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Nematollahi, A. R. and Subba Rao, T.

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On The Spectral Density Estimation of Periodically Correlated (Cyclostationary) Time Series

A. R. Nematollahi and T. Subba Rao
Department of Statistics, Shiraz University and
School of Mathematics, The University of Manchester

Abstract

We consider the estimation of the spectral density matrix of a periodically correlated (PC) time series (also known as cyclostationary time series). We use the well known relation between the spectral density matrix of a periodically correlated time series and a stationary vector time series (Gladyshev, 1961). The spectral matrix of the stationary vector time series is estimated using the eigenvalue decomposition of block Toeplitz matrices. The method of estimation is illustrated with simulated and real time series.

Keywords: periodically correlated (cyclostationary) processes, Capon's estimate, high resolution estimate, eigenvalue decomposition, block-Toeplitz matrix.

1 Introduction

Many time series we come across in the real world exhibit nonstationary and nonlinear property. It has been found that many meteorological variables such as rainfall, global temperatures and precipitation are nonstationary. Most estimate techniques depend on the assumption that the time series is stationary, and the techniques for the identification of linear ARMA models, spectral estimation (both parametric and nonparametric) also depend on this assumption. In recent years attempts have been made to relax the assumption of stationarity. Priestley (1965) defined an oscillatory process and the spectral representation of a nonstationary time series which enabled him to define the spectral density function which depended on both time and frequency. Using Priestley's definition, Subba Rao (1970 and 1997) has considered the estimation of time-dependent time series models.

A special class of nonstationary time series has been defined by Gladyshev (1961), called periodically correlated (PC) time series (also known as cyclostationary time series). These time series are nonstationary, but have periodic

means and covariances. They have been known to be very useful in describing many time series (Hardin and Miamee, 1990, Gardner, 1994, Hurd et al., 1998, Hipel and Mcleod, 1994, Mcleod, 1994, Boshnakov, 1996 and Nematollahi and Soltani, 2000). Though lot of literature is now available on PC time series, we are not aware of any literature on spectral estimation. It is well known that a PC time series with period T can be characterized by a T-dimensional stationary vector time series. We use this characterization for our estimation technique. Our approach is based on the properties of the eigenvalues of block-Toeplitz matrices, and is very similar to the approach given by Subba Rao and Gabr (1989) and heavily based on the results of Hannan and Wahlberg (1989). The results we derive here for T-dimensional vector time series are of general interest, and can be used in the estimation of stationary vector time series. These can also be used for the estimation of vector AR and ARMA models. We wish to pursue these aspects in later publications.

In section 2, we review some necessary results related to the spectral domain theory of a PC time series and also the relation between the PC time series and the stationary vector series. The spectral density matrix of the stationary vector series is expressed in terms of the eigenvalue matrices of the variance covariance matrix of the series, in section 3. The theoretical form of the spectral density matrix is expressed in terms of a truncated form which is an approximation to the theoretical spectral matrix. In section 4, using the results of section 3, we obtain a generalization of the high resolution spectral matrix similar to Capon's estimate. Two estimates for the spectral density function of the PC time series are proposed in section 5. In section 6, the estimates proposed to the spectral density matrix of vector series are calculated for a simulated series and compared with their theoretical forms. We illustrate the above methodology with a real time series in section 7.

2 Periodically correlated time series and their spectra

Let $\{X_t\}$ be a discrete parameter, zero mean and real time series and let $E\left|X_t\right|^2 < \infty$, $t \in Z$, where Z stands for all integers. The time series $\{X_t\}$ is said to be periodically correlated (PC) with period T, if there is a positive integer T, such that the covariance function $R(t,s) = cov(X_t, X_s)$ satisfies

$$R(t,s) = R(t+T, s+T), \tag{2.1}$$

for all t, s, and moreover T is the smallest integer for which (2.1) holds.

Since $\{X_t\}$ is a nonstationary time series, it does not possess a spectral density function in the conventional sense, but Gladyshev (1961) has shown that we may associate with $\{X_t\}$ a Hermitian nonnegative definite $T \times T$ matrix $\mathbf{f}(\omega)$, which we may call the "spectral matrix," defined as follows. The (j,k)th element of $\mathbf{f}(\omega)$, $f_{jk}(\omega)$ is given by

$$f_{jk}(\omega) = \frac{1}{T} f_{k-j}((\omega - 2\pi j)/T),\ j,\ k = 0, ..., T-1,\ 0 \le \omega \le 2\pi, \tag{2.2}$$

where $f_k(\omega)$ satisfies the relation

$$f_k(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} B_k(\tau) \exp(-i\tau\omega), \tag{2.3}$$

with $B_k(\tau) = B_{k+T}(\tau)$ the kth order coefficient of the Fourier series expansion of the periodic function (with respect to t) $R_t(\tau) := E(X_{t+\tau} X_t)$, i.e.

$$B_k(\tau) = \frac{1}{T} \sum_{t=0}^{T-1} R_t(\tau) \exp(-\frac{2\pi ikt}{T}). \tag{2.4}$$

When $k < 0$ and $\omega < 0$ or $\omega > 2\pi$ the functions $\{f_k(\omega)\}$ are completely determined by the identities $f_k(\omega) = f_{k+T}(\omega)$ and $f_k(\omega+2\pi) = f_k(\omega)$, $f_k(0) = 0$ for all k. These $f_k(\omega)$ are in a sense spectra of PC processes. It is well known that the PC time series X_t must be harmonizable in the sense of Loeve (1963), i.e. can be represented as

$$X_t = \int_0^{2\pi} \exp(it\omega) d\mathcal{Z}(\omega), \tag{2.5}$$

where $\{\mathcal{Z}(\omega),\ 0 \le \omega < 2\pi\}$ is a zero mean complex-valued random process. The covariance structure of $\{\mathcal{Z}(\omega),\ 0 \le \omega < 2\pi\}$ is described by the complex-valued bivariate measure μ, restricted to $[0, 2\pi) \times [0, 2\pi)$ with increments $\mu(d\omega_1, d\omega_2) = E[d\mathcal{Z}(\omega_1)\overline{d\mathcal{Z}(\omega_2)}]$. Moreover the original covariance function $R(t, s)$ has the expression

$$R(t, s) = \int_0^{2\pi} \int_0^{2\pi} \exp(i(t\omega_1 - s\omega_2)) g(\omega_1, \omega_2) d\omega_1 d\omega_2,$$

where $g(\omega_1, \omega_2)$ be the spectral density function corresponding to the spectrum μ. Also we have

$$g(\omega_1, \omega_2) = \sum_{k=-T+1}^{T-1} f_k(\omega_1) \delta(\omega_2 - \omega_1 + 2\pi k/T), \tag{2.6}$$

where $\delta(.)$ is the Dirac delta function (see Sakai, 1991).

