The Homotopy Exponent Problem For Certain Classes Of Polyhedral Products

Robinson, Daniel Mark

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THE HOMOTOPY EXPONENT PROBLEM FOR CERTAIN CLASSES OF POLYHEDRAL PRODUCTS

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By
Daniel Robinson
School of Mathematics

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Abstract

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Given a sequence of n topological pairs $A_i \subseteq X_i$ for $i = 1, \ldots, n$ and a simplicial complex K on n vertices, there is a topological space $(\underline{X}, \underline{A})^K$ by a construction of Buchstaber and Panov. Such spaces are called polyhedral products and they generalize the central notion of the moment-angle complex in toric topology. In this thesis, we study certain classes of polyhedral products from a homotopy theoretic point of view.

The boundary of the 2-dimensional n-sided polygon, where $n \geq 3$, may be viewed as a 1-dimensional simplicial complex with n vertices and n faces which we call the n-gon. When K is an n-gon for $n \geq 5$, $(D^2, S^1)^K$ is a hyperbolic space, by a Theorem of Debongnie. We show that there is an infinite basis of the rational homotopy groups $\pi_*((D^2, S^1)^K) \otimes \mathbb{Q}$ represented by iterated Samelson products.

When K is an n-gon, for $n \geq 3$, and $P^m(p^r)$ is a Moore space with $m \geq 3, r \geq 1$, we show that the order of the elements in the p-primary torsion component of the homotopy groups $\pi_*((\mathrm{Cone}\, \Omega P^m(p^r), \Omega P^m(p^r))^K)$ is bounded above by p^{r+1}, adding new evidence to a conjecture of Moore. Moreover, this bound is the best possible. In fact, if a certain conjecture of M.G. Barratt is assumed to be true, then this bound is also valid, and is the best possible, when K is an arbitrary simplicial complex.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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General Notation

Throughout this thesis, unless stated otherwise, all spaces are assumed to be simply connected, based topological spaces which have the homotopy type of finite type CW-complexes, and all maps are basepoint preserving.

The identity map on X is denoted by 1_X or just by 1 if it is clear which space we are referring to. In homotopy commutative diagrams, $X = X$ is also used for the identity. A *left homotopy inverse* for a map $f\colon X \longrightarrow Y$ is a map $g\colon Y \longrightarrow X$ such that the composite $X \xrightarrow{f} Y \xrightarrow{g} X$, denoted by $g \circ f$, is homotopic to the identity on X. Similarly, a *right homotopy inverse* for f is a map $g\colon Y \longrightarrow X$ such that $f \circ g$ is homotopic to the identity map on Y. If g is both a left and right homotopy inverse for f, then f is called a *homotopy equivalence*, and X is said to be *homotopy equivalent* to Y. The notation $\simeq$ is used to denote both a homotopy of maps $f \simeq g$ as well as homotopy equivalent spaces $X \simeq Y$.

Let X, Y be spaces. The *wedge sum* $X \vee Y$ is the disjoint union of X and Y modulo the identification of the basepoints $*_X \sim *_Y$. The *smash product* $X \wedge Y$ is the quotient space $X \times Y/\sim$ where $(x, *) \sim (*, y)$ for each $x \in X$ and $y \in Y$. The wedge sum of n copies of X is denoted nX, the smash product of n copies of X is denoted $X^{\wedge n}$, and the product of n copies of X is denoted X^n.

Let $n \geq 1$ be an integer. For an H-space X, the n-power map defined by the multiplication on X is denoted by $n\colon X \longrightarrow X$. For a co-$H$-space Y, the degree

n map defined via the co-multiplication is denoted $[n]\colon Y \longrightarrow Y$.

The n-dimensional disc $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is denoted by D^n. The boundary of D^{n+1} is called the n-dimensional sphere and is denoted by S^n. A related space of particular interest in this thesis is the (mod p^r) Moore space $P^m(p^r)$ which is defined for $m \geq 2$ and $r \geq 1$ as the homotopy cofibre of the degree map $[p^r]\colon S^{m-1} \longrightarrow S^{m-1}$.

Let $I = [0,1]$ be the unit interval. The *reduced suspension* of X is the quotient space $\Sigma X = X \times I/\sim$ where $(x,0) \sim (x,1) \sim (*,t)$ for $x \in X$ and $t \in I$. The *reduced cone* of X is the quotient space $\mathrm{Cone}\, X = X \times I/((X \times \{1\}) \cup (*_X \times I))$. The *join* of spaces X, Y is the quotient space $X * Y = X \times I \times Y/\sim$ where $(x,0,y) \sim (x,0,y')$ and $(x,1,y) \sim (x',1,y)$ for each $x \in X$ and $y \in Y$. There is a natural homotopy equivalence $X * Y \simeq \Sigma X \wedge Y$. The (right) *half-smash product* is the quotient space $X \rtimes Y = (X \times Y)/* \times Y$ and similarly, the (left) *half smash product* is the space $X \ltimes Y = (X \times Y)/X \times *$.

Let X, Y be topological manifolds of dimension n, and choose a point $x \in X$ and $y \in Y$. Let B_x be an open neighbourhood in X, centered at x, and denote the closure by $\overline{B}_x$. Similarly, let B_y be an open neighbourhood of $y \in Y$, with closure $\overline{B}_y$. Then the *connected sum* of X, Y is the space $X \# Y = (X \backslash B_x \coprod Y \backslash B_y) \coprod (I \times S^{n-1})/\sim$ where $\{0\} \times S^{n-1}$ is identified with the boundary of $\overline{B}_x$, and $\{1\} \times S^{n-1}$ is identified with the boundary of $\overline{B}_y$. It is a standard result that upto homeomorphism, the connected sum does not depend on the choice of the points x, y, the choice of the open neighbourhoods B_x, B_y, or the identification of the boundaries $\overline{B}_x$, $\overline{B}_y$ with the ends of the cylinder $I \times S^{n-1}$.

Let X be a space. The *diagonal* map $\Delta\colon X \longrightarrow X \times X$ is defined by $\Delta(x) = (x,x)$ and the *fold* map $\nabla\colon X \vee X \longrightarrow X$ is the map whose restriction to each wedge summand is the identity map on X.

Let $f\colon X \longrightarrow Y$, $g\colon A \longrightarrow B$ be maps. We use the notation

$$X \vee A \xrightarrow{f\vee g} Y \vee B, \qquad X \wedge A \xrightarrow{f\wedge g} Y \wedge B, \qquad X \times A \xrightarrow{f\times g} Y \times B,$$

where $(f \vee g)(z)$ is $f(z)$ if $z \in X$ or $g(z)$ if $z \in A$; $(f \wedge g)(x \wedge a) = f(x) \wedge g(a)$; and $(f \times g)(x, a) = (f(x), g(a))$.

The *unbased pathspace* of X, denoted X^I, is the set of all (unbased) continuous maps $I \longrightarrow X$, with the compact-open topology. The *(based) path space* of X, denoted $\mathcal{P}X$ is the set of all based maps $I \longrightarrow X$, with the compact-open topology. The subspace $\Omega X = \{f\colon I \longrightarrow X \in \mathcal{P}X \mid f(0) = f(1)\}$ is called the *(based) loopspace* of X.

Given spaces X, Y, the set of homotopy classes of maps $[X, Y]$ is called the *homotopy set* of X, Y. If either X is a co-H-group or Y is an H-group, then $[X, Y]$ can be given a natural group structure. In the case that both X is a co-H-group and Y is an H-group, the group structures coincide and the group $[X, Y]$ is Abelian. For arbitrary spaces X, Y, there is an isomorphism of groups $[\Sigma X, Y] \cong [X, \Omega Y]$, since Σ and Ω are adjoint functors.

