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Asymptotic matching constraints for a boundary-layer flow of a power-law fluid

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We reconsider the three-dimensional boundary-layer flow of a power-law (Ostwald–de Waele) rheology fluid, driven by the rotation of an infinite rotating plane in an otherwise stationary system. Here we address the problem for both shear-thinning and shear-thickening fluids and show that there are some fundamental issues regarding the application of power-law models in a boundary-layer context that have not been mentioned in previous discussions. For shear-thickening fluids, the leading-order boundary-layer equations are shown to have no suitable decaying behaviour in the far field, and the only solutions that exist are necessarily non-differentiable at a critical location and of 'finite thickness'. Higher-order effects are shown to regularize the singularity at the critical location. In the shear-thinning case, the boundary-layer solutions are shown to possess algebraic decay to a free-stream flow. This case is known from the existing literature; however here we shall emphasize the complexity of applying such solutions to a global flow, describing why they are in general inappropriate in a traditional boundary-layer context. Furthermore, previously noted difficulties for fluids that are highly shear thinning are also shown to be associated with the imposition of incorrect assumptions regarding the nature of the far-field flow. Based on Newtonian results, we anticipate the presence of non-uniqueness and through accurate numerical solution of the leading-order boundary-layer equations we locate several such solutions.

1. Introduction

The flow induced by the rotation of an infinite impermeable plane submerged in a viscous incompressible Newtonian fluid is described by a long-standing and classical (exact) solution of the Navier–Stokes equations. The solutions and their extensions have found many uses in industrial devices and processes, and they provide a test bed for the development of three-dimensional cross-flow instability theories and have applications to geophysics in Ekman pumping models and the decay of geostrophic flows.

A similarity reduction of the fully nonlinear governing partial differential equations was provided by von Kármán (1921) for the flow induced by a rotating disk in a stationary fluid. This was later extended by Bödewadt (1940) to the problem of a rigidly rotating fluid above a stationary disk, with the obvious later continuation of the solutions across the intervening range of the parameter space for a rotating disk in a rotating fluid.

In terms of the characteristics of the solutions to the Newtonian rotating-disk equations, it is well known that an enormously rich and detailed structure exists, spanned by the ratio of the rotation of the fluid at infinity to that of the disk. The interested reader is referred to the review article of Zandbergen & Dijsktra (1987) for details; however we note here that infinite degrees of non-uniqueness, singular solutions (at a critical ratio of rotation rates), Hopf bifurcations, and regions of non-existence of solutions are all present. Furthermore, finite-time singularities have been shown to occur in the unsteady development of some initial-value problems by Bodonyi & Stewartson (1977). More recent extensions have shown further non-uniqueness, with a class of exact non-axisymmetric flows also existing, see Hewitt & Duck (2000).

In terms of non-Newtonian fluids, the rotating-disk configuration has attracted some attention in the context of electro-rheological flows as a fundamental boundary layer that has potential industrial/mechanical applications; see Burgess & Wilson (1996). Further interest has been directed towards this flow purely from its fluid-mechanical interest, the long history of Newtonian investigations driving investigators to extend the von Kármán solution to power-law rheology fluids. The analysis of Burgess & Wilson (1996) is notionally for the flow between two parallel planes;† however the large Reynolds number approximation that is applied in their work leads to the analysis being essentially that for an isolated plane (a low Reynolds number expansion is also presented that does take into account the finite axial range).

Recently, Andersson, de Korte & Meland (2001) (subsequently referred to herein as AKM) have revisited the problem of the flow of a non-Newtonian fluid over a rotating disk, which was first considered by Mitschka (1964). The motivation of AKM was to address the 'accuracy' of the numerical solutions presented by Mitschka (1964) in the region of highly shear-thinning fluids and to extend the numerical results to a greater range of shear-thickening fluids. As we shall show, there are some issues to be addressed both in the assumptions and the analysis of the work of AKM.

The flat-plate boundary layer of a non-Newtonian fluids has also received considerable attention. The earliest work appears to be that of Acrivos, Shah & Petersen (1960) who presented results for a Blasius-like flow of a fluid with a constitutive relation given by the classical power-law model. This paper has attracted considerable attention because of its prediction of a 'finite thickness' self-similar boundary layer (on the boundary-layer length scale) for dilatant (or shear-thickening) fluids. Zhizhin & Ufimstev (1977), Pavlov, Fedotov & Shakhorin (1981) and Filipussi, Gratton & Minotti (2001) employed a phase-plane approach to investigate the mathematical structure of the solution to the nonlinear ordinary differential equation which describes the flow. The aforementioned works came to the conclusion that the boundary-layer flow of a dilatant fluid is localized to within a well-defined region. This work was subsequently extended; Denier & Dabrowski (2004) demonstrated that this loose definition of a 'finite width' boundary layer leads to a non-uniqueness in the solution. Denier & Dabrowski (2004) argued that this was due to the inability of the power-law model to adequately capture the behaviour of a dilatant fluid within the boundary layer. An alternative constitutive model for a dilatant fluid, based upon that of Carreau (see Bird, Armstrong & Hassager 1973), was shown to remove all ambiguity from the definition of the boundary layer. This model, however, does not allow similarity solutions (as is the case for the power-law model) and so a full numerical treatment of the governing equations was necessary.

Our aim here is to provide a detailed investigation of the flow of a power-law fluid over a rotating disk with the aim of elucidating some of the numerical and

† In the work presented in Burgess & Wilson (1996) we note that the primary equations (12)–(15) are incorrect and require the sign inverting in the factor $(n-1)/(1+n)$.

asymptotic issues present in the implicit assumption that such boundary-layer flows are appropriate. By ‘appropriate’ we mean that the boundary layer can be matched in a self-consistent manner to an external flow. This role of the boundary layer in the broader context of the global flow appears to have been ignored in the literature up until now. One might speculate that this oversight may have arisen from the fact that, in the Newtonian case, the flow is described by an exact solution and therefore no such matching constraints are necessary for a full solution of the Navier–Stokes equations. However as we shall see, an Ostwald–de Waele power-law fluid no longer allows an exact similarity reduction and the boundary layer must be viewed in the broader context of the full field equations.

There are a number of experimental studies that are relevant to the current work. Although they focused upon flat-plate boundary-layer flows, Wu & Thompson (1996) demonstrated that the boundary-layer equations provide an accurate (and useful) model for the flow of shear-thinning fluids even when the Reynolds is not asymptotically large (for example when the Reynolds number is $O(10^3)$). Hoyt & Fabula (1964) present some of the earliest qualitative experiments on the problem of viscous drag reduction through the use of fluid additives. The majority of their experiments were undertaken using a rotating disk apparatus operated over a wide range of Reynolds numbers (typically greater than $O(10^5)$). Although predominantly concerned with measuring torque reduction in the presence of polymer additives their results clearly demonstrate that large-Reynolds-number flows of shear-thinning fluids (such as guar gum at relatively high weight parts per million) are common and easily achievable experimentally. Interestingly there is no attempt in this early work to model the high-Reynolds-number flow of shear-thinning fluids in rotating disk flows and so no comparison between theoretical and experimental results are available. It is therefore difficult to draw any firm conclusion from Hoyt & Fabula (1964) with regard the effect of the shear-thinning nature of the fluid/polymer mix on the dynamics of the flow.