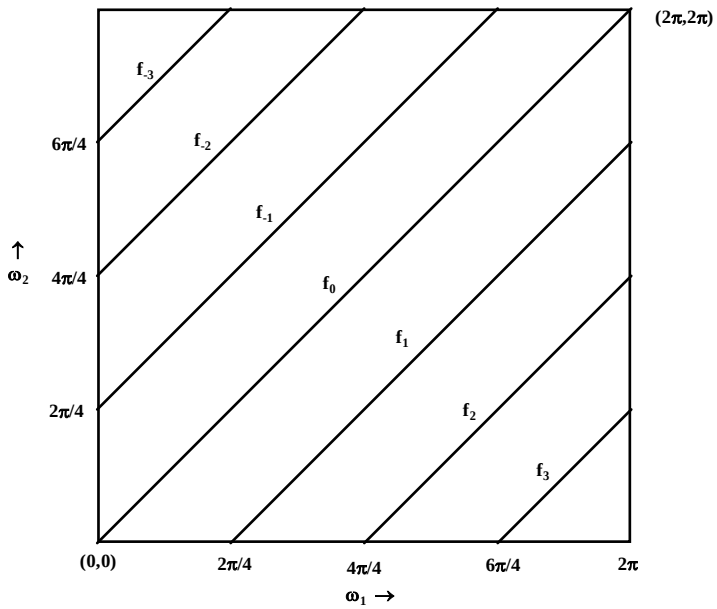

Figure 1: Periodic diagonal nature for T=4

In the other words, the spectrum μ is concentrated on the $2T-1$ parallel diagonal lines $\omega_2 = \omega_1 + 2\pi k/T$, $k = 0, \pm 1, ..., \pm(T-1)$ restricted to $[0, 2\pi) \times [0, 2\pi)$. In this case, we see $\mathcal{Z}(\omega)$ has PC increments, i.e. $\mu(d\omega_1, d\omega_2) = 0$, unless $\omega_2 = \omega_1 + 2\pi k/T$, $k = 0, \pm 1, ..., \pm(T-1)$. When $T = 1$, then $\mathcal{Z}(\omega)$ has orthogonal increments, i.e. $\mu(d\omega_1, d\omega_2) = 0$, unless $\omega_2 = \omega_1$, i.e., μ is concentrated on the main diagonal $\omega_2 = \omega_1$, that is a well known result related to the stationary case. In the general case, the main diagonal component of μ is $\mu(d\omega, d\omega) = E|d\mathcal{Z}(\omega)|^2 = f_0(\omega)d\omega$. The entire mass of the spectra in $f_0(\omega)$, is concentrated on the main diagonal $\omega_2 = \omega_1$, (like the spectrum of a stationary time series), and is a real valued and nonnegative function, and the masses of the other spectra, which can be complex valued, namely f_{k-j}, $j \neq k$, have their masses concentrated on the off diagonals $\omega_1 = \omega_2 + 2\pi k/T$, $k = \pm 1, \pm 2, .., \pm(T-1)$. The periodic diagonal nature of this support set for $T = 4$ is shown in Figure 2.1.

Following Gladyshev (1961), we may construct an alternative, but equivalent definition of a PC process as follows. We say that the series $\{X_t\}$ is periodically correlated with period T if and only if the T-dimensional vector series $(X_{tT}, X_{tT+1}, ..., X_{tT+T-1})'$ is stationary in the wide sense. It can then be shown that (Gladyshev, 1961) that

$$\mathbf{f}(\omega) = \frac{1}{T}\mathbf{U}(\omega)\mathbf{h}(\omega)\mathbf{U}^{-1}(\omega), \tag{2.7}$$

where $\mathbf{f}(\omega) = [f_{jk}(\omega)]_{j,k=0,...,T-1}$ is the spectral density matrix of $\{X_t\}$, $\mathbf{h}(\omega) = [h_{jk}(\omega)]_{j,k=0,...,T-1}$ the spectral density matrix of the stationary vector series $\{(X_{tT}, X_{tT+1}, ..., X_{tT+T-1})'\}$ and $\mathbf{U}(\omega) = [U_{jk}(\omega)]_{j,k=0,...,T-1}$ is a unitary matrix (i.e., $\mathbf{U}(\omega)\mathbf{U}^*(\omega) = \mathbf{I}$) with elements $U_{jk}(\omega) = T^{-1/2}\exp(\frac{2\pi ijk - ik\omega}{T})$.

The relation (2.7) suggests that the estimation of $\mathbf{f}(\omega)$ can be accomplished through the estimation of $\mathbf{h}(\omega)$. One can estimate $\mathbf{h}(\omega)$ using either parametric estimation (say via linear vector ARMA models) or using the kernel approach (Brillinger, 1975, chapter 7 and Hannan, 1970). Here our object is to exploit the special structure of the block-Toeplitz matrix associated with multidimensional series $\{\mathbf{Y}_t\}$ to derive spectral estimates similar to those of Subba Rao and Gabr (1989). The results of Subba Rao and Gabr (1989) also enable us to derive a high resolution estimate of $\mathbf{h}(\omega)$ and hence of the spectral matrix, $\mathbf{f}(\omega)$ of PC time series. We extensively use the result of Hannan and Wahlberg (1989) in the derivation of our estimates and their interpretation.

3 Spectral density matrix of the stationary vector series and the eigenvalue decomposition

We now consider the estimation of the spectral density function of a vector-valued series. Let $\{\mathbf{Y}_t,\ t \in Z\}$ be zero mean, discrete-parameter and second order T-dimensional stationary series with $Y_t(j),\ j = 0, ..., T-1$, as its jth element and $\mathbf{R}(\tau) = E\mathbf{Y}_{t+\tau}\mathbf{Y}_t'\ , \tau \in Z$, be the autocovariance matrix of $\mathbf{Y}_t$. We assume that the series has an absolutely continuous spectrum and let $\mathbf{h}(\omega)$ denotes its spectral density matrix, i.e., Let

$$\mathbf{h}(\omega) \ = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\mathbf{R}(\tau)e^{-i\omega\tau}\ ,\ 0 \le \omega \le 2\pi. \tag{3.1}$$

Let $\{\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n\}$ be a sample of size n from $\{\mathbf{Y}_t\}$ and $\mathbf{P}_n(\omega)$ be the periodogram matrix. Then

$$E\left\{\mathbf{P}_n(\omega)\right\} = \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\mathbf{R}(t-s)e^{-i(t-s)\omega} \tag{3.2}$$

$$= \mathbf{h}_n(\omega) \text{ say,} \tag{3.3}$$

$$= \mathbf{h}(\omega) + \mathbf{O}\left(\frac{\log n}{n}\right), \tag{3.4}$$

under some regularity conditions on $\mathbf{h}(\omega)$ (Priestley, 1981). We shall call $\mathbf{h}_n(\omega)$ the *truncated spectral density matrix.*

Thus $\mathbf{P}_n(\omega)$ is an asymptotically unbiased estimator of $\mathbf{h}(\omega)$, but of course, it is not consistent (Brillinger, 1975). In order to find a consistent estimator of $\mathbf{h}(\omega)$, the usual procedure is to smooth the periodogram by a suitable kernel

(Brillinger, 1975). We first show that the theoretical spectral density matrix $\mathbf{h}(\omega)$ can be written in terms of the eigenvalues of the variance covariance matrix, then we can estimate $\mathbf{h}(\omega)$ using the eigenvalue decomposition of the sample variance covariance matrix and show that it intrinsically makes use of Féjer kernel type of weight functions.

Define a $nT \times 1$ vector $\mathbb{Y}_n = (\mathbf{Y}'_n, \mathbf{Y}'_{n-1}, ..., \mathbf{Y}'_1)'$ and let $\Gamma_n = E\mathbb{Y}_n\mathbb{Y}'_n$ be its variance-covariance matrix. We have

$$\Gamma_n = \begin{pmatrix} \mathbf{R}(0) & \mathbf{R}(1) & ... & \mathbf{R}(n-1) \\ \mathbf{R}(-1) & \mathbf{R}(0) & ... & \mathbf{R}(n-2) \\ . & . & . & . \\ \mathbf{R}(-(n-1)) & \mathbf{R}(-(n-2)) & ... & \mathbf{R}(0) \end{pmatrix}.$$

We note that Γ_n is a block-Toeplitz matrix. Individual matrix elements are not, in general, symmetric ($\mathbf{R}(\tau) \neq \mathbf{R}'(\tau)$), although $\mathbf{R}(-\tau) = \mathbf{R}'(\tau)$.