The *Whitehead product* of maps $f\colon \Sigma X \longrightarrow Z$, $g\colon \Sigma Y \longrightarrow Z$ is denoted by $[f, g]\colon \Sigma X \wedge Y \longrightarrow Z$ and the *Samelson product* of maps $f\colon X \longrightarrow \Omega Z$, $g\colon Y \longrightarrow \Omega Z$ is denoted by $\langle f, g\rangle\colon X \wedge Y \longrightarrow \Omega Z$. We define these products in Section 2.2.2.

Finally, $\mathbb{Z}/n$ denotes the cyclic group of order n and for a prime p, $X_{(p)}$ is the *localization* of X at p.

Chapter 1

Introduction

The calculation of the homotopy groups of topological spaces is a notoriously difficult problem in general. The homotopy groups of spheres for example, despite much progress over the past century still remain a great mystery. In his PhD thesis, Serre showed that the homotopy groups of simply connected finite CW-complexes are finitely generated Abelian groups and showed in [27] that $\pi_k(S^n)$ is a finite group except for $\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z}$, modulo torsion, for all $n \geq 1$. In particular, it is the torsion in the homotopy groups of spheres which has been so problematic to calculate.

This situation led topologists to take a more qualitative approach. In the 50's, James [17], [18] introduced what are now known as James-Hopf invariants and using associated fibrations was able to determine that the 2-primary torsion in $\pi_*(S^{2n+1})$ is annihilated by 4^n. Toda [29],[30] quickly built on the work of James and extended his result to odd primes by defining analogous invariants. In particular, Toda showed that p^{2n} annihilates the p-torsion in $\pi_*(S^{2n+1})$.

It took twenty years for further improvements on the work of James and Toda to appear. In 1978, Selick [23] showed that for odd primes, the p-torsion in $\pi_*(S^3)$ is annihilated by p, improving on Toda's result of p^2, and this was almost immediately extended by Cohen, Moore and Neisendorfer [7] who showed that p^n annihilates the p-torsion in $\pi_*(S^{2n+1})$ for each $n \geq 1$ when p is odd. Gray [12] had earlier shown that for odd primes, and $n \geq 1$, $\pi_*(S^{2n+1})$ contains elements of order p^n. Thus the result of Cohen, Moore and Neisendorfer is the best possible. For the prime 2, it is known that James' result is not the best possible. See for example [24]. In fact the best possible bound in this case is still unknown.

In modern parlance, results such as those outlined above are known as exponent results. The *homotopy exponent* of a space X, (at the prime p), denoted $\exp_p(X)$, is the least p^r which annihilates the p-torsion in the homotopy groups of X.

As well as determining the homotopy exponent for all odd primes of odd dimensional spheres, Cohen, Moore and Neisendorfer [6] were also able to determine the homotopy exponent of another very interesting family of spaces. For $n \geq 1$ and p a prime, the cofibre of the p^r degree map on the $m-1$ sphere $[p^r]\colon S^{m-1} \longrightarrow S^{m-1}$ is called the mod p^r Moore space of dimension m and denoted $P^m(p^r)$. For odd primes, and $r \geq 1$, they proved that $\exp_p(P^m(p^r)) = p^{r+1}$.

At the end of the 70's, Moore made an intriguing conjecture relating the existence of finite homotopy exponents for a space X to the dimension of the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$.

Conjecture 1.1 (Moore's Conjecture)**.** *Let p be a prime and let X be a simply connected finite CW-complex. Then $\pi_*(X) \otimes \mathbb{Q}$ is finite-dimensional if and only if X has a finite homotopy exponent at p.*

Spaces which have finite-dimensional rational homotopy are called elliptic spaces and spaces which are not elliptic are called hyperbolic. Interestingly, Moore's Conjecture is independent of the prime p, and so if correct, the implication is that the finiteness of the homotopy exponent at a specific prime, implies the finiteness of the homotopy exponent at all primes. Moore did not publish this conjecture officially, but for some historical background and illuminating discussion of this conjecture, the reader is advised to consult Selick's expository article [25].

Another conjecture of interest was put forward by Barratt regarding the homotopy exponents of double suspensions, and emerged out of the work in [3]. An expository discussion is also given in the aforementioned article of Selick [25]. We state the weak form of Barratt's Conjecture here:

Conjecture 1.2 (Weak form of Barratt's Conjecture)**.** *Suppose that the p^r degree map $[p^r]\colon \Sigma Y \longrightarrow \Sigma Y$ is null-homotopic. Then* $\exp_p(\Sigma^2 Y) \leq p^{r+1}$.

Moore spaces are one of very few families of spaces which are known to satisfy Barratt's Conjecture.

In this thesis, we study a family of topological spaces called polyhedral products. Given a sequence of pairs of spaces $\{(X_i, A_i)\}_{i=1}^n$ with A_i a subspace of X_i for each i, and a simplicial complex K, the associated polyhedral product, denoted $(\underline{X}, \underline{A})^K$, is the homotopy colimit $\cup_{\sigma \in K} Y^\sigma$ over the face category of K, where $Y^\sigma = \{(x_1, \ldots, x_k) \in \prod_{i=1}^n X_i \mid x_j \in A_j \text{ if } j \notin \sigma\}$. Polyhedral products generalize a construction, due to Buchstaber and Panov [4], of the moment-angle complex $\mathcal{Z}_K$ and Davis-Januszkiewicz space DJ_K, which were first introduced in [8] and are central objects of study in the field of toric topology. In this thesis, inspired by the conjectures of Moore and Barratt, we study certain sub-classes of polyhedral products, and obtain results related to their rational homotopy and their homotopy exponents.

1.1 Summary of the main results

Our first result concerns the classical moment-angle complex $\mathcal{Z}_K$, which is the polyhedral product obtained by taking $(X_i, A_i) = (D^2, S^1)$ for each i.

Theorem A. Suppose $n \geq 5$ and let K be the n-gon. Then there is a subspace W of $(D^2, S^1)^K$, where W is a wedge of spheres, and the homotopy fibre of the inclusion $W \hookrightarrow (D^2, S^1)^K$ is homotopy equivalent to a wedge sum of infinitely many spheres.

As a consequence of Theorem A we prove:

Theorem B. Suppose $n \geq 5$ and let K be the n-gon. Then $\pi_*((D^2, S^1)^K) \otimes \mathbb{Q}$ has an infinite basis of iterated Samelson products as a vector space over $\mathbb{Q}$.

One conclusion of Theorem B is that $(D^2, S^1)^K$ is hyperbolic for such K. Actually this was already known by a special case of a theorem of Debongnie [9], who showed that $(D^2, S^1)^K$ is elliptic if and only if K is a join of simplices and boundaries of simplices. However, our result complements that of Debongnie by providing extra information about the basis.

Our next results concern the polyhedral products $(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^K$ where X_i is a Moore space for each i.

Theorem C. Let $n \geq 3$ and let K be the n-gon. Let p be an odd prime and take $X_i = P^{m_i}(p^{r_i})$ where $m_i \geq 3$ and $r_i \geq 1$ for each i. Then

$$\exp_p((\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^K) = p^{R+1}.$$

where $R = \max\{r_i\}_{i=1}^n$.

Our final result extends the statement of Theorem C, under the hypothesis that Barratt's Conjecture is true, to $m_i = m \geq 4, r_i = r \geq 1$, and all simplicial complexes K, except the full simplex, in which case the associated polyhedral product is contractible.

