MANCHESTER
1824

The University
of Manchester

Flandersâ theorem for many matrices under commutativity assumptions

De Teran, Fernando and Nakatsukasa,
Yuji and Noferini, Vanni

2013

MIMS EPrint: **2013.20**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

Flanders' theorem for many matrices under commutativity assumptions

Fernando de Terán*, Yuji Nakatsukasa†, and Vanni Noferini‡

Abstract

We analyze the relationship between the Jordan canonical form of products, in different orders, of k square matrices $A_1, \ldots, A_k$. Our results extend some classical results by H. Flanders. Motivated by a generalization of Fiedler matrices, we study permuted products of $A_1, \ldots, A_k$ under the assumption that the graph of non-commutativity relations of $A_1, \ldots, A_k$ is a forest. Under such condition, we show that the Jordan structure of all nonzero eigenvalues is the same for all permuted products. For the eigenvalue zero, we obtain an upper bound on the difference between the sizes of Jordan blocks for any two permuted products, and we show that this bound is attainable. For $k = 3$ we show that, moreover, the bound is exhaustive.

AMS subject classification: 05A05, 15A18, 15A21.
Keywords: Eigenvalues, Jordan canonical form, Segré characteristic, product of matrices.

1 Introduction

The *Jordan canonical form* (JCF) is the canonical form under similarity of square matrices. It consists of direct sum of *Jordan blocks* associated with eigenvalues, and it is unique up to permutation of these blocks [7, §3.1]. We assume throughout the paper that, for a given eigenvalue λ, the Jordan blocks at λ in the JCF are given in non-increasing order of their sizes. In 1951 Flanders published the following result [3, Theorem 2]:

Theorem 1.1. *If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, then the JCFs of AB and BA may differ only in the sizes of the Jordan blocks at 0. Moreover, the difference between two corresponding sizes is at most one. Conversely, if the JCFs of $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ satisfy these properties, then $M = AB$ and $N = BA$, for some A, B.*

*Universidad Carlos III de Madrid. Avda Universidad, 30. 28911, Leganés, Spain. fteran@math.uc3m.es. Partially supported by the Ministerio de Economía y Competitividad of Spain through grants MTM-2009-09281 and MTM-2012-32542,

†School of Mathematics, University of Manchester, England, M13 9PL. yuji.nakatsukasa@manchester.ac.uk. Partially supported by EPSRC grant EP/I005293/1

‡School of Mathematics, University of Manchester, England, M13 9PL. vanni.noferini@manchester.ac.uk. Partially supported by ERC Advanced Grant MATFUN-267526

We call such a pair (M, N) as in Theorem 1.1 a *Flanders pair*, and we say that there is a *Flanders bridge* between M and N. Theorem 1.1 has been revisited several times and re-proved using different techniques [1, 8, 9, 10, 11, 13].

In this paper, we investigate what happens if, instead of two matrices, we have products of k matrices, $A_1, A_2, \ldots, A_k \in \mathbb{C}^{n\times n}$. We refer to products of $A_1, \ldots, A_k$, in any order and with no repetitions of the factors, as *permuted products*.

We assume $A_1, \ldots, A_k$ are all $n \times n$ to ensure all permuted products are well defined. An important difference between $k = 2$ and $k > 2$ is that, without any assumption on $A_1, \ldots, A_k$, the products of $A_1, \ldots, A_k$ have, in general, completely different eigenvalues for different permutations. One exception is the eigenvalue 0, as if 0 is an eigenvalue of some product of $A_1, \ldots, A_k$, then it must be an eigenvalue of any other product of $A_1, \ldots, A_k$. Indeed, in Theorem 1.1 the eigenvalue 0 is treated exceptionally, with nontrivial results on the sizes of the Jordan blocks at 0. However, by the simple example $A = \mathrm{diag}(1, 1/2, \ldots, 1/n)$, $B = -J_n(-1)^T$, $C = (AB)^{-1}J_n(0)$, where $J_n(\lambda)$ is the $n \times n$ Jordan block at the eigenvalue λ [7, Def. 3.1.1], it can be seen that the sizes of the Jordan blocks at $\lambda = 0$ in ABC and CBA can be arbitrarily different. Hence, a general statement on the eigenvalues or the sizes of their Jordan blocks for $k \geq 3$ matrices appears to be impossible.

In [2], Fiedler introduced a decomposition of a companion matrix into a product of n matrices, $C = \prod_{i=1}^{n} M_i$, and showed that the product of the matrices M_i in any order is similar to C, hence all permuted products have the same JCF. For the nonzero eigenvalues, this is precisely what happens when $k = 2$. This motivates us to examine general conditions that allow an extension of Theorem 1.1 for nonzero eigenvalues to the case $k > 2$. The Fiedler factors M_i have the following properties:

1. Commutativity: $M_iM_j = M_jM_i$, if $|i - j| > 1$.

2. M_i are all nonsingular, possibly except for M_n.

Fiedler's results suggest that Theorem 1.1 might be naturally extended to three or more matrices when appropriate commutativity conditions on A_i hold. Indeed, we will show that if the graph of non-commutativity relations is a forest (see Section 4), then all permuted products have the same Jordan blocks for nonzero eigenvalues. This commutativity assumption generalizes (1), and imposes no requirement when $k = 2$, i.e., the two matrices can be arbitrary, thus recovering Theorem 1.1. We impose no nonsingularity condition such as (2) above because this imposes similarity, i.e., also the Jordan blocks at zero must be the same: an undesirable restriction given our goal of generalizing Flanders' theorem. Indeed, Theorem 1.1 shows that the difference in the sizes of Jordan blocks at zero is at most 1 when $k = 2$. One key result of this paper is that, for general k, under our commutativity conditions the difference is bounded by $k - 1$, and that this bound is attainable.

For $k = 3$ matrices, our condition reduces to the requirement that one pair commutes, and we prove that the allowable sizes are exhaustive. More precisely, we prove that given two lists of these allowable sizes, there are matrices A, B, C such that the Jordan canonical forms of ABC and CBA consist of Jordan blocks at $\lambda = 0$ whose sizes match those in the respective lists.

Several previous papers have addressed extensions of Flanders' results to many matrices. For example, [6] examines cyclic permutations, and [4] derives conditions for the products to have the same trace (and eigenvalues/Jordan form, though their treatment is less complete) with focus on $k = 3$ or 2×2 matrices. Unlike in previous studies, we deal with *any* permutation and arbitrary n and $k \geq 3$, and work with commutativity conditions guaranteeing that the Jordan structures for nonzero eigenvalues coincide for all permutations.

The paper is organized as follows. Section 2 reviews basic notions and previous results. In Section 3 we analyze permuted products of $k = 3$ matrices. Section 4 discusses the case $k > 3$, which requires the use of permutations and graphs. We conclude, in Section 5, with some open problems related to this work.

2 Notation, definitions and previous results

We follow the standard notation I_n and 0_n to denote, respectively, the $n \times n$ identity and null matrices. Given a square matrix $M \in \mathbb{C}^{n\times n}$, $\Lambda(M)$ denotes the spectrum (set of eigenvalues counting multiplicity) of M; $\operatorname{diag}(A_1, \ldots, A_k)$ is the block-diagonal matrix whose diagonal blocks are $A_1, \ldots, A_k$, in this order (that is, the *direct sum* of $A_1, \ldots, A_k$).

For a given $\lambda \in \mathbb{C}$, the *Segré characteristic of* M *at* λ, denoted by $\mathcal{S}_\lambda(M)$, is the list of the sizes of the Jordan blocks associated with λ in the JCF of M. In this paper we see it as an infinite nonincreasing sequence of nonnegative integers, by attaching an infinite sequence of zeros at the end. Note that the Segré characteristic at any λ is uniquely determined, and that this definition includes also those complex numbers that are not eigenvalues of M, though in this case all entries in the Segré characteristic are zeroes.

We use boldface for lists of nonnegative integers. Given two (possibly infinite) sequences of integers $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$ and $\boldsymbol{\mu}' = (\mu_1', \mu_2', \ldots)$, we will often refer to the standard ℓ^∞ and ℓ^1 norms, which we denote by, respectively, $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

Given k matrices $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$, by a *permuted product of* $A_1, \ldots, A_k$ we mean any of the products of $A_1, \ldots, A_k$ in all possible orders, without repetitions. The set of permuted products of $A_1, \ldots, A_k$ is denoted by $\mathcal{P}(A_1, \ldots, A_k)$, and we use the notation $\Pi(A_1, \ldots, A_k)$ (or just Π when there is no risk of confusion) for the elements in $\mathcal{P}(A_1, \ldots, A_k)$. For instance, for three matrices A, B, C, we have $\mathcal{P}(A, B, C) = \{ABC, ACB, BAC, BCA, CAB, CBA\}$.

The relation $\mathcal{R}$ on $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ given by "$M\mathcal{R}N$ if and only if (M, N) is a Flanders pair" is not an equivalence relation, since $\mathcal{R}$ is not transitive. In Corollary 3.6 we give a characterization of the non-transitivity of this relation for pairs of matrices with the same size.

