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FREE CENTRE-BY-METABELIAN LIE RINGS

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ABSTRACT. We study the free centre-by-metabelian Lie ring, that is, the free Lie ring with the property that the second derived ideal is contained in the centre. We exhibit explicit generating sets for the homogeneous and fine homogeneous components of the second derived ideal. Each of these components is a direct sum of a free abelian group and a (possibly trivial) elementary abelian 2-group. Our generating sets are such that some of their elements generate the torsion subgroup while the remaining ones freely generate a free abelian group. A key ingredient of our approach is the determination of the dimensions of the corresponding homogeneous and fine homogeneous components of the free centre-by-metabelian Lie algebra over fields of characteristic other than 2. For that we exploit a 6-term exact sequence of modules over a polynomial ring that is originally defined over the integers, but turns into a sequence whose terms are projective modules after tensoring with a suitable field. Our results correct a partly erroneous theorem in the literature.

1. INTRODUCTION

The free centre-by-metabelian Lie rings are curious structures with a number of unusual properties: Unlike free nilpotent or free soluble Lie rings, their underlying abelian groups are not free, but contain elements of order 2. Moreover, the 2-torsion occurring in odd degrees is drastically different from the 2-torsion in even degrees. Another peculiar feature is that the free centre-by-metabelian Lie ring is not isomorphic to the Lie ring of the free centre-by-metabelian group (of the same rank). More precisely, let $G = G(X)$ denote the free centre-by-metabelian Lie ring of finite rank r on a free generating set X, $|X| = r > 1$. Then G is a central extension of the free metabelian Lie ring G/G''. The additive structure of the latter is well understood: It is a free abelian group and the simple basic monomials form a $\mathbb{Z}$-basis for it, see, for example, [1, 4.7.1]. In fact, this basis appeared already in [2] where, moreover, a formula for the number of such basis elements in every degree was derived. When studying G itself it is therefore natural to focus on the second derived ideal G''. Let G_n, $n \geqslant 1$ denote the degree n homogeneous component of G, and set $G''_n = G'' \cap G_n$. Then G''_4 is easily seen to be isomorphic to $G_2 \wedge G_2$, a free abelian group, and the situation gets interesting when $n \geqslant 5$. Namely, for $n \geqslant 5$ there is a direct decomposition

$$G''_n = F_n \oplus T_n \tag{1.1}$$

where T_n is a (possibly trivial) elementary abelian 2-group and F_n is a free abelian group. This, together with explicit bases for T_n as a free $\mathbb{Z}/2\mathbb{Z}$-module and for F_n

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as a free $\mathbb{Z}$-module, is the main result (Theorem 4) in the section on Lie rings in Yu.V. Kuz'min's ground-breaking paper [6]. However, it turned out later that some of the details in [6] are in need of correction. Zerck [13] pointed out that the sets asserted to be bases for the torsion subgroups T_n for even $n \geqslant 6$ in [6] are actually not sufficient to generate those groups. Unfortunately, Zerck's preprint [13] was never published properly and is not easily accessible. Moreover, Zerck did not give full proofs, but refered in certain instances to the methods in [6], which is hardly satisfactory as some of the arguments in that paper are compromised. Now it turns out that, apart from the shortcomings related to T_n, the sets claimed in [6] to be bases of F_n for even $n \geqslant 6$ fail to be linearly independent over $\mathbb{Z}$. The aim of the present paper is to put this right.

Closely related to free centre-by-metabelian Lie rings are the lower central quotients of the free centre-by-metabelian groups. These groups are themselves highly curious objects. In 1974 C.K. Gupta [3] discovered that, from rank 4 onwards, they contain elements of order 2 in the centre, a very surprising result at the time. Let $\mathfrak{G}$ denote the free centre-by-metabelian group of rank r on a free generating set X, $|X| = r > 1$, and let $\gamma_n\mathfrak{G}$ denote the nth term of the lower central series of $\mathfrak{G}$. The Lie ring $L(\mathfrak{G})$ of the group $\mathfrak{G}$ is the direct sum of the lower central quotients $\gamma_n\mathfrak{G}/\gamma_{n+1}\mathfrak{G}$ ($n = 1, 2, 3, \ldots$) with Lie bracket induced by the commutator in $\mathfrak{G}$ (see [1, 8.2.4]). Now, $L(\mathfrak{G})$ is a centre-by-metabelian Lie ring. Hence it is a homomorphic image of the free centre-by-metabelian Lie ring G, and the lower central quotients $\gamma_n\mathfrak{G}/\gamma_{n+1}\mathfrak{G}$ are homomorphic images of the respective homogeneous components G_n. Again, in contrast to free nilpotent or free soluble groups and Lie rings, the canonical map $G \to L(\mathfrak{G})$ is not an isomorphism [7, Theorem 2]. However, there are isomorphisms $L_n \otimes \mathbb{Q} \cong (\gamma_n\mathfrak{G}/\gamma_{n+1}\mathfrak{G}) \otimes \mathbb{Q}$ for all $n \geqslant 1$ [7, Theorem 1]. The lower central quotients of $\mathfrak{G}$ were studied in [4]. Let $\mathfrak{G}''_n = (\gamma_n\mathfrak{G} \cap \mathfrak{G}'')\gamma_{n+1}\mathfrak{G}/\gamma_{n+1}\mathfrak{G}$, in other words, $\mathfrak{G}''_n$ is the image of G''_n under the canonical epimorphism $G \to L(\mathfrak{G})$. It was shown that $\mathfrak{G}''_n \cong \mathfrak{F}_n \oplus \mathfrak{T}_n$ where $\mathfrak{T}_n$ is an elementary abelian 2-group and $\mathfrak{F}_n$ is a free abelian group. Moreover, explicit generating sets for both $\mathfrak{T}_n$ and $\mathfrak{F}_n$ were obtained [4, Theorems 1 and 4]. For odd n, the generating sets for $\mathfrak{F}_n$ in [4] are exactly the canonical images of the basis for F_n given in [6]. For even n, however, the generating sets for $\mathfrak{F}_n$ in [4] have fewer elements that the alleged bases for F_n given in [6]. We mention that in their Theorem 1 of [4] the authors use the word basis, rather than generating set. It appears, though, that this is used in the sense of generating set as the question of linear independence over $\mathbb{Z}$ is not addressed in the proof. Also, in [4] no proof is given for part (ii) of Theorem 4, the part relating to $\mathfrak{F}_n$.

In this paper we focus on the torsion-free part of G''. We obtain $\mathbb{Z}$-bases for the free abelian parts of the fine homogeneous components of G'', and we also derive formulae for the ranks of these groups as free $\mathbb{Z}$-modules. Our approach is as follows. First we obtain generating sets for the fine homogeneous components of G''. Then we show that certain subsets of those generating sets span torsion subgroups, more precisely, elementary abelian 2-groups. Finally, we show that the remaining elements in those generating sets are linearly independent over $\mathbb{Z}$, that is, they actually freely generate the components modulo their torsion subgroups as free $\mathbb{Z}$-modules. This yields our main result, Theorem 7.1, which gives a detailed description of the fine homogeneous components of G''. Our approach is based on an isomorphism, due to Kuz'min [6], between G'' and a certain tensor product. More precisely, the adjoint representation of G induces on the quotient $M = G'/G''$ the structure of a module for the polynomial ring $U = \mathbb{Z}[X]$, which is, in fact, the universal envelope for the abelian Lie algebra G/G'. Then G'' is isomorphic to $(M \wedge M) \otimes_U \mathbb{Z}$ where the exterior square $M \wedge M$ is regarded as a U-module with derivation action and $\mathbb{Z}$ is the trivial U-module. Most of the work in this paper is carried out in that tensor product. The methods we employ to find spanning sets for the fine homogeneous components, and to prove that part of the spanning sets generate elementary abelian 2-groups are essentially the same as in [6], except that we take advantage of the fine homogeneous structure. This makes it possible to obtain very simple generating sets for fine homogeneous components in which at least one of the free generators occurs with multiplicity greater than one. Where our approach significantly differs from [6] is our method for proving that the generating sets we obtain for the torsion-free part are linearly independent over $\mathbb{Z}$. Here we use homological methods: The exterior square $M \wedge M$ fits into a 6-term exact sequence. This sequence enables us to work out the dimensions of the fine homogeneous components of $G'' \otimes K$, where K is a field of characteristic other than 2. It turns out that these dimensions coincide with the number of non-torsion elements in our generating sets for the corresponding fine homogeneous components of G'', which implies that these elements freely generate free $\mathbb{Z}$-modules. Our main result confirms the results in [6] on the torsion-free part of G''_n for odd $n \geqslant 5$, see Theorem 7.2 below, and corrects the results for even $n \geqslant 6$. Our results also show that the generating sets for the torsion-free part of $\mathfrak{G}''_n$ given in [4] are linearly independent over $\mathbb{Z}$. Finally, we give a short direct proof for the fact that G'' is a direct sum of a free abelian group and an elementary abelian 2-group.

The paper is organized as follows. Notation and some preliminary notions are set up in Section 2. Section 3 is devoted to deriving generating sets for the fine homogeneous components of G'', and in Section 4 we examine torsion elements in G''. In Section 5 we introduce a 6-term exact sequence that includes the exterior square $M \wedge M$, and we examine the modules occurring in that sequence. Some of the results in this section require to work over a field rather than over the ring of integers. These results are then exploited in Section 6 to work out the dimensions of the fine homogeneous components of $G'' \otimes K$ for an arbitrary field K of characteristic other than 2. Our main result is stated and proved in Section 7, where we also discuss how our results relate to those in [6] and [4]. In the concluding Section 8 we give the above mentioned direct proof of the fact that the torsion subgroup of G'' is annihilated by 2, and that the quotient by it is free abelian.

2. Preliminaries

Let $L = L(X)$ be the free Lie ring of rank $r > 1$ on a set $X = \{x_1, x_2, \ldots, x_r\}$. We assume the set X to be ordered by $x_1 < x_2 < \cdots < x_r$. The free centre-by-metabelian Lie ring $G = G(X)$ of rank r is the quotient $G = L/[L'', L]$ where L'' is the second derived ideal of L. Our aim is to study the second derived ideal $G'' = L''/[L'', L]$.