Theorem D. Let K be a simplicial complex on $n \geq 1$ vertices, $K \neq \Delta^{n-1}$, and let $P = P^m(p^r)$ where $m \geq 4$ and $r \geq 1$. If Barratt's Conjecture holds, then

$$\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}.$$

1.2 Outline of the thesis

A brief description of the organization of this thesis is as follows.

Chapter 2: We introduce basic homotopy theoretic definitions and consider various classical results and methods concerning decompositions of spaces up to homotopy.

Chapter 3: In this chapter we introduce our main of objects of study, polyhedral products, and record some basic facts and useful properties.

Chapter 4: We introduce homotopy exponents and discuss the conjectures of Moore and Barratt. We also review some of the work of Cohen, Moore and Neisendorfer and others on the homotopy exponents of spheres and Moore spaces.

Chapter 5: In this chapter, we begin to discuss our main results. We review what is known about the ellipticity/ hyperbolicity problem, and homotopy exponents, for polyhedral products, and conclude by proving Theorems A and B.

Chapter 6: In the final chapter we study the polyhedral product $(\mathrm{Cone}\,\Omega X, \Omega X)^K$ where X is a Moore space $P^m(p^r)$ for $m \geq 3, r \geq 1$. In the case that K is an n-gon we obtain the value p^{r+1} for the p-primary homotopy exponent, and we show that if Barratt's conjecture holds, then the exponent is also p^{r+1} for arbitrary $K \neq \Delta^n$, when $m \geq 4$. This proves Theorems C and D.

Chapter 2

Preliminaries

2.1 Basic homotopy theory and definitions

In this chapter we introduce our basic homotopy theoretic definitions, constructions and philosophies related which permeate the work in this thesis.

2.1.1 Homotopy pullbacks and pushouts

Two fundamental constructions which appear throughout this thesis are homotopy pushouts and homotopy pullbacks. First of all, we recall the definitions of the standard topological pullback and pushout. Given maps $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$, the *topological pullback* of f and g is the space $P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. Moreover, the maps $P \longrightarrow X$ and $P \longrightarrow Y$ induced by the projection maps fit into a commutative diagram called a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \underset{f}{\longrightarrow} & Z. \end{array}$$

The pullback satisfies the following universal property. If $\alpha\colon A \longrightarrow X$ and $\beta\colon A \longrightarrow Y$ are maps such that $f \circ \alpha = g \circ \beta$, then there exists a unique map $A \longrightarrow P$ such that α equals the composite $A \longrightarrow P \longrightarrow X$ and β equals $A \longrightarrow P \longrightarrow Y$.

Dually the *topological pushout* of maps $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Z$ is the adjunction space $Q = (Y \sqcup Z)/\sim$, where $f(x) \sim g(x)$. The inclusion maps $Y \longrightarrow Q$ and $Z \longrightarrow Q$ fit into a commutative diagram called a pushout diagram, and we have the following universal property. If $\alpha\colon Y \longrightarrow A$ and $\beta\colon Z \longrightarrow A$ are maps, such that $\alpha \circ f = \beta \circ g$, then there exist a unique map $Q \longrightarrow A$ such that α is the composite $Y \longrightarrow Q \longrightarrow A$ and β is the composite $Z \longrightarrow Q \longrightarrow A$.

In the homotopy category, that is, the category of simply-connected CW-complexes with homotopy classes of maps, we have the notion of homotopy pullback and homotopy pushout.

Definition 2.1. The *homotopy pullback* of two maps $f\colon X \to W$, $g\colon Y \to W$ is given by

$$P = \{(x, \omega, y) \in X \times W^I \times Y | f(x) = \omega(0),\ g(y) = w(1)\}$$

where W^I is the unbased pathspace on W. The projection maps $\pi_1\colon P \to X$, $\pi_2\colon P \to Y$ produce a homotopy commutative square

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & W. \end{array}$$

P is universal in the sense that for any two maps $h\colon A \to X$, $k\colon A \to Y$ there exists a map $m\colon A \to P$, unique upto homotopy, such that $\pi_1 \circ m \simeq h$ and $\pi_2 \circ m \simeq k$.

Example 2.2. The product $X \times Y$ is both the topological pullback, and the homotopy pullback of the trivial maps $X \longrightarrow *$ and $Y \longrightarrow *$. Moreover, the induced maps $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ are the projection maps.

The notion of homotopy pushout is dual to that of the homotopy pullback.

Definition 2.3. The *homotopy pushout* of two maps $f\colon X \to Y$, $g\colon X \to Z$ is given by the double mapping cylinder

$$W = Y \sqcup Z \sqcup (X \times I)/\sim$$

where the equivalence relation $\sim$ is defined by $(x, 0) \sim f(x)$, and $(x, 1) \sim g(x)$. The inclusion maps of Y an Z into $Y \cup Z \cup (X \times I)$ induce inclusions $i_1\colon Y \to W$, $i_2\colon Z \to W$ respectively which fit into a homotopy commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_2} \\ Y & \xrightarrow[i_1]{} & W. \end{array}$$

W is universal in the dual sense to Definition (2.1).

A very simple example of a pushout is the wedge of two spaces.

Example 2.4. The topological pushout of the two maps $f\colon * \longrightarrow X$ and $g\colon * \longrightarrow Y$ which include the basepoint, is given by the disjoint union $X \sqcup Y$ with the basepoints identified $*_X \sim *_Y$. This is by definition the wedge $X \vee Y$. The homotopy pushout of f and g on the other hand, is the disjoint union $X \sqcup Y \sqcup I$ modulo the identification of $0 \in I$ with $*_X$ and the identification of $1 \in I$ with $*_Y$. This is clearly homotopy equivalent to the wedge $X \vee Y$.

In general, homotopy pullbacks and homotopy pushouts do not coincide with their topological counterparts upto homotopy. In fact, the topological pushout Q

of maps $f\colon X \longrightarrow Y$, $g\colon X \longrightarrow Z$ is not well-defined in the homotopy category, and similarly for the topological pullback. That is, if we replace f, g with homotopy equivalent maps $\overline{f}, \overline{g}$, it needn't be the case that the topological pushout of $\overline{f}, \overline{g}$ is homotopy equivalent to Q.

The classic example of this is the following:

Example 2.5. The diagram

$$D^2 \hookleftarrow S^1 \hookrightarrow D^2$$

is equivalent upto homotopy to the diagram

$$* \hookleftarrow S^1 \hookrightarrow *.$$

However, the topological pushout defined by the first diagram is homotopy equivalent to S^2, whereas the topological pushout defined by the second diagram is contractible.

On the other hand, if we replace X, Y, Z by homeomorphic spaces X', Y', Z', and replace f, g by maps $f'\colon X' \longrightarrow Y'$ and $g'\colon X' \longrightarrow Z'$ such that there are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ X' & \xrightarrow[f']{} & Y', \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ Y' & \xrightarrow[g']{} & Z' \end{array}$$

then the topological pushout of the maps f', g' is in fact homeomorphic to Q, so the topolgical pushout, is well-defined in the category of topological spaces and continuous maps. Similarly topological pullbacks are well defined in this category.

The homotopy pushout and homotopy pullback are the correct modification required to the definition of their topological counterparts in order to ensure that pushouts and pullbacks exist in the homotopy category.

2.1.2 Homotopy fibrations and cofibrations

Homotopy fibrations and cofibrations are extremely important in homotopy theory and will feature heavily in our work.

Definition 2.6. A map $p\colon E \to B$ is called a *topological fibration*, (or fibration), if the following homotopy lifting property is satisfied. For all spaces W and maps $g_0\colon W \longrightarrow B$, $h_0\colon W \longrightarrow E$, and a homotopy $g_t\colon W \longrightarrow B$ of g_0 such that $p \circ h_0 = g_0$, there exists a homotopy $h_t\colon W \longrightarrow E$ of h_0 such that $p \circ h_t = g_t$.