Now let $\mathbf{W}_n(\omega)$ be the $T \times nT$ matrix given by

$$\mathbf{W}_n(\omega) = n^{-\frac{1}{2}} \left(\mathbf{I}, e^{i\omega}\mathbf{I}, ..., e^{(n-1)i\omega}\mathbf{I}\right), \tag{3.5}$$

where $\mathbf{I}$ is an $T \times T$ identity matrix and let $\omega_j = \frac{2\pi j}{n}$, $j = 0, ..., n-1$. From Hannan and Wahlberg (1989) we have the eigenvalue decomposition of Γ_n,

$$\Gamma_n = \widetilde{\mathbf{W}}^*_n \mathbf{D}_n \widetilde{\mathbf{W}}_n, \tag{3.6}$$

where $\widetilde{\mathbf{W}}_n$ is the $nT \times nT$ matrix, with jth (block) row $\mathbf{W}_n(\omega_j)$ given by (3.5), i.e.,

$$\widetilde{\mathbf{W}}_n = \begin{pmatrix} \mathbf{W}_n(\omega_0) \\ \mathbf{W}_n(\omega_1) \\ . \\ \mathbf{W}_n(\omega_{n-1}) \end{pmatrix},$$

and $\mathbf{D}_n = diag(\Lambda_n(\omega_0), \Lambda_n(\omega_1), ..., \Lambda_n(\omega_{n-1}))$, where $\Lambda_n(\omega)$ is a $T \times T$ Hermitian matrix given by

$$\Lambda_n(\omega) = \mathbf{W}_n(\omega)\Gamma_n\mathbf{W}^*_n(\omega). \tag{3.7}$$

We note that $\widetilde{\mathbf{W}}_n$ is a unitary matrix, i.e., $\widetilde{\mathbf{W}}_n\widetilde{\mathbf{W}}^*_n = \mathbf{I}$. We identify $\Lambda_n(\omega_j)$ as an "eigenvalue-matrix" and also $\mathbf{W}_n(\omega_j)$ as an "eigenvector-matrix" associated with $\Lambda_n(\omega_j)$, (clearly, they are not the eigenvalue and the eigenvector in the usual sense). In the following we show that $\Lambda_n(\omega_j)$ is proportional to $\mathbf{h}_n(\omega_j)$,

and hence asymptotically proportional to $\mathbf{h}(\omega_j)$, the spectral density function of $\mathbf{Y}_t$. From (3.6) we have

$$\Gamma_n = \sum_{j=0}^{n-1} \mathbf{W}_n^*(\omega_j)\Lambda_n(\omega_j)\mathbf{W}_n(\omega_j). \tag{3.8}$$

Collecting the (t,s)th block-matrix element from both sides of (3.8), we obtain

$$\mathbf{R}(t-s) = \sum_{j=0}^{n-1} \mathbf{W}_n^{t*}(\omega_j)\Lambda_n(\omega_j)\mathbf{W}_n^s(\omega_j), \tag{3.9}$$

where $\mathbf{W}_n^s(\omega_j)$ is the s-th block of $\mathbf{W}_n(\omega_j)$. Substituting $\mathbf{R}(t-s)$ from (3.9) into (3.3), we now have

$$\begin{aligned}
\mathbf{h}_n(\omega_l) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\left[\sum_{j=0}^{n-1}\mathbf{W}_n^{t*}(\omega_j)\Lambda_n(\omega_j)\mathbf{W}_n^s(\omega_j)\right]e^{-i(t-s)\omega_l} \\
&= \frac{1}{2\pi n}\sum_{j=0}^{n-1}[\sum_{t=1}^{n}\mathbf{W}_n^{*t}(\omega_j)\exp(-it\omega_l)]\Lambda_n(\omega_j)[\sum_{s=1}^{n}\mathbf{W}_n^s(\omega_j)\exp(is\omega_l)] \\
&= \frac{1}{2\pi n}\sum_{j=0}^{n-1}\mathbf{B}_n^*(\omega_j,\omega_l)\Lambda_n(\omega_j)\mathbf{B}_n(\omega_j,\omega_l),
\end{aligned}$$

where $\mathbf{B}_n(\omega_j,\omega_l) = \sum_{s=1}^{n}\mathbf{W}_n^s(\omega_j)\exp(is\omega_l)$. We thus have

$$\mathbf{h}_n(\omega_l) = \frac{1}{4\pi}\sum_{j=0}^{n-1}\mathbf{A}_n(\omega_j,\omega_l), \tag{3.10}$$

where

$$\mathbf{A}_n(\omega_j,\omega_l) = \frac{2}{n}\mathbf{B}_n^*(\omega_j,\omega_l)\Lambda_n(\omega_j)\mathbf{B}_n(\omega_j,\omega_l) \tag{3.11}$$

$$= \frac{2}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\mathbf{W}_n^{t*}(\omega_j)\Lambda_n(\omega_j)\mathbf{W}_n^s(\omega_j)e^{-i(t-s)\omega_l}. \tag{3.12}$$

To see the significance of writing $\mathbf{h}_n(\omega_l)$ in the form (3.10), we shall first assume that the stationary series $\{\mathbf{Y}_t\}$ is circular, i.e.

$$\mathbf{Y}_1 = \mathbf{Y}_{1+n}, \mathbf{Y}_2 = \mathbf{Y}_{2+n}, ...$$

so that Γ_n can be replaced by the equivalent circulant block-matrix Γ_n^c given by

$$\Gamma_n^c = \begin{pmatrix} \mathbf{R}(0) & \mathbf{R}(1) & ... & \mathbf{R}(n-1) \\ \mathbf{R}(n-1) & \mathbf{R}(0) & ... & \mathbf{R}(n-2) \\ . & . & . & . \\ \mathbf{R}(1) & \mathbf{R}(2) & ... & \mathbf{R}(0) \end{pmatrix}. \tag{3.13}$$

The equivalence of eigenvalues of Γ_n and Γ_n^c in the univariate case are well known (see Gray, 1972). The similar results for the block Toeplitz matrices of the type considered here, have been proved by Nematollahi and Shishebor (2005). This is not crucial for our estimation, we need this only for interpretation. By setting $\mathbf{R}(n-1) = \mathbf{R}^{'}(1)$, $\mathbf{R}(n-2) = \mathbf{R}^{'}(2), ..., \mathbf{R}(1) = \mathbf{R}^{'}(n-1)$ in (3.13), we obtain a circular symmetric block-matrix. Using this matrix, one can show the equivalence of eigenvalues of Γ_n and Γ_n^c, see Nematollahi and Shishebor (2005) for more details. From now on we shall only use the circulant symmetric block-matrix Γ_n^c for studying the properties, and in order to avoid extra notation we shall use the same symbols for the eigenvalue-matrices and eigenvector-matrices of $\Gamma_n$and Γ_n^c. Using Hannan and Wahlberg (1989), it is not difficult to show that

(i) $\Lambda_n(\omega_j) \sim 2\pi \mathbf{h}_n(\omega_j) \sim 2\pi \mathbf{h}(\omega_j)$, *for* $\omega_j = \frac{2\pi j}{n}$, $j = 0, 1, ..., n-1$.

(ii)*For n odd,*

$$\mathbf{W}_n(\omega_0) = n^{-\frac{1}{2}} (\mathbf{I}, \mathbf{I}, ..., \mathbf{I}),$$
$$\mathbf{W}_n(\omega_{2j}) = n^{-\frac{1}{2}} 2^{-\frac{1}{2}} (\mathbf{0}, \sin\omega_j \mathbf{I}, \sin 2\omega_j \mathbf{I}, ..., \sin(n-1)\omega_j \mathbf{I}),$$
$$\mathbf{W}_n(\omega_{2j-1}) = n^{-\frac{1}{2}} 2^{-\frac{1}{2}} (\mathbf{I}, \cos\omega_j \mathbf{I}, \cos 2\omega_j \mathbf{I}, ..., \cos(n-1)\omega_j \mathbf{I}),$$

and also we have $\Lambda_n(\omega_j) = \Lambda_n^{'}(\omega_{n-j})$, $j = 1, 2, ..., \frac{n-1}{2}$.