The derived ideal L' is a free Lie ring, and the Lie monomials

$$[y_1, y_2, \ldots, y_n] \quad \text{with } y_i \in X, \ n \geqslant 2 \text{ and } y_1 > y_2 \leqslant y_3 \leqslant \cdots \leqslant y_n \tag{2.1}$$

(using the left-normed convention for Lie products) form a free generating set for L' (see, for example, [1, Section 4.2.2]). It follows that the elements (2.1) (more precisely, their cosets modulo L'') form a basis of the quotient $M = L'/L''$. The adjoint representation of L induces on M the structure of an L/L'-module, and hence of a module for the universal envelope U of the abelian Lie ring L/L'. This universal envelope is in fact the ring of polynomials on X with integer coefficients: $U = \mathbb{Z}[X]$. At this point we introduce some notation relating to this polynomial ring. We let Δ denote the augmentation ideal of U, that is the ideal of all polynomials with zero constant term. Hence Δ is the kernel of the augmentation map $\varepsilon : U \to \mathbb{Z}$ that maps every polynomial to its constant term. The short exact sequence

$$0 \to \Delta \to U \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{2.2}$$

is known as the augmentation sequence. The ring of integers $\mathbb{Z}$ will be regarded as a trivial U-module, and then the augmentation sequence is a sequence of U-modules. For a (commutative and associative) monomial $u = y_1 y_2 \ldots y_k \in U$ with $y_i \in X$, we let $l(u)$ denote the smallest of the elements $y_1, y_2, \ldots, y_k$ with respect to the ordering of X, and we write $\deg u$ for the degree of u. By $\mathcal{U}$ we denote the set of all monomials,

$$\mathcal{U} = \{y_1 y_2 \ldots y_k; \ \ y_i \in X, \ y_1 \leqslant y_2 \leqslant \cdots \leqslant y_k, \ k \geqslant 0\},$$

in U with the convention that 1 is the only monomial of degree 0. The set $\mathcal{U}$ is a basis for U as a free $\mathbb{Z}$-module. Now we return to our free Lie rings. The quotient M is, as a U-module, generated by the elements $[x_i, x_j]$ with $x_i, x_j \in X$. Using module notation, the images of the elements (2.1) in M can be written as

$$[y_1, y_2]u \quad \text{with } y_i \in X, \ \ u \in \mathcal{U} \text{ and } y_1 > y_2 \leqslant l(u). \tag{2.3}$$

These elements form a $\mathbb{Z}$-basis of M. Let P be a free U-module with free generators $e_1, e_2, \ldots, e_r$. Then the map $[x_i, x_j] \mapsto e_i x_j - e_j x_i$ extends to an embedding $\mu : M \to P$. Moreover, if $\sigma : P \to \Delta$ is the map determined by $e_i \mapsto x_i$, then

$$0 \to M \xrightarrow{\mu} P \xrightarrow{\sigma} \Delta \to 0 \tag{2.4}$$

is a short exact sequence of U-modules. A proof of this can be found in [11]. For U-modules A and B, the tensor product $A \otimes B$ (over $\mathbb{Z}$) will be regarded as a

U-module with derivation action, that is for $a \in A$, $b \in B$ and $y \in X$ we have

$$(a \otimes b)y = ay \otimes b + a \otimes by.$$

Likewise, the exterior and symmetric squares of a U-module A, denoted by $A \wedge A$ and $A \circ A$, respectively, will be regarded as U-modules with derivation action. The adjoint representation of L induces on the lower central quotients of the derived ideal L' the structure of a U-module. For the first lower central quotient $\gamma_1(L')/\gamma_2(L') = L'/L''$ this is precisely the U-module M discussed above. For the second lower central quotient $\gamma_2(L')/\gamma_3(L')$ the module action is, for $m_1, m_2 \in L'$ and $y \in X$, given by

$$([m_1, m_2] + \gamma_3(L'))y = [[m_1, y], m_2] + [m_1, [m_2, y]] + \gamma_3(L'). \tag{2.5}$$

As an abelian group, $\gamma_2(L')/\gamma_3(L')$ is isomorphic to the exterior square of $M = L'/L''$,

$$\gamma_2(L')/\gamma_3(L') \cong M \wedge M, \tag{2.6}$$

via the map $([m_1, m_2] + \gamma_3(L')) \mapsto (m_1 + \gamma_2(L')) \wedge (m_2 + \gamma_2(L'))$. In view of (2.5) this is, in fact, an isomorphism of U-modules (where $M \wedge M$ is regarded as a U-module with derivation action). Since $(\gamma_2(L')/\gamma_3(L'))\Delta = [\gamma_2(L'), L]/\gamma_3(L')$, and

$$\begin{aligned} (\gamma_2(L')/\gamma_3(L')) \otimes_U \mathbb{Z} &= (\gamma_2(L')/\gamma_3(L'))/(\gamma_2(L')/\gamma_3(L'))\Delta \\ &= (\gamma_2(L')/\gamma_3(L'))/([\gamma_2(L'), L]/\gamma_3(L')) \\ &\cong \gamma_2(L')/[\gamma_2(L'), L] \\ &= L''/[L'', L] \\ &= G'', \end{aligned}$$

trivializing the U-action on both sides of the isomorphism (2.6) yields an isomorphism

$$G'' \cong (M \wedge M) \otimes_U \mathbb{Z} \tag{2.7}$$

([6, Lemma 2]). The free centre-by-metabelian Lie ring G has a natural grading by degree. We let G_n denote the degree n homogeneous component of G, and write G''_n for the degree n homogeneous component of the second derived ideal: $G''_n = G'' \cap G_n$. We are also interested in the fine homogeneous components of G. These are the $\mathbb{Z}$-submodules of G spanned by all Lie monomials of the same multidegree in the free generators. More precisely, a composition q of n in r parts ($q \vDash n$) is a sequence $q = (q_1, q_2, \ldots, q_r)$ of non-negative integers such that $\sum_{j=1}^{r} q_j = n$. Note that we do allow zero parts and that throughout we assume the number of parts to be r. For a fixed composition $q = (q_1, q_2, \ldots, q_r)$ of n, let G_q be the $\mathbb{Z}$-submodule of G generated by all Lie products of partial degree q_j with respect to x_j for $1 \leqslant j \leqslant r$. Then

$$G_n = \bigoplus_{q \vDash n} G_q.$$

We write G''_q for $G'' \cap G_q$. A fine homogeneous component G_q with $q = (q_1, q_2, \ldots, q_r)$ is called multilinear if $q_i \leqslant 1$ for all i. The module M is also graded by degree, and so are the exterior square $M \wedge M$ and the tensor product $(M \wedge M) \otimes_U K$. We write $((M \wedge M) \otimes_U K)_n$, $((M \wedge M) \otimes_U K)_q$ for the various homogeneous components. The isomorphism (2.7) is an isomorphism of graded U-modules.

We conclude this section by recording some easy facts about the fine homogeneous structure of G. For any map $f : X \to X$ of the free generating set X to itself we let $\pi_f : G \to G$ denote the unique endomorphism of G that extends f. It is clear that the image of a fine homogeneous component G_q with $q = (q_1, q_2, \ldots, q_r) \vDash n$ under an endomorphism of the form π_f is itself a fine homogeneous component. In fact,

$$G_q \pi_f = G_{q'} \ \text{ where } q' = (q'_1, q'_2, \ldots, q'_r) \ \text{ with } q'_i = \sum_{j : f(j) = i} q_j.$$

It is plain that $G_q \pi_f \cong G_q$ if f is a bijection. Consequently, each fine homogeneous component is isomorphic to a fine homogeneous component of the form G_q where $q = (q_1, q_2, \ldots, q_r)$ is a partition of n, that is, a composition with the additional condition that $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_r$. Finally, note that every fine homogeneous component G_q, $q \vDash n \leqslant r$ is a homomorphic image of the multilinear fine homogeneous component $q = (\underbrace{1, \ldots, 1}_{n}, 0, \ldots, 0)$ under some endomorphism of the form π_f. Moreover, it is easily seen that f can be chosen in such a way that it preserves the order of the free generators, i.e. if $x_i \leqslant x_j$ then $x_i f \leqslant x_j f$. All the easy facts recorded in this paragraph remain true if G_q is replaced by G''_q.

3. Generating sets

We have seen that the elements (2.3) form a generating set of M as a free $\mathbb{Z}$-module. It follows that the exterior square $M \wedge M$ is generated by all elements

$$[y_1, y_2]u_1 \wedge [y_3, y_4]u_2 \qquad (y_i \in X,\ u_1, u_2 \in \mathcal{U}). \tag{3.1}$$

In this section we obtain generating sets for the tensor product $(M \wedge M) \otimes_U \mathbb{Z} = (M \wedge M)/(M \wedge M)\Delta$. To simplify notation and to save space, we denote the canonical image of an exterior product $m_1 \wedge m_2 \in M \wedge M$ $(m_1, m_2 \in M)$ in $(M \wedge M) \otimes_U \mathbb{Z}$ in what follows by $m_1 \wedge_* m_2$ rather than $(m_1 \wedge m_2) \otimes 1$. We now record a number of relations that are satisfied in $(M \wedge M) \otimes_U \mathbb{Z}$. First of all, there are the relations coming from anticommutativity and the Jacobi identity in M:

$$[x_i, x_i] = 0, \qquad [x_i, x_j] = -[x_j, x_i] \tag{3.2}$$

and

$$[x_i, x_j]x_k = -[x_j, x_k]x_i + [x_i, x_k]x_j \tag{3.3}$$

for all $x_i, x_j, x_k \in X$. Then there is anticommutativity coming from the exterior square $M \wedge M$:

$$m \wedge_* m = 0, \qquad m_1 \wedge_* m_2 = -(m_2 \wedge_* m_1) \tag{3.4}$$

for all $m, m_1, m_2 \in M$. Finally, there are relations coming from the trivialization of the U-action:

$$m_1 y \wedge_* m_2 = -(m_1 \wedge_* m_2 y) \tag{3.5}$$

for all $m_1, m_2 \in M, y \in X$. Indeed, since $(m_1 \wedge m_2)y = m_1 y \wedge m_2 + m_1 \wedge m_2 y \in (M \wedge M)\Delta$, it follows that $m_1 y \wedge_* m_2 + m_1 \wedge_* m_2 y = 0$ in $(M \wedge M) \otimes_U \mathbb{Z}$. It will be useful to record the following obvious consequence of (3.4) and (3.5):

$$m_1 \wedge_* m_2 u = (-1)^{\deg u + 1}(m_2 \wedge_* m_1 u) \qquad (m_1, m_2 \in M, u \in \mathcal{U}). \tag{3.6}$$

Of course, the images of the elements (3.1) in $(M \wedge M) \otimes_U \mathbb{Z}$ generate that tensor product as a $\mathbb{Z}$-module. In view of (3.5), each of those images is, up to sign, equal to an element of the form

$$[y_1, y_2] \wedge_* [y_3, y_4](y_5 y_6 \cdots y_n) \qquad (y_i \in X). \tag{3.7}$$

Hence the elements (3.7) form a generating set of $(M \wedge M) \otimes_U \mathbb{Z}$. Our task is to further reduce this generating set. We call elements of the form (3.7) *Kuz'min elements* if $y_1 > y_2$, $y_3 > y_4$, $y_1 \geqslant y_3$, $y_2 \geqslant y_4$ and $y_2 \leqslant l(u)$. We will show that the multilinear Kuz'min elements of degree $n \geqslant 5$ together with one additional element form a generating set for the degree n multilinear fine homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z}$. We start with a simple observation.