If p is a topological fibration, we call $p^{-1}(*)$ the *topological fibre* of p. We generally write fibrations as sequences $F \overset{i}{\hookrightarrow} E \overset{p}{\longrightarrow} B$ to indicate that p is a fibration with topological fibre F, with i being the inclusion.

A homotopy fibration is a sequence of maps which is a fibration up to homotopy:

Definition 2.7. A *homotopy fibration* is a sequence of maps $X \longrightarrow Y \longrightarrow Z$ such that there is a homotopy commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

in which the vertical maps are homotopy equivalences and the bottom row is a topological fibration.

The following construction shows that any map may be viewed as a homotopy fibration.

Construction 2.8 (Replacing a map by a homotopy fibration)**.** Let $f\colon X \to Y$ be a map and let

$$\tilde{X} = \{(x, \omega) \in X \times Y^I \mid f(x) = \omega(1)\}.$$

Then there is a homotopy commutative diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle \simeq}\big\downarrow & \searrow^{f} & \\ \tilde{X} & \xrightarrow[ev_0]{} & Y \end{array}$$

where ev_0 is the evaluation map defined by $(x, \omega) \mapsto \omega(0)$. The homotopy equivalence $X \xrightarrow{\simeq} \tilde{X}$ is given by the inclusion $x \mapsto (x, *_x)$ where $*_x$ is the constant path at x.

The map ev_0 in Construction 2.8 is a topological fibration, with topological fibre $\{(x, w) \in X \times \mathcal{P}Y \mid f(x) = w(1)\}$. This leads us to define the homotopy fibre of a map.

Definition 2.9. Let $f\colon X \longrightarrow Y$ be a map. The *homotopy fibre* of f is the space $\{(x, w) \in X \times \mathcal{P}Y \mid f(x) = w(1)\}$.

Notice that if F is the homotopy fibre of $f\colon X \longrightarrow Y$, and $p\colon F \longrightarrow X$ is the projection $(x, w) \mapsto x$, then there is a homotopy commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{p} & X & \xrightarrow{f} & Y \\ \| & & {\scriptstyle \simeq}\big\downarrow & & \| \\ F & \longrightarrow & \tilde{X} & \xrightarrow[\mathrm{ev}_0]{} & Y \end{array}$$

in which the bottom row is a topological fibration, and thus by Definition 2.7, the sequence $F \xrightarrow{p} X \xrightarrow{f} Y$ is a homotopy fibration.

Next we state some useful properties exhibited by homotopy fibrations.

Proposition 2.10. *Let $F \longrightarrow E \xrightarrow{p} B$ be a homotopy fibration, and $f\colon X \longrightarrow E$ be any map which becomes null-homotopic after composing with p. Then, there exists a lift of f to a map $\tilde{f}\colon X \to F$. Thus, there is a homotopy commutative diagram,*

$$\begin{array}{ccccc} & & X & & \\ & \overset{\exists \tilde{f}}{\swarrow} & \downarrow{\scriptstyle f} & \overset{*}{\searrow} & \\ F & \longrightarrow & E & \longrightarrow & B. \end{array}$$

Proof. Trivial. □

We now go on to prove a useful fibre square property, which is useful for constructing new fibrations. First we need a lemma.

Lemma 2.11. *Let*

$$\begin{array}{ccc} P & \xrightarrow{p_1} & B \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ A & \xrightarrow{p_2} & C \end{array}$$

be a homotopy commutative square and let F_1, F_2 be the homotopy fibres of q_1, q_2 respectively and let G_1, G_2 be the homotopy fibres of p_1, p_2 respectively. Then the following are equivalent:

1. *there exists an induced map $F_1 \longrightarrow F_2$ which is a homotopy equivalence,*
2. *the square is equivalent in the homotopy category to a topological pullback in which q_1 and q_2 are topological fibrations*
3. *there exists an induced map $G_1 \longrightarrow G_2$ which is a homotopy equivalence,*
4. *the square is equivalent in the homotopy category to a topological pullback in which p_1 and p_2 are topological fibrations.*

Proof. See [26], Proposition 7.6.1. □

Now we can prove the fibre square property. The proof we give follows [6].

Proposition 2.12. *Any homotopy commutative square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

can be extended to a homotopy commutative diagram of the following form

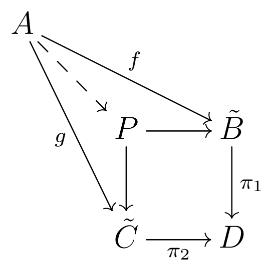
$$\begin{array}{ccccc} H & \longrightarrow & F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & A & \xrightarrow{g} & B \\ \downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ G_2 & \longrightarrow & C & \xrightarrow{j} & D \end{array}$$

in which all rows and columns are homotopy fibrations.

Proof. Construction 2.8 allows us to replace the original square by the following homotopy commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & \tilde{B} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \tilde{C} & \xrightarrow[\pi_2]{} & D \end{array} \tag{2.1}$$

in which the π_i are topological fibrations, homotopy equivalent to i, j respectively. In particular, π_1 satisfies the homotopy lifting property and so f can be replaced, if necessary, by a homotopic map which makes the diagram (2.1) strictly commute. Let P be the topological pullback of π_1 and π_2. By universality, there is a unique map $A \longrightarrow P$ is defined, making the following diagram commute.



(2.2)

After replacing the map $A \longrightarrow P$ with a topological fibration, $\tilde{A} \longrightarrow P$, as in construction 2.8, all maps in diagram (2.2) are now topological fibrations.

Let H be the fibre of $\tilde{A} \longrightarrow P$. Extending the maps in the commutative square

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \tilde{B} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \tilde{C} & \xrightarrow[\pi_2]{} & D \end{array}$$

to fibration sequences, there is a commutative diagram

$$\begin{array}{ccccc} & & F_1 & \longrightarrow & F_2 \\ & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & \tilde{A} & \longrightarrow & \tilde{B} \\ \downarrow & & \downarrow & & \downarrow \\ G_2 & \longrightarrow & \tilde{C} & \longrightarrow & D. \end{array} \tag{2.3}$$

Since P is a pullback, the map $P \longrightarrow \tilde{C}$ in diagram (2.2) has the same fibre as

π_1. Thus there is a commutative diagram

$$\begin{array}{ccccc}
F_1 & \longrightarrow & F_2 & = & F_2 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{A} & \longrightarrow & P & \longrightarrow & \tilde{B} \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{C} & = & \tilde{C} & \longrightarrow & D,
\end{array}$$

in which all columns are fibration sequences. Since the map $\tilde{C} \longrightarrow \tilde{C}$ is the identity, the upper left square is a pullback. Thus, $F_1 \longrightarrow F_2$ is a fibration with the same fibre as $\tilde{A} \longrightarrow P$, namely H. A similar argument also shows that $G_1 \longrightarrow G_2$ is a fibration with fibre H.

Hence diagram (2.3) can be completed to give a commutative diagram

$$\begin{array}{ccccc}
H & \longrightarrow & F_1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow & & \downarrow \\
G_1 & \longrightarrow & \tilde{A} & \longrightarrow & \tilde{B} \\
\downarrow & & \downarrow & & \downarrow \\
G_2 & \longrightarrow & \tilde{C} & \longrightarrow & D
\end{array}$$

in which all rows and columns are fibrations. Replacing $\tilde{A}, \tilde{B}, \tilde{C}$ with the homotopy equivalent spaces A, B, C gives the result. □

We now briefly discuss homotopy cofibrations which are defined dually to homotopy fibrations. First we define topological cofibrations.

