langle A^2_{\ell k}(x_t^n),\ x_t^{n+1}\big\rangle + \big\langle A^2_{\ell k}(x_t^{n+1}),\ x_t^n\big\rangle\Big)^2\Big\} \\
&+ \sum_{\ell=2^n+1}^{2^{n+1}}\Big\{\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big\langle A^1_{\ell k}(x_t^{n+1}),\ x_t^n\big\rangle^2 + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big\langle A^2_{\ell k}(x_t^{n+1}),\ x_t^n\big\rangle^2\Big\}\,dt.
\end{aligned}
$$

Using proposition A.5, we see that

$$
\frac{d}{D_{\ell,i}}\sum_{k=1}^{D_{\ell,i}}\big\langle x_t^n, A^i_{\ell k}(x_t^{n+1})\big\rangle^2 = \sin^2\theta_t^n,
$$

and

$$
\begin{aligned}
d\Theta_t^n = 2(\sin\theta_t^n)^2 &\sum_{\ell=1}^{2^n}\Big\{a_\ell\Big(1-\cos\theta_t^n\gamma_\ell(\cos\theta_t^n) + (\sin\theta_t^n)^2\gamma_\ell'(\cos\theta_t^n)\Big) \\
&+ b_\ell(1-\gamma_\ell(\cos\theta_t^n)\Big\}dt + (\sin\theta_t^n)^2\sum_{\ell=2^n+1}^{2^{n+1}}(a_\ell+b_\ell)dt.
\end{aligned}
$$

Now introduce the function G_n defined by

$$G_n(\theta) = \sum_{\ell=1}^{2^n} \frac{\gamma_{\ell+1}(\cos\theta)}{\ell^3} = \sum_{\ell=1}^{2^n} \frac{\tilde{\gamma}_\ell(\theta)}{\ell^3}. \tag{2.11}$$

Recall that

$$\tilde{\gamma}_\ell(\theta) = \int_0^\pi (\cos\theta - \sqrt{-1}\sin\theta\cos\varphi)^\ell \sin^d\varphi \frac{d\varphi}{c_d}.$$

We have: $\tilde{\gamma}'_\ell(\theta) = \gamma'_{\ell+1}(\cos\theta)(-\sin\theta)$ or $\gamma'_{\ell+1}(\cos\theta) = -\dfrac{\tilde{\gamma}'_\ell(\theta)}{\sin\theta}$. Define

$$\Xi_n(\theta) = \sum_{\ell=2^n+1}^{2^{n+1}} \frac{\tilde{\gamma}_\ell(\theta)}{\ell^3}. \tag{2.12}$$

Using these notations, introduce

$$\begin{aligned} V_n(\theta) &= a\Big[(d-\sin^2\theta)G_n(\theta) + \cos\theta\sin\theta G'_n(\theta)\Big] \\ &\quad + b\Big[d\cos\theta G_n(\theta) + \sin\theta G'_n(\theta)\Big] - d(a+b)\,\cos\theta\, G_n(0) \\ &\quad + \frac{1}{2}\Big\{a\Big[(d-\sin^2\theta)\Xi_n(\theta) + \cos\theta\sin\theta\Xi'_n(\theta)\Big] \\ &\quad + b\Big[d\cos\theta\Xi_n(\theta) + \sin\theta\Xi'_n(\theta)\Big] - d(a+b)\,\cos\theta\,\Xi_n(0)\Big\}. \end{aligned} \tag{2.13}$$

Then

$$V_t^n = V_n(\theta_t^n).$$

Define

$$\begin{aligned} U_n(\theta) = 2\sin^2\theta\,\Big\{a\Big[G_n(0) - \cos\theta\, G_n(\theta) - \sin\theta\, G'_n(\theta)\Big] \\ + b\big(G_n(0) - G_n(\theta)\big)\Big\} + \sin^2\theta\,(a+b)\Xi_n(0). \end{aligned} \tag{2.14}$$

Then

$$\Theta_t^n = U_n(\theta_t^n).$$

Therefore there exists a real Brownian motion $W_n(t)$ defined on the same probability space such that

$$d\cos\theta_t^n = \sqrt{U_n(\theta_t^n)}\, dW_n(t) + V_n(\theta_t^n)\, dt. \tag{2.15}$$

Using the relation $\theta = \cos^{-1}(\cos\theta)$, we obtain

$$d\theta_t^n = -\frac{\sqrt{U_n(\theta_t^n)}}{\sin\theta_t^n}\, dW_n(t) - \Big(\frac{V_n(\theta_t^n)}{\sin\theta_t^n} + \frac{1}{2}\frac{\cos\theta_t^n}{\sin^3\theta_t^n} U_n(\theta_t^n)\Big)\, dt. \tag{2.16}$$

Of course, we have to justify the passage from (2.15) to (2.16), by taking care of the points 0 and π. We shall see below in proposition 2.4 that π is polar for θ_t^n.

Let

$$B_n(\theta) = \frac{V_n(\theta)}{\sin\theta} + \frac{1}{2}\frac{\cos\theta}{\sin^3\theta}\, U_n(\theta). \tag{2.17}$$

By (2.13) and (2.14), we find

$$\begin{aligned} B_n(\theta) &= \frac{d-1}{\sin\theta}\Big\{a(G_n(\theta) - \cos\theta\, G_n(0)) + b\cos\theta(G_n(\theta) - G_n(0))\Big\} \\ &\quad + bG_n'(\theta) + \frac{1}{2}(a\cos\theta + b)\Xi_n'(\theta) - \frac{1}{2}\sin\theta\,\Xi_n(\theta) \\ &\quad + \frac{1}{2\sin\theta}\Big\{(ad + bd\cos\theta)\Xi_n(\theta) - (d-1)(a+b)\cos\theta\Xi_n(0)\Big\}. \end{aligned} \tag{2.18}$$

Lemma 2.1 *There exists a constant $C > 0$ independent of n such that*

$$|G_n'(\theta)| \le C\left(\theta\log\frac{2\pi}{\theta} + 2^{-n}\right), \quad \text{for all } \theta \in [0, \pi]. \tag{2.19}$$

Proof. Let $z(\theta, \varphi) = \cos\theta - \sqrt{-1}\sin\theta\cos\varphi$. Then $\frac{d}{d\theta}z(\theta,\varphi) = -\sin\theta - \sqrt{-1}\cos\theta\cos\varphi$. It is clear that

$$|z(\theta,\varphi)| \le 1, \quad \left|\frac{d}{d\theta}z(\theta,\varphi)\right| \le 1.$$

We have

$$G_n'(\theta) = \sum_{\ell=1}^{2^n}\int_0^\pi \frac{z(\theta,\varphi)^{\ell-1}}{\ell^2}\frac{d}{d\theta}z(\theta,\varphi)\,\sin^d\varphi\frac{d\varphi}{c_d}. \tag{2.20}$$

Let

$$G(\theta) = \sum_{\ell=1}^{+\infty}\frac{\tilde\gamma_\ell(\theta)}{\ell^3}. \tag{2.21}$$

By (2.20), we see that G_n' converge uniformly to G' over $[0,\pi]$. Now as in [6], using the relation $\frac{1}{\ell^2} = \int_0^{+\infty} e^{-\ell s}\, s\, ds$, we express G_n' as

$$\begin{aligned} G_n'(\theta) &= \sum_{\ell=1}^{2^n}\int_0^\pi\int_0^{+\infty} z(\theta,\varphi)^{\ell-1}e^{-\ell s}s\frac{d}{d\theta}z(\theta,\varphi)\,\sin^d\varphi\,\frac{d\varphi}{c_d} \\ &= \int_0^\pi\int_0^{+\infty}\frac{e^{-s}s\frac{d}{d\theta}z(\theta,\varphi)}{1 - z(\theta,\varphi)e^{-s}}\,\sin^d\varphi\,\frac{d\varphi}{c_d} \\ &\quad - \int_0^\pi\int_0^{+\infty}\left(z(\theta,\varphi)e^{-s}\right)^{2^n}\frac{e^{-s}s\frac{d}{d\theta}z(\theta,\varphi)}{1 - z(\theta,\varphi)e^{-s}}\,\sin^d\varphi\,\frac{d\varphi}{c_d}. \end{aligned} \tag{2.22}$$