Lemma 3.1. *Let $n \geqslant 4$ and $a, b \in X$. Any element (3.7) with $y_i = a$, $y_j = b$ for some i, j with $1 \leqslant i, j \leqslant n$, $i \neq j$, belongs to the span of the elements*

$$[z_1, z_2] \wedge_* [b, a](z_3 \cdots z_{n-2}) \quad \text{and} \quad [z_1, b] \wedge_* [z_2, a](z_3 \cdots z_{n-2}) \tag{3.8}$$

of the same multidegree with $z_1, z_2, \ldots, z_{n-2} \in X$.

Proof. First we show that under our assumptions the element (3.7) is in the span of the elements

$$[z_1, z_2] \wedge_* [z_3, a](z_4 \cdots z_{n-1}) \tag{3.9}$$

where $z_1, z_2, \ldots, z_{n-1} \in X$. In view of the relations (3.6) and the anticommutativity of the Lie bracket this is obvious if a is one of y_1, y_2, y_3, y_4 since in this case the element (3.7) is (up to sign) actually equal to one of the elements (3.9). If a is one of $y_5, \ldots, y_n$, we may assume that $a = y_5$, and then, by using (3.3) we get

$$\begin{aligned}[y_1, y_2] \wedge_* [y_3, y_4](a y_6 \cdots y_n) = -[y_1, y_2] \wedge_* [y_4, a](y_3 y_6 \cdots y_n) \\ + [y_1, y_2] \wedge_* [y_3, a](y_4 y_6 \cdots y_n),\end{aligned}$$

as required. It remains to show that any element of the form (3.9) such that b is equal to one of the elements $z_1, z_2, \ldots, z_{n-1}$ is in the span of the elements (3.8). Again, this is clear if b is one of z_1, z_2, z_3. Otherwise b will be one of $z_4, \ldots, z_{n-1}$, and we may assume that $b = z_4$. Then we find, using the relations (3.5) and (3.3),

$$\begin{aligned} & [z_1, z_2] \wedge_* [z_3, a](b z_5 \cdots z_{n-1}) \\ = \; & -[z_1, z_2]b \wedge_* [z_3, a](z_5 \cdots z_{n-1}) \\ = \; & +[z_2, b]z_1 \wedge_* [z_3, a](z_5 \cdots z_{n-1}) - [z_1, b]z_2 \wedge_* [z_3, a](z_5 \cdots z_{n-1}) \\ = \; & -[z_2, b] \wedge_* [z_3, a](z_1 z_5 \cdots z_{n-1}) + [z_1, b] \wedge_* [z_3, a](z_2 z_5 \cdots z_{n-1}),\end{aligned}$$

as required. □

Armed with Lemma 3.1 it is very easy to obtain efficient generating sets for the fine homogeneous components $((M \wedge M) \otimes_U \mathbb{Z})_q$ for partitions $q \vDash n$ with at least one part $\geqslant 2$. We will see later that these generating sets are actually minimal if n is odd, and that they can easily be reduced to minimal generating sets if n is even.

Lemma 3.2. *Let $n \geqslant 5$ and let $q = (q_1, q_2, \ldots, q_r)$ be a composition of n in r parts such that $q_i \geqslant 2$ for some i. Then the elements*

$$[z_1, x_i] \wedge_* [z_2, x_i](z_3 z_4 \cdots z_{n-2}),$$

$z_1, z_2, \ldots, z_{n-2} \in X$, of multidegree q with $z_1 \geqslant z_2$ and $z_1, z_2 \neq x_i$ form a generating set for the fine homogeneous component $((M \wedge M) \otimes_U \mathbb{Z})_q$.

Proof. By Lemma 3.1 with $a = b = x_i$, each element (3.7) of multidegree q in $((M \wedge M) \otimes_U \mathbb{Z})$ is a linear combination of elements of the form $[z_1, x_i] \wedge_* [z_2, x_i](z_3 \cdots z_{n-2})$, and because of the relation (3.6) it is sufficient to take those elements with $z_1 \geqslant z_2$. □

Now we deal with multilinear homogeneous components.

Lemma 3.3. *Suppose $|X| = n \geqslant 5$. Then every element (3.7) of multidegree $q = (1, 1, \ldots, 1) \vDash n$ is a linear combination of Kuz'min elements of multidegree $(1, 1, \ldots, 1)$ and the element $h = [x_3, x_2] \wedge_* [x_4, x_1](x_5 \cdots x_n)$.*

Proof. Suppose we are given an arbitrary element g of the form (3.7) of multidegree $(1, 1, \ldots, 1)$, i.e. each of the elements $x_1, x_2, \ldots, x_n$ appears exactly once. By Lemma 3.1 with $a = x_1$ and $b = x_2$, we may assume that g is either

$$[z_1, z_2] \wedge_* [x_2, x_1](z_3 \cdots z_{n-2}) \tag{3.10}$$

or

$$[z_1, x_2] \wedge_* [z_2, x_1](z_3 \cdots z_{n-2}) \tag{3.11}$$

where $z_1, z_2, \ldots, z_{n-2} \in X$. In the former case (3.10) the element is a Kuz'min element if $z_2 = x_3$. Also, if $z_1 = x_3$, we swap z_1 and z_2 at the expense of a sign change to obtain a Kuz'min element. If neither z_1 nor z_2 are equal to x_3, then x_3 must be one of $z_3, \ldots, z_{n-2}$, and we may assume that $z_3 = x_3$. Let $v = z_4 \cdots z_{n-2}$. Then we have, by using the relations (3.5) and (3.3),

$$\begin{aligned} [z_2, z_1] \wedge_* [x_2, x_1] x_3 v &= -[z_2, z_1] x_3 \wedge_* [x_2, x_1] v \\ &= ([z_1, x_3] z_2 - [z_2, x_3] z_1) \wedge_* [x_2, x_1] v \\ &= -[z_1, x_3] \wedge_* [x_2, x_1] z_2 v + [z_2, x_3] \wedge_* [x_2, x_1] z_1 v. \end{aligned} \tag{3.12}$$

The two elements at the bottom are Kuz'min, as required. Now consider the latter case (3.11). Such elements are Kuz'min if $z_2 = x_3$. If this is not the case, then either $z_1 = x_3$ (Case 1) or x_3 is one of $z_3, \ldots, z_{n-2}$ (Case 2). In Case 1, if $z_2 = x_4$, we get the element h. If this is not the case, x_4 must be one of $z_3, \ldots, z_{n-2}$, and we may assume that $z_3 = x_4$. Then the element in question is of the form

$$[x_3, x_2] \wedge_* [z_2, x_1] x_4 v \tag{3.13}$$

where $v = z_4 \cdots z_{n-2}$. Using the relations (3.3) and (3.2) we get

$$[x_3, x_2] \wedge_* [z_2, x_1]x_4v = [x_3, x_2] \wedge_* [x_4, x_1]z_2v + [x_3, x_2] \wedge_* [z_2, x_4]x_1v.$$

The first element on the right hand side is the element h. It remains to show that the second element on the right hand side is of the required form. By using the relations (3.2)-(3.5) we get

$$\begin{aligned}
& [x_3, x_2] \wedge_* [z_2, x_4]x_1v \\
= \;& -[x_3, x_2]x_1 \wedge_* [z_2, x_4]v \\
= \;& [x_2, x_1]x_3 \wedge_* [z_2, x_4]v - [x_3, x_1]x_2 \wedge_* [z_2, x_4]v \\
= \;& -[z_2, x_4]v \wedge_* [x_2, x_1]x_3 + [z_2, x_4]v \wedge_* [x_3, x_1]x_2 \\
= \;& (-1)^{\deg v+1}[z_2, x_4] \wedge_* [x_2, x_1]x_3v + (-1)^{\deg v}[z_2, x_4] \wedge_* [x_3, x_1]x_2v.
\end{aligned}$$

We have already seen that the first of the two elements at the bottom can be written as a linear combination of Kuz'min elements (see (3.12)). For the second of these elements we have, again by using the relation (3.2),(3.3) and (3.5)

$$\begin{aligned}
& (-1)^{\deg v}[z_2, x_4] \wedge_* [x_3, x_1]x_2v \\
= \;& (-1)^{\deg v+1}[z_2, x_4]x_2 \wedge_* [x_3, x_1]v \\
= \;& (-1)^{\deg v+1}[z_2, x_2]x_4 \wedge_* [x_3, x_1]v + (-1)^{\deg v}[x_4, x_2]z_2 \wedge_* [x_3, x_1]v \\
= \;& (-1)^{\deg v}[z_2, x_2] \wedge_* [x_3, x_1]x_4v + (-1)^{\deg v+1}[x_4, x_2] \wedge_* [x_3, x_1]z_2v.
\end{aligned}$$

The two elements at the bottom are Kuz'min elements.