The following elementary result can be easily verified:

Lemma 2.1. *If $M, N \in \mathbb{C}^{n\times n}$ are similar, then (M, N) is a Flanders pair.*

The converse of Lemma 2.1 is not true in general, as we shall see below. However, if M, N are nonsingular, then (M, N) is a Flanders pair if and only if M is similar to N. This is also an immediate consequence of Theorem 1.1.

Flanders pairs connecting three matrices M, N, Q in the form $(M, N), (N, Q)$ are closely related to our problem. The following direct consequence of [3, Theorem 2] establishes some elementary features of these pairs.

Corollary 2.2. *If $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{q\times q}$ are such that (M, N) and (N, Q) are Flanders pairs, then*

(i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(Q)$, *for all $\lambda \neq 0$, and*

(ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(Q)\|_\infty \leq 2$.

Flanders also shows that, given two lists of nonnegative integers whose distance is at most 1, there is a Flanders pair whose Segré characteristics at zero coincide with these two lists (see also [10, Cor. 3.4]). We provide an extension of this result to three matrices in Theorem 3.4.

3 The case of three matrices

Unlike what happens for two matrices, given three matrices, $A, B, C \in \mathbb{C}^{n\times n}$, the spectra of two different permuted products of A, B, C may be completely different. To verify this, one may just take three random matrices A, B, C and compute the eigenvalues of ABC and ACB. This is related to the fact that two similar matrices, as BC and CB are if one of B, C is nonsingular, may give two matrices with completely different spectra when multiplied on the left by a third matrix A. To what extent may the spectra of different permutation products of three given matrices differ? One restriction is that the determinants must all be the same, which implies that if 0 is an eigenvalue then it must be shared by all permuted products. Below is a restatement, with a more straightforward proof, of [5, Theorem 4], which shows that, without any additional assumptions, this condition gives the only restriction on the spectra for nonsingular A, B, C.

Theorem 3.1. *Let $\Lambda_1 = \{\lambda_{11}, \ldots, \lambda_{n1}\}$ and $\Lambda_2 = \{\lambda_{12}, \ldots, \lambda_{n2}\}$ be two sets of n nonzero complex numbers, eventually repeated. If $\lambda_{11}\cdots\lambda_{n1} = \lambda_{12}\cdots\lambda_{n2}$, then there are three matrices $A, B, C \in \mathbb{C}^{n\times n}$, such that $\Lambda(ABC) = \Lambda_1$ and $\Lambda(ACB) = \Lambda_2$.*

Proof. By Lemma 2.1, the problem reduces to finding two similar matrices $M, N \in \mathbb{C}^{n\times n}$, and a third matrix $A \in \mathbb{C}^{n\times n}$, such that $\Lambda(AM) = \Lambda_1$ and $\Lambda(AN) = \Lambda_2$. This can be done using only diagonal matrices. More precisely, set $r_1 \neq 0$ (arbitrary), $a_1 := \lambda_{11}/r_1$ and, recursively for $i = 2, \ldots, n$, $r_i := \lambda_{i2}/a_{i-1}$, $a_i := \lambda_{i1}/r_i$. Note that, with these definitions, we have

$$\lambda_{n2} = \frac{\lambda_{11}\cdots\lambda_{n1}}{\lambda_{12}\cdots\lambda_{n-1,2}} = \frac{(a_1r_1)(a_2r_2)\cdots(a_nr_n)}{(a_1r_2)(a_2r_3)\cdots(a_{n-1}r_n)} = a_nr_1.$$

Hence, if we set $M = \mathrm{diag}(r_1, r_2, \ldots, r_n)$, $N = \mathrm{diag}(r_2, r_3, \ldots, r_n, r_1)$, and $A = \mathrm{diag}(a_1, \ldots, a_n)$, then M is similar to N, and $AM = \mathrm{diag}(\lambda_{11}, \ldots, \lambda_{n1})$, $AN = \mathrm{diag}(\lambda_{12}, \ldots, \lambda_{n2})$, as required. $\square$

Under the conditions of the statement of Theorem 3.1, by Theorem 1.1 we have $\Lambda(ABC) = \Lambda(CAB) = \Lambda(BCA) = \Lambda_1$, and $\Lambda(ACB) = \Lambda(BAC) = \Lambda(CBA) = \Lambda_2$. Moreover, as a consequence of Theorem 1.1, the set of permuted products is partitioned into two classes, namely: $\mathcal{C}_1 = \{ABC, BCA, CAB\}$, and $\mathcal{C}_2 = \{ACB, BAC, CBA\}$. Any two products in each class are related by a "cyclic permutation", so they form a Flanders pair. Hence, we can relate the JCFs of these permuted products. The remaining question is to relate the JCFs of permuted products in $\mathcal{C}_1$ with the ones in $\mathcal{C}_2$. Theorem 3.1 shows that, if A, B, C are nonsingular, there may be no relationship at all between the spectra of products in different classes.

Motivated by the work of Fiedler, here we require that at least two of A, B, C commute. As we see in Section 4, if we consider symbolic products of an arbitrary number of matrices, commutativity conditions allow us to characterize those cases where any two arbitrary permutations are linked by a sequence of Flanders bridges. In this case, all permuted products have the same Segré characteristic at an arbitrary nonzero complex number.

Proposition 3.2. *Let $A, B, C \in \mathbb{C}^{n\times n}$ be such that at least two of A, B, C commute. Let $\Pi_1, \Pi_2 \in \mathcal{P}(A, B, C)$. Then*

(i) $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, *for all $\lambda \neq 0$, and*

(ii) $\|\mathcal{S}_0(\Pi_1), \mathcal{S}_0(\Pi_2)\|_\infty \leq 2$.

Proof. By Corollary 2.2, it suffices to show that, in the conditions of the statement, one of the following situations occurs:

1. (Π_1, Π_2) is a Flanders pair.
2. There exists $\widetilde{\Pi} \in \mathcal{P}(A, B, C)$ such that $(\Pi_1, \widetilde{\Pi})$ and $(\widetilde{\Pi}, \Pi_2)$ are Flanders pairs.

In the conditions of the statement there are, at most, 4 distinct elements in $\mathcal{P}(A, B, C)$ which give, at most, 6 distinct (non-ordered) pairs of permuted products. Let us assume, without loss of generality, that $AB = BA$. In this case, the elements in $\mathcal{P}(A, B, C)$ (including Π_1 and Π_2) are ABC, ACB, BCA, CAB, and (ABC, BCA), (ABC, CAB), (ACB, CAB) and (BCA, CAB) are Flanders pairs. Hence, one of the situations described above holds for Π_1 and Π_2. $\square$

Lemma 3.3, whose proof is straightforward, is used to prove Theorem 3.4:

Lemma 3.3. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$, $\boldsymbol{\mu}' = (\mu'_1, \mu'_2, \ldots) \in \ell^1$ be two sequences of nonnegative integers. Suppose that*

(i) $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty = 2$, *and*

(ii) $\|\boldsymbol{\mu}\|_1 = \|\boldsymbol{\mu}'\|_1 = n$.

Then we may rearrange $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ in such a way that

$$\boldsymbol{\mu} = (\mu_{i_1}, \mu_{i_2}, \mu_{i_3}; \mu_{i_4}, \mu_{i_5}, \mu_{i_6}; \ldots), \qquad \boldsymbol{\mu}' = (\mu'_{i_1}, \mu'_{i_2}, \mu'_{i_3}; \mu'_{i_4}, \mu'_{i_5}, \mu'_{i_6}; \ldots),$$

with

$$\mu_{i_j} + \mu_{i_{j+1}} + \mu_{i_{j+2}} = \mu'_{i_j} + \mu'_{i_{j+1}} + \mu'_{i_{j+2}}, \quad \textit{for all} \quad j \equiv 1 \pmod 3. \tag{1}$$

The main result of this section is an extension of [10, Cor. 3.4] to three matrices A, B, C under the commutativity condition $AC = CA$.

Theorem 3.4. *Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, 0, \dots)$, $\boldsymbol{\mu}' = (\mu_1', \mu_2', \dots, 0, \dots) \in \ell^1$ be two nonincreasing sequences of nonnegative integers such that*

(i) $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \le 2$, *and*

(ii) $\|\boldsymbol{\mu}\|_1 = \|\boldsymbol{\mu}'\|_1 = n$.