Definition 2.13. A map $f\colon A \longrightarrow X$ is called a *(topological) cofibration* if for any space Z, maps $g_0\colon A \longrightarrow Z$ and $h_0\colon X \longrightarrow Z$ and a homotopy $g_t\colon A \longrightarrow Z$ of g_0 such that $h_0 \circ f = g_0$, there exists a homotopy $h_t\colon X \longrightarrow Z$ of h_0 such that $h_t \circ f = g_t$.

If $f\colon A \longrightarrow X$ is a cofibration, we call the space $X/\operatorname{im}(f)$ the *topological cofibre* of f. The sequence $A \xrightarrow{f} X \xrightarrow{q} C$ is also referred to as a cofibration, where q is the quotient map.

Definition 2.14. A *homotopy cofibration* is a sequence of maps $X \longrightarrow Y \longrightarrow Z$ such that there is a homotopy commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

in which the vertical maps are homotopy equivalences and the bottom row is a topological cofibration.

We remark that by a constructon dual to Construction 2.8, any map may be replaced by a homotopy cofibration. This construction leads to the definition of the homotopy cofibre of a map.

Definition 2.15. Let $f\colon A \longrightarrow X$ be a map. The *homotopy cofibre* of f is defined as $(X \sqcup \operatorname{Cone} A)/\sim$ where $f(a) \sim (a, 0)$ for each $a \in A$.

We omit the details here but it is straightforward to dualize Proposition 2.10, Lemma 2.11 and Proposition 2.12 to the corresponding results for homotopy cofibrations.

An important family of spaces we shall consider in this thesis are Moore spaces. Let $m \geq 2$ and let $r \geq 1$. Denote the homotopy cofibre of the p^r degree map $[p^r]\colon S^{m-1} \longrightarrow S^{m-1}$ by $P^m(p^r)$. We call $P^m(p^r)$ the mod p^r Moore space of dimension m. Moore spaces are to homology as Eilenberg-Maclane spaces are to homotopy, in the sense that they have only one non-trivial integral homology group $H_{m-1}(P^m(p^r); \mathbb{Z}) \cong \mathbb{Z}_{p^r}$.

From now on all fibrations will be homotopy fibrations, unless stated otherwise.

2.1.3 Homotopy actions and principal fibrations

A useful property possessed by certain homotopy fibrations is that there is an action of the fibre on the base.

Definition 2.16. A homotopy fibration $F \xrightarrow{i} E \xrightarrow{p} B$ is called a *principal fibration* if there is a space X, a map $f\colon B \longrightarrow X$ and a homotopy pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{P}X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ B & \xrightarrow[f]{} & X \end{array}$$

It follows from the definition that if $F \xrightarrow{i} E \xrightarrow{p} B$ is a homotopy fibration and $\partial\colon \Omega B \longrightarrow F$ is the connecting map, then $\Omega B \xrightarrow{\partial} F \xrightarrow{i} E$ is principal. Next we define the notion of a left and right homotopy action of an H-space X on a space Y.

Definition 2.17. Let X be an H-space with multiplication $\mu\colon X \times X \longrightarrow X$, and let $f\colon X \longrightarrow Y$ be a map. A *left homotopy action* for f is a map $\theta_l\colon X \times Y \longrightarrow Y$ such that the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ {\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\ X \times Y & \xrightarrow{\theta_l} & Y \end{array}$$

is homotopy commutative.

Similarly, a right homotopy action for f is a map $\theta_r\colon Y \times X \longrightarrow Y$ which fits into a homotopy commutative diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times X & \xrightarrow{\theta_r} & Y. \end{array}$$

Given a principal fibration $\Omega B \xrightarrow{\partial} F \xrightarrow{i} E$, it is a classical and well known result that ∂ has a left homotopy action $\theta_l\colon \Omega B \times F \longrightarrow F$, and similarly, a right homotopy action. Represent a loop in ΩB by a map $\alpha\colon I \longrightarrow B$ such that $\alpha(0) = \alpha(1)$. The action of α on $f \in F$ is given by lifting α to a path $\tilde{\alpha}\colon I \longrightarrow E$ such that $\tilde{\alpha}(0) = f$. The endpoint $\tilde{\alpha}(1)$ belongs to F and we set $\theta(\alpha, f) = \tilde{\alpha}(1)$.

Homotopy actions associated to principal fibrations are natural in the following sense. Suppose there is a morphism of homotopy fibrations

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \longrightarrow & B_1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ F_2 & \longrightarrow & E_2 & \longrightarrow & B_2 \end{array}$$

with associated homotopy actions $\psi_1\colon \Omega B_1 \times F_1 \longrightarrow F_1$ and $\psi_2\colon \Omega B_2 \times F_2 \longrightarrow F_2$ respectively. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega B_1 \times F_1 & \xrightarrow{\psi_1} & F_1 \\ {\scriptstyle (\Omega f_3)\times f_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ \Omega B_2 \times F_2 & \xrightarrow[\psi_2]{} & F_2. \end{array}$$

The following proposition gives conditions for recognizing when a principal fibration splits.

Proposition 2.18. *For any homotopy fibration sequence* $\Omega B \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{p} B$*. The following are equivalent:*

1. *p is null-homotopic,*

2. *i admits a right homotopy inverse $r\colon E \longrightarrow F$,*

3. *there is a homotopy equivalence $\Omega B \times E \xrightarrow{1\times r} \Omega B \times F \xrightarrow{\theta} F$ where r is the right homotopy inverse in (2) and θ is the standard homotopy action for ∂,*

4. *∂ admits a left homotopy inverse $l\colon F \longrightarrow \Omega B$,*

5. *there is a homotopy equivalence $F \xrightarrow{\Delta} F \times F \xrightarrow{l\times i} \Omega B \times E$ where l is the left homotopy inverse in (4).*

Proof (1) $\Rightarrow$ (2). If p is null-homotopic then the identity map on E lifts to a map $r\colon E \longrightarrow F$ and so $i \circ r$ is homotopic to the identity. $\square$

Proof (2) $\Rightarrow$ (3). Suppose r is a right homotopy inverse for i. Let $\theta\colon \Omega B \times F \longrightarrow F$ denote the homotopy action for ∂ and let μ be the loop multiplication on ΩB. Then there is a homotopy commutative diagram

$$\begin{CD}
\Omega B \times * @>{1\times *}>> \Omega B \times \Omega B @>{\mu}>> \Omega B \\
@V{1\times *}VV @VV{1\times\partial}V @VV{\partial}V \\
\Omega B \times E @>{1\times r}>> \Omega B \times F @>{\theta}>> F \\
@V{*\times 1}VV @VV{*\times i}V @VV{i}V \\
* \times E @= * \times E @= E
\end{CD} \tag{2.4}$$

which yields a morphism of homotopy fibrations

$$\begin{CD}
\Omega B @= \Omega B \\
@V{\text{inc}}VV @VV{\partial}V \\
\Omega B \times E @>{\theta\circ(1\times r)}>> F \\
@V{\text{proj}}VV @VV{i}V \\
E @= E.
\end{CD}$$

Looking at the induced long exact sequences of homotopy groups, the Five Lemma shows that θ is a weak homotopy equivalence and hence by Whitehead's Theorem is a homotopy equivalence. $\square$