It remains to deal with Case 2, where x_3 is one of $z_3, \dots, z_{n-2}$. We may assume that $x_3 = z_3$. Then the element in question is of the form

$$[z_1, x_2] \wedge_* [z_2, x_1]x_3v$$

where $v = z_4 \cdots z_{n-2}$. If $z_1 > z_2$, this element is Kuz'min. If $z_1 < z_2$, we use (3.3) and (3.2) to get

$$[z_1, x_2] \wedge_* [z_2, x_1]x_3v = [z_1, x_2] \wedge_* [x_3, x_1]z_2v + [z_1, x_2] \wedge_* [z_2, x_3]x_1v.$$

The first element on the right hand side is Kuz'min. For the second element we get by using (3.5), (3.3) and (3.6), respectively,

$$\begin{aligned}
& [z_1, x_2] \wedge_* [z_2, x_3]x_1v \\
= \;& -[z_1, x_2]x_1 \wedge_* [z_2, z_3]v \\
= \;& [x_2, x_1]z_1 \wedge_* [z_2, x_3]v + [x_1, z_1]x_2 \wedge_* [z_2, x_3]v \\
= \;& (-1)^{\deg v+1}[z_2, x_3] \wedge_* [x_2, x_1]z_1v + (-1)^{\deg v}[z_2, x_3] \wedge_* [z_1, x_1]x_2v.
\end{aligned}$$

The first element on the bottom line is Kuz'min (up to sign), and the second element can be rewritten by using (3.5), (3.3) and (3.6) as

$$\begin{aligned} & (-1)^{\deg v}[z_2, x_3] \wedge_* [z_1, x_1]x_2v \\ = \;& (-1)^{\deg v+1}[z_2, x_3]x_2 \wedge_* [z_1, x_1]v \\ = \;& (-1)^{\deg v}[x_3, x_2]z_2 \wedge_* [z_1, x_1]v + (-1)^{\deg v+1}[z_2, x_2]x_3 \wedge_* [z_1, x_1]v \\ = \;& (-1)^{\deg v+1}[x_3, x_2] \wedge_* [z_1, x_1]z_2v + (-1)^{\deg v}[z_2, x_2] \wedge_* [z_1, x_1]x_3v. \end{aligned}$$

Now the second summand on the bottom line is a Kuz'min element (up to sign). As to the first summand, if $z_1 = x_4$ then this is h. If $z_1 \neq x_4$, then x_4 is one of the elements $z_2, z_4, z_5, \ldots, z_{n-2}$, and in this case the first summand is (up to sign) of the form (3.13). Such elements have been dealt with in Case 1. This completes the proof of the lemma. □

4. t-elements

Apart from the Kuz'min elements, there is a second kind of elements that will play a crucial role in studying the homogeneous components of $(M \wedge M) \otimes_U \mathbb{Z} = G''$ in odd degree $n \geqslant 5$. We call elements of the form

$$w(y_1, y_2, y_3, y_4; u) = [y_1, y_2] \wedge_* [y_3, y_4]u + [y_2, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_2, y_4]u,$$

where $y_1, \ldots, y_4 \in X$ and $u \in \mathcal{U}$, *t-elements.* These elements feature prominently in [6], where they are called Jacobian elements, and, slightly earlier, in [5]. The results in this section are due to Kuz'min [6], and some have been obtained independently in [5].

Lemma 4.1. *Let $n \geqslant 5$ be an odd integer, $y_1, \ldots, y_n \in X$ and $u = y_5 \cdots y_n$. Then the following holds for the t-element $w = w(y_1, y_2, y_3, y_4; u) \in (M \wedge M) \otimes_U \mathbb{Z}$.*

(i) *If any two of the elements y_1, y_2, y_3, y_4 are equal, then $w = 0$. In particular, w is antisymmetric in y_1, y_2, y_3, y_4.*

(ii) $w(y_1, y_2, y_3, y_4; y_5y_6 \cdots y_n) = w(y_1, y_2, y_3, y_5; y_4y_6 \cdots y_n)$.

Proof. Suppose $y_1 = y_2$. Then

$$\begin{aligned} & w(y_1, y_1, y_3, y_4; u) \\ = \;& [y_1, y_1] \wedge_* [y_3, y_4]u + [y_1, y_3] \wedge_* [y_1, y_4]u + [y_3, y_1] \wedge_* [y_1, y_4]u \\ = \;& 0. \end{aligned}$$

The proof for the cases where $y_1 = y_3$ and $y_2 = y_3$ is similar. Hence w is antisymmetric with respect to the first three entries y_1, y_2, y_3. To complete the proof of part (i) it is now sufficient to show that $w = 0$ if $y_3 = y_4$. We mention that so far we have not used the assumption that n is odd. This condition, however, is required for the case where $y_3 = y_4$. If this holds we have

$$\begin{aligned} & w(y_1, y_2, y_3, y_3; u) \\ = \;& [y_1, y_2] \wedge_* [y_3, y_3]u + [y_2, y_3] \wedge_* [y_1, y_3]u + [y_3, y_1] \wedge_* [y_2, y_3]u \\ = \;& 0. \end{aligned}$$

Indeed, the first summand is zero by anticommutativity, and the sum of the second and third is zero by anticommutativity and by (3.6) since $\deg u$ is odd. This proves (i). For (ii), set $u = y_5 v$ where $v = y_6 \cdots y_n$. Then

$$\begin{aligned} & w(y_1, y_2, y_3, y_4; y_5 v) \\ = \; & [y_1, y_2] \wedge_* [y_3, y_4] y_5 v + [y_2, y_3] \wedge_* [y_1, y_4] y_5 v + [y_3, y_1] \wedge_* [y_2, y_4] y_5 v \\ = \; & -[y_1, y_2] \wedge_* [y_4, y_5] y_3 v + [y_1, y_2] \wedge_* [y_3, y_5] y_4 v \\ & -[y_2, y_3] \wedge_* [y_4, y_5] y_1 v + [y_2, y_3] \wedge_* [y_1, y_5] y_4 v \\ & -[y_3, y_1] \wedge_* [y_4, y_5] y_2 v + [y_3, y_1] \wedge_* [y_2, y_5] y_4 v. \end{aligned}$$

The sum of the second summands in the last three rows is equal to $w(y_1, y_2, y_3, y_5; y_4 v)$, and the sum of the first summands in these rows is zero. Indeed, by using (3.5) we get

$$\begin{aligned} & -[y_1, y_2] \wedge_* [y_4, y_5] y_3 v - [y_2, y_3] \wedge_* [y_4, y_5] y_1 v - [y_3, y_1] \wedge_* [y_4, y_5] y_2 v \\ = \; & [y_1, y_2] y_3 \wedge_* [y_4, y_5] v + [y_2, y_3] y_1 \wedge_* [y_4, y_5] v + [y_3, y_1] y_2 \wedge_* [y_4, y_5] v \\ = \; & 0 \end{aligned}$$

(by the Jacobi identity). This completes the proof of part (ii). □

Corollary 4.1. ([6, Lemma 29]) *For any t-element $w(y_1, y_2, y_3, y_4; u)$ with $u = y_5 \cdots y_n$ of odd degree,*

$$2w(y_1, y_2, y_3, y_4; u) = 0.$$

Moreover, if any two of the elements $y_1, \ldots, y_n$ are equal, then

$$w(y_1, y_2, y_3, y_4; u) = 0.$$

Proof. Let $u = y_5 v$ where $v = y_6 \cdots y_n$. By using parts (i) and (ii) of Lemma 4.1 we have

$$\begin{aligned} & w(y_1, y_2, y_3, y_4; y_5 v) = w(y_1, y_2, y_3, y_5; y_4 v) = -w(y_1, y_2, y_5, y_3; y_4 v) \\ = & -w(y_1, y_2, y_5, y_4; y_3 v) = w(y_1, y_2, y_4, y_5; y_3 v) = w(y_1, y_2, y_4, y_3; y_5 v) \\ = & -w(y_1, y_2, y_3, y_4; y_5 v). \end{aligned}$$

Hence $2w(y_1, y_2, y_3, y_4; u) = 0$, as required. Finally, suppose that $y_i = y_j$ for $1 \leqslant i < j \leqslant n$. Then Lemma 4.1 implies that

$$w(y_1, y_2, y_3, y_4; u) = \pm w(y_i, y_j, \ldots; u')$$

for some suitable $u' \in \mathcal{U}$. But the right hand side is zero by part (i) of Lemma 4.1. □

5. The 6-term exact sequence

In this Section we introduce a 6-term exact sequence that will be our main tool for working out dimensions of the homogeneous components of $G'' \otimes K$ where K is a field of characteristic other than 2. This exact sequence is very similar to the exact sequence obtained in Section 7.1 of [12]. However, for the convenience of the reader we include full details. Recall the short exact sequences (2.4) and (2.2) from Section 2:

$$M \rightarrowtail P \twoheadrightarrow \Delta \qquad \text{and} \qquad \Delta \rightarrowtail U \twoheadrightarrow \mathbb{Z}.$$

These two short exact sequences give rise to a chain complex

$$0 \to M \wedge M \to P \wedge P \to \Delta \otimes P \to U \circ U \to U \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{5.1}$$

of abelian groups. The maps in (5.1) are (from left to right) given by

$$\begin{aligned} &m_1 \wedge m_2 \mapsto m_1\mu \wedge m_2\mu && (m_1, m_2 \in M) \\ &p_1 \wedge p_2 \mapsto p_1\sigma \otimes p_2 - p_2\sigma \otimes p_1 && (p_1, p_2 \in P) \\ &\delta \otimes p \mapsto \delta \circ p\sigma && (\delta \in \Delta, p \in P) \\ &f \circ g \mapsto (f\varepsilon)g + (g\varepsilon)f && (f, g \in U) \\ &f \mapsto f\varepsilon + 2\mathbb{Z} && (f \in U) \end{aligned}$$

with μ, σ and ε as defined in Section 2.

Lemma 5.1. *The chain complex* (5.1) *is an exact sequence of U-modules.*

Proof. The short exact sequence (2.4) splits over $\mathbb{Z}$. Let $\iota : \Delta \to P$ be a splitting map. Then $P = M\mu \oplus \Delta\iota$ as a $\mathbb{Z}$-module. Consequently,

$$P \wedge P = (M\mu \wedge M\mu) \oplus (\Delta\iota \otimes M\mu) \oplus (\Delta\iota \wedge \Delta\iota)$$

and

$$\Delta \otimes P = (\Delta \otimes M\mu) \oplus (\Delta \otimes \Delta\iota).$$

Now, the map $M \wedge M \to P \wedge P$ maps $M \wedge M$ isomorphically onto the direct summand $M\mu \wedge M\mu$ of $P \wedge P$, while the map $P \wedge P \to \Delta \otimes P$ maps the direct summand $\Delta\iota \otimes M\mu$ of $P \wedge P$ isomorphically onto the direct summand $\Delta \otimes M\mu$ of $\Delta \otimes P$, and it maps the direct summand $\Delta\iota \wedge \Delta\iota$ injectively into the direct summand $\Delta \otimes \Delta\iota$ of $\Delta \otimes P$: For $\delta_1, \delta_2 \in \Delta$ we get

$$\delta_1\iota \wedge \delta_2\iota \mapsto \delta_1 \otimes \delta_2\iota - \delta_2 \otimes \delta_1\iota.$$

In fact, the image of $\Delta\iota \wedge \Delta\iota$ in $\Delta\iota \otimes \Delta\iota$ is exactly the kernel of the canonical projection $\Delta\iota \otimes \Delta\iota \to \Delta \circ \Delta$. It follows that we have a 4-term exact sequence

$$0 \to M \wedge M \to P \wedge P \to \Delta \otimes P \to \Delta \circ \Delta \to 0,$$

where the first two maps are as in (5.1) and the third, $\Delta \otimes P \to \Delta \circ \Delta$ is given by $\delta \otimes p \mapsto \delta \circ p\sigma$ $(\delta \in \Delta, p \in P)$.