Then, there exist three matrices $A, B, C \in \mathbb{C}^{n\times n}$, such that $AC = CA$ and

$$\mathcal{S}_0(ABC) = \boldsymbol{\mu}, \qquad \text{and} \qquad \mathcal{S}_0(CBA) = \boldsymbol{\mu}'.$$

Proof. First, notice that if $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \le 1$, then by Theorem 1.1 there exist $A, B \in \mathbb{C}^{n\times n}$ such that $\mathcal{S}_0(AB) = \boldsymbol{\mu}$ and $\mathcal{S}_0(BA) = \boldsymbol{\mu}'$. In this case, we may take $C = I_n$ and we are done. Hence, we may assume that $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty = 2$. The proof reduces to showing that the statement is true in the following two cases:

(A1) $\boldsymbol{\mu} = (m, n, 0, \dots)$, $\boldsymbol{\mu}' = (m-2, n+2, 0, \dots)$

(A2) $\boldsymbol{\mu} = (m, n, q, 0, \dots)$, $\boldsymbol{\mu}' = (m-2, n+1, q+1, 0, \dots)$,

with $m, n, q \ge 0$ and $m \ge 2$. Indeed, let us assume that the result is true for both (A1) and (A2), and let $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ be as in the statement. By Lemma 3.3, we can rearrange $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ in such a way that they are partitioned as

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_\alpha, 0, \dots), \qquad \text{and} \qquad \boldsymbol{\mu}' = (\boldsymbol{\mu}_1', \dots, \boldsymbol{\mu}_\alpha', 0, \dots),$$

where the pairs $(\boldsymbol{\mu}_i, \boldsymbol{\mu}_i')$, for $i = 1, \dots, \alpha$, are such that $\|\boldsymbol{\mu}_i\|_1 = \|\boldsymbol{\mu}_i'\|_1 =: n_i$ and they either satisfy $\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_i'\|_\infty \le 1$ or are of one the forms (A1), (A2). Now, since the result is true for both (A1) and (A2), and also for tuples of distance at most 1, there are matrices $A_1, B_1, C_1 \in \mathbb{C}^{n_1\times n_1}, \dots, A_\alpha, B_\alpha, C_\alpha \in \mathbb{C}^{n_\alpha \times n_\alpha}$, such that $A_iC_i = C_iA_i$, and $\mathcal{S}_0(A_iB_iC_i) = (\boldsymbol{\mu}_i, 0, \dots), \mathcal{S}_0(C_iB_iA_i) = (\boldsymbol{\mu}_i', 0, \dots)$, for $i = 1, \dots, \alpha$. Then the matrices $A = \operatorname{diag}(A_1, \dots, A_\alpha)$, $B = \operatorname{diag}(B_1, \dots, B_\alpha)$ and $C = \operatorname{diag}(C_1, \dots, C_\alpha)$ satisfy $AC = CA$ and $\mathcal{S}_0(ABC) = \boldsymbol{\mu}, \mathcal{S}_0(CBA) = \boldsymbol{\mu}'$.

It remains to prove that the result is true in cases (A1) and (A2). Consider (A1) first. Denote by E_{ij} the matrix, of the appropriate size, whose (i, j) entry is equal to 1 and the remaining entries are zero. Set $A = \operatorname{diag}(I_{m-1}, 0, I_n), B = J_{m+n}(0) + E_{m+n,1}$, and $C = \operatorname{diag}(0, I_{m+n-1})$. Clearly we have $AC = CA$. Direct computation gives $ABC = \operatorname{diag}(J_m(0), J_n(0))$, and $CBA = \operatorname{diag}(0, J_{m-2}(0), J_{n+1}(0)) + E_{m+n,1}$. Now, $\operatorname{diag}(0, J_{n+1}(0)) + E_{n+2,1}$ is similar to $J_{n+2}(0)$, because its only eigenvalue is 0 and its rank deficiency is one. Consequently, the JCF of CBA is $\operatorname{diag}(J_{m-2}(0), J_{n+2}(0))$, so $\mathcal{S}_0(ABC) = (m, n, 0, \dots)$ and $\mathcal{S}_0(CBA) = (m-2, n+2, 0, \dots)$, as required.

Next consider (A2). Set $A = \operatorname{diag}(0, I_{m+n+q-1}), C = \operatorname{diag}(I_{n+q+1}, 0, I_{m-2})$, and $B = \operatorname{diag}(J_{q+1}(0), J_{m+n-1}(0)) + E_{m+n+q,1}$, for which $AC = CA$. Direct computation gives $ABC = \operatorname{diag}(0, J_q(0), J_n(0), J_{m-1}(0)) + E_{m+n+q,1}$, and $CBA = \operatorname{diag}(J_{q+1}(0), J_{n+1}(0), J_{m-2}(0))$. Again, we see that $\operatorname{diag}(0, J_q(0), J_n(0), J_{m+1}(0)) +$

$E_{m+n+q,1}$ is permutation similar to $\mathrm{diag}(J_q(0), J_n(0), \mathrm{diag}(0, J_{m-1}(0)) + E_{m,1})$. Since, as before, $\mathrm{diag}(0, J_{m-1}(0)) + E_{m,1}$ is similar to $J_m(0)$, we conclude that $\mathcal{S}_0(ABC) = (m, n, q, 0, \ldots)$ and $\mathcal{S}_0(CBA) = (m-2, n+1, q+1, 0, \ldots)$, as wanted. $\square$

Remark 3.5. *If $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty = 2$, then the matrices A, B, C constructed in the proof of Theorem 3.4 have the property that none of the pairs (A, B) and (B, C) commute, so there is exactly one commutativity relation in this case. In graph theoretical terminology (see Section 4), the graph of non-commutativity relations is a tree.*

Our last result in this section concerns the "non-trasitivity" of Flanders pairs.

Corollary 3.6. *Let $M, Q \in \mathbb{C}^{n\times n}$. Then, the following conditions are equivalent:*

(a) *There exists $N \in \mathbb{C}^{n\times n}$ such that (M, N) and (N, Q) are Flanders pairs.*

(b) *$\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(Q)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(Q)\|_\infty \leq 2$.*

(c) *There are three matrices $A, B, C \in \mathbb{C}^{n\times n}$ such that $AC = CA$, M is similar to ABC, and Q is similar to CBA.*

Proof. The implication (a) $\Rightarrow$ (b) is Corollary 2.2. Suppose that (b) holds. Without loss of generality, we may assume that M and Q are given in JCF, so that $M = \mathrm{diag}(M_r, M_s)$, and $Q = \mathrm{diag}(Q_r, Q_s)$ where M_r, Q_r contain Jordan blocks associated with nonzero eigenvalues, and M_s, Q_s are Jordan blocks for $\lambda = 0$. By hypothesis, we have $M_r = Q_r$ and $\|\mathcal{S}_0(M_s) - \mathcal{S}_0(Q_s)\|_\infty \leq 2$. Using Theorem 3.4 with $\boldsymbol{\mu} = \mathcal{S}_0(M_s)$ and $\boldsymbol{\mu}' = \mathcal{S}_0(Q_s)$, we see that there exist A_s, B_s, C_s such that $A_sC_s = C_sA_s$, $A_sB_sC_s = M_s$, and $C_sB_sA_s = Q_s$. The block diagonal matrices $A = \mathrm{diag}(I_m, A_s), B = \mathrm{diag}(M_r, B_s), C = \mathrm{diag}(I_m, C_s)$, where m is the size of both M_r and Q_r, fulfill the conditions in (c).

Finally, suppose that (c) holds. Let $N = BCA$. Then (M, N) is clearly a Flanders pair and, by the condition $AC = CA$, so is the pair (N, Q). $\square$

4 More than three matrices

For permutations in Σ_k, the symmetric group of $\{1, \ldots, k\}$, we use the cyclic notation $\sigma = (i_1 i_2 \ldots i_s)$ to mean that $\sigma(i_j) = i_{j+1}$, for $j = 1, \ldots, s-1$, $\sigma(i_s) = i_1$, and $\sigma(i) = i$ for $i \neq i_1, \ldots, i_s$.

An element in $\mathcal{P}(A_1, \ldots, A_k)$ is related to a permutation $\sigma \in \Sigma_k$ in the form $A_{\sigma^{-1}(1)} A_{\sigma^{-1}(2)} \cdots A_{\sigma^{-1}(k)}$, that is, $\sigma(i)$ is the position of the factor A_i in the permuted product. In this case, we write $\Pi_\sigma := A_{\sigma^{-1}(1)} A_{\sigma^{-1}(2)} \cdots A_{\sigma^{-1}(k)}$.

Definition 4.1. *Given a permutation $\sigma \in \Sigma_k$, a* cyclic permutation *of σ is another permutation of the form $(12\ldots k)^\ell \sigma$, for some $\ell \geq 0$. We say that σ, τ* are in the same equivalence class up to cyclic permutations *if τ is a cyclic permutation of σ.*

Accordingly, given a permuted product $\Pi_\sigma \in \mathcal{P}(A_1, \ldots, A_k)$, a cyclic permutation *of Π_σ is another permuted product of the form $\Pi_\tau \in \mathcal{P}(A_1, \ldots, A_k)$, with $\tau = (12\ldots k)^\ell \sigma$, for some $\ell \in \mathbb{N}$. If Π_σ is a cyclic permutation of Π_τ, then Π_σ and Π_τ are* cyclically equivalent*, and we write $\Pi_\sigma \sim_C \Pi_\tau$.*

We note that $\sim_C$ is, indeed, an equivalence relation. Moreover, if $\Pi_{\sigma_1} \sim_C \Pi_{\sigma_2}$, then $(\Pi_{\sigma_1}, \Pi_{\sigma_2})$ is a Flanders pair. Conversely, if $(\Pi_{\sigma_1}, \Pi_{\sigma_2})$ is a Flanders pair for *all* $A_1, \ldots, A_k$ (that is, "symbolically"), then Π_{σ_1} is a cyclic permutation of Π_{σ_2}.