Let I_n be the last term in (2.22). We have the estimate

$$\begin{aligned} |I_n| &\le \int_0^\pi\int_0^{+\infty}\frac{e^{-2^n s}e^{-s}s}{1 - |z(\theta,\varphi)|e^{-s}}\,\sin^d\varphi\,\frac{d\varphi}{c_d} \\ &\le \int_0^\pi\int_0^1\frac{e^{-2^n s}e^{-s}s}{1 - |z(\theta,\varphi)|e^{-s}}\,\sin^d\varphi\,\frac{d\varphi}{c_d} + \int_0^\pi\int_1^{+\infty}\frac{e^{-2^n s}e^{-s}s}{1 - |z(\theta,\varphi)|e^{-s}}\,\sin^d\varphi\,\frac{d\varphi}{c_d} \\ &= I_{n,1} + I_{n,2}. \end{aligned}$$

For $s \geq 1$, using the inequality $\dfrac{e^{-2^n s} e^{-s} s}{1-|z(\theta,\varphi)|e^{-s}} \leq e^{-2^n} \dfrac{e^{-s}s}{1-e^{-s}}$, we get

$$I_{n,2} \leq e^{-2^n} \int_1^{+\infty} \frac{e^{-s}s}{1-e^{-s}}\, ds \leq C\, 2^{-n}. \tag{2.23}$$

For $0 \leq s \leq 1$,

$$\begin{aligned}
\frac{e^{-s}s}{1-|z(\theta,\varphi)|e^{-s}} &= \frac{e^{-s}s}{1-e^{-s}\sqrt{\cos^2\theta+\sin^2\theta\cos^2\varphi}} \\
&= \frac{e^{-s}s(1+e^{-s}\sqrt{\cos^2\theta+\sin^2\theta\cos^2\varphi})}{1-(\cos^2\theta+\sin^2\theta\cos^2\varphi)e^{-2s}} \\
&\leq \frac{2se^{-s}}{1-e^{-2s}+\sin^2\theta\sin^2\varphi\, e^{-2s}} \\
&= \frac{2se^{s}}{e^{2s}-1+\sin^2\theta\sin^2\varphi} \leq \frac{2es}{2s+\sin^2\theta\sin^2\varphi} \leq e.
\end{aligned}$$

It follows that

$$I_{n,1} \leq \int_0^\pi \int_0^1 e^{-2^n s} e \frac{\sin^d\varphi}{c_d} d\varphi\, ds \leq e 2^{-n}. \tag{2.24}$$

Combining (2.23) and (2.24), we get $|I_n| \leq C\, 2^{-n}$. Now going back to (2.22) and letting $n \to +\infty$, we get that

$$G'(\theta) = \int_0^\pi \int_0^{+\infty} \frac{e^{-s}s \frac{d}{d\theta}z(\theta,\varphi)}{1-z(\theta,\varphi)e^{-s}}\ \sin^d\varphi\, \frac{d\varphi}{c_d}. \tag{2.25}$$

Moreover

$$\sup_{\theta\in[0,\pi]} |G'_n(\theta) - G'(\theta)| \leq C\, 2^{-n}.$$

Now the estimate (2.19) follows from the following main result. ∎

Theorem 2.2

$$|G'(\theta)| \leq C\,\theta \log\frac{2\pi}{\theta}, \quad \theta \in [0,\pi]. \tag{2.26}$$

Proof. We compute the term

$$\frac{\frac{d}{d\theta}z(\theta,\varphi)}{1-z(\theta,\varphi)e^{-s}} = \frac{-\sin\theta - \sqrt{-1}\cos\theta\, \cos\varphi}{1-\cos\theta\, e^{-s} + \sqrt{-1}\sin\theta\, \cos\varphi\, e^{-s}},$$

which has the real part

$$\frac{-\sin\theta + \sin\theta\cos\theta\sin^2\varphi\, e^{-s}}{(1-\cos\theta\, e^{-s})^2 + e^{-2s}\sin^2\theta\cos^2\varphi}.$$

Since G is a real valued function, it follows from (2.25) that

$$G'(\theta) = -\int_0^\pi\int_0^{+\infty} \frac{s\sin\theta(e^s - \cos\theta\sin^2\varphi)}{(e^s-\cos\theta)^2 + \sin^2\theta\cos^2\varphi}\ \sin^d\varphi\frac{d\varphi}{c_d}\,ds \tag{2.27}$$

which can be written as two parts $I_1(\theta) + I_2(\theta) = \int_0^\pi\int_0^1 + \int_0^\pi\int_1^{+\infty}$. For $0 < s \le 1$,

$$\begin{aligned}\frac{|e^s - \cos\theta\sin^2\varphi|}{(e^s-\cos\theta)^2 + \sin^2\theta\cos^2\varphi} &\le \frac{e+1}{(e^s-1)^2 + \sin^2\theta\cos^2\varphi}\\ &\le \frac{e+1}{s^2 + \sin^2\theta\cos^2\varphi}.\end{aligned}$$

But

$$\begin{aligned}\int_0^1 \frac{2s\,ds}{s^2 + \sin^2\theta\cos^2\varphi} &= \log\left(1 + \sin^2\theta\cos^2\varphi\right) - \log\left(\sin^2\theta\cos^2\varphi\right)\\ &\le \log 2 - \log\left(\sin^2\theta\cos^2\varphi\right).\end{aligned}$$

It follows that

$$|I_1(\theta)| \le \frac{e+1}{2}|\sin\theta|\Big(\log 2 + \log\frac{1}{\sin^2\theta} - \int_0^\pi \log\left(\cos^2\varphi\right)\sin^d\varphi\frac{d\varphi}{c_d}\Big).$$

Therefore there exists a constant $C > 0$ such that

$$|I_1(\theta)| \le C\,\theta\log\frac{2\pi}{\theta}, \quad \theta\in[0,\pi].$$

For the estimate of I_2, it is sufficient to remark that

$$\frac{s(e^s - \cos\theta\sin^2\varphi)}{(e^s-\cos\theta)^2 + \sin^2\theta\cos^2\varphi} \le \frac{se^s}{(e^s-1)^2}.$$

The proof of (2.26) is complete. ∎

Let $\sigma_n(\theta) = -\frac{\sqrt{U_n(\theta)}}{\sin\theta}$. We have the following key estimates.

Theorem 2.3 *There exist $N > 0$ and a constant $C > 0$ such that for all $n \ge N$*

$$\sigma_n^2(\theta) \le C\left(\theta^2\log\frac{2\pi}{\theta} + 2^{-n}\right), \tag{2.28}$$

$$-B_n(\theta) \le C\left(\theta\,\log\frac{2\pi}{\theta} + 2^{-n}\right). \tag{2.29}$$

Proof. Using (2.14), σ_n^2 has the expression

$$\begin{aligned}\sigma_n^2(\theta) = 2a\Big(&G_n(0) - G_n(\theta) + 2\sin^2(\frac{\theta}{2})\,G_n(\theta) - \sin\theta\,G_n'(\theta)\Big)\\ &+ 2b(G_n(0) - G_n(\theta)) + (a+b)\Xi_n(0).\end{aligned} \tag{2.30}$$

Since $\Xi_n(0) \leq \dfrac{\pi^2}{6}\, 2^{-n}$, (2.28) follows from (2.19). The estimate for (2.29) is much more delicate. Remark first that $\theta \to B_n(\theta)$ is smooth over $]0,\pi[$, but explodes at 0 and π. More precisely, let

$$B_{n,1}(\theta) = \frac{d-1}{\sin\theta}\Big\{a(G_n(\theta) - \cos\theta\, G_n(0)) + b\cos\theta\,(G_n(\theta) - G_n(0))\Big\},$$

$$B_{n,2}(\theta) = bG_n'(\theta) - \frac{a}{2}\sin\theta\,\Xi_n(\theta) + \frac{1}{2}(a\cos\theta + b)\Xi_n'(\theta).$$

$$B_{n,3}(\theta) = \frac{1}{2\sin\theta}\Big\{(ad + bd\cos\theta)\Xi_n(\theta) - (d-1)(a+b)\cos\theta\,\Xi_n(0)\Big\}.$$

We have:

$$|\Xi_n'(\theta)| \leq \sum_{\ell=2^n+1}^{2^{n+1}} \frac{1}{\ell^2} \leq 2^{-2n}(2^{n+1} - 2^n) = 2^{-n}.$$