Furthermore, the short exact sequence (2.2) too splits over $\mathbb{Z}$. Here $U = \Delta \oplus \mathbb{Z}$ where we identify $\mathbb{Z}$ with the constant polynomials in $U = \mathbb{Z}[X]$. Consequently,

$$U \circ U = (\Delta \circ \Delta) \oplus (\Delta \otimes \mathbb{Z}) \oplus (\mathbb{Z} \circ \mathbb{Z}) \cong (\Delta \circ \Delta) \oplus \Delta \oplus \mathbb{Z}.$$

It is easily seen that the map $U \circ U \to U$ in (5.1) maps the direct summand $\Delta \otimes \mathbb{Z} \cong \Delta$ of $U \circ U$ isomorphically onto the $\mathbb{Z}$-direct summand Δ of U, while $\mathbb{Z} \circ \mathbb{Z} \cong \mathbb{Z}$ is mapped isomorphically onto $2\mathbb{Z}$. This yields a 4-term exact sequence

$$0 \to \Delta \circ \Delta \to U \circ U \to U \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

By combining this sequence with the 4-term exact sequence obtained above, we get the 6-term exact sequence (5.1). This is an exact sequence of abelian groups, but since all the maps in (5.1) agree with the derivation module action, we have actually an exact sequence of U-modules. This completes the proof of the lemma. □

Next we examine the modules in that exact sequence. Of course U itself is a free U-module. It turns out that the same is true for $\Delta \otimes P$. In fact, the tensor product of a $\mathbb{Z}$-free U-module and free U-module under derivation action is always a free module. This is a well-known general fact. It holds for both tensor products of modules for groups with diagonal action and tensor products of modules for Lie algebras with derivation action (see [10, Theorem 1.9.4]). For the convenience of the reader we include an elementary proof for the current setting. The proof provides an explicit free generating set that will be useful later.

Lemma 5.2. *Let N be an arbitrary U-module that is free as a $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\mathcal{N}$. Then the tensor product $N \otimes U$ is a free U-module and the elements $m \otimes 1$ with $m \in \mathcal{N}$ form a free generating set for $N \otimes U$ as a U-module.*

Proof. Recall that $\mathcal{U}$ denotes the $\mathbb{Z}$-basis of U consisting of all monomials. Then the elements $n \otimes u$ with $n \in \mathcal{N}$ and $u \in \mathcal{U}$ form a $\mathbb{Z}$-basis of $N \otimes U$. To prove that the elements $m \otimes 1$ form a generating set of $N \otimes U$ as a U-module, it is sufficient to show that each basis element $n \otimes u$ is a linear combination of those elements with coefficients in U. This is obviously the case when $\deg u = 0$, that is $u = 1$. Now let $\deg u = k > 0$. Then $u = vy$ for some $v \in \mathcal{U}$ with $\deg v = k - 1$ and some $y \in X$. Now,

$$n \otimes vy = (n \otimes v)y - ny \otimes v,$$

and by the inductive hypothesis both $n \otimes v$ and $ny \otimes v$ can be expressed as a linear combination of the elements $m \otimes 1$ ($m \in \mathcal{N}$) with coefficients in U. Consequently $n \otimes vy$ can be expressed in this way. Hence the elements $m \otimes 1$ are a generating set. It remains to show that this set is actually a free generating set. Suppose that this is not the case. Then there exists a finite subset $\{n_1, n_2, \ldots, n_k\}$ of $\mathcal{N}$ and non-zero polynomials $f_1, f_2, \ldots, f_k \in U$ such that

$$\sum_i (n_i \otimes 1) f_i = 0. \tag{5.2}$$

Observe that for any monomial $u \in \mathcal{U}$ and any $n \in \mathcal{N}$ we have

$$(n \otimes 1)u = n \otimes u + \sum c_{m,v} m \otimes v \tag{5.3}$$

where the sum runs over some basis elements $m \otimes v$ with $m \in \mathcal{N}$, $v \in \mathcal{U}$ with $\deg v < \deg u$, and $c_{m,v} \in \mathbb{Z}$. Note that the $\mathcal{U}$-components of the basis elements under the sum in (5.3) are of degree strictly less than the degree of u. Now consider the dependence (5.2). We may assume that f_1 is of maximal degree among the f_i. Let $w \in \mathcal{U}$ be a monomial that occurs with non-zero coefficient $a_{1,w}$ in the leading term of f_1. Then, if we expand the left hand side of (5.2) as a $\mathbb{Z}$-linear combination of the basis elements $n \otimes u$, (5.3) implies that the coefficient at the basis element $n_1 \otimes w$ is precisely $a_{1,w}$. But this gives that the left hand side of (5.2) is not zero, and the resulting contradiction completes the proof of the lemma. □

Corollary 5.1. *Let N be an arbitrary U-module that is free as a $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\mathcal{N}$, and let P be a free U-module with free generators $e_1, e_2, \ldots, e_r$. Then the tensor product $N \otimes P$ is a free U-module and the elements $m \otimes e_i$ with $m \in \mathcal{N}$ and $i = 1, 2, \ldots, r$ form a free generating set for $N \otimes P$ as a U-module.*

Proof. Since

$$N \otimes P \cong \bigoplus_i N \otimes e_i U,$$

this follows immediately from Lemma 5.2. □

The exterior and symmetric squares in (5.1) are not free U-modules. This generates considerable problems with using (5.1) for obtaining information about the tensor product $(M \wedge M) \otimes_U \mathbb{Z} = G''$. However, if we tensor these squares with a field K of characteristic other than 2, then both become free modules for the polynomial ring $K[X]$. This, in turn, allows us to exploit the exact sequence (5.1) to obtain information on the second derived algebra of the free centre-by-metabelian Lie algebra over the field K. In order to take advantage of this approach, we now work with the free centre-by-metabelian algebra over a field K. *We keep the notation introduced so far, but for the rest of this section and the next section we adopt the standing assumption that the ring of integers $\mathbb{Z}$ has been replaced as the ground ring by a field K.* Thus L is now the free Lie algebra over K with free generating set X, $G = L/[L'', L]$ is the free centre-by-metabelian Lie algebra over K, $U = K[X]$ etc. It is plain that all the results proved so far in this paper remain valid after tensoring with K, that is for the free centre-by-metabelian Lie algebra and associated structures over K.

Proposition 5.1. *If K is a field of characteristic other than 2, then the exterior and symmetric squares $U \wedge U$ and $U \circ U$ are free U-modules. The elements $u \wedge 1$ with $u \in \mathcal{U}$ and $\deg u$ odd form a free generating set for $U \wedge U$, and the elements $u \circ 1$ with $u \in \mathcal{U}$ and $\deg u$ even form a free generating set for $U \circ U$.*

Proof. The exterior square $U \wedge U$ is a homomorphic image of the tensor square $U \otimes U$ via the projection map $\pi : U \otimes U \to U \wedge U$ given by $f_1 \otimes f_2 \mapsto f_1 \wedge f_2$ $(f_1, f_2 \in U)$. By Lemma 5.2 the elements $u \otimes 1$ with $u \in \mathcal{U}$ form a generating set of $U \otimes U$ as a U-module. Consequently, the elements $u \wedge 1$ form a generating set of $U \wedge U$ as a U-module. Consider the trivialization homomorphism $U \wedge U \to (U \wedge U) \otimes_U K$. In the tensor product $(U \wedge U) \otimes_U K$ we have the relations

$$(f_1 x \wedge_* f_2) = -(f_1 \wedge_* f_2 x) \qquad (f_1, f_2 \in U,\ x \in X),$$

and, consequently, for a monomial $u \in \mathcal{U}$ we have

$$u \wedge_* 1 = (-1)^{\deg u}(1 \wedge_* u) = (-1)^{\deg u + 1}(u \wedge_* 1).$$

Since the characteristic of the ground field K is not 2, this implies that $(u \wedge_* 1) = 0$ in $(U \wedge U) \otimes_U K$ if $\deg u$ is even. Hence in this case $u \wedge 1 \in (U \wedge U)\Delta$. But this means that the elements $u \wedge 1$ with u of even degree belong to the submodule of $U \wedge U$ that is generated by the elements $v \wedge 1$ with $\deg v < \deg u$. It follows that these elements can be removed from the generating set of $U \wedge U$ as a U-module. In other words, the elements $u \wedge 1$ with $u \in \mathcal{U}$ and $\deg u$ odd form a generating set of $U \wedge U$ as a U-module. Now we show that this is actually a free generating set. To this end we consider the images of these generators in $U \otimes U$ under the embedding $\nu : U \wedge U \to U \otimes U$ that is given by $(f_1 \wedge f_2) \mapsto f_1 \otimes f_2 - f_2 \otimes f_1$ $(f_1, f_2 \in U)$. Suppose $u = y_1 y_2 \dots y_k$. Then

$$\begin{aligned} 1 \otimes u &= 1 \otimes y_1 y_2 \cdots y_k \\ &= (1 \otimes y_1 y_2 \cdots y_{k-1}) y_k - (y_k \otimes y_1 y_2 \cdots y_{k-2}) y_{k-1} \\ &\quad + (y_{k-1} y_k \otimes y_1 y_2 \cdots y_{k-3}) y_{k-2} - \cdots \\ &\quad \cdots + (-1)^{k-1} (y_2 \cdots y_k \otimes 1) y_1 + (-1)^k y_1 y_2 \cdots y_k \otimes 1. \end{aligned}$$

Consequently, if k, the degree of u, is odd, we have

$$(u \wedge 1)\nu = u \otimes 1 - 1 \otimes u = 2(u \otimes 1) + w \tag{5.4}$$

where w belongs to the submodule of $U \otimes U$ that is generated by the elements $v \otimes 1$ with $v \in \mathcal{U}$ and $\deg v < \deg u$. Since the elements $u \otimes 1$ as free generators of $U \otimes U$ are linearly independent over U, it follows easily from (5.4) that the elements $u \wedge 1$ with $\deg u$ odd are also linearly independent over U, and hence they are free generators for $U \wedge U$ as a U-module.