Definition 4.2. *Given two permutations $\sigma, \tau \in \Sigma_k$, we say that $i_1, \ldots, i_g$, with $1 \le i_1, \ldots, i_g \le k$,* have the same order in σ_1 and σ_2 up to cyclic permutations *if $i_1, \ldots, i_g$ appear in the same order in both $\widetilde{\sigma}_1 := (12\ldots k)^\alpha \sigma_1$ and $\widetilde{\sigma}_2 := (12\ldots k)^\beta \sigma_2$, for some $\alpha, \beta \ge 0$.*

Accordingly, given $\Pi_{\sigma_1}, \Pi_{\sigma_2} \in \mathcal{P}(A_1, \ldots, A_k)$, we say that $A_{i_1}, \ldots, A_{i_g}$ have the same cyclic order *in both Π_{σ_1} and Π_{σ_2} if $i_1, \ldots, i_g$ have the same order in σ_1 and σ_2 up to cyclic permutations.*

4.1 Inverse eigenvalue problem

We start with an observation that characterizes Σ_k up to cyclic permutations.

Lemma 4.3. *Let $\sigma, \tau \in \Sigma_k$ be two permutations. Then σ and τ are in the same class up to cyclic permutations if and only if all triples i_1, i_2, i_3, with $1 \le i_1, i_2, i_3 \le k$ have the same order in σ and τ up to cyclic permutations.*

Proof. If $\sigma = (12\ldots k)^\ell \tau$, for some $\ell \ge 0$, then it is clear that each triple i_1, i_2, i_3 has the same order up to cyclic permutations in both σ and τ.

Conversely, assume that every triple i_1, i_2, i_3 has the same order up to cyclic permutations in σ and τ. Let $\alpha, \beta \ge 0$ be such that $\widetilde{\sigma} := (12\ldots k)^\alpha \sigma$ and $\widetilde{\tau} := (12\ldots k)^\beta \tau$ satisfy $\widetilde{\sigma}(1) = 1 = \widetilde{\tau}(1)$. Suppose $\widetilde{\sigma} \ne \widetilde{\tau}$ and let $\nu = \min\{i : \widetilde{\sigma}(i) \ne \widetilde{\tau}(i)\}$. Then $1, \widetilde{\sigma}(\nu), \widetilde{\tau}(\nu)$ do not have the same order up to cyclic permutations in $\widetilde{\sigma}$ and $\widetilde{\tau}$, a contradiction. Hence, σ and τ are in the same class up to cyclic permutations. $\square$

We next show that it is possible that *any* two permuted products Π_1, Π_2 have different spectra unless $\Pi_1 \sim_C \Pi_2$.

Proposition 4.4. *For each $k \ge 3$, there exist matrices, $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$ such that for any two permuted products Π_1 and Π_2 belonging to different equivalence classes of $\mathcal{P}(A_1, \ldots, A_k)$ under $\sim_C$, $\Lambda(\Pi_1)$ and $\Lambda(\Pi_2)$ are different.*

Proof. First, let us order all the $\binom{k}{3}$ triples (i_1, i_2, i_3), with $1 \le i_1 < i_2 < i_3 \le k$ using, for instance, the lexicographic order. This order induces an ordered list of length $3 \cdot \binom{k}{3} = \frac{k(k-1)(k-2)}{2}$, denoted by $\mathcal{L}$, after adjoining all triples in the given order. For instance, for $k = 4$ we get the list: $\mathcal{L} = (1,2,3;1,2,4;1,3,4;2,3,4)$. Now, let $\gamma : \left\{1, 2, \ldots, \frac{k(k-1)(k-2)}{2}\right\} \to \{1, 2, \ldots, k\}$ be the map defined by $\gamma(i) = \mathcal{L}_i$ (the ith number in $\mathcal{L}$). For each $j = 1, \ldots, \binom{k}{3}$, by Theorem 3.1, there are three matrices $\widetilde{A}_{3j-2}, \widetilde{A}_{3j-1}, \widetilde{A}_{3j} \in \mathbb{C}^{2\times 2}$, such that $\Lambda(\widetilde{A}_{3j-2}\widetilde{A}_{3j-1}\widetilde{A}_{3j}) \ne \Lambda(\widetilde{A}_{3j}\widetilde{A}_{3j-1}\widetilde{A}_{3j-2})$. For $i = 1, \ldots, k$, define $A_i = \operatorname{diag}\left(A_{i1}, A_{i2}, \ldots, A_{i,\binom{k}{3}}\right) \in \mathbb{C}^{2\binom{k}{3}\times 2\binom{k}{3}}$, where

$$A_{ij} = \begin{cases} \widetilde{A}_{3(j-1)+r}\,, & \text{if there is some } 1 \le r \le 3 \text{ such that } \gamma(3(j-1)+r) = i, \\ I_2\,, & \text{otherwise.} \end{cases}$$

For instance, for $k=4$ we have $A_1=\operatorname{diag}(\widetilde{A}_1,\widetilde{A}_4,\widetilde{A}_7,I_2)$, $A_2=\operatorname{diag}(\widetilde{A}_2,\widetilde{A}_5,I_2,\widetilde{A}_{10})$, $A_3=\operatorname{diag}(\widetilde{A}_3,I_2,\widetilde{A}_8,\widetilde{A}_{11})$, $A_4=\operatorname{diag}(I_2,\widetilde{A}_6,\widetilde{A}_9,\widetilde{A}_{12})$. Let Π_{σ_1} and Π_{σ_2} be two permuted products in $\mathcal{P}(A_1,\ldots,A_k)$ that are not cyclically equivalent. By Lemma 4.3, there is a triple (i_1,i_2,i_3), with $1\leq i_1,i_2,i_3\leq k$, such that i_1,i_2,i_3 appear in this order in σ_1, and they appear in the order i_3,i_2,i_1 in σ_2, up to cyclic permutations. The triple (i_1,i_2,i_3) corresponds to a triple $(3j-2,3j-1,3j)$ in $\mathcal{L}$ for some $j=1,\ldots,\binom{k}{3}$, such that $\Lambda\left(\widetilde{A}_{3j-2}\widetilde{A}_{3j-1}\widetilde{A}_{3j}\right)\neq\Lambda\left(\widetilde{A}_{3j}\widetilde{A}_{3j-1}\widetilde{A}_{3j-2}\right)$. The result follows from the inclusions $\Lambda\left(\widetilde{A}_{3j-2}\widetilde{A}_{3j-1}\widetilde{A}_{3j}\right)\subseteq\Lambda\left(\Pi_{\sigma_1}\right)$ and $\Lambda\left(\widetilde{A}_{3j}\widetilde{A}_{3j-1}\widetilde{A}_{3j-2}\right)\subseteq\Lambda\left(\Pi_{\sigma_2}\right)$. $\square$

It is worth noting that, in the construction of the proof of Proposition 4.4, the spectra of Π_{σ_1} and Π_{σ_2} are not necessarily disjoint. Note also that the size of the matrices, namely $n=\frac{k(k-1)(k-2)}{2}$, depends on k.

All permuted products in $\mathcal{P}(A_1,\ldots,A_k)$ have the same determinant. Equivalently, the product of their eigenvalues is the same for all permuted products. One may wonder whether or not this is the only restriction on the eigenvalues of permuted products belonging to different classes under cyclic permutations, as it was for three matrices. More generally, we may pose the following problem. Here, for a given set Λ of complex numbers, we use the notation $\operatorname{prod}(\Lambda)$ to denote the product of all numbers in Λ.

> **Inverse eigenvalue problem for permuted products of k matrices:** *Given $(k-1)!$ sets of n nonzero complex numbers, $\Lambda_1,\ldots,\Lambda_{(k-1)!}$, such that $\operatorname{prod}(\Lambda_i)=\operatorname{prod}(\Lambda_j)$, for all $1\leq i,j\leq(k-1)!$, find matrices $A_1,\ldots,A_k$, with $A_i\in\mathbb{C}^{n\times n}$, for $i=1,\ldots,k$, such that $\Lambda\left(\Pi_j\right)=\Lambda_j$, for $j=1,\ldots,(k-1)!$, where $\Pi_j\in\mathcal{P}(A_1,\ldots,A_k)$ belongs to the jth equivalence class under $\sim_C$.*

In Section 3 we have seen that the "Inverse eigenvalue problem for permuted products of $k=3$ matrices" is always solvable. The following result states that, for k large enough, this is no longer true.