Proof (4) $\Rightarrow$ (5). Let $l\colon F \longrightarrow \Omega B$ be a left homotopy inverse for ∂. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Omega B & \xrightarrow{\ \Delta\ } & \Omega B \times \Omega B & \xrightarrow{1\times *} & \Omega B \times * \\
\Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial\times\partial} & & \Big\downarrow{\scriptstyle 1\times *} \\
F & \xrightarrow{\ \Delta\ } & F \times F & \xrightarrow{l\times i} & \Omega B \times E \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle *\times i} & & \Big\downarrow{\scriptstyle *\times 1} \\
E & \longrightarrow & * \times E & \xrightarrow[*\times 1]{} & * \times E
\end{array}$$

which yields a morphism of homotopy fibrations

$$\begin{array}{ccc}
\Omega B & = & \Omega B \\
\Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \mathrm{inc}} \\
F & \xrightarrow{(l\times i)\circ\Delta} & \Omega B \times E \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \mathrm{proj}} \\
E & = & E
\end{array}$$

Appealing to the Five Lemma and Whitehead's Theorem establishes that $(l\times i)\circ\Delta$ is a homotopy equivalence. $\square$

We conclude this section by noting without giving explicit details that Definition 2.16 dualizes to give the notion of a principal cofibration, and moreover, for any homotopy cofibration $A \longrightarrow B \longrightarrow C$, the extended cofibration sequence $B \longrightarrow C \longrightarrow \Sigma A$ is principal. There is also a notion of homotopy coaction, dual to that of the homotopy action and for any homotopy cofibration $A \longrightarrow B \longrightarrow C$, there is an associated coaction $C \longrightarrow \Sigma X \vee C$. See for example [1] for details. Then Proposition 2.18 dualizes as follows:

Proposition 2.19. *For any homotopy cofibration sequence* $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{j} \Sigma X$*. The following are equivalent:*

1. *f is null-homotopic,*
2. *i admits a left homotopy inverse $l\colon C \longrightarrow Y$,*
3. *there is a homotopy equivalence $C \xrightarrow{\psi} \Sigma X \vee C \xrightarrow{1\vee l} \Sigma X \vee Y$ where ψ the standard coaction for j.*
4. *j admits a right homotopy inverse $r\colon \Sigma X \longrightarrow C$,*
5. *there is a homotopy equivalence $\Sigma X \vee Y \xrightarrow{r\vee i} C \vee C \xrightarrow{fold} C$.*

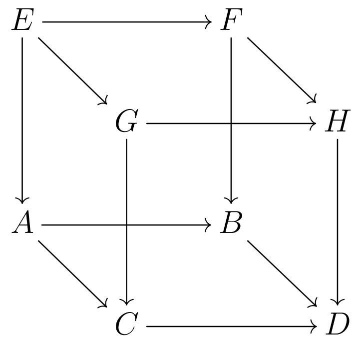
2.2 Homotopy decompositions

In this section we introduce various methods for decomposing spaces upto homotopy. Usually we are interested in decomposing a space as a product or as a wedge of simpler spaces. Such decompositions are especially desirable owing to fact that the homology of wedge sum splits as a direct sum of the homology of the summands, and similarly the homotopy groups of a product splits as the direct sum of homotopy groups of the factors. So such decompositions of a space can be helpful in understanding and calculating their algebraic invariants. Conversely, being able to decompose the homology or homotopy groups of a space can occasionally be a good indication that the space admits an analogous homotopy decomposition.

2.2.1 Mather's Cube Lemma

In [19], Mather studied various properties of the homotopy pullback and pushout. One result which will recur in our work is the following cube lemma.

Lemma 2.20 (Mather's Cube Lemma)**.** *Suppose there is a homotopy commutative diagram*



such that

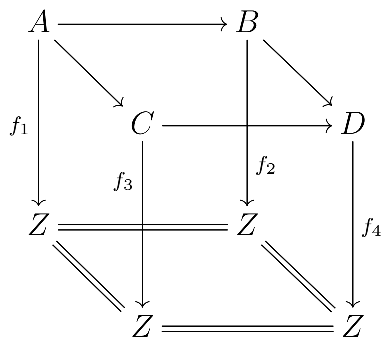
1. *the bottom square $A - B - C - D$ is a homotopy pushout;*

2. *and each of the four vertical squares $E - G - A - C$, $G - H - C - D$, $F - H - B - D$ and $E - F - A - B$ is a homotopy pullback.*

Then the top square $E - F - G - H$ is a homotopy pushout.

Proof. See [19], Theorem 25. □

An interesting special case of Lemma 2.20 can often be helpful in order to calculate the homotopy fibres of certain maps by decomposing the fibre as a homotopy pushout of simpler spaces. Suppose there is a space Z, and maps from each corner of the homotopy pushout $A - B - C - D$ into Z making the diagram

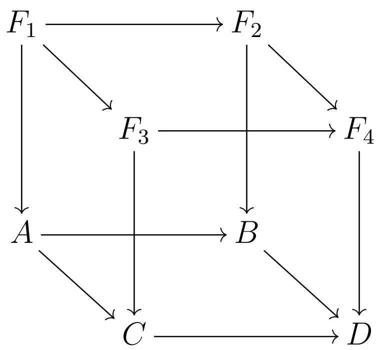


(2.5)

homotopy commutative. Let F_i be the homotopy fibre of the map f_i for each $i = 1, \ldots, 4$. If we consider the face $A - B - Z - Z$, we have a morphism of homotopy fibrations

$$\begin{array}{ccccc} F_1 & \longrightarrow & A & \xrightarrow{f_1} & Z \\ \downarrow & & \downarrow & & \| \\ F_2 & \longrightarrow & B & \xrightarrow[f_2]{} & Z \end{array}$$

and since the map on the base space is the identity, the left hand square is a homotopy pullback. Thus by extending each of the f_i to homotopy fibrations, we obtain a homotopy commutative cube



which satisfies both conditions 1 and 2 of Lemma 2.20. Therefore there is a homotopy pushout of the homotopy fibres

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow \\ F_3 & \longrightarrow & F_4. \end{array}$$

2.2.2 The Hilton-Milnor Theorem

Proposition 2.21. *For spaces X, Y there is a homotopy equivalence*

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y)$$

Proof. Using the group structure in $[\Sigma(X \times Y), \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)]$ we can add together the maps

$$\Sigma(X \times Y) \xrightarrow{\Sigma\pi_1} \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \tag{2.6}$$

$$\Sigma(X \times Y) \xrightarrow{\Sigma\pi_2} \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

$$\Sigma(X \times Y) \xrightarrow{\Sigma q} \Sigma(X \wedge Y) \hookrightarrow \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

where the π_i are the projections and q is the quotient map. This map induces isomorphisms in homology at every degree and since we are working with simply connected CW complexes, it is therefore a homotopy equivalence by Whitehead's theorem. □

Definition 2.22. An H-group is an H-space with homotopy associative multiplication and homotopy inverse.

In particular, loop spaces are H-groups. Let X, Y, Z be spaces and let $X \xrightarrow{f} \Omega Z$, $Y \xrightarrow{g} \Omega Z$ be maps. Since ΩZ is an H-group there is a well defined commutator map $\Omega Z \times \Omega Z \xrightarrow{c} \Omega Z$ defined by the following commutative diagram

$$\begin{array}{ccccc} \Omega Z \times \Omega Z & & \xrightarrow{\quad c \quad} & & \Omega Z \\ \downarrow \Delta & & & & \uparrow \mu \\ (\Omega Z \times \Omega Z) \times (\Omega Z \times \Omega Z) & \xrightarrow[1\times 1\times\iota\times\iota]{} & (\Omega Z \times \Omega Z) \times (\Omega Z \times \Omega Z) & \xrightarrow[\mu\times\mu]{} & \Omega Z \times \Omega Z \end{array}$$

Since the restriction of c to $\Omega Z \vee \Omega Z$ is nullhomotopic, there exists a factorization

$$\begin{array}{ccc} \Omega Z \times \Omega Z & \xrightarrow{c} & \Omega Z \\ \downarrow & \nearrow_{\hat{c}} & \\ \Omega Z \wedge \Omega Z & & \end{array}$$

where the map $\Omega Z \times \Omega Z \longrightarrow \Omega Z \wedge \Omega Z$ is the quotient map and the map $\hat{c}$ is unique up to homotopy. The composite

$$\langle f, g\rangle \colon X \wedge Y \xrightarrow{f\wedge g} \Omega Z \wedge \Omega Z \xrightarrow{\hat{c}} \Omega Z$$

is called the *Samelson product* of the maps f, g.