From (2.19), we see that

$$|B_{n,2}(\theta)| \leq C\,(\theta\log\frac{2\pi}{\theta} + 2^{-n}), \quad \theta \in [0,\pi]. \tag{2.31}$$

When $\theta \to 0$, $B_{n,1}$ behaves as $(d-1)(a+b)G_n'(\theta)$, which is dominated by $C\,(\theta\log\frac{2\pi}{\theta} + 2^{-n})$; and $B_{n,3}$ will behave as

$$\frac{1}{2\sin\theta}(a+b)\Xi_n(0) \to +\infty.$$

When $\theta \to \pi$, $B_{n,1}$ will behave as

$$\frac{d-1}{\sin\theta}\Big[a(G_n(\pi) + G_n(0)) + b(G_n(0) - G_n(\pi))\Big] \to +\infty, \tag{2.32}$$

while $B_{n,3}$ will behave as

$$\frac{1}{2\sin\theta}\Big[d(a-b)\Xi_n(\pi) + (d-1)(a+b)\Xi_n(0)\Big]. \tag{2.33}$$

Since for $k \geq 2^n$,

$$\frac{1}{k^3} - \frac{1}{(k+1)^3} = \frac{3k^2+3k+1}{k^3(k+1)^3} \leq 6\cdot 2^{-n}\frac{1}{k^3},$$

it is clear that

$$|\Xi_n(\pi)| \leq 6\cdot 2^{-n}\Xi_n(0). \tag{2.34}$$

This together with (2.33) shows that $B_{n,3}$ will go to $+\infty$ as $\theta \to \pi$. However in order to get the uniform estimate (2.29), we have to prove that the change of signs near 0 and π will be done independently of n.

By the mean-value formula, there exists $\alpha \in]0,\theta[$ such that $\Xi_n(\theta) = \Xi_n(0) + \theta\,\Xi_n'(\alpha)$. Write $B_{n,3}$ near 0 in the form

$$B_{n,3}(\theta) = \frac{\Xi_n(0)}{2\sin\theta}\Big\{ad(1-\cos\theta) + (a+b)\cos\theta\Big\} + \frac{\theta}{2\sin\theta}(ad + bd\cos\theta)\,\Xi_n'(\alpha).$$

The last term in the above equality is bounded by $C\,2^{-n}$. The first term in the above equality is always positive. So (2.29) holds for $B_{n,3}$ near π uniformly for n. Now we shall deal with the problem at π. Replacing $\Xi_n(\theta)$ by $\Xi_n(\pi)+(\theta-\pi)\Xi_n'(\beta)$ in expression of $B_{n,3}$, we get

$$\begin{aligned} B_{n,3}(\theta) &= \frac{1}{2\sin\theta}\Big\{(ad+bd\cos\theta)\Xi_n(\pi)-(d-1)(a+b)\cos\theta\Xi_n(0)\Big\} \\ &+\frac{\theta-\pi}{2\sin\theta}\,(ad+bd\cos\theta)\Xi_n'(\beta). \end{aligned} \tag{2.35}$$

Using (2.34), we have

$$\begin{aligned} &(ad+bd\cos\theta)\Xi_n(\pi)-(d-1)(a+b)\cos\theta\Xi_n(0) \\ &\geq (a+b)\Big[-(d-1)\cos\theta-6d\cdot 2^{-n}\Big]\,\Xi_n(0) \end{aligned}.$$

Let $N\geq 1$ such that $2^{-N}\leq\dfrac{d-1}{6d}$. Then there exists $\theta_o\in]\frac{\pi}{2},\pi[$ such that

$$-(d-1)\cos\theta-6d\cdot 2^{-n}\geq 0\quad\text{for all }\theta\in[\theta_o,\pi],\ n\geq N.$$

Combining the above discussions, we arrive at

$$-B_{n,3}(\theta)\leq C\,(\theta\log\frac{2\pi}{\theta}+2^{-n}),\quad \theta\in[0,\pi].$$

For the behavior of $B_{n,1}$ near π, consider

$$\varphi(\theta)=a\big(G(\theta)-\cos\theta\,G(0)\big)+b\cos\theta\big(G(\theta)-G(0)\big).$$

We have

$$\delta=\varphi(\pi)=a(G(\pi)+G(0))+b(G(0)-G(\pi))>0.$$

There exists $\theta_0<\pi$ such that $\varphi(\theta)\geq\delta/2$ for $\theta\in[\theta_0,\pi]$. Since

$$\varphi_n(\theta)=a\big(G_n(\theta)-\cos\theta\,G_n(0)\big)+b\cos\theta\big(G_n(\theta)-G_n(0)\big)$$

converges to φ uniformly over $[0,\pi]$. There exists a big enough N such that

$$|\varphi(\theta)-\varphi_n(\theta)|\leq\delta/4,\quad\theta\in[0,\pi],\ n\geq N.$$

It follows that for $\theta\in[\theta_0,\pi]$ and $n\geq N$,

$$\varphi_n(\theta)\geq\delta/4.$$

This implies that

$$-B_{n,1}(\theta)\leq C\,(\theta\log\frac{2\pi}{\theta}+2^{-n}),\quad \theta\in[0,\pi].$$

Combining above facts with (2.18) yields (2.29). ∎

Proposition 2.4 *Let $d\geq 3$. Then the process θ_t^n does not hit the point π.*

Proof. By (2.31) and (2.32) and the expression of σ_n, we see that $\dfrac{2B_n}{\sigma_n^2}$ at the neighborhood of π behaves as

$$\frac{d-1}{\pi-\theta}\,\frac{a(G_n(\pi)+G_n(0))+b(G_n(0)-G_n(\pi))+(a+b)\Xi_n(0)/2+d(a-b)\Xi_n(\pi)/(d-1)}{a(G_n(\pi)+G_n(0))+b(G_n(0)-G_n(\pi))+(a+b)\Xi_n(0)/2}$$

which is $> \frac{1}{\pi-\theta}$ for $d\geq 3$ and for n big enough. Therefore

$$\int_{\theta_0}^{\pi}\exp\Big[\int_{\theta_0}^{y}\frac{2B_n(\theta)}{\sigma_n^2(\theta)}d\theta\Big]\,dy=+\infty.$$

By Breiman's criterion (see [6], p. 856), θ_t^n does not hit the point π. ∎

Remark 2.5 The process θ_t^n could meet the point 0.

3. Flow of homeomorphisms

Let x_t^n be the solution of the stochastic differential equation (2.1) with the initial point x, x_t^{n+1} with the initial point $y\neq x$. Set

$$\theta_n(t)=d(x_t^n,x_t^{n+1}).$$

Let $p\geq 2$. By Itô formula, we have

$$d\theta_n^p(t)=p\theta_n^{p-1}(t)\,d\theta_n(t)+\frac{1}{2}p(p-1)\theta_n^{p-2}(t)\,d\theta_n(t)\cdot d\theta_n(t).$$

By (2.16), we have

$$\begin{aligned}d\theta_n^p(t)&=p\theta_n^{p-1}(t)\sigma_n(\theta_n(t))\,dW_n(t)-p\theta_n^{p-1}(t)B_n(\theta_n(t))\,dt\\&\quad+\frac{1}{2}p(p-1)\theta_n^{p-2}(t)\sigma_n^2(\theta_n(t))\,dt.\end{aligned}\tag{3.1}$$

Using (2.28), $M_n(t)=p\displaystyle\int_0^t\theta_n^{p-1}(s)\sigma_n(\theta_n(s))\,dW_n(s)$ is a martingale. Let $u(t)=\Big(\dfrac{\theta_n(t)}{2\pi}\Big)^p$. Since the coefficient $\theta^{p-1}B_n(\theta)$ explodes at the point π, to insure that the function $t\to\mathbf{E}(u(t))$ is differentiable, we introduce for $\delta>0$,

$$\tau_\delta=\inf\{t>0,\ \theta_n(t)\geq\pi-\delta\}.$$

By proposition 2.4, we see that $\tau_\delta\uparrow+\infty$ as $\delta\downarrow 0$. Now by (2.29), we have

$$\begin{aligned}-\frac{p}{(2\pi)^p}\theta_n^{p-1}(t)B_n(\theta_n(t))&\leq Cp\frac{\theta_n^p(t)}{(2\pi)^p}\log\frac{2\pi}{\theta_n(t)}+Cp\frac{\theta_n^{p-1}}{(2\pi)^p}\cdot 2^{-n}\\&\leq C\,u(t)\log\frac{1}{u(t)}+Cp(\frac{1}{2})^{p-1}\,2^{-n}\end{aligned}$$

and in the same way,

$$\frac{1}{2}\frac{p(p-1)}{(2\pi)^p}\theta_n^{p-2}(t)\sigma_n^2(\theta_n(t))\leq C\frac{p-1}{2}\,u(t)\log\frac{1}{u(t)}+Cp(p-1)(\frac{1}{2})^{p-1}\cdot 2^{-n}.$$