Theorem 4.5. *Let n,k be two integers such that $(k-1)!(n-1)+1>kn^2$. Then, there exist $(k-1)!$ sets of nonzero complex numbers $\Lambda_1,\ldots,\Lambda_{(k-1)!}$, with $|\Lambda_i|=n$ and $\operatorname{prod}(\Lambda_i)=\operatorname{prod}(\Lambda_j)$, such that there are no matrices $A_1,\ldots,A_k\in\mathbb{C}^{n\times n}$ satisfying $\Lambda\left(\Pi_j\right)=\Lambda_j$, for $j=1,\ldots,(k-1)!$, where $\Pi_j\in\mathcal{P}(A_1,\ldots,A_k)$ belongs to the jth equivalence class under $\sim_C$.*

Proof. We first note that prescribing the eigenvalues of a matrix A is equivalent to prescribing the coefficients of the characteristic polynomial $p_A(\lambda):=\det(\lambda I-A)$. Let $A_1,\ldots,A_k\in\mathbb{C}^{n\times n}$ be arbitrary matrices and X be the vector containing the entries of the matrices $A_1,\ldots,A_k$, in some order. Let us denote by $\Pi_1,\ldots,\Pi_{(k-1)!}$ the representatives of each of the equivalence classes of $\mathcal{P}(A_1,\ldots,A_k)$ under $\sim_C$. Define the map $P:\mathbb{C}^M\longrightarrow\mathbb{C}^N$ given by $X\mapsto P(X)=(P_1(X),\ldots,P_N(X))$, where $P(X)$ is the vector containing the coefficients of the characteristic polynomials of $\Pi_1,\ldots,\Pi_{(k-1)!}$, in a certain pre-fixed order. P is a polynomial map, since the coefficients of the characteristic polynomial of a matrix are polynomial functions of the entries of the matrix. Moreover, we have $M=kn^2$ and $N=(k-1)!(n-1)+1$. Indeed, the necessary condition $\operatorname{prod}(\Lambda_i)=\operatorname{prod}(\Lambda_j)$, for $1\leq i,j\leq(k-1)!$, is equivalent to the fact that the zero-degree coefficient of all characteristic polynomials of Π_j, for $j = 1, \dots, (k-1)!$, coincide. We may just slightly modify the definition of P, in such a way that, instead of n coefficients for each characteristic polynomial, we just have $(n-1)$ coefficients. Together with the choice of the determinant, this gives the $(k-1)!(n-1)+1$ coordinates in $P(X)$.

Now, the "Inverse eigenvalue problem for permuted products of k matrices with size $n \times n$" is solvable, for k and n, if and only if P is surjective for these k and n. As is well known, a polynomial map from $\mathbb{C}^M$ to $\mathbb{C}^N$ is not surjective when $N > M$ [12, Th. 7, Ch. I, §6], so the result follows. □

4.2 Commutativity conditions and distance of Segré characteristics

In the following, and unless otherwise stated, when dealing with equivalence relations in $\mathcal{P}(A_1, \dots, A_k)$, we consider the elements of $\mathcal{P}(A_1, \dots, A_k)$ as "formal products", that is, like words of k letters, $A_1, \dots, A_k$.

As we saw in Proposition 4.4, when there is more than one equivalence class in $\mathcal{P}(A_1, \dots, A_k)$ under $\sim_C$, it is pointless to ask about the change in the JCF of different permuted products, since the spectrum can be completely different. On the other hand, when there is only one equivalence class, all permuted products have the same nonzero eigenvalues with the same Segré characteristic.

Motivated by Fiedler matrices, we will impose certain commutativity conditions such that any two elements of $\mathcal{P}(A_1, \dots, A_k)$ are connected by a sequence of Flanders bridges. Under these conditions, we will also analyze the change in the Segré characteristic of the eigenvalue zero for different permuted products.

Our commutativity condition involves the associated graph, denoted by $\mathcal{G} = (V, E)$, with $V = \{v_1, \dots, v_k\}$ the set of vertices and E the set of edges. Edges are given by pairs of indices (i, j), with $1 \leq i \neq j \leq k$, meaning that there is an edge joining v_i with v_j. In this case, we say that the edge (i, j) *connects* v_i and v_j. We deal only with undirected graphs, so we identify the pairs (i, j) and (j, i).

A *cycle of length* $m \geq 3$ is a sequence of edges $\{(1, 2), (2, 3), \dots, (m-1, m), (m, 1)\}$, and a sequence $\{(1, 2), (2, 3), \dots, (m-1, m), (m, m+1)\}$ is called a *path* of length m. We say that a graph has a cycle if a subset of its vertices and edges is a cycle. A graph $\mathcal{G} = (V, E)$ is *connected* if there is at least one path between any pair of vertices in V. A *forest* is a graph that has no cycles, and a *tree* is a connected forest. The *degree* of a vertex $v_i \in V$ in the graph $\mathcal{G} = (V, E)$ is the number of edges connected to this vertex. A *leaf* is a vertex of degree one, and the *predecessor* of a leaf v_j is the only vertex v_i such that $(i, j) \in E$.

Definition 4.6. *Given a graph $\mathcal{G} = (V, E)$ with k vertices, and $\Pi_\sigma \in \mathcal{P}(A_1, \dots, A_k)$, a* free swap *on Π_σ is a permuted product of the form $\Pi_{\tau\sigma}$, where τ is the transposition $(i, i+1)$, with $(\sigma^{-1}(i), \sigma^{-1}(i+1)) \notin E$.*

Given two permuted products $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \dots, A_k)$, we say that Π_1 and Π_2 are $\mathcal{G}$-equivalent*, and we write $\Pi_1 \sim_{\mathcal{G}} \Pi_2$, if Π_2 is obtained from Π_1 by a sequence of cyclic permutations of the factors and free swaps.*

In other words, a free swap exchanges two consecutive factors in Π_σ, $A_{\sigma^{-1}(i)}A_{\sigma^{-1}(i+1)} \mapsto A_{\sigma^{-1}(i+1)}A_{\sigma^{-1}(i)}$, provided that $(\sigma^{-1}(i), \sigma^{-1}(i+1)) \notin E$.

The next result allows us to characterize those commutativity relations in $A_1, \ldots, A_k$ that, when added to cyclic permutations in $\mathcal{P}(A_1, \ldots, A_k)$, guarantee that all permuted products are connected by a sequence of Flanders bridges. The key idea that relates graphs and products of matrices is that, if Π_1 can be obtained from Π_2 by a sequence of cyclic permutations and free swaps, then there is a sequence of Flanders bridges between Π_1 and Π_2.

Theorem 4.7. *Let $\mathcal{G} = (V, E)$ be a graph with k vertices. Then, there is only one equivalence class in $\mathcal{P}(A_1, \ldots, A_k)$ under $\sim_{\mathcal{G}}$ if and only if $\mathcal{G}$ is a forest.*

Proof. Throughout this proof, we denote the vertices of V by $A_1, \ldots, A_k$, to indicate that the ith vertex corresponds to the matrix A_i.

Let us first assume that there is a cycle of length ℓ in $\mathcal{G}$. Without loss of generality, we may assume that this cycle is $(1,2), (2,3), \ldots, (\ell-1, \ell), (\ell, 1)$. Now, the products $(A_1A_2 \cdots A_\ell)A_{\ell+1} \cdots A_k$ and $(A_\ell \cdots A_2A_1)A_{\ell+1} \cdots A_k$ are not $\mathcal{G}$-equivalent. To see this, just note that, since no free swaps can be applied to $A_1A_2 \cdots A_\ell$, the cyclic order $A_1, A_2, \ldots, A_\ell$ is invariant under cyclic permutations and free swaps. It is, in particular, different from the cyclic order $A_\ell, \ldots, A_2, A_1$. Hence, there are, at least, two different classes under $\sim_{\mathcal{G}}$.

Conversely, let us assume that $\mathcal{G}$ is a forest. Let us first see that the proof can be reduced to the case where $\mathcal{G}$ is a tree. Suppose that the result is true when $\mathcal{G}$ is a tree. If $\mathcal{G}$ has t trees, we may partition the set $\{A_1, \ldots, A_k\}$ into t subsets, $\{A_1^{(j)}, \ldots, A_{i_j}^{(j)}\}$, for $j = 1, \ldots, t$, in such a way that each matrix in a given subset commutes with all matrices in the remaining subsets. By assumption, the set $\mathcal{P}(A_1^{(j)}, \ldots, A_{i_j}^{(j)})$ has only one equivalence class under $\sim_{\mathcal{G}}$ (although by slight abuse of notation we are writing $\sim_{\mathcal{G}}$, in the jth tree we consider the equivalence relation in $\mathcal{P}(A_1^{(j)}, \ldots, A_{i_j}^{(j)})$ induced by $\mathcal{G}$). Let us denote by $\Pi_1, \ldots, \Pi_t$ the representatives of these classes. Now, by using free swaps, we may reorder any permuted product $\Pi_\sigma \in \mathcal{P}(A_1, \ldots, A_k)$ as $\Pi_\sigma = \Pi_\sigma^{(1)} \cdots \Pi_\sigma^{(t)}$, where $\Pi_\sigma^{(j)} \in \mathcal{P}(A_1^{(j)}, \ldots, A_{i_j}^{(j)})$, for $j = 1, \ldots, t$. After this, we may use free swaps and cyclic permutations to get $\Pi_\sigma \sim_{\mathcal{G}} \Pi_1\Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)}$. To see this, notice that, after performing the cyclic permutation

$$(A_1^{(1)} \cdots A_s^{(1)} A_{s+1}^{(1)} \cdots A_{i_1}^{(1)})\Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)} \sim_{\mathcal{G}} (A_{s+1}^{(1)} \cdots A_{i_1}^{(1)}\Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)}) \cdot (A_1^{(1)} \cdots A_s^{(1)}),$$

we may use free swaps to get

$$(A_{s+1}^{(1)} \cdots A_{i_1}^{(1)}\Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)}) \cdot (A_1^{(1)} \cdots A_s^{(1)}) \sim_{\mathcal{G}} (A_{s+1}^{(1)} \cdots A_{i_1}^{(1)} A_1^{(1)} \cdots A_s^{(1)}) \cdot \Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)}.$$