In § 2 of this paper we formulate the boundary-layer equations for a power-law fluid. In § 3 we shall provide a full and detailed analysis of the leading-order boundary-layer equations, paying particular attention to the asymptotic behaviour of the solution in the far field. We shall show that the results of AKM and Mitschka (1964) are in fact only accurate over a range of shear-thinning fluids and miss the fact that their solutions for shear-thickening fluids are non-differentiable at a critical location in the layer. Furthermore, the inconsistencies of the numerical work of AKM and Mitschka (1964) for highly shear-thinning fluids are again shown to be associated with an incorrect consideration of the far-field behaviour. Throughout § 3 we shall comment on the appropriateness of the boundary-layer solutions obtained in the broader context of the global flow field, that is, on the boundary-layer free-stream matching process. Of particular significance will be the role played by boundary layers with algebraic decay in the free stream, of which more details and discussion will be given in section three. Our final section will draw some conclusions regarding this and other boundary-layer flows with power-law rheology models.

2. Formulation

We consider the flow of a non-Newtonian fluid whose constitutive relation is governed by the Ostwald-de Waele power-law model, so that the stress tensor is given by

$$\tau_{ij} = k(e_{lm}e_{lm})^{(n-1)/2}e_{ij}, \tag{2.1}$$

where k is the fluid consistency constant, e_{ij} is the rate-of-strain tensor and n is a real parameter, with $n > 1$, $n < 1$ corresponding to shear-thickening and shear-thinning fluids respectively. The classical Newtonian viscosity law is obtained by setting $n = 1$.

We seek an axisymmetric steady solution of the non-dimensionalized Navier–Stokes equations (extended to a power-law rheology model) in a cylindrical-polar coordinate system (r, θ, Z), with associated velocity field $(U(r, Z), V(r, Z), W(r, Z))$ and pressure $P(r, Z)$. In anticipation of a boundary-layer structure arising on the rotating plane we shall introduce the re-scalings

$$z = ZRe^{1/(n+1)} = O(1), \tag{2.2}$$

and

$$(U(r, Z), V(r, Z), W(r, Z)) = \left(u(r, z), v(r, z), w(r, z)/Re^{1/(n+1)}\right), P(r, Z) = p(r, z), \tag{2.3}$$

where the modified Reynolds number is defined as $Re = \rho \bar{U}^{2-n}\bar{L}^n/k$, with $\bar{U}$ and $\bar{L}$ being natural velocity and length scales respectively.

Substitution of (2.2) and (2.3) yields the following system of equations:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.4a}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + Re^{-2/(n+1)}\left\{\frac{2}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial r}\right) - 2\mu\frac{u}{r^2}\right\}, \tag{2.4b}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) + Re^{-2/(n+1)}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\mu r^3\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right), \tag{2.4c}$$

$$\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -Re^{2/(n+1)}\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial z}\right) + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) + Re^{-2/(n+1)}\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial w}{\partial r}\right), \tag{2.4d}$$

where the dimensionless viscosity function μ is defined as

$$\mu = \left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + Re^{-4/(n+1)}\left(\frac{\partial w}{\partial r}\right)^2 + Re^{-2/(n+1)}\left[2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{2u^2}{r^2} + \left(\frac{\partial w}{\partial z}\right)^2 + \left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right)^2 + 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\right]\right\}^{(n-1)/2}. \tag{2.5}$$

As noted by AKM, when $n \neq 1$ these equations do not admit the exact solution first determined by von Kármán (1921). However, in the limit of large Reynolds number some progress can be made as there is a readily available self-similar form which is a simple generalization of that presented by von Kármán (1921) to the case of a power-law fluid. This is the direction chosen by Mitschka (1964) and AKM and we follow the same approach here, albeit somewhat more formally.

We shall proceed with a familiar boundary-layer methodology, assuming that $Re \gg 1$. Traditionally, this leads to the neglect of the viscous terms in an outer region (here defined by $Z = O(1)$). This outer flow is then required to match asymptotically with a corresponding boundary-layer region (here defined by $z = O(1)$). In this power-law model, to neglect the leading-order viscous terms in an outer region we also require the additional constraint that $\mu Re^{-2/(n+1)}$ remains small uniformly in this region. Clearly this latter constraint places bounds on the level of shear for a given Reynolds number.

In the boundary layer region we shall seek a solution to (2.4) in the form

$$u(r, Z) = u_0(r, z) + Re^{-2/(n+1)} u_1(r, z) + \cdots, \tag{2.6a}$$

$$v(r, Z) = v_0(r, z) + Re^{-2/(n+1)} v_1(r, z) + \cdots, \tag{2.6b}$$

$$w(r, Z) = w_0(r, z) + Re^{-2/(n+1)} w_1(r, z) + \cdots, \tag{2.6c}$$

$$p(r, Z) = p_0(r) + Re^{-2/(n+1)} p_1(r, z) + \cdots, \tag{2.6d}$$

$$\mu(r, Z) = \mu_0(r, z) + Re^{-2/(n+1)} \mu_1(r, z) + \cdots, \tag{2.6e}$$

where, in the absence of a far-field rotation (since we shall consider only the flow of a rotating plane in an otherwise stationary fluid), the leading-order centrifugal pressure balance is $p_0 \equiv 0$.

Thus the boundary-layer equations at lowest order are

$$u_0 \frac{\partial u_0}{\partial r} - \frac{v_0^2}{r} + w_0 \frac{\partial u_0}{\partial z} = \frac{\partial}{\partial z}\left(\mu_0 \frac{\partial u_0}{\partial z}\right),$$

$$u_0 \frac{\partial v_0}{\partial r} + \frac{u_0 v_0}{r} + w_0 \frac{\partial v_0}{\partial z} = \frac{\partial}{\partial z}\left(\mu_0 \frac{\partial v_0}{\partial z}\right),$$

$$u_0 \frac{\partial w_0}{\partial r} + w_0 \frac{\partial w_0}{\partial z} = -\frac{\partial p_1}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu_0 r \frac{\partial u_0}{\partial z}\right) + 2\frac{\partial}{\partial z}\left(\mu_0 \frac{\partial w_0}{\partial z}\right),$$

where the leading-order viscosity function within the boundary layer is defined by

$$\mu_0 = \left\{\left(\frac{\partial u_0}{\partial z}\right)^2 + \left(\frac{\partial v_0}{\partial z}\right)^2\right\}^{(n-1)/2}. \tag{2.8}$$

We note that, in contrast to the statement of AKM (p. 80), this is a standard boundary-layer system, with the transverse variation of the pressure being determined *a posteriori* once the velocity field is known.

As noted by Mitschka (1964), the leading-order boundary-layer equations admit a similarity solution in the form

$$u_0 = rF(\eta), \quad v_0 = rG(\eta), \quad w_0 = r^{(n-1)/(n+1)} H(\eta),$$

where the similarity variable η is given by

$$\eta = z r^{(1-n)/(n+1)}.$$

The amplitude functions F, G and H satisfy

$$F^2 - G^2 + \left(H + \frac{1-n}{n+1}\eta F\right) F' = \left[(F'^2 + G'^2)^{(n-1)/2} F'\right]', \tag{2.9a}$$

$$2FG + \left(H + \frac{1-n}{n+1}\eta F\right) G' = \left[(F'^2 + G'^2)^{(n-1)/2} G'\right]', \tag{2.9b}$$

$$H' = -2F - \frac{1-n}{n+1}\eta F', \tag{2.9c}$$

subject to the boundary conditions

$$F = 0, \quad G = 1, \quad H = 0 \qquad \text{on} \quad \eta = 0, \tag{2.9d}$$

and

$$F \to 0, \quad G \to 0 \qquad \text{as} \quad \eta \to \infty. \tag{2.9e}$$

Previous investigations of these similarity forms have neglected the pressure correction term p_1 which can be written as $p_1(r, z) = r^{2(n-1)/(n+1)}Q(\eta)$; the equation for Q, together with its large-η asymptotic form, is given in the Appendix. We note that the expression for p_1 is singular at $r = 0$ for shear-thinning fluids whose index lies in the range $0 < n < 1$. The leading-order viscosity function μ_0 is also infinite at $r = 0$ (for $0 < n < 1$). This behaviour is also encountered in spin-coating flows of non-Newtonian fluids (see Jenekhe 1984; Jenekhe & Schuldt 1984; Slavtchev, Miladinova & Kalitzova-Kurteva 1996). In such applications it is customary to replace the power-law model by a different rheological model such as the Carreau model (see Bird *et al.* 1973) which predicts a finite viscosity at a point of zero shear. We will return to this point later.