Consider

$$(3.2) \qquad u_\delta(t) = u(t\wedge\tau_\delta) = \Big(\frac{\theta_n(t\wedge\tau_\delta)}{2\pi}\Big)^p.$$

Let $\varphi(t) = \mathbf{E}(u_\delta(t))$. Then $t \to \varphi(t)$ is differentiable. Using the above computations, we get

$$\begin{aligned} u_\delta(t+\eta) - u_\delta(t) \le & \frac{1}{(2\pi)^p}\Big(M_n((t+\eta)\wedge\tau_\delta) - M_n(t\wedge\tau_\delta)\Big) \\ & + C\,\frac{p+1}{2}\int_t^{t+\eta} u_\delta(s)\log\frac{1}{u_\delta(s)}\,ds + Cp^2(\frac{1}{2})^{p-1}2^{-n}\eta. \end{aligned}$$

It follows that

$$\begin{aligned} \varphi'(t) &\le C\,\frac{p+1}{2}\mathbf{E}\Big(u_\delta\log\frac{1}{u_\delta}\Big) + Cp^2(\frac{1}{2})^{p-1}\,2^{-n} \\ &\le C\,\frac{p+1}{2}\,\varphi(t)\log\frac{1}{\varphi(t)} + C\,p^2(\frac{1}{2})^{p-1}2^{-n}. \end{aligned}$$

Using the inequality (see [3])

$$-\xi\log\xi + K \le -(\xi+K)\log(\xi+K), \quad \text{for } 0<\xi\le 2^{-4}, 0<K\le 2^{-4},$$

and letting $\psi(t) = \varphi(t) + 2^{-n}$, we get

$$(3.3) \qquad \psi'(t) \le C\,\frac{p+1}{2}\psi(t)\log\frac{1}{\psi(t)}, \quad \psi(0) = d^p(x,y) + 2^{-n}.$$

It follows that

$$\varphi(t) \le \psi(t) \le (\psi(0))^{e^{-\frac{C(p+1)}{2}t}}, \quad t>0.$$

Or

$$\mathbf{E}\Big(\theta_n^p(t\wedge\tau_\delta)\Big) \le (2\pi)^p\,(\psi(0))^{e^{-\frac{C(p+1)}{2}t}}.$$

Letting $\delta\downarrow 0$ and by Fatou lemma, we get

$$(3.4) \qquad \mathbf{E}\Big(d(x_t^n, x_t^{n+1})^p\Big) \le (2\pi)^p\,(\psi(0))^{e^{-\frac{C(p+1)}{2}t}}.$$

Now write $x_t^n(x)$ for x_t^n with initial data x. Using the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $0<\alpha\le 1,\ a>0, b>0$, we get for $x\ne y$,

$$(3.5) \qquad \mathbf{E}\Big(d(x_t^n(x), x_t^{n+1}(y))^p\Big) \le (2\pi)^p\Big[2^{-ne^{-C(p+1)t/2}} + d(x,y)^{pe^{-C(p+1)t/2}}\Big].$$

By continuity, the inequality (3.5) holds for all $x, y\in S^d$.

Proposition 3.1 *Let $T>0$. There exists a constant $C>0$ independent of n such that*

$$(3.6) \qquad \mathbf{E}\Big(d(x_t^n(x), x_s^n(x))^p\Big) \le C\,|t-s|^{p/2}, \quad s,t\in[0,T].$$

Proof. Fix $u \in S^d$ and consider $\eta_t = \langle u, x_t^n \rangle$. We have

$$d\eta_t = \sum_{\ell=1}^{2^n} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} \langle u, A_{\ell,k}^1(x_t^n) \rangle \circ dB_{\ell,k}^1(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \langle u, A_{\ell,k}^2(x_t^n) \rangle \circ dB_{\ell,k}^2(t) \Big\}.$$

Let $\Lambda_t = Q_{x_t^n} u$ be the orthogonal projection of u onto $T_{x_t^n} S^d$. We have

$$\begin{aligned} d\langle u, A_{\ell,k}^1(x_t^n) \rangle &= d\langle \Lambda_t, A_{\ell,k}^1(x_t^n) \rangle \\ &= \langle \frac{D}{dt}\Lambda_t, A_{\ell,k}^1(x_t^n) \rangle + \langle \Lambda_t, (\frac{D}{dt} A_{\ell,k}^1)(x_t^n) \rangle \end{aligned}$$

where $\frac{D}{dt}$ denotes the covariant derivative along $\{x_t^n; t > 0\}$. The Itô contraction $\langle \Lambda_t, \frac{D}{dt} A_{\ell,k}^1 \rangle \cdot dB_{\ell k}^1(t)$ is given by

$$\sqrt{\frac{da_\ell}{D_{\ell,1}}} \langle \Lambda_t, \nabla^1_{A_{\ell,k}} A_{\ell,k}^1 \rangle \, dt.$$

For the computation of $\frac{D}{dt}\Lambda_t$, using expression $\Lambda_t = u - \langle u, x_t^n \rangle x_t^n$, it gives

$$d\Lambda_t \cdot dB_{\ell,k}^1(t) = -\sqrt{\frac{da_\ell}{D_{\ell,1}}} \Big[\langle u, A_{\ell,k}^1(x_t^n) \rangle x_t^n + \langle u, x_t^n \rangle A_{\ell,k}^1(x_t^n) \Big] \, dt.$$

Therefore

$$\langle \frac{D}{dt}\Lambda_t, A_{\ell,l}^1(x_t^n) \rangle \cdot dB_{\ell,k}^1(t) = -\sqrt{\frac{da_\ell}{D_{\ell,1}}} \langle u, x_t^n \rangle \langle A_{\ell,k}^1(x_t^n), A_{\ell,k}^1(x_t^n) \rangle \, dt.$$

Let M_t be the martingale part of η_t and V_t be the drift part. Then V_t has the expression

$$\begin{aligned} V_t = \frac{1}{2} \sum_{\ell=1}^{2^n} \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \Big\{ \langle u, (\nabla_{A_{\ell k}^1} A_{\ell k}^1)(x_t^n) \rangle - \langle u, x_t^n \rangle \langle A_{\ell k}^1(x_t^n), A_{\ell k}^1(x_t^n) \rangle \Big\} \\ + \frac{1}{2} \sum_{\ell=1}^{2^n} \frac{db_\ell}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \Big\{ \langle u, (\nabla_{A_{\ell k}^2} A_{\ell k}^2)(x_t^n) \rangle - \langle u, x_t^n \rangle \langle A_{\ell k}^2(x_t^n), A_{\ell k}^2(x_t^n) \rangle \Big\}. \end{aligned}$$

By $(A.6)$, we get

$$V_t = -\frac{1}{2} \sum_{\ell=1}^{2^n} d(a_\ell + b_\ell) \langle u, x_t^n \rangle = -\frac{1}{2} d(a+b) G_n(0) \eta_t. \tag{3.7}$$

Now the quadratic variation $dM_t \cdot dM_t$ is given by

$$dM_t \cdot dM_t = \sum_{\ell=1}^{2^n} \Big\{ \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \langle u, A_{\ell k}^1(x_t^n) \rangle^2 + \frac{db_\ell}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \langle u, A_{\ell k}^2(x_t^n) \rangle^2 \Big\} \, dt.$$

Using $(A.13)$, we obtain

$$dM_t \cdot dM_t = \sum_{\ell=1}^{2^n} (a_\ell + b_\ell) \sin^2\theta = (a+b) G_n(0)(1 - \cos^2\theta),$$

where θ is the angle between a and x_t^n: $\cos\theta = \langle u, x_t^n \rangle = \eta_t$.