Hence, we may perform cyclic permutations and free swaps on $\mathcal{P}(A_1^{(1)}, \ldots, A_{i_1}^{(1)})$ to get Π_1 on the left and keep the remaining factors $\Pi_\sigma^{(2)} \cdots \Pi_\sigma^{(t)}$ to the right. Proceeding in the same way with $\Pi_\sigma^{(2)}, \ldots, \Pi_\sigma^{(t)}$, we see that $\Pi_\sigma \sim_{\mathcal{G}} \Pi_1 \cdots \Pi_t$.

So let us prove the statement when $\mathcal{G}$ is a tree with k vertices. The proof is carried out by induction on k. In particular, we want to prove that every $\Pi_\sigma \in \mathcal{P}(A_1, \ldots, A_k)$

is $\mathcal{G}$-equivalent to $A_1 \cdots A_k$. For $k = 3$, the result follows from the analysis in Section 3.

Suppose that the result is true for $k-1$ vertices. Let $\Pi \in \mathcal{P}(A_1, \ldots, A_k)$, and let A_j be a leaf of $\mathcal{G}$. By using a suitable cyclic permutation, we have $\Pi \sim_{\mathcal{G}} A_j \cdot \widetilde{\Pi}$, for some $\widetilde{\Pi} \in \mathcal{P}(A_1, \ldots, \widehat{A}_j, \ldots, A_k)$, where the notation $\widehat{A}_j$ means that this element is missing. The graph $\widetilde{\mathcal{G}}$, obtained from $\mathcal{G}$ after removing A_j and all the edges that are connected to A_j, is still a tree. Hence, by induction we have $\widetilde{\Pi} \sim_{\widetilde{\mathcal{G}}} A_1 \cdots \widehat{A}_j \cdots A_k$. We claim that from this it follows that

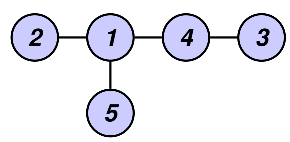
$$A_j \cdot \widetilde{\Pi} \sim_{\mathcal{G}} A_j(A_1 \cdots \widehat{A}_j \cdots A_k). \tag{2}$$

Indeed, perform on the whole $A_j \cdot \widetilde{\Pi}$ all free swaps and cyclic permutations needed to take $\widetilde{\Pi}$ into $A_1 \cdots \widehat{A}_j \cdots A_k$. After applying a cyclic permutation, we take A_j either to the first or to the last position of the product, by using only free swaps. This can be done because A_j is a leaf. By doing this, A_j does not interfere with the free swaps performed within $\widetilde{\Pi}$ between cyclic permutations.

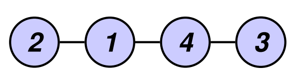
From (2) we easily get $A_j \cdot \widetilde{\Pi} \sim_{\mathcal{G}} A_j(A_1 \cdots \widehat{A}_j \cdots A_k) \sim_{\mathcal{G}} A_1 \cdots A_j \cdots A_k$ in the following way: if the predecessor of A_j in $\mathcal{G}$ is A_i, with $i \geq j+1$, then we use only free swaps in $A_j(A_1 \cdots \widehat{A}_j \cdots A_k)$. Otherwise, we use a cyclic permutation $A_j(A_1 \cdots \widehat{A}_j \cdots A_k) \sim_{\mathcal{G}} (A_1 \cdots \widehat{A}_j \cdots A_k)A_j$, and then free swaps. □

The following example illustrates the construction in the proof of Theorem 4.7

Example 4.8. *Let $k = 5$ and $\mathcal{G} = (V, E)$, with $V = \{1, 2, 3, 4, 5\}$, be the following graph:*



Note that $\mathcal{G}$ is a tree. We associate the jth vertex with the matrix A_j, as in the proof of Theorem 4.7. Set $\Pi = A_4A_3A_5A_1A_2$, and let us apply the procedure described in the proof of Theorem 4.7 to achieve $A_1A_2A_3A_4A_5$ from Π using free swaps and cyclic permutations. We start by choosing the leaf A_5. Using a cyclic permutation, we obtain $\Pi \sim_{\mathcal{G}} A_5(A_1A_2A_4A_3)$. Now A_3 is a leaf of the graph $\widetilde{\mathcal{G}}$:



Following the same procedure, we have $(A_1A_2A_4)A_3 \sim_{\widetilde{\mathcal{G}}} A_3(A_1A_2A_4)$, using a cyclic permutation. When we perform this cyclic permutation in the whole product obtained in the previous step, we get $A_5(A_1A_2A_4A_3) \sim_{\mathcal{G}} A_3A_5(A_1A_2A_4) \sim_{\mathcal{G}} A_5(A_3A_1A_2A_4)$ and, by using free swaps, this is $\mathcal{G}$-equivalent to $A_5(A_1A_2A_3A_4)$. Finally, one cyclic permutation on this permuted product gives $A_1A_2A_3A_4A_5$.

We note that this is not the only way to get $A_1A_2A_3A_4A_5$ from Π. We may, for instance, start with any other leaf in $\mathcal{G}$ instead of A_5, and, even starting with A_5, we may also write, in the first step, $\Pi \sim_{\mathcal{G}} A_5(A_4A_3A_1A_2)$ using free swaps.

Given k matrices $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$, the *graph of non-commutativity relations of* $A_1, \ldots, A_k$ is the graph $\mathcal{G} = (V, E)$ with k vertices, such that $(i, j) \in E$ if and only if $A_iA_j \neq A_jA_i$, for $1 \leq i \neq j \leq k$. As a consequence of Theorem 4.7, if the graph of non-commutativity relations of $A_1, \ldots, A_k$ is a forest, then $\mathcal{P}(A_1, \ldots, A_k)$ has only one equivalence class under $\sim_C$. This is precisely the case in the Fiedler matrices [2], as mentioned in the Introduction. Indeed, the graph of non-commutativity relations of $M_1, \ldots, M_n$ is just a path from M_1 to M_n.

By Theorem 4.7 and Corollary 2.2, when the graph $\mathcal{G}$ of non-commutativity relations of $A_1, \ldots, A_k$ is a forest, all permuted products in $\mathcal{P}(A_1, \ldots, A_k)$ have the same nonzero eigenvalues, with the same Segré characteristics. The remaining question is to analyze what happens to the zero eigenvalue. We now derive an upper bound for the distance of the Segré characteristics at zero of two permuted products of $A_1, \ldots, A_k$ when $\mathcal{G}$ is a forest, and show that the bound is attainable.

Theorem 4.9. *Let $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$ and $\mathcal{G}$ be the graph of non-commutativity relations of $A_1, \ldots, A_k$. Assume that $\mathcal{G}$ is a forest and let d be the length of the longest path in $\mathcal{G}$. Then, given $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$, we have*

$$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq d. \tag{3}$$

Moreover, this bound is attainable in the following sense: Let $\mathcal{G}$ be any forest with k vertices, and let $d \leq k$ be the length of the longest path in $\mathcal{G}$. Then there exist k matrices $A_1, \ldots, A_k$ such that $\mathcal{G}$ is the graph of non-commutativity relations of $A_1, \ldots, A_k$, and there are $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$ with

$$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty = d. \tag{4}$$

Proof. For the first part of the statement, we treat the case where $\mathcal{G}$ is a tree: the extension to a forest is straightforward. By Theorem 1.1, it suffices to show that any two permuted products $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$ are $\mathcal{G}$-equivalent via free swaps and, at most, d cyclic permutations. We prove this first part of the statement by induction on k, and through the following steps.

Step 0: For $k = 2$ the result is Flanders' theorem, and for $k = 3$ it is part (ii) in Proposition 3.2.