The qualitative physical interpretation of the solution in the Newtonian case ($n = 1$) is straightforward and holds as n deviates from unity. As the disk spins, viscous effects near to the boundary force a corresponding rotation to develop in the fluid. In the absence of a leading-order radial pressure gradient, there must be an associated radial flow and therefore continuity of mass requires an axial flow. The spinning boundary therefore acts like a centrifugal fan, with a flow that is uniformly directed towards the boundary, together with a radial outflow in the viscous-dominated region. In general applications this flow towards the boundary is often a significant physical feature (the reader is referred to the large literature on spin-up flows, for example). As we shall see here, in the case of shear-thinning fluids, one must pay careful attention to the implicit assumption that there is a finite mass transported in the layer.

The system of nonlinear ODEs (2.9) requires a full numerical solution. A number of methods are available, the simplest being a shooting method which employs a fourth-order Runge–Kutta quadrature routine to implement the numerical integration of the equations and Newton iteration to determine the unknowns $F'(0)$ and $G'(0)$. Such a solution procedure has been implemented by AKM where they demonstrated that there is an ambiguity in the solution for fluid index $n < \frac{1}{2}$ and, for $n > 1$, there is a reduction in the value of $-H'(\infty)$ for increasing n. They obtained their results by applying the far-field boundary conditions (2.9*e*) at some large value of $\eta = \eta_\infty$. However, it is well-known that calculations that are based upon this form of truncated boundary condition can produce inaccurate and sometime spurious solutions; see, for example, Lentini & Keller (1980). It is therefore necessary to employ the correct asymptotic form for the boundary conditions. We now turn our attention to this matter, with the aim of shedding some light on the results obtained by AKM and the general question of the boundary-layer flow of a power-law fluid.

3. The asymptotic form of the solution for $\eta \gg 1$

Before presenting results regarding the numerical solution of system (2.9) we first consider the details of the asymptotic form for the functions F, G and H in the limit $\eta \to \infty$. These are crucial in order to develop accurate solutions of system (2.9).

3.1. *Shear-thinning fluids: the case* $\frac{1}{2} < n < 1$

To develop the asymptotic forms it proves useful to first integrate the equation for H to give

$$H = -\frac{(3n+1)}{(n+1)} \int_0^\eta F \, \mathrm{d}\eta - \frac{1-n}{n+1} \eta F. \tag{3.1}$$

The function H is bounded as $\eta \to \infty$ provided F is $o(\eta^{-1})$ as $\eta \to \infty$ (this condition guarantees that the definite integral appearing on the right-hand-side of (3.1) converges as $\eta \to \infty$). To proceed let us therefore assume that $H \to H_\infty$ as $\eta \to \infty$. In this case the dominant terms on the left-hand-side of equations (2.9*a,b*) are $H_\infty F'$ and $H_\infty G'$ respectively. Thus in the limit $\eta \to \infty$ we have

$$H_\infty F' \sim \left[(F'^2 + G'^2)^{(n-1)/2} F'\right]', \quad H_\infty G' \sim \left[(F'^2 + G'^2)^{(n-1)/2} G'\right]'.$$

These are most easily solved by setting $(F', G') = R(\cos\theta, \sin\theta)$ (or equivalently, by considering the complex quantity $\Xi(\eta) = F(\eta) + \mathrm{i}G(\eta)$), where the amplitude R and phase θ are given by

$$H_\infty R^{2-n} = nR', \quad \theta' = 0.$$

Integrating the equation for R gives the leading-order asymptotic forms for F and G as

$$(F, G) \sim \eta^{n/(n-1)} \quad \text{as} \quad \eta \to \infty. \tag{3.2}$$

These predict asymptotic decay for F and G provided the fluid index $n < 1$. The exponent in this asymptotic form for F and G has a singularity at $n = 1$ indicating that in this case the decay is faster than algebraic. The case $n = 1$, corresponding to a Newtonian fluid, was solved by Cochran (1934) to give $(F, G) \sim \exp(H_\infty \eta)$, where $H_\infty = -2\int_0^\infty F \, \mathrm{d}\eta < 0$.

The asymptotic form for F and G is obtained on the assumption that H is bounded in the limit $\eta \to \infty$. We therefore require that the exponent appearing in (3.2) satisfies

$$\frac{n}{n-1} < -1.$$

Expression (3.2) is only valid for fluid indices in the range $\frac{1}{2} < n < 1$, and $H \to \infty$ as $\eta \to \infty$ for fluid indices in the range $0 < n \leqslant \frac{1}{2}$. This simple analysis explains why AKM were unable to determine a unique value for the von Kármán pumping rate, H_∞, as no such value exists for fluid index $n < \frac{1}{2}$. The consequences of this for the behaviour of the axial velocity W are considered in § 3.2 and § 3.3.

The large-η form for F and G can be used to determine asymptotic boundary conditions for use in the numerical integration of system (2.9). These are most readily applied by differentiating the asymptotic condition (3.2) to arrive at

$$(F', G') = \frac{n}{\eta(n-1)}(F, G) \quad \text{as} \quad \eta \to \infty. \tag{3.3}$$

These mixed boundary conditions can then be used at some suitably large value of $\eta = \eta_\infty \gg 1$ to ensure correct algebraic decay.

At this point it is worth emphasizing that the function H has a rather slow decay to its far-field value. Indeed, expanding the integral appearing in (3.1) in the limit $\eta \to \infty$, and making use of expression $F = A\eta^{n/(n-1)} + \cdots$, we find that

$$H = H_\infty + \frac{(n+2)(1-n)}{(n+1)(2n-1)} A\eta^{(2n-1)/(n-1)} + \cdots \quad \text{as} \quad \eta \to \infty, \tag{3.4}$$

n	η_∞	$-H(\eta_\infty)$ Present	AKM
0.9	55	0.9698	0.969
0.8	100	1.0957	1.089
0.7	175	1.3051	–
0.6	645	1.7329	1.364

TABLE 1. Comparison of values of H_∞ to those of AKM.