Now by Bürkhölder inequality, for $t_1 > t_2$,

$$\begin{aligned}\mathbf{E}((M_{t_1} - M_{t_2})^p) &\leq C_p\, \mathbf{E}\Big[\Big(\int_{t_2}^{t_1} (a+b)^2 G_n(0)^2 (1-\eta_s^2)^2\, ds\Big)^{p/2}\Big] \\ &\leq C_p\, (a+b)^p G(0)^p |t_1 - t_2|^{p/2}.\end{aligned}$$

Combining with (3.7), there exists a constant C_p independent of n such that

$$\mathbf{E}(|\eta_{t_1} - \eta_{t_2}|^p) \leq C_p\, |t_1 - t_2|^{p/2},$$

or

$$\mathbf{E}(|\langle u, x_{t_1}^n - x_{t_2}^n \rangle|^p) \leq C_p\, |t_1 - t_2|^{p/2}.$$

Using $|x_{t_1} - x_{t_2}|^2 = \sum_{i=1}^{d+1} |\langle u_i, x_{t_1}^n - x_{t_2}^n \rangle|^2$, where $\{u_i,\ i = 1, \cdots d+1\}$ is an orthonormal basis of $\mathbf{R}^{d+1}$, we get the estimate (3.6). ∎

Theorem 3.2 *Let $x \in S^d$ and $T > 0$ be fixed. Then almost surely, as $n \to +\infty$,*

(3.8) $\qquad x_t^n(x)$ converges uniformly with respect to $t \in [0, T]$.

Proof. Seeing $x_t^n(x)$ as an element of $\mathbf{R}^{d+1}$, and using (3.5) and (3.6), we have for $s, t \in [0, T]$,

$$\mathbf{E}\Big(|x_t^n(x) - x_s^{n+1}(x)|^p\Big) \leq C_p\, \Big(|t-s|^{p/2} + 2^{-(n+1)\delta}\Big). \tag{3.9}$$

where $\delta = e^{-C(p+1)T/2}$. Let $c = 2^{-\delta/p} < 1$. Define $\alpha_0 = 0, \alpha_n = \sum_{k=1}^{n} c^k$. Then $\alpha_\infty = \lim_{n\to+\infty} \alpha_n$ is finite. Define

$$X(s,t,x) = x_t^n(x) \frac{\alpha_{n+1} - s}{\alpha_{n+1} - \alpha_n} + x_t^{n+1}(x) \frac{s - \alpha_n}{\alpha_{n+1} - \alpha_n}, \qquad s \in [\alpha_n, \alpha_{n+1}].$$

Using (3.9), we get for $(s_1, s_2) \in [0, \alpha_\infty]^2$ and $(t_1, t_2) \in [0, T]^2$,

$$\mathbf{E}\Big(|X(s_1, t_1, x) - X(s_2, t_2, x)|^p\Big) \leq C_p \Big(|s_1 - s_2|^p + |t_1 - t_2|^{p/2}\Big). \tag{3.10}$$

By Kolmogoroff modification theorem, X has a continuous version $\tilde{X}$. But we have

$$x_t^n(x) = X(\alpha_n, t, x) = \tilde{X}(\alpha_n, t).$$

This last term converges uniformly with respect to $t \in [0, T]$, which finishes the proof. ∎

Let $\{x_t(x),\ t \in [0, T]\}$ be the uniform limit of $\{x_t^n(x),\ t \in [0, T]\}$.

Theorem 3.3 $\{x_t(x),\ t \geq 0\}$ *is the unique solution of the equation:*

$$dx_t = \sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t)\circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t)\circ dB^2_{\ell,k}(t)\Big\} \tag{3.11}$$

Proof. We first show that $\{x_t(x),\ t \geq 0\}$ satisfies the equation (3.11). It suffices to show that for any $u \in S^d$,

$$d\big\langle u, x_t\big\rangle = \sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(x_t)\big\rangle\circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(x_t)\big\rangle\circ dB^2_{\ell,k}(t)\Big\} \tag{3.12}$$

Set $\eta_t = \big\langle u, x_t\big\rangle, \eta^n_t = \big\langle u, x^n_t\big\rangle$. From the proof of Proposition 3.1 that

$$\eta^n_t = \big\langle u, x\big\rangle + M^n_t + \int_0^t V^n_s ds, \tag{3.13}$$

where

$$dM^n_t = \sum_{\ell=1}^{2^n}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(x^n_t)\big\rangle\, dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(x^n_t)\big\rangle dB^2_{\ell,k}(t)\Big\}$$

and $V^n_t = -\dfrac{1}{2}d(a+b)G_n(0)\eta^n_t$. Put

$$dM_t = \sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(x_t)\big\rangle\, dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(x_t)\big\rangle dB^2_{\ell,k}(t)\Big\},$$

and $V_t = -\dfrac{1}{2}d(a+b)G(0)\eta_t$. Clearly, $V^n_t \to V_t$. Fix any positive integer N_0, when n is big enough M^n_t can be split into two parts:

$$\begin{aligned} M^n_t = &\sum_{\ell=1}^{N_0}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\int_0^t\big\langle u, A^1_{\ell,k}(x^n_s)\big\rangle\, dB^1_{\ell,k}(s) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\int_0^t\big\langle u, A^2_{\ell,k}(x^n_s)\big\rangle dB^2_{\ell,k}(s)\Big\} \\ &+ \sum_{\ell=N_0+1}^{2^n}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\int_0^t\big\langle u, A^1_{\ell,k}(x^n_s)\big\rangle\, dB^1_{\ell,k}(s) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\int_0^t\big\langle u, A^2_{\ell,k}(x^n_s)\big\rangle dB^2_{\ell,k}(s)\Big\} \\ &:= M^{(n,1)}_t + M^{(n,2)}_t. \end{aligned}$$

Similarly,

$$\begin{aligned} M_t = &\sum_{\ell=1}^{N_0}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\int_0^t\big\langle u, A^1_{\ell,k}(x_s)\big\rangle\, dB^1_{\ell,k}(s) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\int_0^t\big\langle u, A^2_{\ell,k}(x_s)\big\rangle dB^2_{\ell,k}(s)\Big\} \\ &+ \sum_{\ell=N_0+1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\int_0^t\big\langle u, A^1_{\ell,k}(x_s)\big\rangle\, dB^1_{\ell,k}(s) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\int_0^t\big\langle u, A^2_{\ell,k}(x_s)\big\rangle dB^2_{\ell,k}(s)\Big\} \end{aligned}$$

$$:= M_t^{(1)} + M_t^{(2)}.$$

By $(A.13)$, as in the proof of Proposition 3.1 we have

$$\mathbf{E}\big[\big(M_t^{(n,2)}\big)^2\big] \le \sum_{\ell=N_0+1}^{\infty} (a_\ell + b_\ell), \quad \mathbf{E}\big[\big(M_t^{(2)}\big)^2\big] \le \sum_{\ell=N_0+1}^{\infty} (a_\ell + b_\ell),$$

both of which tend to zero uniformly with respect to n as $N_0 \to \infty$. On the other hand, for fixed N_0,

$$M_t^{(n,1)} \to M_t^{(1)}, \quad \text{as } n \to \infty.$$

Combining above arguments with the triangle inequality, we conclude

$$M_t^n \to M_t, \quad \text{as } n \to \infty.$$

Letting $n \to \infty$ in (3.13) proves (3.12).