Step 1: Let $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$ be two permuted products. By using free swaps, we group all leaves in Π_1 and Π_2 in the rightmost and the leftmost place of the products in the following way: if a given leaf is to the right of its corresponding predecessor, then it is taken to the rightmost position; otherwise, it is taken to the leftmost one. Then, we have $\Pi_1 \sim_\mathcal{G} (A_{i_1} \cdots A_{i_r})\Pi_1'(A_{j_1} \cdots A_{j_s})$, and $\Pi_2 \sim_\mathcal{G} (A_{i_1} \cdots A_{i_d} A_{j_1} \cdots A_{j_e})\Pi_2'(A_{i_{d+1}} \cdots A_{i_r} A_{j_{e+1}} \cdots A_{j_s})$, for some $0 \leq d \leq r, 0 \leq e \leq s$, where $A_{i_1}, \ldots, A_{i_r}, A_{j_1}, \ldots, A_{j_s}$ are all leaves in $\mathcal{G}$, and Π_1', Π_2' are permuted products containing those vertices of $\mathcal{G}$ that are not leaves. Note that these two equivalences only involve free swaps. Now, with at most two cyclic permutations, we get $\Pi_2 \sim_\mathcal{G} \Pi_2'(A_{j_1} \cdots A_{j_s})(A_{i_1} \cdots A_{i_r}) \sim_\mathcal{G} (A_{i_1} \cdots A_{i_r})\Pi_2'(A_{j_1} \cdots A_{j_s})$. Hence,

$$\Pi_1 \sim_\mathcal{G} \widehat{\Pi}_1 := (A_{i_1} \cdots A_{i_r})\Pi_1'(A_{j_1} \cdots A_{j_s}),\ \Pi_2 \sim_\mathcal{G} \widehat{\Pi}_2 := (A_{i_1} \cdots A_{i_r})\Pi_2'(A_{j_1} \cdots A_{j_s}),$$

where these two equivalences involve, at most, two cyclic permutations.

Step 2: We remove all leaves in $\mathcal{G}$ to get $\widetilde{\mathcal{G}}$, where we keep the numbering of the remaining vertices from $\mathcal{G}$. We also group in both $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ all leaves with their corresponding predecessor, to get $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, respectively. This can be done using only free swaps. By Step 1, each group of predecessor+leaves has the same inner order in both $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$. We number each group of predecessor+leaves in both $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ with the index of the predecessor. We keep the notation $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ for the permuted products obtained after doing this.
Step 3 (induction): Since we have removed all leaves from $\mathcal{G}$ to get $\widetilde{\mathcal{G}}$, the longest path in $\widetilde{\mathcal{G}}$ has length $d-2$. By the induction hypothesis, $\widetilde{\Pi}_1$ is $\widetilde{\mathcal{G}}$-equivalent to $\widetilde{\Pi}_2$ using at most $d-2$ cyclic permutations. This $\widetilde{\mathcal{G}}$-equivalence is also a $\mathcal{G}$-equivalence, because the leaves in each group commute with all matrices in any other group. Note also that cyclic permutations in $\widetilde{\Pi}_1$ do not affect the relative position of each predecessor in Π_1 with its corresponding leaves.
Step 4: After Steps 1–3 we get the equivalences: $\Pi_1 \sim_{\mathcal{G}} \widetilde{\Pi}_1 \sim_{\mathcal{G}} \widetilde{\Pi}_2 \sim_{\mathcal{G}} \Pi_2$, where the first equivalence involves only free swaps, the second one involves, at most, $d-2$ cyclic permutations, and the third one involves, at most, two cyclic permutations. Hence, $\Pi_1 \sim_{\mathcal{G}} \Pi_2$ with, at most, d cyclic permutations. This concludes the proof of the first part of the statement.

For the second part, regarding the attainability of the bound (3), we first consider the case where $\mathcal{G}$ is a path of length d, and define the $(d+1)\times(d+1)$ matrices

$$\begin{aligned} \widetilde{A}_i &= \operatorname{diag}(I_{d-i}, J_2(0), I_{i-1}), \quad \text{for } i = 1,\dots,d, \\ \widetilde{A}_{d+1} &= \operatorname{diag}(0, I_d). \end{aligned} \tag{5}$$

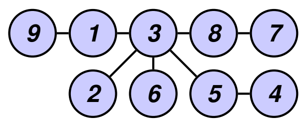
The graph of non-commutativity relations of $\widetilde{A}_1,\dots,\widetilde{A}_{d+1}$ is a single path of length d from $\widetilde{A}_1$ to $\widetilde{A}_{d+1}$. Moreover, we have $\Pi_1 = \widetilde{A}_1\widetilde{A}_2\cdots\widetilde{A}_{d+1} = J_{d+1}(0)$, and $\Pi_2 = \widetilde{A}_{d+1}\cdots\widetilde{A}_2\widetilde{A}_1 = 0_{d+1}$, so $\|\mathcal{S}_0(\Pi_1)-\mathcal{S}_0(\Pi_2)\|_\infty = d$.

If $\mathcal{G}$ is a tree with k vertices, numbered from 1 to k, let us assume, without loss of generality, that $(1,2),(2,3),\dots,(d,d+1)$ is a path of length d in $\mathcal{G}$. Now, let $\widetilde{A}_1,\dots,\widetilde{A}_{d+1}$ be as in (5), and $\widetilde{A}_{d+2} = \cdots = \widetilde{A}_k = I_{d+1}$. Let us number the edges in $\mathcal{G}$ different from $(1,2),(2,3),\dots,(d,d+1)$, as $e_1,\dots,e_g$. For each of these edges we build up the following k matrices: for the edge $e_s = (i,j)$, with $1 \le s \le g$, let $D_1^{(s)},\dots,D_k^{(s)}$ be k nonsingular matrices of size 2×2 such that $D_i^{(s)}D_j^{(s)} \neq D_j^{(s)}D_i^{(s)}$, and $D_\ell^{(s)} = I_2$ for $\ell \neq i,j$. Now, set $A_i = \operatorname{diag}(\widetilde{A}_i, D_i^{(1)},\dots,D_i^{(g)})$, for $i = 1,\dots,k$. The graph of non-commutativity relations of $A_1,\dots,A_k$ is $\mathcal{G}$, by construction. Moreover, since $D_i^{(s)}$ is nonsingular, for all $s = 1,\dots,g$ and $i = 1,\dots,k$, we have $\Pi_1 := A_1A_2\cdots A_k = \operatorname{diag}(J_{d+1}(0), M_1)$ and $\Pi_2 := A_k\cdots A_2A_1 = \operatorname{diag}(0_{d+1}, M_2)$, with M_1, M_2 nonsingular, so $\mathcal{S}_0(\Pi_1) = (d+1)$, and $\mathcal{S}_0(\Pi_2) = (1,\dots,1)$ (containing $d+1$ ones), hence $\|\mathcal{S}_0(\Pi_1)-\mathcal{S}_0(\Pi_2)\|_\infty = d$.

Finally, let $\mathcal{G}$ be a forest with t trees. Let $k_1,\dots,k_t$ be the number of vertices in each tree, with $k_1+\cdots+k_t = k$, and let $d_1,\dots,d_t$ be the lengths of the longest path in each tree. By hypothesis, we have $\max\{d_j : j = 1,\dots,t\} = d$. For each tree, say the jth one, we define matrices $A_1^{(j)},\dots,A_{k_j}^{(j)} \in \mathbb{C}^{n_j\times n_j}$ as before, such that the graph of non-commutativity of $A_1^{(j)},\dots,A_{k_j}^{(j)}$ is precisely this tree, and such that $\|\mathcal{S}_0(A_1^{(j)}A_2^{(j)}\cdots A_{k_j}^{(j)}) - \mathcal{S}_0(A_{k_j}^{(j)}\cdots A_2^{(j)}A_1^{(j)})\|_\infty = d_j$. Now, we set $A_i = \operatorname{diag}(\widehat{A}_1^{(i)},\ldots,\widehat{A}_t^{(i)})$, for $i=1,\ldots,k$, where $\widehat{A}_j^{(i)} = A_h^{(j)}$, if $i = k_1+\cdots+k_{j-1}+h$, for some $1 \le h \le k_j$ (where we set $k_0 := 0$), and $\widehat{A}_j^{(i)} = I_{n_j}$ otherwise. For these matrices, we have $\|\mathcal{S}_0(A_1A_2\cdots A_k) - \mathcal{S}_0(A_k\cdots A_2A_1)\|_\infty = \max_{j=1,\ldots,t}\|\mathcal{S}_0(A_1^{(j)}A_2^{(j)}\cdots A_{k_j}^{(j)}) - \mathcal{S}_0(A_{k_j}^{(j)}\cdots A_2^{(j)}A_1^{(j)})\|_\infty = d$, $\|\mathcal{S}_0(A_1A_2\cdots A_k) - \mathcal{S}_0(A_k\cdots A_2A_1)\|_\infty = \max_{j=1,\ldots,t} d_j = d$, and the graph of non-commutativity relations of $A_1,\ldots,A_k$ is $\mathcal{G}$, by construction. □

The construction in the proof of Theorem 4.9 does not necessarily give the minimum size of $A_1,\ldots,A_k$ that satisfy the second part of the statement. Also note that $d \le k-1$, and equality holds if and only if $\mathcal{G}$ is a path of length $k-1$.