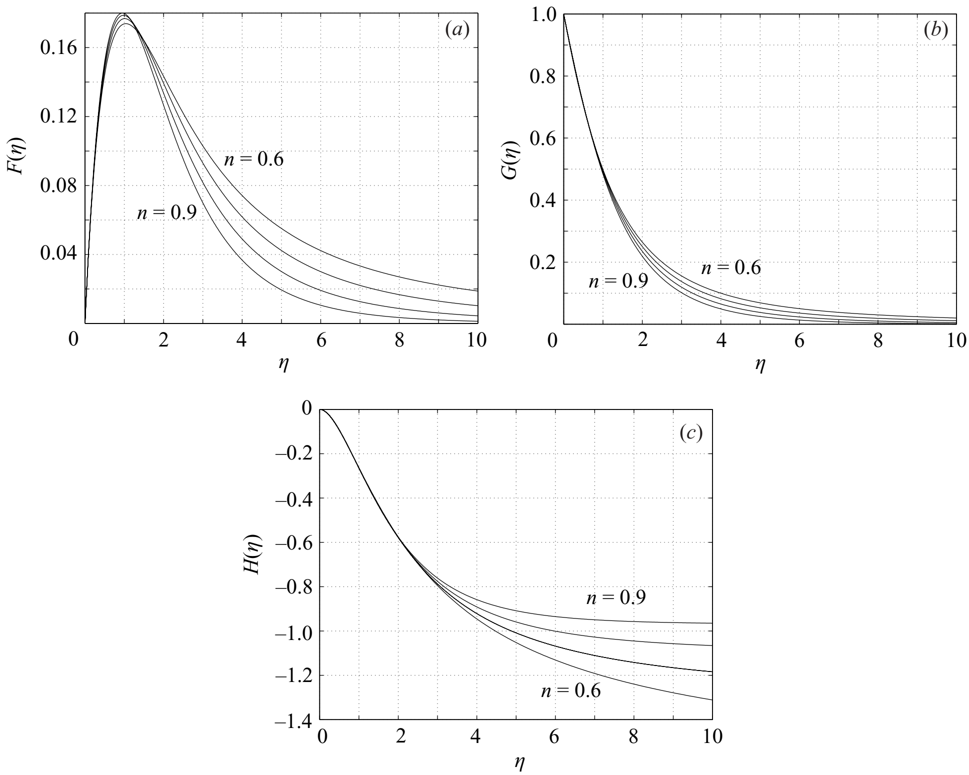

FIGURE 1. Plot of $F(\eta)$, $G(\eta)$ and $H(\eta)$ versus η for $n = 0.9, 0.8, 0.7, 0.6$. The η-axis has been truncated; the value of η_∞ employed for each calculation is given in table 1.

where

$$H_\infty = -\frac{(3n+1)}{(n+1)} \int_0^\infty F \, \mathrm{d}\eta,$$

which is finite (and negative). The rate of decay of H to its limiting value of H_∞ therefore depends strongly upon the fluid index n. For $n = 0.9$ this decay is $O(\eta^{-8})$ whereas for $n = 0.6$ we have decay like $O(\eta^{-1/2})$.

Numerical solutions of (2.9) are presented in table 1 and figure 1. These results were obtained using a fourth-order Runge–Kutta quadrature routine. The procedure iterates on the unknowns $F'(0)$ and $G'(0)$ using simple Newton iteration until the Neumann boundary conditions (3.3) imposed at some η_∞ were satisfied to within some desired tolerance (typically $O(10^{-8})$). To ensure results which are converged with respect to η_∞ we varied this quantity, keeping the step-size constant, until the results presented in figure 1 had converged to (at least) three decimal places. The value of η_∞ employed in each calculation is given in table 1. As a final test, the same calculations were performed using the same formulation, but applying a central-differencing scheme.

Comparison with the results reported by AKM are given in table 1. The most noticeable discrepancy between our results and those reported earlier is in the value of H_∞ for the smaller values of fluid index, this discrepancy being significant for $n = 0.6$. Again, this is directly attributable to the slow decay of the function H to its asymptotic value and highlights the need to ensure that the correct asymptotic boundary conditions are employed in such calculations. Given the comments above regarding divergence of the quantity H, we present no results or comparisons for $n < 1/2$.

3.1.1. *Matching with the outer flow*

The role of algebraically decaying boundary-layer flows in Newtonian fluids has been described by a number of researchers; the interested reader is directed to the recent discussion and references provided by Hewitt & Duck (2002). The essential feature is that, given algebraic decay (say, $z^{-\gamma}$ for large z) of the velocity components within the boundary layer, it is not possible to match such solutions to an inviscid, vorticity-free external flow since this would require solutions to Laplace's equation that behave like $Z^{-\gamma}$ for $Z \ll 1$. This is contrary to the well-known properties of harmonic functions if such behaviour is required over a finite streamwise extent.

We must point out that the situation here is somewhat more complex. The arguments outlined above, which rule out algebraic decay of the field components as one leaves the boundary layer, do not apply if the external flow is instead a full Euler problem (see Brown & Stewartson 1965) or, more likely in this instance, if one cannot neglect the viscous terms.

Simple examination of the viscosity function (2.8) shows that when $\eta = O(Re^{1/(n+1)})$, then all terms depending on u and v in the viscosity coefficient are formally of equal magnitude. Therefore, the outer flow remains fully viscous as one might expect since the fluid in this case is shear thinning. However, the presence of these additional terms in the viscosity function will necessarily remove the possibility of any self-similar solution. Thus the only possible formal application of this similarity form is as a large-radius ($r \gg 1$) (inner) asymptotic form for a more general, non-self-similar, solution of the full field equations.

3.2. *Shear-thinning fluids: the case* $0 < n < \frac{1}{2}$

The results of the previous subsection provide some insight into the nature of the asymptotic form of solution for the case of a fluid index $n < \frac{1}{2}$. The guiding feature is the fact that the function H is no longer bounded in the limit $\eta \to \infty$. This coupled with the severe nonlinearity appearing on the right-hand side of equations (3.2*a*,*b*) suggests that the function F still decays algebraically, as required by the boundary conditions, but not rapidly enough to ensure that the integral (3.1) for H converges as $\eta \to \infty$. These observations suggest seeking an asymptotic form for F and G as

$$F = A\eta^\alpha + \cdots, \quad G = B\eta^\beta + \cdots, \tag{3.5a}$$

where $-1 < \alpha < 0$ and $\beta \leqslant \alpha$ and the ellipsis indicates smaller terms omitted from the series.

We first make use of (3.1) to give

$$H = \frac{n-3}{2n-1} A\eta^{\alpha+1} + \cdots . \tag{3.5b}$$

By substituting these expansions into (3.1) and applying a simple dominant balance argument between the terms involving F on the left-hand side and the higher-order derivatives we find that the exponent α is given by

$$\alpha = -\frac{(n+1)}{2-n}.$$

The requirement that $-1 < \alpha$ gives the constraint $n < \frac{1}{2}$. The index β is undetermined at this stage. Assuming $\beta = \alpha$ leads to the result that $B \equiv 0$ thus suggesting that the function G decays faster than $O(\eta^{\alpha})$. Therefore we assume that $\beta < \alpha$. In this case equations (2.9*a*,*b*) become, to leading order

$$F^2 + \left(H + \frac{1-n}{n+1}\eta F \right) F' = \left[(F'^2)^{(n-1)/2} F' \right]',$$

$$2FG + \left(H + \frac{1-n}{n+1}\eta F \right) G' = \left[(F'^2)^{(n-1)/2} G' \right]'.$$

Substituting the forms (3.5*a*) for F and G into these equations yields, after some simplification,

$$A^2 + \left[\frac{3-n}{2n-1} + \frac{1-n}{n+1} \right] \alpha A^2 = n\alpha(\alpha-1)A[(\alpha A)^2]^{(n-1)/2}, \tag{3.6a}$$

$$2A + \left[\frac{3-n}{2n-1} + \frac{1-n}{n+1} \right] \beta A = \beta(\alpha+\beta-1)[(\alpha A)^2]^{(n-1)/2}. \tag{3.6b}$$

System (3.6) serves to determine the index β and the amplitude A (the amplitude B is determined at higher order in the expansion; due to the complexity of the equations we do not pursue this here). The index β is determined from the solution of the equation

$$\beta(\alpha+\beta+1) + \alpha\beta(\beta-\alpha) \left[\frac{3-n}{2n-1} + \frac{1-n}{n+1} \right] - 2n\alpha(\alpha-1) = 0.$$

This quadratic equation for β has two real solutions, only one of which is negative:

$$\beta = -\frac{7n^2 + 2 + \sqrt{n^4 + 4n^2 + 4 - 120n^3 + 48n}}{2(n+2)(2-n)}$$

A plot of the exponents β and α and the amplitude A versus n is given in figure 2 confirming that $-1 < \beta < \alpha < 0$.