A closely related construction is that of the Whitehead product. Now suppose that X, Y, Z are spaces and $\Sigma X \xrightarrow{f} Z$, $\Sigma Y \xrightarrow{g} Z$ are maps. Let $X \xrightarrow{\tilde{f}} \Omega Z$ and $Y \xrightarrow{\tilde{g}} \Omega Z$ be the adjoints of f and g. Then the *Whitehead product* $[f, g] \colon \Sigma X \wedge Y \longrightarrow Z$ of the maps f, g is defined to be the adjoint of the Samelson product $\langle \tilde{f}, \tilde{g}\rangle$.

A Whitehead product of special interest is the *universal Whitehead product.* Let X, Y be spaces and let $i_X \colon X \longrightarrow X \vee Y$ and $i_Y \colon Y \longrightarrow X \vee Y$ be the inclusions of X, Y into the wedge sum, and for each of $A = X, Y$, define ζ_A to be the composite $\Sigma\Omega A \xrightarrow{\text{ev}} A \xrightarrow{i_A} X \vee Y$. The Whitehead product

$$[\zeta_X, \zeta_Y] \colon \Sigma\Omega X \wedge \Omega Y \longrightarrow X \vee Y$$

is called the *universal Whitehead product* for $X \vee Y$.

One of the most celebrated decomposition results in homotopy theory is the Hilton-Milnor Theorem which shows that after looping, a wedge of suspension spaces splits as a weak infinite product of the domains of certain iterated Whitehead products.

Let i_1, i_2 be the inclusions of ΣX and ΣY into $\Sigma X \vee \Sigma Y$ respectively. Let $L(i_1, i_2)$ be the free Lie algebra genereated by i_1, i_2 and suppose that B is a vector space basis for $L(i_1, i_2)$. For a given iterated Lie bracket $b \in B$, suppose that i_1 occurs n_1 times and i_2 occurs n_2 times. Then we can form the corresponding iterated Whitehead product $w_b \colon \Sigma X^{\wedge n_1} \wedge Y^{\wedge n_2} \longrightarrow \Sigma X \vee \Sigma Y$.

Theorem 2.23 (Hilton-Milnor)**.** *There is a homotopy equivalence*

$$\Omega\Sigma(X \vee Y) \simeq \prod_{b \in B} \Omega(\Sigma X^{\wedge n_1} \wedge Y^{\wedge n_2})$$

such that the restriction to the factor $\Omega\Sigma(X^{\wedge n_1} \wedge Y^{\wedge n_2})$ *is given by the corresponding looped Whitehead product* Ωw_b*.*

2.2.3 James' Splitting

In this section we recall James' classical result on loop suspensions. Namely, after suspension, the loop suspension $\Omega\Sigma X$ of a space X decompses upto homotopy as a wedge sum of spaces of the form $\Sigma X^{\wedge k}$. First of all, let us recap the James construction which gives a combinatorial model for loop suspensions.

Let X be a space, and define $J(X)$ to be the free monoid generated by X, with unit given by the basepoint. Intuitively, $J(X)$ is the set of formal products $\{x_1 x_2 \dots x_k \mid k \geq 1, x_i \in X\}$, subject to the follwing equivalence relations. First of all, an elementary equivalence is defined by replacing the product $x_1 x_2 \dots x_k$ with $x_1 x_2 \dots x_{i-1} * x_i \dots x_k$ for any $1 \leq i \leq k+1$, or vice versa. Two products $x_1 \dots x_k$ and $y_1 \dots y_l$ are then equivalent in $J(X)$ if one can be obtained from the other by a finite sequence of elementary equivalences. The multiplication of two elements $x_1 \dots x_k$ and $y_1 \dots y_l$ in $J(X)$ is given by juxtaposition $x_1 \dots x_k y_1 \dots y_l$.

Alternatively, $J(X)$ can be defined as a certain colimit. Let $J_k(X)$ be the quotient of the k-fold product X^k by the equivalence relations

$$\begin{aligned}(x_1, x_2, \ldots, x_{k-1}, *) &\sim (x_1, x_2, \ldots, x_{k-2}, *, x_{k-1}) \\ &\sim \ldots \\ &\sim (x_1, *, x_3, \ldots, x_{k-2}, x_{k-1}) \\ &\sim (*, x_1, x_2, \ldots, x_{k-2}, x_{k-1}).\end{aligned}$$

Then there are well defined maps $J_{k-1}(X) \longrightarrow J_k(X)$ induced by the inclusions $X^{k-1} \longrightarrow X^k$, sending $(x_1, \ldots, x_{k-1})$ to $(x_1, \ldots, x_{k-1}, *)$. Taking the colimit of the sequence of inclusions

$$* = J_0(X) \subseteq J_1(X) \subseteq \ldots \subseteq J_k(X) \subseteq \ldots$$

defines $J(X)$.

Theorem 2.24. *If X is path-connected, there is a weak homotopy equivalence $J(X) \simeq \Omega\Sigma X$.*

Proof. The original source is [17]. A proof is also given in [16], Chapter 4.J. □

Recall from the proof of Proposition 2.21 there is a homotopy cofibration

$$\Sigma(X \vee Y) \xrightarrow{\Sigma i} \Sigma(X \times Y) \xrightarrow{\Sigma q} \Sigma(X \wedge Y) \tag{2.7}$$

where i is the inclusion and q is the quotient map. The cofibration splits to yield the homotopy decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, for connected spaces X, Y. In particular, Σq admits a right homotopy inverse s. This situation generalizes easily to give the following proposition.

Proposition 2.25. *For connected spaces $X_1, \ldots, X_k$, the quotient map $X_1 \times \ldots X_k \longrightarrow X_1 \wedge \ldots \wedge X_k$ admits a right homotopy inverse after suspending.*

Proof. The quotient map decomposes as

$$\begin{aligned} (X_1 \times \ldots \times X_{k-1}) \times X_k &\xrightarrow{q_{k-1}} (X_1 \times \ldots \times X_{k-1}) \wedge X_k \\ &\xrightarrow{q_{k-2}} (X_1 \times \ldots \times X_{k-2}) \wedge X_{k-1} \wedge X_k \\ &\longrightarrow \ldots \\ &\xrightarrow{q_1} X_1 \wedge \ldots \wedge X_k \end{aligned}$$

where q_i is the map

$$q \wedge 1 \colon (X_1 \times \ldots \times X_i) \wedge (X_{i+1} \wedge \ldots \wedge X_k) \longrightarrow (X_1 \times \ldots \times X_{i-1}) \wedge X_i \wedge (X_{i+1} \wedge \ldots \wedge X_k).$$

Since Σq has a right homotopy inverse s, then Σq_i also has a right homotopy inverse s_i, for each i, given by $s \wedge 1$. The composite $s_{k-1} \circ \ldots \circ s_1$ then gives a right homotopy inverse for $\Sigma(q_1 \circ \ldots \circ q_{k-1})$. □

Lemma 2.26. *For each $k \geq 1$ there is a homotopy equivalence*

$$\Sigma J_k(X) \simeq \Sigma J_{k-1}(X) \vee \Sigma X^{\wedge k}.$$

Proof. Notice that there is a homotopy cofibration

$$J_{k-1}(X) \xhookrightarrow{i} J_k(X) \xrightarrow{q} X^{\wedge k}. \tag{2.8}$$

The map $q \colon J_k(X) \longrightarrow X^{\wedge k}$, preceeded by the quotient map $p \colon X^k \longrightarrow J_k(X)$ is the quotient map $X^k \longrightarrow X^{\wedge k}$. Applying the suspension functor we see that the composite $\Sigma X^k \xrightarrow{\Sigma p} \Sigma J_k(X) \xrightarrow{\Sigma q} \Sigma X^{\wedge k}$ has a right homotopy inverse s by

Proposition 2.25. It follows therefore that $(\Sigma p) \circ s$ is a right homotopy inverse for Σq. Thus the homotopy cofibration (2.8) splits, after suspending, giving the desired homotopy equivalence. □

Iterating Lemma 2.26 produces the well known James spliiting.