Example 4.10. *Let $\mathcal{G} = (V,E)$, with $V = \{1,2,\ldots,9\}$, be the following graph:*



The length of the longest path in $\mathcal{G}$ is $d=4$, which corresponds, for instance, to the path $(9,1)-(1,3)-(3,8)-(8,7)$.

As in Example 4.8, we identify the jth vertex of $\mathcal{G}$ with the matrix A_j so that $\mathcal{G}$ is the graph of non-commutativity relations of $A_1,\ldots,A_9$. Set $\Pi_1 = A_9A_1A_3A_8A_7A_6A_2A_5A_4$ and $\Pi_2 = A_7A_8A_3A_1A_9A_6A_2A_5A_4$, and let us apply the procedure described in the proof of Theorem 4.9 to achieve Π_2 from Π_1 via free swaps and cyclic permutations.

Step 1: *By free swaps, move all the leaves in Π_1 and Π_2 to the left and right ends:*

$$\Pi_1 \sim_{\mathcal{G}} \widehat{\Pi}_1 := (A_9)A_1A_3A_8A_5(A_4A_6A_2A_7), \qquad \Pi_2 \sim_{\mathcal{G}} (A_7)A_8A_3A_1A_5(A_4A_6A_2A_9).$$

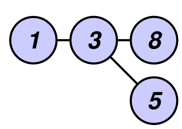
Using free swaps and two *cyclic permutations, we obtain*

$$\Pi_2 \sim_{\mathcal{G}} (A_9)A_8A_3A_1A_5(A_4A_6A_2A_7) := \widehat{\Pi}_2.$$

Step 2: *By free swaps, we group all leaves with their corresponding predecessor in each $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$, and then relabel each group using the index of the predecessor:*

$$\begin{aligned}\widehat{\Pi}_1 &\sim_{\mathcal{G}} (A_9A_1)(A_3A_6A_2)(A_8A_7)(A_5A_4) =: \widetilde{A}_1\widetilde{A}_3\widetilde{A}_8\widetilde{A}_5 \; (:= \widetilde{\Pi}_1),\\ \widehat{\Pi}_2 &\sim_{\mathcal{G}} (A_8A_7)(A_3A_6A_2)(A_9A_1)(A_5A_4) =: \widetilde{A}_8\widetilde{A}_3\widetilde{A}_1\widetilde{A}_5 \; (:= \widetilde{\Pi}_2).\end{aligned}$$

This gives us two permuted products associated with the graph $\widetilde{\mathcal{G}}$:



Step 3: *Proceeding as before, with* $\widetilde{\Pi}_1, \widetilde{\Pi}_2$*, and* $\widetilde{\mathcal{G}}$ *instead of* Π_1, Π_2 *and* $\mathcal{G}$*, we get*

$$\widetilde{\Pi}_1 = (\widetilde{A}_1)\widetilde{A}_3(\widetilde{A}_8\widetilde{A}_5) \sim_{\mathcal{G}} \widetilde{A}_3(\widetilde{A}_8\widetilde{A}_5\widetilde{A}_1), \qquad \widetilde{\Pi}_2 = (\widetilde{A}_8)\widetilde{A}_3(\widetilde{A}_1\widetilde{A}_5) \sim_{\mathcal{G}} \widetilde{A}_3(\widetilde{A}_8\widetilde{A}_5\widetilde{A}_1),$$

where each $\mathcal{G}$*-equivalences uses one cyclic permutation. Since there is only one predecessor (namely,* $\widetilde{A}_3$*), the forward part of the process is finished.*

Step 4: *Put together the operations in Steps* 1–3 *to get the chain of* $\mathcal{G}$*-equivalences:*

$$\begin{aligned}\Pi_2 &\sim_{\mathcal{G}} (A_7)A_8A_3A_1A_5(A_4A_6A_2A_9) \sim_{\mathcal{G}} (A_9)A_8A_3A_1A_5(A_4A_6A_2A_7)\\ &\sim_{\mathcal{G}} (A_8A_7)(A_3A_6A_2)(A_9A_1)(A_5A_4) = \widetilde{\Pi}_2 \sim_{\mathcal{G}} (A_3A_6A_2)(A_8A_7)(A_5A_4)(A_9A_1)\\ &\sim_{\mathcal{G}} (A_9A_1)(A_3A_6A_2)(A_8A_7)(A_5A_4) \sim_{\mathcal{G}} A_9A_1A_3A_8A_7A_6A_2A_5A_4 = \Pi_1.\end{aligned}$$

The $\mathcal{G}$*-equivalences involving cyclic permutations are the second one, which uses two cyclic permutations, and the fourth and fifth ones, both of which use one. The total number of cyclic permutations is four, so* $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq 4$*, and equality is attained by setting* A_i *as constructed in the proof of Theorem* 4.9*.*

Theorem 4.7 shows that if the graph of non-commutativity relations of $A_1, \ldots, A_k$ has no cycles, then all permuted products in $\mathcal{P}(A_1, \ldots, A_k)$ have the same eigenvalues with their corresponding Segré characteristics. The reverse implication, however, is not true. For example, take $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$ to be upper triangular such that no pair commutes and the products of the (i,i) diagonal entries of all matrices, $\pi_i = A_1(i,i)A_2(i,i)\cdots A_k(i,i)$, satisfy $\pi_i \neq \pi_j$ for $i \neq j$. Then, all permuted products have the same eigenvalues, with the same Segré characteristic (they are all simple eigenvalues). However, the graph of non-commutativity relations is the complete graph with k vertices, which is far from a forest.

5 Open problems

We conclude with open problems that arise as a natural continuation.

- **1:** Is it always possible to prescribe the n eigenvalues of the $(k-1)!$ classes under cyclic permutations, provided that the product of all eigenvalues is the same, for k, n satisfying $(k-1)!(n-1) + 1 \leq kn^2$ and $k \geq 4$?
- **2:** Given $d \geq 0$ and two nondecreasing sequences $\boldsymbol{\mu}, \boldsymbol{\mu}'$ of nonnegative integers such that $\|\boldsymbol{\mu}, \boldsymbol{\mu}'\| \leq d$, is it always possible to find d matrices, $A_1, \ldots, A_d$, such that their graph of non-commutativity relations is a path, and such that $\mathcal{S}_0(A_1 \cdots A_d) = \boldsymbol{\mu}$ and $\mathcal{S}_0(A_d \cdots A_1) = \boldsymbol{\mu}'$? (The extension of Theorem 3.4 to $n \geq 4$).
- **3:** If $M, Q \in \mathbb{C}^{n\times n}$ are such that $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(Q)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(Q)\|_\infty \leq 2$, are there three matrices $A, B, C \in \mathbb{C}^{n\times n}$ such that $M = ABC$ and $Q = CBA$?
- **4:** Obtain necessary and sufficient conditions for all products of a given set of k matrices to have the same nonzero eigenvalues with their corresponding Segré characteristic (in the notation of the paper: $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, for all $\lambda \neq 0$, and for all $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$).

Acknowlegdments: We thank Enrique Arrondo for pointing out reference [12, Th. 7, Ch. I §6], Richard A. Brualdi for recommending the Latex packages TikZ and PGF, and James Hook for useful comments on the exposition.

References

[1] S. Bernau and A. Abian. Jordan canonical forms of matrices AB and BA. *Rend. Inst. Mat. Univ. Trieste*, 20(1):101–108, 1988.

[2] M. Fiedler. A note on companion matrices. *Linear Algebra Appl.*, 372:325–332, 2003.

[3] H. Flanders. Elementary divisors of AB and BA. *Proc. Amer. Math. Soc*, 2(6):871–874, 1951.

[4] S. Furtado and C. R. Johnson. Order invariant spectral properties for several matrices. *Linear Algebra Appl.*, 432:1950–1960, 2010.

[5] J. Gelonch and C. R. Johnson. Generalization of Flanders' theorem to matrix triples. *Linear Algebra Appl.*, 380:151–171, 2004.

[6] J. Gelonch, C. R. Johnson, and P. Rubió. An extension of Flanders theorem to several matrices. *Lin. Multilin. Algebra*, 43:181–200, 1997.

[7] R. A. Horn, C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[8] R. A. Horn and D. I. Merino. Contragredient Equivalence: A canonical form and some applications. *Linear Algebra Appl.*, 214:43–92, 1995.

[9] C. R. Johnson and E. S. Schreiner. The relationship between AB and BA. *Amer. Math. Monthly*, 103(7):578–581, 1996.

[10] R. A. Lippert and G. Strang. The Jordan forms of AB and BA. *Electron. J. Linear Algebra*, 18:281–288, 2009.

[11] W. V. Parker and B. E. Mitchell. Elementary divisors of certain matrices. *Duke Mathematical Journal*, 19(3):483–485, 1952.

[12] I. R. Safarevich. *Basic Algebraic Geometry.* Springer Verlag, Berlin, 1974.

[13] R. C. Thompson. On the matrices AB and BA. *Linear Algebra Appl.*, 1(1):43 – 58, 1968.