The proof for $U \circ U$ is similar with the embedding $U \circ U \to U \otimes U$ given by $(f_1 \circ f_2) \mapsto f_1 \otimes f_2 + f_2 \otimes f_1$ $(f_1, f_2 \in U)$ being used instead of ν.

□

Corollary 5.2. *If K is a field of characteristic other than 2 and P is a free U-module with free generators $e_1, \dots, e_r$, then*

- (i) *$P \wedge P$ is a free U-module and the elements $e_i u \wedge e_i$ with $i = 1, 2, \dots, r$, $u \in \mathcal{U}$ and* $\deg u$ *odd together with the elements $e_i u \wedge e_j$ with $1 \leqslant i < j \leqslant r$, $u \in \mathcal{U}$ form a free generating set of $P \wedge P$ as a U-module,*
- (ii) *$P \circ P$ is a free U-module and the elements $e_i u \circ e_i$ with $i = 1, 2, \dots, r$, $u \in \mathcal{U}$ and* $\deg u$ *even together with the elements $e_i u \circ e_j$ with $1 \leqslant i < j \leqslant r$, $u \in \mathcal{U}$ form a free generating set of $P \circ P$ as a U-module.*

Proof. We have that $P = \bigoplus_i e_i U$, and then

$$P \wedge P \cong \bigoplus_i (e_i U \wedge e_i U) \oplus \bigoplus_{i<j} (e_i U \otimes e_j U),$$

and the result follows. The proof for $P \circ P$ is similar. □

6. Dimensions

Throughout this section we work with the free centre-by-metabelian Lie algebra over a field K of characteristic other than 2. Then $K/2K = 0$, and the 6-term exact sequence (5.1) turns into

$$0 \to M \wedge M \to P \wedge P \to P \otimes \Delta \to U \circ U \to U \to 0. \tag{6.1}$$

Note that all the modules in (6.1) have a natural grading (stemming from the grading by degree in $U = K[X]$), and that the sequence is an exact sequence of graded modules. Moreover, since all the modules to the right of $M\wedge M$ are free U-modules, this exact sequence stays exact after tensoring with the trivial U-module K:

$$\begin{aligned}0\to&(M\wedge M)\otimes_U K\to(P\wedge P)\otimes_U K\to\\&\to(P\otimes\Delta)\otimes_U K\to(U\circ U)\otimes_U K\to U\otimes_U K\to 0.\end{aligned}\tag{6.2}$$

This is an exact sequence of graded K-spaces, and it can be used to work out the dimensions of the homogeneous components of $(M\wedge M)\otimes_U K$ which are, in view of (2.7), the dimensions of the homogeneous components of G''. In fact, for any homogeneous or fine homogeneous component $((M\wedge M)\otimes_U K)_*$ the exactness of (6.2) yields

$$\begin{aligned}\dim((M\wedge M)\otimes_U K)_*=&\dim((P\wedge P)\otimes_U K)_*-\dim((P\otimes\Delta)\otimes_U K)_*\\&+\dim((U\circ U)\otimes_U K)_*-\dim(U\otimes_U K)_*.\end{aligned}\tag{6.3}$$

It remains to work out the terms on the right hand side of (6.3). First of all, $U\otimes_U K$ is a K-space of dimension one with free generator $1\otimes 1$ of degree 0. Hence this term does not contribute to homogeneous components of degree other than 0. For the other three terms on the right hand side of (6.3), the modules in question are free and free generating sets for them have been obtained in Section 5. Now if $\mathcal{F}$ is a homogeneous free generating set for a graded U-module F, then $\{f\otimes 1:\ f\in\mathcal{F}\}$ is a basis for the tensor product $F\otimes_U K$ as a K-space. Hence, in order to determine the dimensions on the right hand side of (6.3), we need to count the number of free module generators of a given degree or multidegree. This is carried out in the following three lemmas.

Lemma 6.1. *If K is a field of characteristic other than 2, then the following holds.*

(i) *If $n\geqslant 3$ is odd, then*

$$\dim((P\wedge P)\otimes_U K)_n=\binom{r+1}{2}\binom{n+r-3}{n-2}.$$

Moreover, if $q\vDash n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim((P\wedge P)\otimes_U K)_q=\binom{k}{2}+k-m.$$

(ii) *If $n\geqslant 2$ is even, then*

$$\dim((P\wedge P)\otimes_U K)_n=\binom{r}{2}\binom{n+r-3}{n-2}.$$

Moreover, if $q\vDash n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim((P\wedge P)\otimes_U K)_q=\binom{k}{2}.$$

Proof. If n is odd, Corollary 5.2 implies that the elements $e_iu\otimes e_j$ $(1\leqslant i\leqslant j\leqslant r)$ where $u\in\mathcal{U}$ with $\deg u=n-2$ form a basis of $((P\wedge P)\otimes_U K)_n$. The number of monomials of degree $n-2$ in r variables is $\binom{n+r-3}{n-2}$ and the number of possible pairs e_i,e_j with $i\leqslant j$ is $\binom{r+1}{2}$. This yields the dimension formula for $((P\wedge P)\otimes_U K)_n$.

In order to determine the dimension of $((P \wedge P) \otimes_U K)_q$, we need to count the number of basis elements of multidegree q. For each possible choice of the pair e_i, e_j in $e_i u \otimes e_j$ there is precisely one such basis element of multidegree q. As to the possible choices of e_i, e_j, there are $\binom{k}{2}$ with $e_i \neq e_j$, and $k - m$ with $e_i = e_j$. This yields the dimension formula for $((P \wedge P) \otimes_U K)_q$. If n is even, Corollary 5.2 implies that the elements $e_i u \otimes e_j$ $(1 \leqslant i < j \leqslant r)$ where $u \in \mathcal{U}$ with $\deg u = n - 2$ form a basis of $((P \wedge P) \otimes_U K)_n$. An easy count gives the corresponding dimension formulae. □

Lemma 6.2. *For any field K and for all $n \geqslant 2$,*

$$\dim((\Delta \otimes P) \otimes_U K)_n = r\binom{n+r-2}{n-1}.$$

Moreover, if $q \vDash n$ is a composition of n in r parts such that k of the parts are non-zero, then

$$\dim((\Delta \otimes P) \otimes_U K)_q = k.$$

Proof. Corollary 5.1 implies that the elements $u \otimes e_i$ with $i = 1, 2, \ldots, r$, $u \in \mathcal{U}$ and $\deg u = n - 1$ form a basis of $\Delta \otimes P$. An easy count of those basis elements and such among them of a particular multidegree confirms the dimension formulae in Lemma 6.2. □

Lemma 6.3. *If K is a field of characteristic other than 2, then the following holds.*

(i) *If $n \geqslant 1$ is odd, then*

$$((U \circ U) \otimes_U K)_n = \{0\}.$$

(ii) *If $n \geqslant 2$ is even, then*

$$\dim((U \circ U) \otimes_U K)_n = \binom{n+r-1}{n}.$$

Moreover, if $q \vDash n$ is a composition of n in r parts, then

$$\dim((U \circ U) \otimes_U K)_q = 1.$$

Proof. This follows from Proposition 5.1 by an argument similar to those used in the proofs of the previous two lemmas. □

To get the main result of this section, it remains to substitute the dimension formulae of the previous three lemmas into (6.3), and use the isomorphism (2.7).

Theorem 6.1. *Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field K of characteristic other than 2. Then the dimensions of the homogeneous and fine homogeneous components of the second derived algebra G'' are as follows.*

(i) *If $n \geqslant 5$ is odd, then*

$$\dim(G'')_n = \frac{r(n-3)}{2}\binom{n+r-3}{n-1}.$$

Moreover, if $q \vDash n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim(G'')_q = \binom{k}{2} - m.$$

(ii) *If $n \geqslant 6$ is even, then*

$$\dim(G'')_n = \binom{n-1}{2}\binom{n+r-3}{n}.$$

Moreover, if $q \vDash n$ is a composition of n in r parts such that k of the parts are non-zero

$$\dim(G'')_q = \binom{k-1}{2}.$$

□

We mention that the formula for $\dim G''_5$ has been derived in our earlier paper [9].

7. The Basis Theorem

In this section we return to the free centre-by-metabelian Lie ring. We need another combinatorial result.

Lemma 7.1. *Let $X = \{x_1, x_2, \ldots, x_r\}$ with $r \geqslant 2$ and let n be a positive integer with $n \geqslant 5$.*

(i) *The number of Kuz'min elements of degree n with entries in X is*

$$k(n,r) = \frac{r(n-3)}{2}\binom{n+r-3}{n-1}.$$

(ii) *If $n = r$, the number multilinear Kuz'min elements of degree n, that is Kuz'min elements of multidegree $(1,1,\ldots,1)$, with entries in X is*

$$\tilde{k}(n) = \frac{n(n-3)}{2}.$$

Proof. For part (i) we use induction on r. The assertion is true for $r = 2$ as the only Kuz'min elements in this case are of the form

$$[x_2, x_1] \wedge_* [x_2, x_1] x_1^k x_2^{n-4-k} \qquad (k = 0, \ldots, n-4).$$

Hence there is $n-3$ of them, which is the required number.