3.2.1. *Matching with the outer flow*

For the case of a power-law fluid with index n in the range $0 < n < \frac{1}{2}$ the solution for the boundary-layer flow over a rotating disk predicts an infinite axial velocity. This then induces an infinite radial pressure component in the limit $\eta \to \infty$ (see the Appendix). Turning our attention to the original scalings for the field components (2.6) we see that the algebraically growing form for p_1 suggests that the structure noted above will become disordered when the terms p_0 and $Re^{-2/(n+1)} p_1(r, Z)$ are comparable. This occurs when $\eta = O(Re^{\sigma})$ where $\sigma = (2-n)/[(n+1)(1-2n)]$. Note

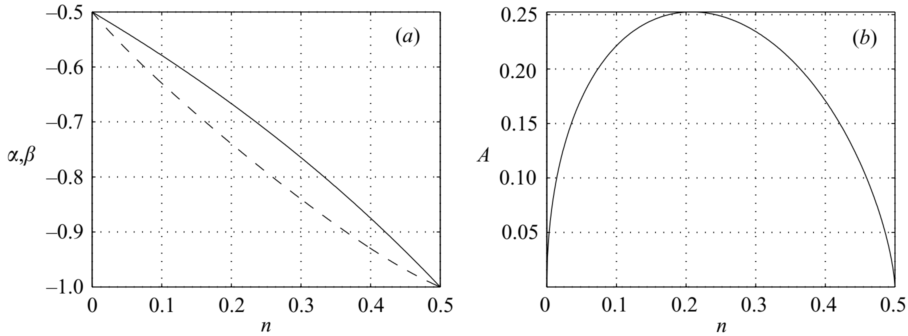

FIGURE 2. Plot of (*a*) the indices α (solid) and β (dashed) and (*b*) the amplitude A versus n.

however that $Re^{\sigma} \gg Re^{1/(n+1)}$, and so the boundary-layer thickness, and thus the expansion for the pressure remains well-ordered within the boundary layer.

However, a more serious issue arises when we consider the problem of matching the boundary-layer flow, in which the velocity components have asymptotic forms (3.5), to a suitable outer flow. In this case, the algebraically growing nature of the axial velocity, coupled with the fact that F and G decay algebraically, suggests that there will be a new distinguished limit that will occur when the $O(Re^{-2/(n+1)})$ terms appearing in the viscosity function (2.5) are of the same magnitude as those that were retained in the leading-order viscosity function. It is a relatively simple task to show that this occurs when $\eta = O(Re^{1/(n+1)})$, that is, in the outer flow all terms appearing in the viscosity function are of equal importance. Thus the outer flow is necessarily viscous.

Furthermore, it is easy to see that, in terms of the outer coordinate Z (where $Z = zRe^{-1/(n+1)}$, and z is the boundary-layer coordinate), the axial flow in the outer region must be $O(Re^{-1/(2-n)})$ and behave like $Z^{(1-2n)/(2-n)}$ for small Z. Similarly, one must have a comparable radial and azimuthal flow, also of $O(Re^{-1/(2-n)})$, that behave like $Z^{(n+1)/(n-2)}$. Thus in the outer flow, all terms are comparable and the solution must be fully viscous and non-self-similar. This result contradicts the traditional view of a boundary layer, namely that viscosity is only important within the boundary layer.

3.3. *Shear-thinning fluids: the degenerate case* $n = \frac{1}{2}$

The analysis presented in § 3.1 and § 3.2 is not valid for a fluid index $n = \frac{1}{2}$. In this case, the results above suggest that a new far-field asymptotic form develops for H. We briefly turn our attention to this case here.

In order to develop the correct asymptotic form for F, G and H when $n = \frac{1}{2}$ we first note that the analysis of § 3.2 suggests that $F \sim \eta^{-1}$ as $\eta \to \infty$. This in turn suggests that $H \sim \ln \eta$ as $\eta \to \infty$. This however, ignores the fact that both exponents α and β are equal to -1 for $n = \frac{1}{2}$ and that the amplitude A appearing in (3.6) also tends to zero as $n \to \frac{1}{2}$. This suggests that the functions F, G and H have a degenerate large-η asymptotic form. To determine the correct far-field expansion for the field variables we first note that, when $n = \frac{1}{2}$, the only sensible dominant balance (that is, one that gives a non-trivial result for the amplitudes of F and G) is found from balancing the viscous derivatives with the terms HF' and HG' in (2.9*a*,*b*). In this case we find that

$$F = \frac{A}{\eta}(\ln\eta)^{-2/3} + \cdots, \quad G = \frac{A}{\eta}(\ln\eta)^{-2/3} + \cdots, \quad H = -5A(\ln\eta)^{1/3} + \cdots,$$

where

$$A^3 = \frac{1}{25\sqrt{2}}.$$

3.4. *Shear-thickening fluids: the case $n > 1$*

The case of a shear-thickening fluid (in which the index satisfies $n > 1$) warrants some detailed comment. The results presented in § 3.1 and § 3.2 show that the asymptotic form of the solutions of (2.9) can be derived on the basis that either (i) H is bounded as $\eta \to \infty$ or (ii) H is unbounded as $\eta \to \infty$. For the first of these a dominant balance yields $F \sim \eta^{\alpha}$ where $\alpha = n/(n-1)$ and is only valid for $\frac{1}{2} < n < 1$. The second case, with H unbounded, yields $F \sim \eta^{\alpha}$ where now $\alpha = (n+1)/(n-2)$ and $\alpha > -1$ for the assumption that H is unbounded to hold. Thus this second asymptotic form is only valid for $0 < n < \frac{1}{2}$.

However both ÅKM and Mitschka (1964) present results for a fluid index with $n > 1$. We believe that these results are spurious in that, although they appear to satisfy the far-field boundary conditions, closer inspection shows they represent solutions to the system (2.9) over a truncated domain. These truncated solutions are non-differentiable at a critical location within the boundary layer and therefore defeat any naive finite-difference numerical scheme applied over an arbitrary computational domain $\eta \in [0, \eta_\infty]$. As we shall see, the only appropriate manner to compute these states (without allowing for non-differentiable solutions) is by treating the governing equations as a free-boundary problem.

An analogous problem has been encountered in the boundary-layer flow of a power-law fluid, of index $n > 1$, over a semi-infinite flat plate and here we have a similar phenomenon, albeit for a three-dimensional flow. Acrivos *et al.* (1960) proposed that the boundary layer for $n > 1$ had 'finite extent', an explanation that has since been taken up by a number of other authors. This however overlooks the real requirement of the solutions of the boundary-layer equations: not that they simply satisfy some far-field boundary conditions but that they do so asymptotically and smoothly.

In order to explore this possibility (and formulate a free-boundary approach) we shall first re-write system (2.9) as an autonomous system by defining

$$\hat{H} = H + \frac{1-n}{n+1}\eta F.$$

We then have

$$F^2 - G^2 + \hat{H}F' = \left[(F'^2 + G'^2)^{(n-1)/2}F'\right]', \tag{3.7a}$$

$$2FG + \hat{H}G' = \left[(F'^2 + G'^2)^{(n-1)/2}G'\right]', \tag{3.7b}$$

$$\hat{H}' = \frac{3n+1}{n+1}F. \tag{3.7c}$$

We seek solutions to this system that satisfy

$$F = F' = G = G' = 0 \quad \text{at} \quad \eta = \eta_c, \tag{3.7d}$$

and

$$F = 0, \quad G = 1, \quad \hat{H} = 0 \quad \text{on} \quad \eta = 0. \tag{3.7e}$$

At first glance the system (3.7) seems over specified; however one of the boundary conditions (3.7*d*) (for example $G = 0$) is redundant and follows directly from the governing equations. The rationale behind the introduction of these conditions lies in it leading to a zero of the viscosity function at a critical location, around which it is possible to construct a local solution.