Next we prove the pathwise uniqueness for the equation (3.11). Let $x_t, y_t, t \ge 0$ be two solutions to equation (3.11) such that $x_0 = y_0$. Fix $u \in S^d$. Seeing x_t and y_t as elements in $\mathbf{R}^{d+1}$, consider $\eta_t = \big\langle u, x_t - y_t \big\rangle$. Put

$$\begin{aligned} M_t = \sum_{\ell=1}^{+\infty} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} \int_0^t \big\langle u, A^1_{\ell,k}(x_s) - A^1_{\ell,k}(y_s) \big\rangle \, dB^1_{\ell,k}(s) \\ + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \int_0^t \big\langle u, A^2_{\ell,k}(x_s) - A^2_{\ell,k}(y_s) \big\rangle dB^2_{\ell,k}(s) \Big\}. \end{aligned}$$

We have $d\eta_t = dM_t - \dfrac{d}{2}(a+b)G(0)\eta_t \, dt$ and

$$\begin{aligned} d\Theta_t := dM_t \cdot dM_t = \sum_{\ell=1}^{+\infty} \Big\{ \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \big| \big\langle u, A^1_{\ell,k}(x_s) - A^1_{\ell,k}(y_s) \big\rangle \big|^2 \\ + \frac{db_\ell}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \big| \big\langle u, A^2_{\ell,k}(x_s) - A^2_{\ell,k}(y_s) \big\rangle \big|^2 \Big\} \, dt. \end{aligned} \tag{3.14}$$

Hence

$$d\eta_t^2 = 2\eta_t \, dM_t - d(a+b)G(0)\eta_t^2 \, dt + d\Theta_t. \tag{3.15}$$

Let $\xi_t = |x_t - y_t|^2$. Using $(A.10)$ and $(A.11)$, we have

$$\begin{aligned} &\frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \big| A^1_{\ell,k}(x_s) - A^1_{\ell,k}(y_s) \big|^2 \\ &= \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \Big\{ |A^1_{\ell,k}(x_t)|^2 + |A^1_{\ell,k}(y_t)|^2 - 2\big\langle A^1_{\ell,k}(x_t), A^1_{\ell,k}(y_t) \big\rangle \Big\} \\ &= 2da_\ell - 2a_\ell \Big((d - 1 + \cos^2\theta_t)\gamma_\ell(\cos\theta_t) - \cos\theta_t \sin^2\theta_t \gamma'_\ell(\cos\theta_t) \Big), \end{aligned}$$

and in the same way

$$\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,2}}\left|A^2_{\ell,k}(x_s)-A^2_{\ell,k}(y_s)\right|^2$$
$$=2db_\ell-2b_\ell\Big(d\cos\theta_t\gamma_\ell(\cos\theta_t)-\sin^2\theta_t\gamma'_\ell(\cos\theta_t)\Big)$$

where $\theta_t=d(x_t,y_t)$. It follows from (3.15) that

$$d\xi_t=\text{Martingale}-d(a+b)G(0)\xi_t\,dt+B(\theta_t)\,dt \tag{3.16}$$

with

$$\begin{aligned}B(\theta)=2\Big[&daG(0)-a\Big((d-1+\cos^2\theta)G(\theta)+\cos\theta\sin\theta G'(\theta)\Big)\\&+dbG(0)-b\big(d\cos\theta G(\theta)+\sin\theta\,G'(\theta)\big)\Big].\end{aligned} \tag{3.17}$$

Let $\varphi(t)=\mathbf{E}(\xi_t)$. By (3.16), we get

$$\varphi'(t)=-d(a+b)G(0)\varphi(t)+\mathbf{E}(B(\theta_t)).$$

By (2.26), $|B(\theta)|\le C\,\theta^2\log\dfrac{2\pi}{\theta}$. It follows, according to (2.3), that

$$\varphi'(t)\le C\varphi(t)\log\frac{1}{\varphi(t)},\quad \varphi(0)=0$$

which implies that $\varphi(t)=0$. Therefore for each t, $x_t=y_t$. The two processes are indistinguishable. ∎

Theorem 3.4 $\{x_t(x),\ t\in[0,T]\}$ *has a version* $\tilde{x}_t(x)$ *such that almost surely, for all* $t\in[0,T]$,

$$x\to\tilde{x}_t(x)\quad\text{is a homeomorphism of } S^d.$$

Proof. Let $X(s,t,x)$ be defined as in the proof of Theorem 3.2. Using (3.5) and (3.6), we have

$$\mathbf{E}\Big(|X(s_1,t_1,x)-X(s_2,t_2,y)|^p\Big)\le C_p\Big(|s_1-s_2|^p+|t_1-t_2|^{p/2}+d(x,y)^{p\delta}\Big). \tag{3.18}$$

Recall that $\delta=e^{-C(p+1)T/2}$. Therefore it exists a small $T_0>0$ and a big enough $p\ge 2$, so that we can apply the Kolmogoroff modification theorem. In this way, $x^n_t(x)$ converges uniformly to $x_t(x)$, with respect to $(t,x)\in[0,T_0]\times S^d$. Remark that T_0 is not dependent of the particular Brownian motion, but of its law. Now consider the family of the Brownian motion $\{(\theta_{T_0}B)(t),\ t\ge 0\}$, where $(\theta_{T_0}B)^i_{\ell k}(t)=B^i_{\ell k}(t+T_0)-B^i_{\ell k}(T_0)$ and $i=1,2$. Denote explicitly $x^n_t(x,\omega)$ the solution of (2.1), with respect to the given family of Brownian motion $\{B^i_{\ell k}(t),\ t\ge 0\}$, and $x^n_t(x,\theta_{T_0}\omega)$ with respect to $\{(\theta_{T_0}B)(t),\ t\ge 0\}$. Then we have

$$x^n_{t+T_0}(x,\omega)=x^n_t(x^n_{T_0}(x,\omega),(\theta_{T_0}\omega)). \tag{3.19}$$

Now letting $n \to +\infty$ in (3.19), the right hand side tends to $x_t(x_{T_0}(x,\omega),(\theta_{T_0}\omega))$ uniformly with respect to $(t,x) \in [0,T_0] \times S^d$, while the left hand side tends to $x_{t+T_0}(x,\omega)$. It follows that we have a continuous version

$$(t,x) \to x_t(x,\omega) \quad \text{over } [0,2T_0] \times S^d.$$

Proceeding in this way, we get a continuous version $(t,x) \to x_t(x,\omega)$ over $[0,+\infty[\times S^d$. Now we shall prove that for any t given, $x \to x_t(x,\omega)$ is a homeomorphism of S^d. First consider $T \in [0,T_0]$ and define $\{B^T(t),\ t \in [0,T]\}$ by

$$B^{T,i}_{\ell k}(t) = B^i_{\ell k}(T-t) - B^i_{\ell k}(T),$$

which are time reversed independent Brownian motions. Let $x^n_t(x,\omega^T)$ be the solution of (2.1), but with $\{B^T(t),\ t \in [0,T]\}$. It is well known that

$$x^n_{T-t}(x,\omega) = x^n_t(x^n_T(x,\omega),\omega^T), \quad x^n_{T-t}(x,\omega^T) = x^n_t(x_T(x,\omega^T),\omega).$$

Letting $n \to +\infty$ in the above equality, we get

$$x_{T-t}(x,\omega) = x_t(x_T(x,\omega),\omega^T), \quad x_{T-t}(x,\omega^T) = x_t(x_T(x,\omega^T),\omega).$$

Taking $t = T$, we see that the inverse of $x \to x_T(x,\omega)$ is $x_T(x,\omega^T)$. Now we complete the proof by using the method in [1] or in [3, p.174] to find a common version such that for all $t > 0$, $x \to x_t(x,\omega)$ is a homeomorphism. ∎

4. Appendix: Eigen-vector fields

In this section, we shall collect some notations and useful properties of the eigen vector fields of Δ for readers convenience. Here we follow closely the exposition in [9]. Fix the point $P_o = (0,\cdots,0,1) \in S^d$. The group SO$(d{+}1)$ acts transitively on S^d. The subgroup leaving P_o fixed is SO(d) so that $S^d = \mathrm{SO}(d{+}1)/\mathrm{SO}(d)$. Let χ_g be the action of $g \in \mathrm{SO}(d{+}1)$ on S^d, $\chi_g : x \to gx$. We have

$$d\chi_h(P_o) : T_{P_o}S^d \to T_{P_o}S^d \qquad \text{for} \quad h \in H$$

where $T_P S^d$ denotes the tangent space at the point $P \in S^d$. Therefore $U : h \to d\chi_h(P_o)$ is a representation of SO(d); it is irreducible when $d \geq 3$. Let $\{\varepsilon_1,\cdots,\varepsilon_d,\varepsilon_{d+1}\}$ be the canonical basis of $\mathbf{R}^{d+1}$ with $P_o = \varepsilon_{d+1} \in S^d$. For $1 \leq i \leq d$, consider $\varphi_i(t) = \sin t\,\varepsilon_i + \cos t\,\varepsilon_{d+1}$. Then φ_i is a curve on S^d starting from P_o, having ε_i as the tangent vector at P_o. In this way, we shall identify $T_{P_o}S^d$ with $\mathbf{R}^d$

Let $\{T^\lambda;\ \lambda \in \Lambda\}$ be the family of equivalence class of unitary irreducible representations of SO$(d{+}1)$.