When n is even, a similar result holds, see Nematollahi and Shishebor (2005). We shall now consider the behaviour of the function $\mathbf{h}_n(\omega)$ given by (3.10) when we substitute the eigenvalue-matrices and eigenvector-matrices of the circular symmetric matrix Γ_n^c. For this we change variables from t, s to t, s-t in (3.12), and then have

$$\begin{aligned} \mathbf{A}_n(\omega_j, \omega_l) &= \frac{2}{n} \sum_{t=1}^{n-1} \sum_{s=-t}^{n-t} \mathbf{W}_n^{t*}(\omega_j) \Lambda_n(\omega_j) \mathbf{W}_n^{t+s}(\omega_j) e^{-is\omega_l}, \\ &= 2 \sum_{s=-(n-1)}^{n-1} \Delta_{n,j}(s) e^{-is\omega_l}, \end{aligned}$$

where for $s > 0$,

$$\Delta_{n,j}(s) = \frac{1}{n}\sum_{t=1}^{n-s} \mathbf{W}_n^{t*}(\omega_j)\Lambda_n(\omega_j)\mathbf{W}_n^{t+s}(\omega_j).$$

When $j = 0$, using the above result, we have

$$\begin{aligned}\Delta_{n,0}(s) &= \frac{1}{n}\sum_{t=1}^{n-s} \mathbf{W}_n^{t*}(\omega_0)\Lambda_n(\omega_0)\mathbf{W}_n^{t+s}(\omega_0)\\ &= \frac{n-|s|}{n^2}\Lambda_n(\omega_0),\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{A}_n(\omega_0,\omega_l) &= 2\sum_{s=-(n-1)}^{n-1} \frac{n-|s|}{n^2}\Lambda_n(\omega_0)e^{-is\omega_l}\\ &= \frac{2}{n}2\pi F_{n-1}(\omega_l)\Lambda_n(\omega_0),\end{aligned}$$

where $F_n(\theta) = \frac{1}{2\pi n}\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}$ which is the Fejér kernel. Also we have

$$\begin{aligned}\Delta_{n,2j}(s) &= \frac{1}{n}\sum_{t=1}^{n-s} \mathbf{W}_n^{t*}(\omega_{2j})\Lambda_n(\omega_{2j})\mathbf{W}_n^{t+s}(\omega_{2j})\\ &= \frac{1}{n^2}\sum_{t=1}^{n-s} \sin t\omega_j \Lambda_n(\omega_{2j})\sin(t+s)\omega_j\\ &= \Lambda_n(\omega_{2j})\frac{n-s}{2n^2}\cos(\omega_j s), \qquad (s>0),\end{aligned}$$

so that

$$\begin{aligned}\mathbf{A}_n(\omega_{2j},\omega_l) &= \mathbf{A}_n(\omega_{2j-1},\omega_l)\\ &= \sum_{s=-(n-1)}^{n-1} 2\frac{n-|s|}{n^2}\Lambda_n(\omega_{2j})e^{-is\omega_l}e^{-is\omega_j}\\ &= \frac{1}{n}\Lambda_n(\omega_{2j})\left\{2\pi F_{n-1}(\omega_j+\omega_l) + 2\pi F_{n-1}(\omega_j-\omega_l)\right\}.\end{aligned}$$

We note that $F_n(\theta)$ tends to a Dirac delta function as $n \to \infty$, and $\{2\pi F_n(\theta)\}$ tends to n as $\theta \to 0$. Hence $\{\mathbf{A}_n(\omega_0,\omega_l)\} \to 2\Lambda_n(\omega_0)$, as $\omega_l \to 0$ and $\mathbf{A}_n(\omega_{2j},\omega_l) \to \Lambda_n(\omega_{2j})$, as $\omega_l \to \omega_j$, .

Now let us consider the truncated spectral matrix $\mathbf{h}_n(\omega_l)$ given by (3.10) and replace the eigenvalue-matrices and eigenvector-matrices given by these new calculations. We then have

$$\begin{aligned}
\mathbf{h}_n(\omega_l) &= \frac{1}{4\pi}\{\mathbf{A}_n(\omega_0,\omega_l) + 2\sum_{j=1}^{\frac{n-1}{2}} \mathbf{A}_n(\omega_{2j},\omega_l)\} \\
&= \frac{1}{4\pi}\{\frac{2}{n}\mathbf{B}_n^*(\omega_0,\omega_l)\Lambda_n(\omega_0)\mathbf{B}_n(\omega_0,\omega_l) + \ 2\sum_{j=1}^{\frac{n-1}{2}} \frac{2}{n}\mathbf{B}_n^*(\omega_{2j},\omega_l)\Lambda_n(\omega_{2j})\mathbf{B}_n(\omega_{2j},\omega_l)\} \\
&= \frac{1}{4\pi}\{\frac{2}{n}2\pi F_{n-1}(\omega_l)\Lambda_n(\omega_0) + 2\sum_{j=1}^{\frac{n-1}{2}} \frac{1}{n}\Lambda_n(\omega_{2j})[2\pi F_{n-1}(\omega_j+\omega_l) + 2\pi F_{n-1}(\omega_j-\omega_l)]\}.
\end{aligned} \tag{3.14}$$

The approximate relation (3.14) tells us that the spectral density function $\mathbf{h}_n(\omega_l)$ given by (3.10) is in fact a smooth function of $\Lambda_n(\omega_j)$ and the smoothing function is the well known Fejér kernel. We observe that $\mathbf{h}_n(\omega_l)$ is linear in $\Lambda_n(\omega_j)$ and that $\mathbf{B}_n(\omega_j,\omega_l)$ does not depend on the time series $\{\mathbf{Y}_t\}$. This suggests that these eigenvalue-matrices can be replaced by any nonlinear function of $\Lambda_n(\omega_j)$ and by suitably defining an inverse function, we can, in the limiting form, recover the original spectrum. To be more precise, consider a strictly monotonic continuous function $G(.)$ and $g(.)$ be an inverse function, i.e. $g(G(x)) = x$. Then we can consider the function

$$\mathbf{h}_{n,P}(\omega_l) = g\left\{\sum_{j=0}^{n-1} \frac{2}{n}\mathbf{B}_n^*(\omega_j,\omega_l)G(\Lambda_n(\omega_j))\mathbf{B}_n(\omega_j,\omega_l)\right\}, \tag{3.15}$$

as an approximation to $\mathbf{h}_n(\omega_l)$. In fact, this is a generalization of the way that Pisarenko (1972) derived his estimate in the univariate case, and (3.15) is the generalization of the theoretical form of the Pisarenko's estimator, to the multivariate case. In section 5 we consider the estimation of $\mathbf{h}_{n,P}(\omega_l)$, from the data. Also, note that (3.15) must be multiplied by an appropriate scale factor to recover $\mathbf{h}_n(\omega_l)$, and this factor is independent of the form of $G(.)$.

4 High resolution spectral density matrix of $\mathbf{Y}_t$

High resolution estimation of the spectral density function was introduced by Capon (1969) in the univariate case. In this section we give a multivariate generalization and also give an explicit expression for the spectral density matrix of the vector series $\mathbf{Y}_t$. We shall show later that this form can be used for the estimation of the spectrum of a PC time series. For this let us consider the linear combination of the vector $\left(\mathbf{Y}_t', \mathbf{Y}_{t-1}', ..., \mathbf{Y}_{t-p}'\right)'$,

$$\mathbf{S}_t = \sum_{k=0}^{p} \mathbf{G}(k)\mathbf{Y}_{t-k} = \mathbf{G}'\widetilde{\mathbb{Y}}_t, \tag{4.1}$$

where $\mathbf{G}'=(\mathbf{G}(0), \mathbf{G}(1), ..., \mathbf{G}(p))$ and $\widetilde{\mathbb{Y}}_t = \left(\mathbf{Y}'_t, \mathbf{Y}'_{t-1}, ..., \mathbf{Y}'_{t-p}\right)'$ denotes the matrix of coefficients and the vector of $(p+1)$ observations, respectively (the choice of p is important in practical situations, but we do not consider this problem here).