As the system is autonomous we can readily apply a shift of coordinates and define the origin to be at the critical point within the boundary layer,

$$y = \eta_c - \eta,$$

and set $\tilde{H} = -\hat{H}$. The system (3.7) is invariant under this transformation, but the boundary conditions become

$$F = F' = G = G' = 0 \quad \text{at} \quad y = 0, \tag{3.8a}$$

and

$$F = 0, \quad G = 1, \quad \hat{H} = 0 \quad \text{on} \quad y = \eta_c. \tag{3.8b}$$

In order to start the calculation at $y = 0$ it is first necessary to develop the small-y asymptotic solution of (3.7). This is most readily done by assuming that as $y \to 0$

$$F = \alpha y^{m+1} + \cdots, \qquad G = \beta y^{m+1} + \cdots, \qquad \tilde{H} = H_0 + \cdots, \tag{3.9}$$

and applying a simple balance between the leading-order terms on the left-hand side of (3.7*a*,*b*) and the second-order derivatives on the right-hand side. This leads us to the conclusion that $m = 1/(n-1)$. The constants α and β must be determined as part of the solution process. The form for $\tilde{H}$ can be written as

$$\tilde{H} = mn\big[(m+1)(\alpha^2+\beta^2)\big]^{(n-1)/2} - \frac{(3n+1)(n-1)}{(n+1)(2n-1)}\alpha y^{(2n-1)/(n-1)} + \cdots. \tag{3.10}$$

The system was solved using a fourth-order Runge–Kutta routine coupled to Newton–Raphson iteration to determine the values of α, β and η_c such that the boundary conditions (3.8) are satisfied (to within some desired tolerance). The results of this numerical procedure were confirmed using a similarly formulated finite-difference calculation, with agreement below the tolerance of the two schemes.

The results for η_c versus n are presented in figure 3. System (3.7*a*–*c*) has multiple solutions satisfying the boundary conditions (3.7*d*,*e*), and figure 3 shows the first two. Plots of the functions F, G and H for the first of these, which we will refer to as mode 1, are given in figure 4. Here we have plotted the function versus y and η. Without further details, one could readily be convinced that the plots of $F(\eta)$ and $G(\eta)$ in figure 4 are indeed solutions of the full equations (2.9) satisfying the correct asymptotic boundary conditions (2.9*e*). However, as the corresponding plots in figure 4 for $F(y)$ and $G(y)$ demonstrate, these are solutions of the problem (3.7) on the finite interval $0 \leqslant y \leqslant \eta_c$. The corresponding values of $H(\eta_c)$ are given in table 2 along with the results (where available) from AKM.

The results presented in table 2 clearly demonstrate that the solutions found by AKM are in fact approximations to the true solutions that we present herein. In contrast to the statement of AKM, they do not correspond to solutions on the full semi-infinite domain with the asymptotic constraints as stated, but instead are approximations to the true truncated-domain solutions shown here and must therefore be non-differentiable at some point within their domain of computation.

The results presented above demonstrate that this three-dimensional flow suffers from a ‘finite width’ crisis analogous to that in the flat-plate boundary layer.

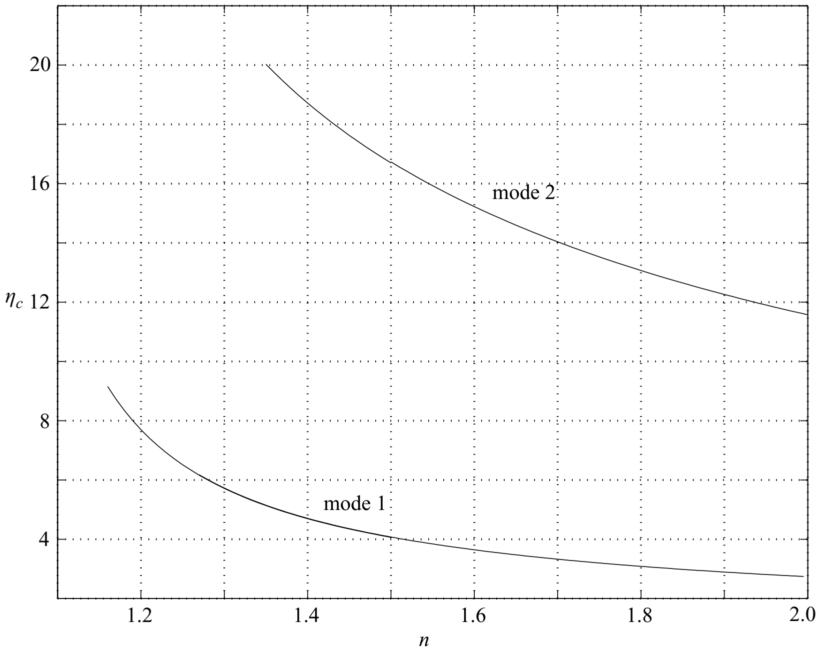

FIGURE 3. Plot of the critical location η_c as a function of n.

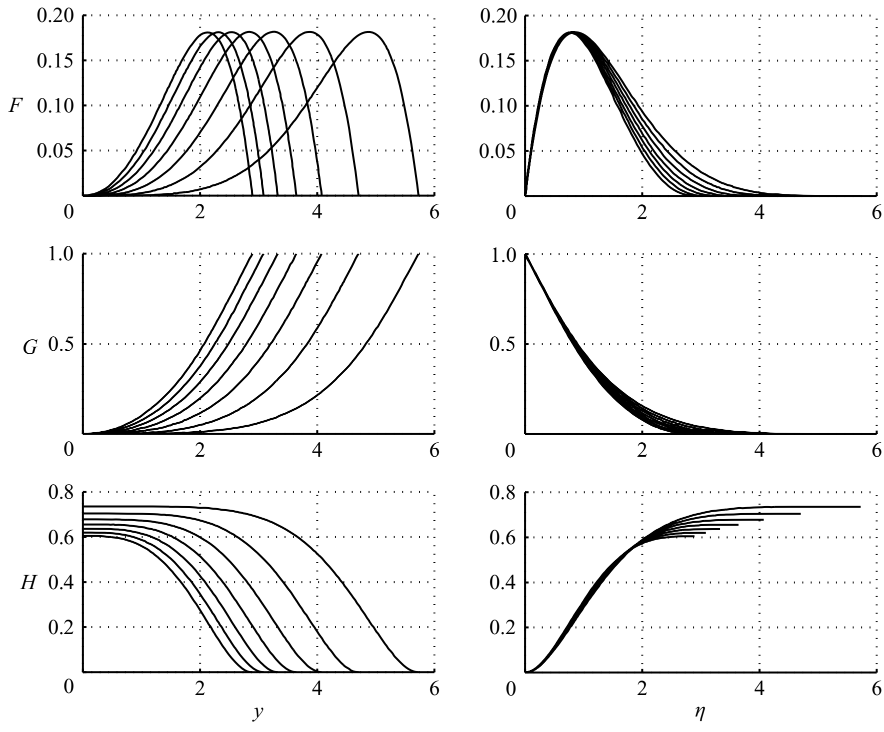

FIGURE 4. Plot of the first modal solution. Shown are F, G, and H versus y (left-hand series of plots) and versus η (right-hand series of plots) for $n = 1.3, 1.4, 1.5, \ldots, 1.9$.

n	η_c	$-H(\eta_c)$ Present	$-H(\eta_\infty)$ AKM
1.9	2.8932	0.6049	–
1.8	3.0855	0.6198	–
1.7	3.3277	0.6366	0.633
1.6	3.6436	0.6559	–
1.5	4.0754	0.6783	0.676
1.4	4.7064	0.7046	–
1.3	5.7292	0.7359	0.735

TABLE 2. Comparison of values of $H(\eta_c)$ from the first modal solution of (3.7) to those obtained by AKM. The results of AKM correspond to spurious solutions of system (2.4) with boundary conditions $F = G = 0$ imposed at $\eta_\infty \gg 1$. Here we show that a non-trivial solution only exists for $\eta \in [0, \eta_c]$.