Definition A.1 *We say that T^λ contains a copy of U if there exists a subspace W_λ of the base space V_λ of T^λ, which is invariant by all $\{T^\lambda(h);\ h \in H\}$ and such that the restriction of T^λ to W_λ is equivalent to U.*

Denote by Λ_o such sub-family of T^λ having this property. By theory of representation (see [9], [10]),

$$\{T^\lambda;\ \lambda \in \Lambda_o\} = \{T^{(d+1)\ell}, Q^{(d+1)\ell};\ \ell \geq 1\};$$

the base space of $T^{(d+1)\ell}$ is the space $\mathcal{H}_{d+1,\ell}$ of homogeneous harmonic polynomials on $\mathbf{R}^{d+1}$ of degree $\ell \geq 1$ and the base space of $Q^{(d+1)\ell}$ is the space of 2-differential forms $\mathcal{F}_{d+1,\ell}$ considered in [9]. Let's describe the subspace W_ℓ of $\mathcal{H}_{d+1,\ell}$ and $\hat{W}_\ell$ of $\mathcal{F}_{d+1,\ell}$, which are invariant by $h \in \mathrm{SO}(d)$. Let $\mathcal{R}_{d+1,\ell}$ be the space of homogeneous polynomials on $\mathbf{R}^{d+1}$ of degree ℓ, equipped with the inner product $< P, Q >= \int_{S^d} P(x)Q(x)\,dx$. Let H be the orthogonal projection from $\mathcal{R}^{d+1,\ell}$ onto $\mathcal{H}_{d+1,\ell}$. Then $W_\ell = H\big(x_{d+1}^{\ell-1} \cdot \mathcal{H}_{d,1}\big)$. Set $\Theta_i(x) = C_i\, H(x_{d+1}^{\ell-1} x_i)$ for $i = 1, \cdots, d$, where C_i are chosen so that we have an orthonormal basis of W_ℓ. The space $\hat{W}_\ell$ is the vector space spanned by $\{\hat{\Theta}_i = C_i\, H\big(dx_{d+1} \wedge d\Theta_i\big);\ i = 1, \cdots, d\}$. Completing $\{\Theta_i;\ i = 1, \cdots, d\}$ and $\{\hat{\Theta}_i;\ i = 1, \cdots, d\}$ into an orthonormal basis of $\mathcal{H}_{d+1,\ell}$ and $\mathcal{F}_{d+1,\ell}$, we denote by (T_{ij}^ℓ) and (Q_{ij}^ℓ) the associated matrices. For further discussions on this topic, we refer to the book [11]. The following result is taken from [9].

Proposition A.2 *We have for* $1 \leq i, j \leq d,\ g \in \mathrm{SO}(d+1)$,

$$(A.1) \qquad T_{ij}^\ell(g) = \gamma_\ell(t)\, g_{ij} + \gamma_\ell'(t)\, g_{i,d+1} g_{d+1,j}\ ,$$

$$(A.2) \qquad Q_{ij}^\ell(g) = \Big(t\gamma_\ell(t) - \frac{1-t^2}{d-1}\gamma_\ell'(t)\Big)\, g_{ij} + \Big(-\gamma_\ell(t) - \frac{t}{d-1}\gamma_\ell'(t)\Big)\, g_{i,d+1} g_{d+1,j}$$

where $t = g_{d+1,d+1}$ *and*

$$(A.3) \qquad \gamma_\ell(\cos\theta) = \int_0^\pi (\cos\theta - \sqrt{-1}\sin\theta\cos\varphi)^{\ell-1}\ \sin^d\varphi \frac{d\varphi}{c_d}$$

with $c_d = \displaystyle\int_0^\pi \sin^d\varphi\, d\varphi.$

Remark that $\gamma_\ell(t)$ is real and $|\gamma_\ell(t)| \leq 1$. For $g = h \in \mathrm{SO}(d)$, $t = g_{d+1,d+1} = 1,\ g_{i,d+1} = 0$. By $(A.1)$, we see that $T_{ij}^\ell(h) = h_{ij}$. On the other hand, by the choice of basis,

$$T_{ki}^\ell(h) = 0 \quad \text{if} \quad k > d \quad \text{and} \quad 1 \leq i \leq d.$$

The same results hold for Q^ℓ. Now using the Peter-Weyl theorem, we get the spectral expansion for eigen vector fields $\{A_{\ell k}^1, A_{\ell k}^2\}$:

$$(A.4) \qquad A_{\ell k}^1(gP_0) = \sqrt{\frac{D_{\ell,1}}{d}} \sum_{i=1}^d T_{ki}^\ell(g)\, d\chi_g(P_0)\varepsilon_i, \quad A_{\ell k}^2(gP_0) = \sqrt{\frac{D_{\ell,2}}{d}} \sum_{i=1}^d Q_{ki}^\ell(g)\, d\chi_g(P_0)\varepsilon_i,$$

where $D_{\ell,1} = \dim(\mathcal{H}_{d+1,\ell})$ and $D_{\ell,2} = \dim(\mathcal{F}_{d+1,\ell})$. Now we will compute the covariant derivative of a vector field A on S^d. Let $Q_x : \mathbf{R}^{d+1} \to T_x S^d$ be the orthogonal projection. Let $u \in T_x S^d$ and $\{\eta_s\}_{s\geq 0}$ be the curve on S^d such that $\eta_0 = x$ and $u = \Big\{\dfrac{d\eta_s}{ds}\Big\}_{s=0}$. Then

$$(A.5) \qquad (\nabla_u A)(x) = Q_x \Big\{\frac{dA_{\eta_s}}{ds}\Big\}_{s=0}$$

Proposition A.3 *We have*

$$(A.6) \qquad \sum_{k=1}^{D_{\ell,i}} \nabla_{A_{\ell k}^i} A_{\ell k}^i = 0, \quad \text{for } i = 1, 2.$$

Proof. We shall only prove $(A.6)$ for $i=1$ because of the similarity. Fix a point $gP_0 \in S^d$. Let $E_j = d\chi_g(P_o)\varepsilon_j$. Then E_j is a tangent vector at the point gP_0.

$$(A.7)\qquad (\nabla_{A^1_{\ell,k}} A^1_{\ell k})(gP_0) = \sqrt{\frac{D_{\ell,1}}{d}} \sum_{j=1}^{d} T^{\ell}_{kj}(g)(\nabla_{E_j} A^1_{\ell k})(gP_0).$$

Let $g_{j,d+1}(s) \in \mathrm{SO}(d+1)$ defined by $g_{j,d+1}(s)\varepsilon_\alpha = \varepsilon_\alpha$ for $\alpha \neq j, \alpha \neq d+1$ and

$$(A.8)\qquad g_{j,d+1}(s)\varepsilon_j = \cos s\, \varepsilon_j - \sin s\, \varepsilon_{d+1},\ g_{j,d+1}(s)\varepsilon_{d+1} = \sin s\, \varepsilon_j + \cos s\, \varepsilon_{d+1}.$$

Then $g_{j,d+1}(0) = \mathrm{Id}$, $\Big\{\frac{d}{ds} g_{j,d+1}(s)P_o\Big\}_{s=0} = \varepsilon_j$ and $E_j = \Big\{\frac{d}{ds} g g_{j,d+1}(s)P_o\Big\}_{s=0}$. Moreover

$$\Big\{\frac{d}{ds} g_{j,d+1}(s)\varepsilon_i\Big\}_{s=0} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \Big\{\frac{d}{ds} g_{j,d+1}(s)\varepsilon_j\Big\}_{s=0} = -\varepsilon_{d+1};$$

therefore

$$(A.9)\qquad Q_{gP_0}\Big\{\frac{d}{ds} g_{j,d+1}(s)\varepsilon_j\Big\}_{s=0} = -Q_{gP_0}(g\varepsilon_{d+1}) = 0.$$

According to $(A.4)$,

$$A^1_{\ell k}(g g_{j,d+1}(s)P_0) = \sqrt{\frac{D_{\ell,1}}{d}} \sum_{i=1}^{d} \sum_{\beta=1}^{D_{\ell,1}} T^{\ell}_{k\beta}(g) T^{\ell}_{\beta i}(g_{j,d+1}(s))(g g_{j,d+1}(s)\varepsilon_i).$$