The variance-covariance matrix of $\mathbf{S}_t$ is

$$\begin{aligned} Var(\mathbf{S}_t) &= E\mathbf{S}_t\mathbf{S}'_t = E\mathbf{G}'\widetilde{\mathbb{Y}}_t\widetilde{\mathbb{Y}}'_t\mathbf{G} \\ &= \mathbf{G}'E\widetilde{\mathbb{Y}}_t\widetilde{\mathbb{Y}}'_t\mathbf{G} \\ &= \mathbf{G}'\widetilde{\Gamma}_p\mathbf{G}, \end{aligned} \tag{4.2}$$

where the $(p+1)T \times (p+1)T$ block-Toeplitz autocovariance matrix $\widetilde{\Gamma}_p = E\widetilde{\mathbb{Y}}_t\widetilde{\mathbb{Y}}'_t$ is given by

$$\widetilde{\Gamma}_p = \begin{pmatrix} \mathbf{R}(0) & \mathbf{R}(1) & ... & \mathbf{R}(p) \\ \mathbf{R}(-1) & \mathbf{R}(0) & ... & \mathbf{R}(p-1) \\ . & . & . & . \\ \mathbf{R}(-p) & \mathbf{R}(-(p-1)) & ... & \mathbf{R}(0) \end{pmatrix}.$$

The coefficient matrices $\{\mathbf{G}(k)\}$, are to be chosen so that, at a frequency ω under consideration, the frequency response of the filter has unit gain. This constraint can be represented as

$$\sum_{k=0}^{p} \mathbf{G}(k)e^{-ki\omega}\mathbf{I} = \mathbf{G}'\mathbf{L}_p(-\omega) = \mathbf{I}, \tag{4.3}$$

where the matrix $\mathbf{L}_p(\omega)$ is of dimension $(p+1)T \times T$ and is given by $\mathbf{L}_p(\omega) = \left(\mathbf{I}, e^{i\omega}\mathbf{I}, ..., e^{pi\omega}\mathbf{I}\right)'$. We use Lagrange multiplier $(\mathbf{K})$ for minimizing the trace of the $Var(\mathbf{S}_t)$, subject to (4.3). Let

$$Q = tr(Var(\mathbf{S}_t)) + 2\mathbf{K}'(\mathbf{I} - \mathbf{G}'\mathbf{L}_p(-\omega)).$$

By minimizing Q with respect to $\mathbf{G}$ and $\mathbf{K}$ and then substituting for $\mathbf{K}$ we obtain

$$\mathbf{G}_{MV} = \widetilde{\Gamma}_p^{-1}\mathbf{L}_p(\omega)(\mathbf{L}_p^*(\omega)\widetilde{\Gamma}_p^{-1}\mathbf{L}_p(\omega))^{-1},$$

(see Rogers, 1980, p.107 and Balestra, 1976). Substitution of this into (4.2) yields the minimum variance

$$Var(\mathbf{S}_t)_{MV} = (\mathbf{L}_p^*(\omega_l)\widetilde{\Gamma}_p^{-1}\mathbf{L}_p(\omega_l))^{-1}$$

If we consider $\mathbf{S}_t$ given by (4.1) as the output of the linear filter of the vector $\left(\mathbf{Y}_t', \mathbf{Y}_{t-1}', ..., \mathbf{Y}_{t-p}'\right)'$, $Var(\mathbf{S}_t)$ is the measure of power function and is related to the spectral density matrix of $\mathbf{Y}_t$. An appropriate minimum variance (MV) spectral estimator that may be deduced from the above is

$$\mathbf{h}_{p,cap}(\omega) = \frac{1}{\pi}[\frac{2}{p}\mathbf{L}_p^*(\omega)\widetilde{\Gamma}_p^{-1}\mathbf{L}_p(\omega)]^{-1} \tag{4.4}$$

This is a multivariate generalization of the *minimum variance spectral (MVS) estimator* due to Capon.

The general form (3.15) also includes the above MVS estimate as shown below. To obtain this, substitute $G(\Lambda_n(\omega_j)) = \Lambda_n^{-1}(\omega_j)$ in (3.15) to obtain

$$\begin{aligned}\widetilde{\mathbf{h}}_{n,P}(\omega_l) &= \left\{\sum_{j=0}^{n-1}\frac{2}{n}\mathbf{B}_n^*(\omega_j,\omega_l)\Lambda_n^{-1}(\omega_j)\mathbf{B}_n(\omega_j,\omega_l)\right\}^{-1} \\ &= \left\{\sum_{j=0}^{n-1}\widetilde{\mathbf{A}_n}(\omega_j,\omega_l)\right\}^{-1}\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathbf{A}_n}(\omega_j,\omega_l) &= \sum_{j=0}^{n-1}\frac{2}{n}\mathbf{B}_n^*(\omega_j,\omega_l)\Lambda_n^{-1}(\omega_j)\mathbf{B}_n(\omega_j,\omega_l) \\ &= \frac{2}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\mathbf{W}_n^{t*}(\omega_j)\Lambda_n^{-1}(\omega_j)\mathbf{W}_n^{s}(\omega_j)e^{-i(t-s)\omega_l}.\end{aligned}$$

As $\omega_j \to \omega_l$, $\widetilde{\mathbf{A}_n}(\omega_{2j},\omega_l) \to \Lambda_n(\omega_{2j})$, and hence (when $\omega_l \neq 0$)

$$\widetilde{\mathbf{h}}_{n,P}(\omega_l) \simeq \frac{1}{2}\Lambda_n(\omega_l) \simeq \pi\mathbf{h}_n(\omega_l).$$

Hence the theoretical form of generalized Capon's estimator is given by

$$\begin{aligned}\widetilde{\mathbf{h}}_{n,cap}(\omega_l) &= \frac{1}{\pi}\widetilde{\mathbf{h}}_{n,P}(\omega_l) \\ &= \frac{1}{\pi}\left\{\sum_{j=0}^{n-1}\widetilde{\mathbf{A}_n}(\omega_j,\omega_l)\right\}^{-1}.\end{aligned} \tag{4.5}$$

Note that from (4.4) we have

$$\mathbf{h}_{n,cap}(\omega_l) = \frac{1}{\pi}[\frac{2}{n}\mathbf{L}_n^*(\omega_l)\Gamma_n^{c^{-1}}\mathbf{L}_n(\omega_l)]^{-1}, \tag{4.6}$$

where $\mathbf{L}_n(\omega_l) = \left(\mathbf{I}, e^{i\omega_l}\mathbf{I}, ..., e^{ni\omega_l}\mathbf{I}\right)'$. But

$$\begin{aligned}
\mathbf{L}_n^*(\omega_l)\Gamma_n^{c^{-1}}\mathbf{L}_n(\omega_l) &= \mathbf{L}_n^*(\omega_l)\sum_{j=0}^{n-1}\mathbf{W}_n^*(\omega_j)\Lambda_n^{-1}(\omega_j)\mathbf{W}_n(\omega_j)\mathbf{L}_n(\omega_l) \\
&= \sum_{j=0}^{n-1}\mathbf{L}_n^*(\omega_l)\mathbf{W}_n^*(\omega_j)\Lambda_n^{-1}(\omega_j)\mathbf{W}_n(\omega_j)\mathbf{L}_n(\omega_l) \\
&= \sum_{j=0}^{n-1}[\sum_{t=1}^{n}\mathbf{W}_n^{*^t}(\omega_j)\exp(-it\omega_l)]\Lambda_n^{-1}(\omega_j)[\sum_{s=1}^{n}\mathbf{W}_n^s(\omega_j)\exp(is\omega_l)] \\
&= \sum_{j=0}^{n-1}\mathbf{B}_n^*(\omega_j,\omega_l)\Lambda_n^{-1}(\omega_j)\mathbf{B}_n(\omega_j,\omega_l),
\end{aligned}$$

and so

$$\frac{2}{n}\mathbf{L}_n^*(\omega_l)\Gamma_n^{c^{-1}}\mathbf{L}_n(\omega_l) = \sum_{j=0}^{n-1}\widetilde{\mathbf{A}}_n(\omega_j,\omega_l),$$

and therefore we have again (4.5). As seen above, the theoretical form of Capon's estimator $\widetilde{\mathbf{h}}_{n,cap}(\omega_l)$ given by (4.5) is consistent with our definition of the power spectral density matrix $\mathbf{h}_{n,cap}(\omega_l)$ given by (4.6) .