Furthermore there are, at least, three solution branches which exhibit this behaviour (we only show modes 1 and 2 here), and based on our knowledge of the Newtonian flow, we conjecture that an infinity of such states exist. Importantly, these finite-domain solutions only exist when m in (3.9) is positive which is equivalent to $n > 1$. Thus finite-domain solutions cannot exist for shear-thinning fluids.

3.4.1. *Regularization of the singularity*

The critical point η_c is associated with a zero of the leading-order viscosity function μ_0. As usual, we must return to the full field equations to review the nature of the solution in the neighbourhood of the critical point.

In the vicinity of the singularity we know that the velocity components have local expansions given by (3.9)–(3.10). Given this local behaviour, we may return to the viscosity function μ and observe that it has the form

$$\mu = \left(\bar{\mu}_0 r^{4/(n+1)} y^{2/(n-1)} + \bar{\mu}_1 Re^{-2/(n+1)} y^{2/(n-1)} + \bar{\mu}_2 Re^{-4/(n+1)} r^{-4/(n+1)} + \cdots\right)^{(n-1)/2}, \quad (3.11)$$

for constants $\bar{\mu}_{0,1,2\ldots}$. Therefore, on approaching the singularity $\eta \to \eta_c$ (or equivalently $y \to 0$) a disordering in the expansion for the viscosity function can occur. Away from the axis of rotation, the only balance to be struck is between the first and third terms in (3.11) and a disordering occurs when

$$y \sim (r^2 Re)^{-2(n-1)/(n+1)} \ll 1 .$$

Thus, for a fixed radial location and Reynolds number, the shear term corresponding to the non-parallel development of the axial flow, $(\partial w/\partial r)^2$, must be retained in the viscosity function as the leading-order terms (as given by μ_0) vanish at the critical location. Therefore, in terms of the boundary-layer coordinate, z, there is a further free-standing sublayer at

$$z = \eta_c r^{(n-1)/(n+1)} ,$$

of width $(r^3 Re^2)^{(1-n)/(n+1)}$.

To describe the flow in the sublayer we can introduce a scaled coordinate $Y = O(1)$ that spans this new region according to

$$z = \eta_c r^{(n-1)/(n+1)} + Y[r^3 Re^2]^{-(n-1)/(n+1)} . \quad (3.12)$$

The velocity components in this region are then given by

$$u = U_0(Y) r^{(1-3n)/(n+1)} Re^{-2n/(n+1)} + \cdots, \tag{3.13}$$

$$v = V_0(Y) r^{(1-3n)/(n+1)} Re^{-2n/(n+1)} + \cdots, \tag{3.14}$$

$$w = W_0 r^{(n-1)/(n+1)} + \cdots, \tag{3.15}$$

where W_0 is the (constant) leading-order term as given in (3.10). Substitution and examination of the leading-order system yields the following equation for the correction U_0 which acts to smooth out the previously described singularity:

$$W_0 U_0' = \left(\left[\left(\frac{n-1}{n+2} \right)^2 W_0^2 + (U_0')^2 + (V_0')^2 \right]^{(n-1)/2} U_0' \right)', \tag{3.16}$$

with the same equation for V_0.

The solution in this layer must be obtained subject to matching with the lower boundary-layer structure highlighted above as $Y \to -\infty$ and to the appropriate free-stream conditions as $Y \to \infty$. Clearly, as $Y \to -\infty$, we require $U_0 \sim Y^{n/(n-1)}$, whereas as $Y \to \infty$ we see that the appropriate asymptotic behaviour is

$$U_0 \sim \exp\Big(W_0 / \big(\gamma^2 W_0^2\big)^{(n-1)/2} Y \Big),$$

where $\gamma = (n-1)/(n+1)$. Furthermore, we note that $W_0 < 0$ (since axial mass transfer is directed towards the boundary) and so U_0 is exponentially decaying as $Y \to \infty$. The behaviour for V_0 is the same, and the correction to W_0 arises at higher order, being determined via the continuity equation.

Thus, although the leading-order boundary-layer equations possess a singular point, this singularity can be regularized entirely within the context of the power-law model. The resulting full solution can therefore be considered in a global context, with matching to an external inviscid flow via the exponential decay relationships described above.

4. Discussion and conclusions

Here we have considered the structure of the boundary-layer flow of a power-law fluid over a rotating disk. We have shown that the Ostwald–de Waele model leads to difficulties when applied naively in a boundary-layer context.

For $n < 1$ (shear-thinning fluids) the traditional concept of a boundary layer as a viscous sublayer matched to an inviscid external bulk flow is obviously no longer appropriate for flows with zero-shear free streams. The application of the 'boundary-layer approximation' in this case merely acts to simplify the viscosity function near the boundary (accounting for the dominant lateral shear components), whilst the external flow remains governed by the full (viscous) field equations. A consequence of this is that whilst the boundary-layer flow admits a self-similar solution, the external flow is necessarily not of the same self-similar form; see the discussion in § 3.1.1 and § 3.2.1. Another way of viewing this solution is as a large-radius asymptotic form of a general, non-self-similar, state; that is, as a possible solution at a fixed vertical distance from the boundary, at a sufficiently large radius.

For $n > 1$ (shear-thickening fluids) the mathematical structure of the solutions of the governing equations is more interesting. The solutions to the leading-order boundary-layer equations exist only over a finite range (unusually, compared to their Newtonian counterparts), terminating at a critical distance from the boundary. Regularization of the solution in the neighbourhood of the critical point is achieved by balancing otherwise higher-order terms in the viscosity function that arise through the non-parallel nature of the flow. Clearly, a deviation of the constitutive relation from the power-law model may also act to remove the singularity; however the question of which of these effects comes into play first is dependent on the exact nature of the flow and fluid under consideration. It is nonetheless interesting that in this shear-thickening case the singularity can be regularized entirely within the confines of the model.

The phenomenon of a finite-thickness boundary layer also occurs in hypersonic boundary layers (see Bush 1996; Lee & Cheng 1969; Mikhailov, Neiland & Sychev 1971). In this case the finite width of the boundary layer arises due to the vanishing nature of the temperature, and consequently the fluid viscosity which is a function of temperature. For the case of a fluid whose viscosity–temperature relation is given by Sutherland's law (that is, a nonlinear dependence of viscosity upon temperature) Bush (1966) demonstrated that this singularity is smoothed out in a thin viscous transition layer which allows uniform matching with an outer inviscid shock layer. Lee & Cheng (1969) further extended this analysis to the case when the viscosity–temperature relation is given by Chapman's law (that is, a linear dependence of viscosity upon temperature). Although there are some subtle differences between the two cases, both result in a viscous transition layer at the outer extent of the finite-width boundary layer. The parallels between the structure of the hypersonic boundary layer and that of the shear-thickening boundary layer are obvious. In the latter case the underlying cause behind the existence of the finite-width boundary layer is the vanishing of the leading-order viscosity as $\eta \to \infty$. The regularization of the resulting singularity is accomplished through the re-introduction of lower-order terms in the viscosity function.

It is accepted within the rheological literature that the classical Ostwald–de Waele power-law model is only valid over a finite range of shear. Nevertheless, the publications referenced in the introduction of this work have applied the model to boundary-layer flows, which necessarily cover shears ranging from zero to some finite value. In applying this model there is an implicit assumption that the solutions obtained still have a role to play in the more general flows; that is, with a general constitutive relation that correctly describes the fluid over the entire shear range realized in the boundary layer. However, we have shown that there are a number of non-trivial issues that have been overlooked in earlier studies that have focused solely upon self-similar flows.