Taking the derivative with respect to s in two sides and using $(A.5)$ and $(A.9)$, we get

$$(\nabla_{E_j} A^1_{\ell k})(gP_0) = \sqrt{\frac{D_{\ell,1}}{d}} \sum_{i=1}^{d} \sum_{\beta=1}^{D_{\ell,1}} T^{\ell}_{k\beta}(g) \Big\{\frac{d}{ds} T^{\ell}_{\beta i}(g_{j,d+1}(s))\Big\}_{s=0} E_i.$$

Therefore, by expression $(A.7)$,

$$\begin{aligned}\sum_{k=1}^{D_{\ell,1}} \nabla_{A^1_{\ell k}} A^1_{\ell k} &= \frac{D_{\ell,1}}{d} \sum_{i,j=1}^{d} \sum_{k,\beta=1}^{D_{\ell,1}} T^{\ell}_{kj}(g) T^{\ell}_{k\beta}(g) \Big\{\frac{d}{ds} T^{\ell}_{\beta i}(g_{j,d+1}(s))\Big\}_{s=0} E_i \\ &= \frac{D_{\ell,1}}{d} \sum_{i,j=1}^{d} \Big\{\frac{d}{ds} T^{\ell}_{ji}(g_{j,d+1}(s))\Big\}_{s=0} E_i.\end{aligned}$$

By $(A.8)$ the term $g_{d+1,d+1}$ in formula $(A.1)$ and $(A.2)$ is equal to $\cos s$; the term $g_{ji} = 0$, $g_{d+1,i} = 0$ for $i \neq j$ and $g_{jj} = \cos s$, $g_{j,d+1} g_{d+1,j} = -\sin^2 s$. Therefore $\Big\{\frac{d}{ds} T^{\ell}_{ji}(g_{j,d+1}(s))\Big\}_{s=0} = 0$. We prove $(A.6)$. ∎

Proposition A.4 *Let $x, y \in S^d$ and θ the angle between x, y. Then*

$$(A.10)\qquad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \langle A^1_{\ell k}(x), A^1_{\ell k}(y)\rangle_{\mathbf{R}^{d+1}} = (d-1+\cos^2\theta)\gamma_\ell(\cos\theta) - \cos\theta \sin^2\theta\, \gamma'_\ell(\cos\theta),$$

$$(A.11) \qquad \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \big\langle A^2_{\ell k}(x), A^2_{\ell k}(y) \big\rangle_{\mathbf{R}^{d+1}} = d\cos\theta\gamma_\ell(\cos\theta) - \sin^2\theta\gamma'_\ell(\cos\theta)$$

Proof. In order to prove $(A.10)$ and $(A.11)$ in a unified way, we denote by $A_{\ell k}$ the vector field $\sqrt{\frac{D_{\ell,1}}{d}} A^1_{\ell k}$ or $\sqrt{\frac{D_{\ell,2}}{d}} A^2_{\ell k}$ and by Z^ℓ_{ki} the term T^ℓ_{ki} or Q^ℓ_{ki} and D_ℓ the associated dimension. Let $x = g_o P_o,\ y = g_o P_1$. We have $\big\langle x, y \big\rangle = \big\langle P_o, P_1 \big\rangle = \cos\theta$. Then

$$(A.12) \qquad A_{\ell k}(gx) = \sum_{\alpha=1}^{D_\ell} Z^\ell_{k\alpha}(g) \cdot d\chi_g(x) A_{\ell\alpha}(x) \quad \text{for} \quad g \in SO(d+1).$$

Using $(A.12)$, we have $\sum_{k=1}^{D_\ell} \big\langle A_{\ell k}(x), A_{\ell k}(y) \big\rangle = \sum_{\alpha=1}^{D_\ell} \big\langle A_{\ell\alpha}(P_o), A_{\ell\alpha}(P_1) \big\rangle$. Up to a rotation in SO(d), we can suppose that P_1 is in the plan spanned by $\{\varepsilon_d, \varepsilon_{d+1}\}$. Let $P_1 = g(\theta)P_o$ with $g(\theta) \in$ SO(d+1) given by $g(\theta)\varepsilon_i = \varepsilon_i$ for $1 \le i \le d-1$ and

$$g(\theta)\varepsilon_d = \cos\theta\,\varepsilon_d - \sin\theta\,\varepsilon_{d+1},\ \ g(\theta)\varepsilon_{d+1} = \sin\theta\,\varepsilon_d + \cos\theta\,\varepsilon_{d+1}.$$

We have $A_{\ell k}(P_o) = \sum_{j=1}^d Z^\ell_{kj}(e)\varepsilon_j = \varepsilon_k$ if $1 \le k \le d$ and $A_{\ell k}(P_o) = 0$ otherwise. On other hand, $A_{\ell k}(P_1) = \sum_{j=1}^{d} g(\theta)\varepsilon_j Z^\ell_{kj}(g(\theta))$. Therefore

$$\big\langle A_{\ell k}(P_o), A_{\ell k}(P_1) \big\rangle = \sum_{j=1}^{d} \big\langle g(\theta)\varepsilon_j, \varepsilon_k \big\rangle\, Z^\ell_{kj}(g(\theta)), \quad 1 \le k \le d$$

which is equal to $Z^\ell_{kk}(g(\theta))$ for $1 \le k \le d-1$ and to $\cos\theta\, Z^\ell_{dd}(g(\theta))$ for $k = d$. Hence

$$\sum_{k=1}^{D_\ell} \big\langle A_{\ell k}(x), A_{\ell k}(y) \big\rangle = \sum_{k=1}^{d-1} Z^\ell_{kk}(g(\theta)) + \cos\theta\, Z^\ell_{dd}(g(\theta)).$$

Now using the explicit formula $(A.1)$ and $(A.2)$, we get $(A.10)$ and $(A.11)$. ∎

Proposition A.5 *Let $x, y \in S^d$ and θ the angle between them. Then*

$$(A.13) \qquad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \big\langle A^1_{\ell k}(x), y \big\rangle^2 = \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \big\langle A^2_{\ell k}(x), y \big\rangle^2 = \sin^2\theta$$

$$(A.14) \quad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \Big(\big\langle A^1_{\ell k}(x),\ y \big\rangle + \big\langle A^1_{\ell k}(y),\ x \big\rangle \Big)^2 = 2\sin^2\theta \Big[1 - \cos\theta\,\gamma_\ell(\cos\theta) + \sin^2(\theta)\,\gamma'_\ell(\cos\theta) \Big],$$

$$(A.15) \qquad \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \Big(\big\langle A^2_{\ell k}(x),\ y \big\rangle + \big\langle A^2_{\ell k}(y),\ x \big\rangle \Big)^2 = 2\sin^2\theta \Big[1 - \gamma_\ell(\cos\theta) \Big].$$

Proof. In a similar way, we get the results. ∎

References

[1] J.M. Bismut: *Mécanique aléatoire*, Lect. notes in Math., Vol. 866 (1981), Springer-Verlag, Berlin, New-York.
[2] K.D. Elworthy: *Stochastic differential equations on manifolds*, London Mathematical Society. Lect. Note Series 70 (1982), Cambridge University Press, Cambridge.
[3] S. Fang: Canonical Brownian motion on the diffeomorphism group of the circle, *J. Funct. Anal.*, **196** (2002), p.162-179.
[4] S. Fang and T. Zhang: A study of a class of stochastic differential equations with non-Lipschitzian coefficients, to appear in *Prob. Theory Rel. Fields*.
[5] H. Kunita: *Stochastic flows*, Cambridge Univ. Press, Cambridge, UK, 1988.
[6] Y. LeJan and O. Raimond: Integration of Brownian vector fields, *Ann. of Prob.*, **30** (2002), p. 826-873.
[7] P. Malliavin: The canonical diffusion above the difeomorphism group of the circle, *C. R. Acad. Sci.*, **329**, p. 325-329.
[8] P. Malliavin: *Stochastic Analysis*, Grunlehren des Math. **313**, Springer, Berlin, 1997.
[9] O. Raimond: Flots browniens isotropes sur la sphère, *Ann. Inst. H. Poincaré*, **35** (1999), p. 313-354.
[10] E. Stein: *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Study 63 (1970), Princeton.
[11] N.J. Vilenkin: *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, Paris, 1969.