In the previous sections we defined various characterizations of the spectral density matrix of $\mathbf{Y}_t$, using the eigenvalue-matrices of the block-Toeplitz matrix Γ_n. We note that all these forms depend on the eigenvalue-matrices, $\Lambda_n(\omega_j)$, $j = 0, ..., T-1$, which are the eigenvalue-matrices of Γ_n. In the following section we briefly discuss the estimation of Γ_n and hence the estimation of $\Lambda_n(\omega_j)$. We proceed as in Subba Rao and Gabr (1989).

5 Estimation of the spectral density matrix of $\mathbf{Y}_t$ and the PC time series

Let $\{\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n\}$ be a sample of size n from $\{\mathbf{Y}_t\}$. We assume that $E\mathbf{Y}_t = 0$. Let $n = Mm$, where M and m are integers. Divide the data into M groups, where each group consists of m observations, and let the observations in the l-th group ($l = 1, ..., M$) be denoted by the $mT \times 1$ vector $\widetilde{\mathbf{Y}}_l$, where

$$\widetilde{\mathbf{Y}}_l = (\mathbf{Y}'_{lm}, \mathbf{Y}'_{lm-1}, ..., \mathbf{Y}'_{(l-1)m+1})' \ , \qquad l = 1, ..., M. \tag{5.1}$$

We estimate the $mT \times mT$ block-Toeplitz covariance matrix Γ_m of order m by

$$\widehat{\Gamma}_m = \frac{1}{M} \sum_{j=1}^{M} \widetilde{\mathbf{Y}}_j \widetilde{\mathbf{Y}}_j'. \tag{5.2}$$

Let $\widehat{\Lambda}_m(\omega_j)$, $j = 0, ..., m-1$ be the eigenvalue-matrices of $\widehat{\Gamma}_m$ and assume that m is odd. We consider

$$\widehat{\mathbf{h}}_m(\omega_l) = \frac{1}{4\pi} \sum_{j=0}^{m-1} \widehat{\mathbf{A}}_m(\omega_j, \omega_l), \tag{5.3}$$

$$\widehat{\mathbf{h}}_{m,cap}(\omega_l) = \frac{1}{\pi} \left\{ \sum_{j=0}^{m-1} \widetilde{\mathbf{A}_n}(\omega_j, \omega_l) \right\}^{-1}. \tag{5.4}$$

as estimators of $\mathbf{h}_m(\omega_l)$ and $\mathbf{h}_{m,cap}(\omega_l)$, respectively, where

$$\widehat{\mathbf{A}}_m(\omega_j, \omega_l) = \frac{2}{m} \sum_{t=1}^{m} \sum_{s=1}^{m} \mathbf{W}_m^{t*}(\omega_j) \widehat{\Lambda}_m(\omega_j) \mathbf{W}_m^{s}(\omega_j) e^{-i(t-s)\omega_l}$$

and

$$\widetilde{\mathbf{A}_m}(\omega_j, \omega_l) = \frac{2}{m} \sum_{t=1}^{m} \sum_{s=1}^{m} \mathbf{W}_m^{t*}(\omega_j) \widehat{\Lambda}_m^{-1}(\omega_j) \mathbf{W}_m^{s}(\omega_j) e^{-i(t-s)\omega_l}.$$

Using (2.7) we propose two estimators $\mathbf{f}_m(\omega_l) = \frac{1}{T}\mathbf{U}(\omega_l)\mathbf{h}_m(\omega_l)\mathbf{U}^{-1}(\omega_l)$ (truncated spectral density matrix) and $\mathbf{f}_{m,cap}(\omega_l) = \frac{1}{T}\mathbf{U}(\omega_l)\mathbf{h}_{m,cap}(\omega_l)\mathbf{U}^{-1}(\omega_l)$ (Capon estimator), as the estimates for the spectral density matrix of PC time series, $\mathbf{f}(\omega_l)$, namely,

$$\widehat{\mathbf{f}}_m(\omega_l) = \frac{1}{T}\mathbf{U}(\omega_l)\widehat{\mathbf{h}}_m(\omega_l)\mathbf{U}^{-1}(\omega_l), \tag{5.5}$$

$$\widehat{\mathbf{f}}_{m,cap}(\omega_l) = \frac{1}{T}\mathbf{U}(\omega_l)\widehat{\mathbf{h}}_{m,cap}(\omega_l)\mathbf{U}^{-1}(\omega_l). \tag{5.6}$$

The sampling properties, such as bias and consistency of $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$ need to be investigated. These depend on the sampling behaviour of the eigenvalue-matrices $\widehat{\Lambda}_m(\omega_j)$, which are the eigenvalues of $\widehat{\Gamma}_m$.

6 Comparison of the truncated estimates and the high resolution estimates

An appropriate way to compare the two estimates $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$ is to compare their mean square errors. Since we do not have expressions for

their mean square errors, we calculated $\widehat{\mathbf{h}}_m(\omega_l)$ and $\widehat{\mathbf{h}}_{m,cap}(\omega_l)$ for a simulated series and compared them with their theoretical spectral values. In view of the relationships (5.5) and (5.6) this is equivalent to the comparison of $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$.

Let $\{\mathbf{Y}_t\}$ be a bivariate stationary series generated from the model

$$\mathbf{Y}_t + \mathbf{A}_1\mathbf{Y}_{t-1} = \mathbf{e}_t \tag{6.1}$$

where $\mathbf{A}_1 = \begin{bmatrix} -0.16 & 0.15 \\ -0.14 & -0.15 \end{bmatrix}$ and $\mathbf{e}_t$ is a bivariate Gaussian white noise with mean zero and variance covariance matrix $\Sigma = \begin{bmatrix} 1.19 & 0 \\ 0 & 2.15 \end{bmatrix}$. It is well known that the theoretical spectral density matrix of $\mathbf{Y}_t$ is given by

$$\mathbf{h}(\omega) = [\mathbf{I} + \mathbf{A}_1 e^{-i\omega}]^{-1}\Sigma[\mathbf{I} + \mathbf{A}_1 e^{i\omega}]^{'-1},\ 0 \le \omega \le 2\pi. \tag{6.2}$$

Let $\mathbf{h}(\omega) = [h_{jk}(\omega)]_{j,k=0,1}$. We note $h_{00}(\omega)$ and $h_{11}(\omega)$ are real valued functions and $h_{01}(\omega) = \overline{h_{10}(\omega)}$, is a complex valued function as the cross spectral density function.

First we generated 500 observations $\mathbf{Y}_t,\ t = 1, 2, ..., 500$ from model (6.1) and collected them in 100 groups, with 5 elements in each group, i.e., $M = 100$, $m = 5$. The estimates $\widehat{\mathbf{h}}_m(\omega_l)$ and $\widehat{\mathbf{h}}_{m,cap}(\omega_l)$ are calculated using the formulae (5.3) and (5.4) with $\omega_j = \frac{2\pi j}{m}$, $j = 0, ..., m-1$. The above estimates are computed at the frequencies $\omega_l = l\pi$, $l = 0(0.1)1$. Figure 2 shows the logarithm of theoretical density, truncated estimate and Capon estimate of $h_{00}(\omega_l)$, respectively. The graph related to $h_{11}(\omega)$ is similar.

As we see in Figure 2, the truncated estimate is closer to the theoretical form for most all frequencies. We may use the following criteria for comparison

$$Q = \frac{1}{11}\sum_{l=0(0.1)1}[(h_{00}(\omega_l) - \widehat{h}_{00}(\omega_l))^2 + (h_{11}(\omega_l) - \widehat{h}_{11}(\omega_l))^2] \tag{6.3}$$

After calculating Q for two types of estimates, we obtain that $Q_{truncate} = 4.0319$ and $Q_{Capon} = 4.1430$. On the basis of this criterion, truncated estimate seems to be preferable to Capon's estimate.