Now let $r > 2$. By induction, the number of Kuz'min polynomials of degree n in X that do not involve x_1 is $k(n, r-1)$. To that we need to add the number of Kuz'min polynomials (3.7) that do involve x_1. If x_1 is present, we must have that $y_4 = x_1$. Hence these polynomials are of the form

$$[y_1, y_2] \wedge_* [y_3, x_1] y_5 \cdots y_n \tag{7.1}$$

with

$$y_1 \geqslant y_3 > x_1, \quad x_1 \leqslant y_2 \leqslant y_5 \leqslant \cdots \leqslant y_n \tag{7.2}$$

and

$$y_1 > y_2. \tag{7.3}$$

First we count the polynomials satisfying condition (7.2). In these polynomials (y_1, y_3) can be any pair of elements in $X \setminus \{x_1\}$ with $y_1 \leqslant y_3$. The number of such pairs is $(r-1)r/2$. The entries $y_2, y_5, \ldots, y_n$ can be any elements of X with

$y_2 \leqslant y_5 \leqslant \cdots \leqslant y_n$. The number of such sequences of $n-3$ elements is $\binom{n+r-4}{n-3}$. Hence the number of polynomials (7.1) satisfying the conditions (7.2) is

$$\frac{(r-1)r}{2}\binom{n+r-4}{n-3}. \tag{7.4}$$

In order to find the number of Kuz'min polynomials involving x_1, we need to subtract from (7.4) the number of polynomials satisfying the conditions (7.2) but not (7.3). These are precisely the polynomials (7.1) where the entries $y_1, y_2, y_3, y_5, \ldots, y_n$ satisfy the condition $x_1 < y_3 \leqslant y_1 \leqslant y_2 \leqslant y_5 \cdots \leqslant y_n$. The number of such sequences of $n-1$ elements is $\binom{n+r-3}{n-1}$. Thus the number of Kuz'min polynomials involving x_1 is

$$\frac{(r-1)r}{2}\binom{n+r-4}{n-3} - \binom{n+r-3}{n-1}. \tag{7.5}$$

Now we get the total number of Kuzmin polynomials in X by adding (7.5) to $k(n, r-1)$:

$$k(n,r) \quad = \quad \frac{(r-1)(n-3)}{2}\binom{n+r-4}{n-1} + \frac{(r-1)r}{2}\binom{n+r-4}{n-3} - \binom{n+r-3}{n-1}.$$

An elementary calculation shows that this is equal to the number given in part (i) of the lemma.

For part (ii), observe first that if an element of the form (3.7) is Kuz'min and of multidegree $(1, 1, \ldots, 1)$, then we must have that $y_4 = x_1$, and either $y_2 = x_2$ or $y_3 = x_2$. In the former case the element is of the form

$$[y_1, x_2] \wedge_* [y_3, x_1] y_5 \cdots y_n.$$

Such elements are Kuz'min if and only if $y_1 > y_3$, and there are precisely $\binom{n-2}{2}$ such elements of multidegree $(1, 1, \ldots, 1)$. In the latter case we must have $y_2 = x_3$, so the element is of the form

$$[y_1, x_3] \wedge_* [x_2, x_1] y_5 \cdots y_n.$$

All multilinear elements of this form are Kuz'min elements, and hence there are $n-3$ of them. Thus altogether we have

$$\tilde{k}(n) = \binom{n-2}{2} + n - 3 = \frac{n(n-3)}{2}.$$

This completes the proof of the lemma. □

Now we are ready for our main Theorem. We find it convenient to state it in terms of Lie rings. In this context the relevant elements are Lie monomials in X of the form

$$[[y_1, y_2], [y_3, y_4, y_5, \ldots, y_n]] \qquad (y_i \in X). \tag{7.6}$$

Then Kuz'min elements are Lie monomials of the form (7.6) such that

$$y_1 > y_2, \;\; y_3 > y_4, \;\; y_1 \geqslant y_3, \;\; y_4 \leqslant y_2 \leqslant y_5 \leqslant \cdots \leqslant y_n,$$

and t-elements are defined as

$$w(y_1, y_2, y_3, y_4; y_5 \dots y_n) = [[y_1, y_2],[y_3, y_4, y_5, \dots, y_n]] + [[y_2, y_3], [y_1, y_4, y_5, \dots, y_n]] + [[y_3, y_1], [y_2, y_4, y_5, \dots, y_n]] \qquad (y_i \in X).$$

These correspond under the isomorphism (2.7) to the Kuz'min and t-elements introduced in Sections 3 and 4.

Theorem 7.1. *Let G be the free centre-by-metabelian Lie ring of rank $r > 1$ on a free generating set $X = \{x_1, x_2, \dots, x_r\}$, let $q = (q_1, q_2, \dots, q_r) \vDash n$ be a composition of $n \geqslant 5$, and let G''_q denote the fine homogeneous component of multidegree q of the second derived ideal $G'' \subseteq G$.*

(i) *Suppose that $q = (q_1, \dots, q_r) \vDash n$ is multilinear with $q_i = 1$ for $i = i_1, i_2, \dots, i_n$, where $1 \leqslant i_1 < \cdots < i_n \leqslant r$. Then,*
   (a) *if n is odd, G''_q is generated by the Kuz'min elements of multidegree q and the t-element $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \cdots x_{i_n})$. The former freely generate a free abelian group of rank $\frac{1}{2}n(n-3)$ and the latter generates a cyclic group of order at most 2,*
   (b) *if n is even, then G''_q is a free abelian group of rank $\binom{n-1}{2}$, and the Kuz'min elements of multidegree q together with the element*

$$[[x_{i_3}, x_{i_2}], [x_{i_4}, x_{i_1}, x_{i_5}, \dots, x_{i_n}]]$$

   form a free generating set for it.

(ii) *Suppose that $q = (q_1, \dots, q_r) \vDash n$ is a composition of n with k non-zero parts, such that $q_i \geqslant 2$ for some i with $1 \leqslant i \leqslant n$, and m of the parts of q are 1. Then*
   (a) *if n is odd, G''_q is a free abelian group of rank $\binom{k}{2} - m$, and the elements (7.6) of multidegree q with $y_2 = y_4 = x_i$, $y_1 > y_3$ and $y_1, y_3 \neq x_i$, form a free generating set for it,*
   (b) *if n is even, then G''_q is a direct sum of a free abelian group of rank $\binom{k-1}{2}$ that is freely generated by the elements (7.6) of multidegree q with $y_2 = y_4 = x_i$, $y_1 > y_3$ and $y_1, y_3 \neq x_i$, and an elementary abelian 2-group generated by the elements (7.6) of multidegree q such that $y_2 = y_4 = x_i$, $y_1 = y_3 \neq x_i$. If all parts of q are even, then all of the latter elements are zero, and the torsion subgroup of G''_q is trivial.*

Proof. It is clearly sufficient to prove part (i) in the case where $r = n$ and $q = (1, 1, \dots, 1)$. Then, by Lemma 3.3, the Kuz'min elements of multidegree q together with the element $h = [[x_3, x_2], [x_4, x_1, x_5, \dots, x_n]]$ form a generating set of G''_q as a $\mathbb{Z}$-module. Assume that n is odd. Consider the t-element

$$w(x_1, x_2, x_3, x_4; x_5 \dots x_n) = [[x_1, x_2],[x_3, x_4, x_5, \dots, x_n]] + [[x_2, x_3], [x_1, x_4, x_5, \dots, x_n]] + [[x_3, x_1], [x_2, x_4, x_5, \dots, x_n]].$$

The second summand on the right hand side is equal to h, and the first and third summands on the right hand side are (up to sign) equal to Kuz'min elements:

$$[[x_1, x_2], [x_3, x_4, x_5, \dots, x_n]] = [[x_4, x_3], [x_2, x_1, x_5, \dots, x_n]]$$

$$[[x_3, x_1], [x_2, x_4, x_5, \dots, x_n]] = -[[x_4, x_2], [x_3, x_1, x_5, \dots, x_n]].$$

It follows that the element h in our generating set for G''_q can be replaced by the t-element $w(y_1, y_2, y_3, y_4; y_5 \cdots y_n)$. In other words, the multilinear fine homogeneous component G''_q is generated by the Kuz'min elements of multidegree q and the single t-element $w(x_1, x_2, x_3, x_4; x_5 \cdots x_n)$. By Lemma 4.1, this t-element is a torsion element that is annihilated by 2. On the other hand, by Lemma 7.1 (ii), the number of Kuz'min elements of multidegree q is $\frac{1}{2}n(n-3)$. By Theorem 6.1 (i), applied to the case where $r = n = m$, this is exactly the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2. It follows that the Kuz'min elements of multidegree q in G''_q freely generate a free abelian group of rank $\frac{1}{2}n(n-3)$. Hence G''_q is the direct sum of this free abelian group and the torsion subgroup generated by w. This completes the proof of (i.a).

Now assume that n is even. Again, by Lemma 3.3, G''_q is generated by the Kuz'min elements and the element $h = [[x_3, x_2], [x_4, x_1, x_5, \dots, x_n]]$. The number of those Kuz'min elements has been calculated in Lemma 7.1 (ii), and hence the number of elements in our generating set is

$$\tilde{k}(n) + 1 = \frac{1}{2}n(n-3) + 1 = \binom{n-1}{2}.$$

By Theorem 6.1 (ii), applied to the case where $r = n = k$, this is exactly the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2. It follows that the elements in our generating set freely generate a free $\mathbb{Z}$-module of rank $\binom{n-1}{2}$. This proves (i.b).

Now suppose the assumptions of part (ii) are satisfied. Then, by Lemma 3.2, the elements (7.6) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geqslant y_3$ and $y_1, y_3 \neq x_i$, form a generating set of G''_q. The number of such elements is $\binom{k}{2} - m$. If n is odd, this is precisely the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2 (see Theorem 6.1 (i)). It follows that these elements freely generate a free abelian group of rank $\binom{k}{2} - m$ in G''_q. There is no torsion part in this fine homogeneous component. Now suppose that n is even. Then we split the elements (7.6) of multidegree q with $y_2 = y_4 = x_i$, $y_1 \geqslant y_3$ and $y_1, y_3 \neq x_i$ into the disjoint union of those with $y_1 = y_3$ and those with $y_1 > y_3$. The former generate an elementary abelian 2-group. Indeed, in view of (3.6), translated into the setting of G'', we have

$$[[y_1, x_i], [y_1, x_i, y_5, \dots, y_n]] = -[[y_1, x_i], [y_1, x_i, y_5, \dots, y_n]],$$

and hence

$$2[[y_1, x_i], [y_1, x_i, y_5, \dots, y_n]] = 0.$$

So the elements (7.6) with $y_2 = y_4 = x_i$ and $y_1 = y_3$ generate a torsion group. If all free generators involved in such an element occur even multiplicity, the images of such elements in $(M \wedge M) \otimes_U \mathbb{Z}$ are of the form $[y_1, x_i] \wedge_* [y_1, x_i](z_1^2 \dots z_{(n-4)/2}^2)$ for some $z_1, \dots, z_{(n-4)/2} \in X$. But then, by using (3.5),

$$[y_1, x_i] \wedge_* [y_1, x_i](z_1^2 \dots z_{(n-4)/2}^2) = \\ \pm [y_1, x_i](z_1 \dots z_{(n-4)/2}) \wedge_* [y_1, x_i](z_1 \dots z_{(n-4)/2}) = 0,$$

and hence we can delete these elements from our generating set. The number of elements with $y_1 > y_3$ among the elements (7.6) is $\binom{k-1}{2}$. By Theorem 6.1 (ii), this is exactly the dimension of $G''_q \otimes K$, where K is a field of characteristic other than 2. It follows that these elements in our generating set freely generate a free $\mathbb{Z}$-module of rank $\binom{k-1}{2}$. This completes the proof of the theorem. □

Our theorem asserts, inter alia, that each of the homogeneous components G''_n $(n \geqslant 5)$ is a direct sum of a free abelian group and a (possibly trivial) elementary abelian 2-group, that is, we obtain the direct decomposition (1.1) in Section 1. The rank of the free abelian group F_n in this decomposition is equal to the dimension of $G'' \otimes K$ where K is a field of characteristic other than 2. This dimension was calculated in Section 6.