Placing the similarity solutions for a power-law model in the broader context of a general rheological model is a non-trivial issue. The full non-self-similar solution to the three-dimensional flow considered here is yet to be described; however Denier & Dabrowski (2004) have addressed a similar issue in the case of flat-plate boundary layers. In that case the problems encountered above were resolved by employing a modified version of the Carreau rheological model as described in Bird *et al.* (1973). That model removes the difficulty encountered in shear-thinning fluids having an infinite viscosity in regions of zero shear and in shear-thickening fluids have zero viscosity in regions of zero shear. This model does not admit similarity solutions and therefore the full parabolic partial differential equations must be solved using a suitable marching scheme. However, for a large distance x from the leading edge of the plate the flow can be described in terms of a uniform series expansion in inverse powers of x; the leading-order term for the streamwise velocity is a modified form of the Blasius boundary layer. For the present case of the rotating-disk boundary layer, the adoption of a similar model may also alleviate the difficulties (or inconsistencies) noted above but would present a considerably more challenging numerical task.

Finally we should emphasize that all the rheological models mentioned in this paper are empirical and none appear to have been adequately validated against experiment (in the context of three-dimensional boundary-layer flows). This would appear to be an area where there is a distinct need for careful experiments on typical non-Newtonian fluids in order to provide some insight for the further development of physically correct models as well as giving a benchmark against which further theoretical developments can be compared. Such experiments are planned for the future.

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Appendix. The pressure correction

The pressure correction term p_1 is written as $p_1(r,z) = r^{2(n-1)/(n+1)}Q(\eta)$ where Q satisfies

$$\frac{\partial Q}{\partial \eta} = \frac{(1-n)}{(n+1)}F(H-\eta H')-HH'+\frac{(3n+1)}{(n+1)}\hat{\mu}F'+\frac{(n-1)}{(n+1)}\left(\hat{\mu}F'\right)'+2\left(\hat{\mu}H'\right)', \quad \text{(A 1)}$$

with

$$\hat{\mu} = (F'^2 + G'^2)^{(n-1)/2}.$$

In the case $0 < n < \frac{1}{2}$ the asymptotic forms for F and H found in § 3.2 indicate that in the large-η limit the pressure correction Q behaves like

$$Q \sim \eta^{2(1-2n)/(2-n)}.$$

Given that this term is algebraically growing, the asymptotic expansion (2.6d) becomes disordered when

$$\eta^{2(1-2n)/(2-n)} = O\left(Re^{2/(n+1)}\right),$$

that is, when

$$\eta = O\left(Re^{\sigma}\right) \quad \text{where} \quad \sigma = \frac{(2-n)}{(n+1)(1-2n)}. \quad \text{(A 2)}$$

However, $Re^{\sigma} \gg Re^{1/(n+1)}$, the thickness of the boundary layer. Thus the large Re expansion (2.6d) remains well ordered within the boundary layer.

REFERENCES

Acrivos, A., Shah, M. J. & Petersen, E. E. 1960 Momentum and heat transfer in laminar boundary-layer flows of non-Newtonian fluids past external bodies. *AIChE J.* **6**, 312.

Andersson, H. I., de Korte, E. & Meland, R. 2001 Flow of a power-law fluid over a rotating disk revisited. *Fluid Dyn. Res.* **28**, 75 (referred to herein as AKM).

Bird, R. B., Armstrong, R. C. & Hassager, O. 1977 *Dynamics of Polymeric Liquids*. Wiley.

Bödewadt, U. T. 1940 Die Drehströmung über festem Grund. *Z. Angew. Math. Mech.* **20**, 241.

BODONYI, R. J. & STEWARTSON, K. 1977 The unsteady laminar boundary layer on a rotating disk in a counter-rotating fluid. *J. Fluid Mech.* **79**, 669.

BROWN, S. N. & STEWARTSON, K. 1965 On similarity solutions of the boundary-layer equations with algebraic decay. *J. Fluid Mech.* **23**, 673.

BURGESS, S. & WILSON, S. 1996 Steady Ostwald flow between two differentially rotating co-axial discs. *Rheol. Acta.* **35**, 48.

BUSH, W. B. 1966 Hypersonic strong-interaction similarity solutions for flow past a flat plate. *J. Fluid Mech.* **25**, 51.

COCHRAN, W. G. 1934 The flow due to a rotating disk. *Proc. Camb. Phil. Soc.* **30**, 365.

DENIER, J. P. & DABROWSKI, P. P. 2004 On the boundary-layer equations for power-law fluids. *Proc. R. Soc. Lond.* A (in press).

FILIPUSSI, D., GRATTON, J. & MINOTTI, F. 2001 The self-similar laminar boundary layer of power law non-Newtonian fluids. *Il Nuovo Cim.* **116**, 393.

HEWITT, R. E. & DUCK, P. W. 2000 Non-axisymmetric rotating-disk flows: nonlinear travelling-wave states. *J. Fluid Mech.* **413**, 169.

HEWITT, R. E. & DUCK, P. W. 2002 Continua of States in boundary-layer flows. *J. Fluid Mech.* **468**, 121.

HOYT, J. W. & FABULA, A. G. 1964 The effects of additives on fluid friction. In *Proc. 5th Symp. Naval Hydrodynamics* (ed. J. K. Lunde & S. W. Doroff), pp. 947–974. Office of Naval Research.

JENEKHE, A. S. 1984 Effects of solvent mass transfer on flow of polymer solutions on a flat rotating disk. *Ind. Engng. Chem. Fundam.* **23**, 425.

JENEKHE, A. S. & SCHULDT, S. B. 1984 Coating flow on non-Newtonian fluids on a flat rotating disk. *Ind. Engng. Chem. Fundam.* **23**, 432.

VON KÁRMÁN, T. 1921 Über laminare und turbulente Reibung. *Z. Angew. Math. Mech.* **1**, 233.

LEE, R. S. & CHENG, H. K. 1969 On the outer-edge problem of a hypersonic boundary layer. *J. Fluid Mech.* **38**, 161.

LENTINI, M. & KELLER, H. B. 1980 Boundary-value problems on semi-infinite intervals and their numerical solution. *SIAM J. Numer. Anal.* **17**, 577.

MIKHAILOV, V. V., NEILAND, V. YA. & SYCHEV, V. V. 1971 The theory of viscous hypersonic flow. *Annu. Rev. Fluid Mech.* **3**, 371.

MITSCHKA, P. 1964 Nicht-Newtonsche Flüssigkeiten II. Drehstromungen Ostwald-de Waelescher Nicht-Newtonscher Flüssigkeiten. *Coll. Czech. Chem. Comm.* **29**, 2892.

PAVLOV, K. B., FEDOTOV, I. A. & SHAKHORIN, A. P. 1981 Structure of laminar boundary layer in non-Newtonian dilatant fluids. *Izv. Akad. Nauk SSSR, Meck. Zhidk. Gaza* **4**, 142.

SLAVTCHEV, S., MILADINOVA, S. & KALITZOVA-KURTEVA, P. 1996 Unsteady film flow of power-law fluids on a rotating disk. *J. Non-Newtonian Fluid Mech.* **66**, 117.

WU, J. & THOMPSON, M. C. 1996 Non-Newtonian shear-thinning flows past a flat plate. *J. Non-Newtonian Fluid Mech.* **66**, 127.

ZANDBERGEN, P. J. & DIJKSTRA, D. 1987 Von Kármán swirling flows. *Annu. Rev. Fluid Mech.* **19**, 465.

ZHIZHIN, G. V. & UFIMTSEV, A. A. 1977 Flow in the plane laminar boundary layer of dilatant liquids. *Izv. Akad. Nauk SSSR, Meck. Zhidk. Gaza* **5**, 164.