7 Numerical examples

In this section we illustrate above methodology with two examples. First, we consider a periodic autoregression (PAR) model with period T. These models are introduced by Jones and Brelsford (1967). The following example is a PAR model with period $T = 2$,

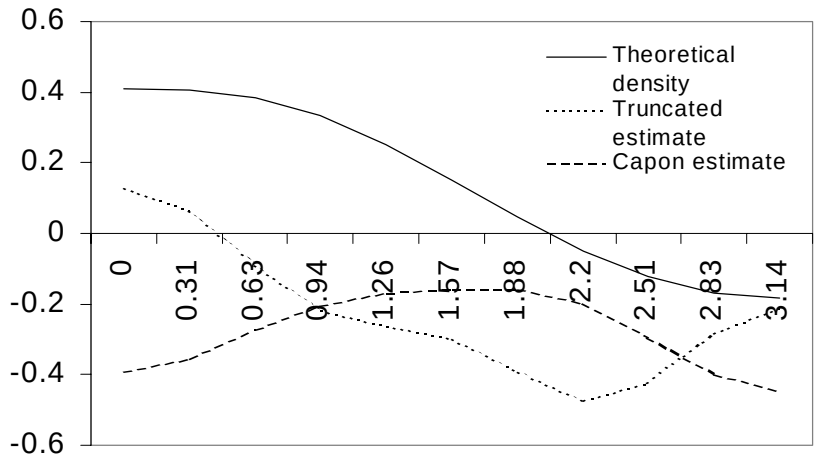

Figure 2: The logarithm of theoretical density, truncated estimate and Capon estimate of $h_{00}(\omega_l)$, $\omega_l = l\pi$, $l = 0(0.1)1$

$$X_t = a(t)X_{t-1} + e_t, \tag{7.1}$$

where $\{e_t\}$ is a sequence of independent random variables (periodic white noise), where for each t,

$$e_t \sim \begin{cases} N(0,3), & \textit{when } t \textit{ is odd} \\ N(0,4), & \textit{when } t \textit{ is even} \end{cases} \qquad \text{and} \qquad a(t) = \begin{cases} 0.8, & \textit{when } t \textit{ is odd} \\ 0.1, & \textit{when } t \textit{ is even} \end{cases}$$

Hurd, Makagon and Miamee (1998) derived an explicit form of the spectral density of the time series X_t satisfying the (7.1). For $k = 0, ..., T-1$, the spectral density that concentrated on diagonals is given by

$$f_k(\omega) = \sum_{l=0}^{T-1} |1 - P\exp(iT(\omega + 2l\pi/T)|^{-2}\, \widehat{J}_l(\omega + 2l\pi/T)\overline{\widehat{J}_{l-k}(\omega + 2l\pi/T)}\sigma_l^2, \tag{7.2}$$

where $J_n(\omega) = \sum_{k=0}^{T-1} C_{n-k+1}^n \exp(-ik\omega)$, $\widehat{J}_j(\omega) = \frac{1}{T}\sum_{n=0}^{T-1} J_n(\omega)\exp(2i\pi jn/T)$, $j \in Z$, $C_r^s = \prod_{j=r}^{s} a(j)$ for $r \leq s$ and equal to 1 for $r > s$, $P = \prod_{j=0}^{T-1} a(j)$ and $\sigma_l^2 = Ee_l^2$.

After some simplification for $T = 2$, for $k = 0, 1$,we obtain

$$f_k(\omega) = 3\left|1 - .08\exp(2i\omega)\right|^{-2} \widehat{J}_0(\omega)\overline{\widehat{J}_k(\omega)} + 4\left|1 - .08\exp(2i(\omega+\pi))\right|^{-2} \widehat{J}_1(\omega+\pi)\overline{\widehat{J}_{1-k}(\omega+\pi)}. \tag{7.3}$$

When $k = 0$, $\widehat{J}_0(\omega) = 1 + 0.45\exp(-i\omega)$, $\widehat{J}_0(\omega+\pi) = 1 - 0.45\exp(-i\omega)$ and when $k = 1$, we have $\widehat{J}_1(\omega+\pi) = \widehat{J}_{-1}(\omega) = 0.35\exp(-i\omega)$. So we have

$$f_0(\omega) = 3\left|1 - .08\exp(2i\omega)\right|^{-2} \left|1 + 0.45\exp(-i\omega)\right|^2 + 4\left|1 - .08\exp(2i(\omega+\pi))\right|^{-2} \left|0.35\exp(-i\omega)\right|^2. \tag{7.4}$$

We compare this theoretical form with our estimates. As noted by Pagano (1978), there is a two-dimensional stationary autoregression process, $\mathbf{Y}_t = \begin{pmatrix} X_{tT} \\ X_{tT+1} \end{pmatrix}$, associated with this periodic autoregression (PAR) model . First we generated 1000 observations from X_t and then using these observations we constructed a 2-dimensional stationary time series $\mathbf{Y}_t$ $(t = 1, 2, ..., 500)$. We consider these in 100 groups, with 5 elements in each group, i.e., $M = 100$, $m = 5$. The estimates $\widehat{\mathbf{h}}_m(\omega_l)$ and $\widehat{\mathbf{h}}_{m,cap}(\omega_l)$ are calculated using the formulae (5.3) and (5.4) with $\omega_j = \frac{2\pi j}{m}$, $j = 0, ..., m-1$. The above estimates are computed at the frequencies $\omega_l = l\pi$, $l = 0(0.1)1$. And then using relations (5.5) and (5.6) we computed $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$, the truncated estimate and Capon estimate of spectral density matrix of periodically correlated X_t. The estimates of $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$ are given in the following table

ω_l	$\widehat{\mathbf{f}}_m(\omega_l)$	$\widehat{\mathbf{f}}_{m,cap}(\omega_l)$
0	$\begin{pmatrix} 2.3871 & .0082 \\ .0082 & .8001 \end{pmatrix}$	$\begin{pmatrix} 1.7929 & -.2618 \\ -.2618 & .6021 \end{pmatrix}$
0.1π	$\begin{pmatrix} 2.2446 & -.0406-.1316i \\ -.0406+.1316i & .9183 \end{pmatrix}$	$\begin{pmatrix} 1.6528 & -.2311-.1121i \\ -.2311+.1121i & .6601 \end{pmatrix}$
0.2π	$\begin{pmatrix} 1.9660 & -.1559-.2403i \\ -.1559+.2403i & 1.1724 \end{pmatrix}$	$\begin{pmatrix} 1.4468 & -.1825-.2312i \\ -.1825+.2312i & .8465 \end{pmatrix}$
0.3π	$\begin{pmatrix} 1.7823 & -.2701-.3535i \\ -.2701+.3535i & 1.3844 \end{pmatrix}$	$\begin{pmatrix} 1.3724 & -.1669-.3689i \\ -.1669+.3689i & 1.1077 \end{pmatrix}$
0.4π	$\begin{pmatrix} 1.6773 & -.3300-.4749i \\ -.3300+.4749i & 1.4980 \end{pmatrix}$	$\begin{pmatrix} 1.3545 & -.1815-.4872i \\ -.1815+.4872i & 1.3017 \end{pmatrix}$
0.5π	$\begin{pmatrix} 1.4712 & -.3261-.4742i \\ -.3261+.4742i & 1.5562 \end{pmatrix}$	$\begin{pmatrix} 1.1811 & -.1594-.4292i \\ -.1594+.4292i & 1.3768 \end{pmatrix}$
0.6π	$\begin{pmatrix} 1.1570 & -.2813-.2462i \\ -.2813+.2462i & 1.5679 \end{pmatrix}$	$\begin{pmatrix} .9326 & -.0955-.1779i \\ -.0955+.1779i & 1.4268 \end{pmatrix}$
0.7π	$\begin{pmatrix} .9470 & -.2229+.0817i \\ -.2229-.0817i & 1.5197 \end{pmatrix}$	$\begin{pmatrix} .8480 & -.0587+.1190i \\ -.0587-.1190i & 1.5245 \end{pmatrix}$
0.8π	$\begin{pmatrix} .9746 & -.1684+.2647i \\ -.1684-.2647i & 1.4670 \end{pmatrix}$	$\begin{pmatrix} .8805 & -.0920+.2717i \\ -.0920-.2717i & 1.4485 \end{pmatrix}$
0.9π	$\begin{pmatrix} 1.1238 & -.1291+.2502i \\ -.1291-.2502i & 1.4736 \end{pmatrix}$	$\begin{pmatrix} .8878 & -.1520+.2273i \\ -.1520-.2273i & 1.2320 \end{pmatrix}$
π	$\begin{pmatrix} 1.2131 & -.1145+.2189i \\ -.1145-.2189i & 1.4814 \end{pmatrix}$	$\begin{pmatrix} .8976 & -.1775+.1935i \\ -.1775-.1935i & 1.1322 \end{pmatrix}$