Corollary 7.1. *For each $n \geqslant 5$, G''_n is a direct sum of a free abelian group F_n and a (possibly trivial) elementary abelian 2-group T_n. The rank of F_n is equal to the dimension of $G'' \otimes K$, where K is a field of characteristic other than two. A formula for this dimension is given in Theorem 6.1.* □

Remark. For certain compositions q the generating sets described in part (ii) of the main theorem turn out to be empty, and then the corresponding parts of G''_q are zero. For example, $G''_{(4,1)} = 0$, and, more interestingly, $G''_{(3,1,1,1)}$ is torsion-free and $G''_{(3,3)}$ is a torsion group.

Theorem 7.1 does not address the question of whether or not the torsion subgroups featuring in parts (i.a) and (ii.b) are actually non-trivial. According to [6], the t-element in part (i.a) is a non-trivial element of order 2. This is confirmed in [4], which in its turn relies on [5, Lemma 3.8] for the crucial fact that this element is not zero. In fact, in [5] the relevant part of Lemma 3.8 is attributed to Hurley (unpublished). According to [13], the torsion elements described in part (ii.b) are not only non-zero, but form a basis of T_q as a $\mathbb{Z}/2\mathbb{Z}$-module. However, no proof is given, and instead the author says that this can be established by the methods used in [6]. The latter is hardly satisfactory since we know that some of the arguments in [6] are flawed. However, the torsion part of G'' is beyond the scope of the present paper. We focus exclusively on the determination of the ranks and the construction of explicit $\mathbb{Z}$-bases of the free abelian groups F_q. Our results contradict what is claimed in parts 3) and 4) of Theorem 4 in [6]. For $r = 3$, for example, this theorem asserts that the elements $[[x_3, x_2], [x_3, x_1, x_2, x_3]]$ and $[[x_3, x_2], [x_2, x_1, x_3, x_3]]$ are linearly independent over $\mathbb{Z}$ in $G''_{(1,2,3)}$, while Theorem 7.1 (ii.b) says that $G''_{(1,2,3)}$ modulo its torsion subgroup is an infinite cyclic group. Our results are in keeping with parts 1) and 2) of Theorem 4 in [6]. Moreover, we can confirm the assertions of this Theorem regarding the free abelian part of G''_n for odd n:

Theorem 7.2. [6, Theorem 4, Parts 1) and 2)] *For G as in Theorem 7.1, and all odd $n \geqslant 5$, the homogeneous component G''_n is a direct sum*

$$G''_n = F_n \oplus T_n,$$

where F_n is a free abelian group that is freely generated by the Kuz'min elements (7.6) *of degree n and T_n is an elementary abelian 2-group generated by the multilinear t-elements $w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}; x_{i_5} \cdots x_{i_n})$ with $x_{i_1}, x_{i_2}, \dots x_{i_n} \in X$ and $i_1 < i_2 < \cdots < i_n$.*

Proof. Let n be odd with $n \geqslant 5$. It is sufficient to prove the theorem for the case where $n = r$. In the proof of Theorem 7.1 (i.a) we have seen that the multilinear component $G''_{(1,1,\dots,1)}$ is generated by the multilinear Kuz'min elements and the t-element $w(x_1, x_2, x_3, x_4; x_5 \dots x_n)$ of degree n. Since n is odd, Corollary 4.1 tells us that the t-element has order at most 2. Any other fine homogeneous component G''_q of total degree n is a homomorphic image of the multilinear one under a suitable endomorphism of the form π_f (see Section 2). Moreover, this endomorphism can be chosen in such a way that it preserves the order of the free generators. In this case, the images of Kuz'min elements under π_f will be either Kuz'min elements themselves or zero. Moreover, every Kuz'min element of multidegree q is a homomorphic image of a multilinear Kuz'min element. Furthermore, the image of the multilinear t-element will be a t-element. But the latter will have at least two equal entries, and hence, by Corollary 4.1, it will be zero. It follows that for odd $n \geqslant 5$ every fine homogeneous component G_q with $q \vDash n$ and $q \neq (1, 1, \dots, 1)$ is generated by the Kuz'min elements multidegree q. By Lemma 7.1 (i) and Theorem 6.1 (i), the total number of Kuz'min elements of degree n is equal to the dimension of $G''_n \otimes K$, where K is an arbitrary field of characteristic other than 2. It follows that the Kuz'min elements of odd degree are linearly independent over $\mathbb{Z}$, and freely generate a free abelian group F_n. □

Finally, our results imply that the generating sets for the free abelian part of the lower central quotients $\gamma_n \mathfrak{G}/\gamma_{n+1}\mathfrak{G}$ of the free centre-by-metabelian group $\mathfrak{G}$ are optimal. We use the notation introduced in Section 1.

Corollary 7.2. *The generating sets for $\mathfrak{F}_n$, the free abelian part of the group $\mathfrak{G}''_n = (\gamma_n \mathfrak{G} \cap \mathfrak{G}'')\gamma_{n+1}\mathfrak{G}/\gamma_{n+1}\mathfrak{G}$ with $n \geqslant 5$ given in* [4, Theorems 1 and 4] *are optimal, i.e. linearly independent over $\mathbb{Z}$.*

Proof. By [7, Theorem 1], we have that $G'' \otimes \mathbb{Q} \cong \mathfrak{G}''_n \otimes \mathbb{Q}$. In view of this, the result for odd n follows immediately from Theorem 7.2 since the generating set in Theorem 1 of [4] is exactly the set of Kuz'min commutators. For even n it is not hard to verify that the number of commutators of a given multidegree $q \vDash n$ in the generating sets in Theorem 4 of [4] is equal to the dimension of $G'' \otimes \mathbb{Q}$ as given in Theorem 6.1. The result follows. □

8. A direct decomposition

The decomposition of each homogeneous and fine homogeneous component of G'' into a direct sum of a free abelian group and an elementary abelian 2-group was a by-product of the proof of Theorem 7.1. In conclusion we give a short direct proof of this result. We mention that this is actually a special case of a far more general result proved in [13]. However, since this paper is not easily accessible, and since our proof is short and does not require the full force of the arguments used in [13], we felt it is justified to include it for completeness.

Theorem 8.1. *The second derived ideal G'' of the free centre-by-metabelian Lie ring of rank $r \geqslant 2$ is a direct sum of a free abelian group and an elementary abelian 2-group.*

Proof. In view of the isomorphism (2.7) it is sufficient to prove the result for the tensor product $(M \wedge M) \otimes_U \mathbb{Z}$. For the exterior square $M \wedge M$ there is an embedding $\nu : M \wedge M \to M \otimes M$ given by $m_1 \wedge m_2 \mapsto m_1 \otimes m_2 - m_2 \otimes m_1$ $(m_1, m_2 \in M)$. On the other hand, there is the epimorphism $\pi : M \otimes M \to M \wedge M$ given by $m_1 \otimes m_2 \mapsto m_1 \wedge m_2$. The composite $\nu\pi$ amounts to multiplication by 2 on $M \wedge M$. Consequently, the composite $\nu\pi \otimes 1$,

$$(M \wedge M) \otimes_U \mathbb{Z} \xrightarrow{\nu \otimes 1} (M \otimes M) \otimes_U \mathbb{Z} \xrightarrow{\pi \otimes 1} (M \wedge M) \otimes_U \mathbb{Z},$$

too amounts to multiplication by 2 on $(M \wedge M) \otimes_U \mathbb{Z}$. It follows that the kernel of $\nu \otimes 1$ is annihilated by 2, i.e. it is an elementary abelian 2-group. We claim that the tensor product $(M \otimes M) \otimes_U \mathbb{Z}$ is a free abelian group. Once established, this will prove the theorem, as $(M \wedge M) \otimes_U \mathbb{Z}$ will be the direct sum of the kernel of $\nu \otimes 1$ and the image of $\nu \otimes 1$ in $(M \otimes M) \otimes_U \mathbb{Z}$. Tensoring the exact sequence (2.4) with M yields an exact sequence

$$0 \to M \otimes M \to M \otimes P \to M \otimes \Delta \to 0.$$

Recall that $M \otimes P$ is a free U-module (see Corollary 5.1). Then part of the long exact homology sequence associated with that short exact sequence looks as follows.

$$0 \to \mathrm{Tor}_1^U(M \otimes \Delta, \mathbb{Z}) \to (M \otimes M) \otimes_U \mathbb{Z} \to (M \otimes P) \otimes_U \mathbb{Z} \to \cdots .$$

Here $(M \otimes P) \otimes_U \mathbb{Z}$, the trivialization of a free U-module, is a free abelian group. But $\mathrm{Tor}_1^U(M \otimes \Delta, \mathbb{Z})$ too is a free abelian group. Indeed, dimension shifting using the short exact sequences (2.4) and (2.2) gives

$$\mathrm{Tor}_1^U(M \otimes \Delta, \mathbb{Z}) = \mathrm{Tor}_4^U(\mathbb{Z}, \mathbb{Z}),$$

and the latter is a free abelian group of rank $\binom{r}{4}$ [8, Section VII.2]. Now the long exact homology sequence gives that $(M \otimes M) \otimes_U \mathbb{Z}$ is free abelian, and this completes the proof of the theorem.

□

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