Finally, one can determine the total amount of measure that concentrated on each diagonal in Figure 1, by using the following relations (see (2.2)),

$$\widehat{f}_0(\omega) = \begin{cases} 2\widehat{f}_{00}(2\omega) & ;\ 0 \le \omega < \pi \\ 2\widehat{f}_{11}(2(\omega-\pi)) & ;\ \pi \le \omega \le 2\pi \end{cases} \tag{7.5}$$

and

$$\widehat{f}_1(\omega) = \begin{cases} 2\widehat{f}_{01}(2\omega) & ;\ 0 \le \omega < \pi \\ 2\widehat{f}_{10}(2(\omega-\pi)) & ;\ \pi \le \omega \le 2\pi. \end{cases} \tag{7.6}$$

Also, note that we have $f_{-1}(\omega) = f_1(\omega)$.

The regularity condition of X_t is determined by the diagonal part of the spectral measure μ of X_t, or its spectral density $f_0(\omega)$. Miamee (1990) has shown that the PC time series X_t is deterministic if

$$\int_0^{2\pi} \ln f_0(\omega) d\omega = -\infty.$$

In Figure 3, the logarithm of two estimates of the $f_0(\omega)$ with the logarithm of the theoretical spectral density $f_0(\omega)$ given by (7.4) are plotted. If we approximate the above integral by $\mathcal{L} = \sum_{l=0(0.1)1} \ln f_0(\omega_l)\Delta\omega_l$, and compute this

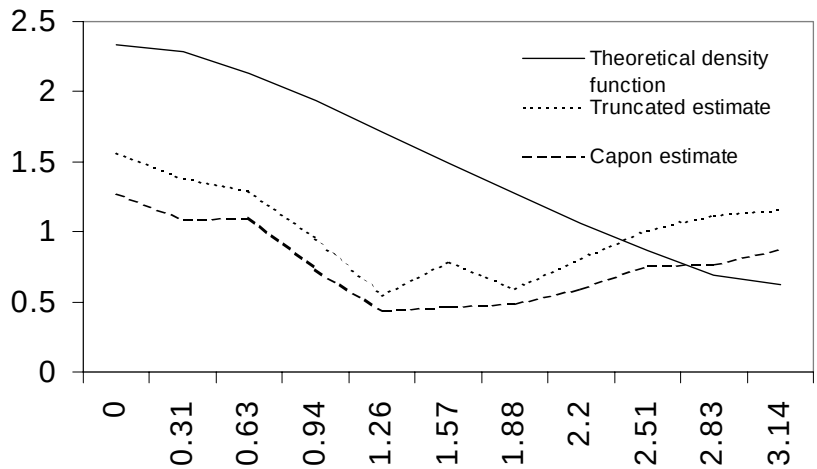

Figure 3: The logarithm of theoretical density, truncated estimate and Capon estimate of $f_0(\omega_l)$, $\omega_l = l\pi$, $l = 0(0.1)1$

for theoretical, truncated and Capon estimate, then we have $\mathcal{L}_{Theoretical} = 10.3042$, $\mathcal{L}_{Truncated} = 7.0474$, $\mathcal{L}_{Capon} = 5.3832$, and so we conclude that this process is purely nondeterministic. By virtue of these calculations, we see that the truncated estimate is closer to the theoretical form.

As an application of these methods for estimating the spectral density matrix of a real PC time series, let us consider time series of mean monthly flows of Fraser River at Hope, BC, from January 1913 to December 1990. This time series has been used by several authors for fitting PC time series models, and is known to have periodicity of 12 months, $T = 12$, (Vecchia and Ballerini (1991), Mcleod, (1994) and see Hipel and Mcleod (1994)). The data is plotted in Figure 4.

There is a twelve-dimensional stationary time series $\mathbf{Y}_t$, associated with this time series. The number of observations are 936 and so we have 78 observations from $\mathbf{Y}_t$, that we consider in $M = 26$ groups, with $m = 3$ elements in each group. The estimates $\widehat{\mathbf{h}}_m(\omega_l)$ and $\widehat{\mathbf{h}}_{m,cap}(\omega_l)$ are calculated using the formulas (5.3) and (5.4) with $\omega_j = \frac{2\pi j}{m}$, $j = 0, ..., m-1$. The above estimates are computed at the frequencies $\omega_l = l\pi$, $l = 0(0.1)1$. And then using relation (5.5) and (5.6) we computed $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$, the truncated estimate and Capon estimate of spectral density matrix of periodically correlated X_t. The estimates of $\widehat{\mathbf{f}}_m(\omega_l)$ and $\widehat{\mathbf{f}}_{m,cap}(\omega_l)$ are both 12×12 matrices and for reasons of space, we don't include these, but we plot the logarithm of $\widehat{f}_0(\omega)$ given by (7.5) in Figure 5.

From these illustrations, we see that the estimators are satisfactory, and the truncated estimate seems to have more interesting features correspond to the

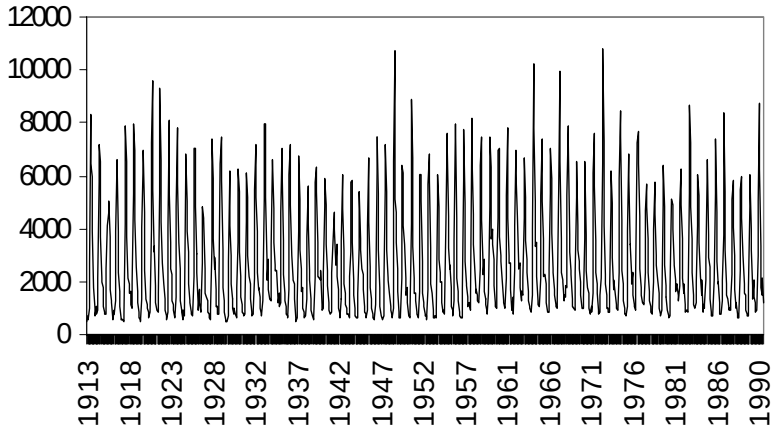

Figure 4: Mean monthly flows of Fraser River at Hope, BC, from January 1913 to December 1990

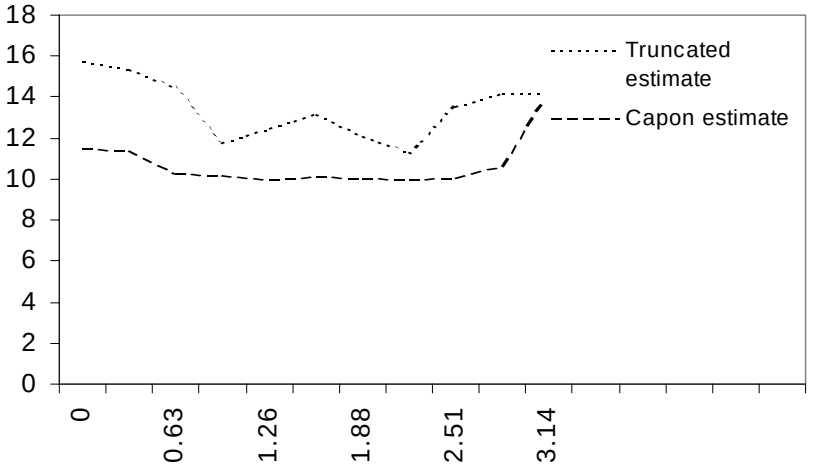

Figure 5: The logarithm of two estimates $\widehat{f}_0(\omega_l)$, $\omega_l = l\pi$, $l = 0(0.1)1$

Capon estimate, as we discussed in section 6.

Acknowledgements

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