Theorem 2.27 (James, [17]). *For any space X, there are homotopy equivalences*

a) $\Sigma J_n(X) \simeq \bigvee_{k=1}^{n} \Sigma X^{\wedge k}$, *for each* $n \geq 1$,

b) $\Sigma J(X) \simeq \Sigma\Omega\Sigma X \simeq \bigvee_{k=1}^{\infty} \Sigma X^{\wedge k}$.

2.2.4 Porter's Theorem

Now we introduce some notation. Let $X_1, \ldots, X_n$ be path-connected spaces and for $1 \leq k \leq n-1$ define T_k^n to be the following subspace of $X_1 \times \ldots \times X_n$

$$T_k^n = \{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n \mid \text{ at least } k \text{ co-ordinates are the basepoint}\}. \tag{2.9}$$

The space T_0^n is the product $X_1 \times \ldots \times X_n$. Denote the homotopy fibre of the inclusion $T_k^n \longrightarrow T_0^n$ by F_k^n. Porter [22] determined that there is a homotopy decomposition of F_k^n into a wedge of suspensions of various smash products:

Theorem 2.28 (Porter [22], Theorem 1). *Suppose $X_1, \ldots, X_n$ are path-connected spaces and suppose $1 \leq k \leq n-1$. Then there is a homotopy equivalence*

$$F_k^n \simeq \bigvee_{j=n-k+1}^{n} \left(\bigvee_{1 \leq i_1 < \ldots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k}\Omega X_{i_1} \wedge \ldots \wedge \Omega X_{i_j} \right).$$

where $\binom{j-1}{n-k}$ is the binomial coefficient.

2.2.5 More homotopy decompositions

We conclude this chapter by recording some useful results which follow from the material in this chapter, and will be needed later in the thesis.

Proposition 2.29. *The homotopy pushout of the projection maps* $\pi_1 \colon X \times Y \longrightarrow X$ *and* $\pi_2 \colon X \times Y \longrightarrow Y$ *is homotopy equivalent to* $\Sigma X \wedge Y$*. Moreover the induced maps* $X \longrightarrow \Sigma X \wedge Y$ *and* $Y \longrightarrow \Sigma X \wedge Y$ *are null-homotopic.*

Proof. Denote the homotopy pushout of π_1, π_2 by Q. By extending π_1 and the induced map $Y \longrightarrow Q$ to homotopy cofibration sequences, we obtain a homotopy commutative diagram

$$\begin{array}{ccccccccc}
X \times Y & \xrightarrow{\pi_1} & X & \longrightarrow & C & \longrightarrow & \Sigma(X \times Y) & \xrightarrow{\Sigma\pi_1} & \Sigma X \\
\downarrow{\scriptstyle \pi_2} & & \downarrow & & \| & & \downarrow & & \\
Y & \longrightarrow & Q & \longrightarrow & C & \xrightarrow[f]{} & \Sigma Y. & &
\end{array}$$

By Proposition 2.21 there is a homotopy splitting $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, and so $\Sigma\pi_1$ admits a right homotopy inverse given by $\Sigma i_1 \colon \Sigma X \longrightarrow \Sigma(X \times Y)$, where i_1 is the inclusion of the first factor. By Proposition 2.19 it follows that $\Sigma(X \times Y) \simeq C \vee \Sigma X$ and consequently, $C \simeq \Sigma Y \vee \Sigma(X \wedge Y)$.

From the diagram, f is homotopic to the composite $C \xrightarrow{\simeq} \Sigma Y \vee \Sigma(X \wedge Y) \hookrightarrow \Sigma X \vee (\Sigma Y \vee \Sigma(X \wedge Y)) \xrightarrow{\text{pinch}} \Sigma Y$ and thus has a right homotopy inverse given by the inclusion of the ΣY summand. It follows that $C \simeq Q \vee \Sigma Y$ and hence $Q \simeq \Sigma X \wedge Y$. The fact that $Y \longrightarrow Q$ is null-homotopic follows from Proposition 2.19 and the existence of the right homotopy inverse for f. The same reasoning shows that $X \longrightarrow Q$ is null-homotopic. □

As a consequence of Proposition 2.29 we can give an alternative proof of the following classical result which is a special case of Porter's Theorem.

Proposition 2.30. *a) The homotopy fibre of the inclusion map $i\colon X \vee Y \longrightarrow X \times Y$ is homotopy equivalent to $\Sigma\Omega X \wedge \Omega Y$.*

b) The inclusion of the homotopy fibre is homotopic to the Whitehead product $[\zeta_X, \zeta_Y]\colon \Sigma\Omega X \wedge \Omega Y \longrightarrow X \vee Y$, where ζ_W is the composite $\Sigma\Omega W \xrightarrow{ev_W} W \hookrightarrow X \vee Y$, for $W = X, Y$.

Proof. a) By Proposition 2.4 there is a homotopy pushout square

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle i_2} \\ X & \xrightarrow[i_1]{} & X \vee Y. \end{array}$$

where i_1, i_2 are the canonical inclusion maps. Mapping each corner of the diagram into $X \times Y$ by inclusion and taking homotopy fibres produces a homotopy pushout

$$\begin{array}{ccc} \Omega X \times \Omega Y & \xrightarrow{\pi_2} & \Omega X \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow \\ \Omega Y & \longrightarrow & F \end{array}$$

where π_1, π_2 are the projection maps and F is the homotopy fibre of i. It follows from Proposition 2.29 that $F \simeq \Sigma\Omega X \wedge \Omega Y$.

b) See [26], Theorem 7.7.4a'. $\square$

Corollary 2.31. *For spaces X, Y, there is a homotopy equivalence*

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y).$$

Proof. Consider the homotopy fibration $\Sigma\Omega X \wedge \Omega Y \longrightarrow X \vee Y \hookrightarrow X \times Y$ from Proposition 2.30. Let $i_1\colon X \longrightarrow (X \vee Y)$ and $i_2\colon Y \longrightarrow (X \vee Y)$ be the canonical inclusion maps and let μ be the loop multiplication on $\Omega(X \vee Y)$. Then the composite

$$m\colon \Omega X \times \Omega Y \xrightarrow{\Omega i_1 \times \Omega i_2} \Omega(X \vee Y) \times \Omega(X \vee Y) \xrightarrow{\mu} \Omega(X \vee Y)$$

is a right homotopy inverse for $\Omega i\colon \Omega(X \vee Y) \longrightarrow \Omega(X \times Y)$ where i is the inclusion. To see this, let $\alpha \in \Omega X$, $\beta \in \Omega Y$. Then $\Omega i \circ m(\alpha, \beta) = \beta \circ \alpha$. Under the homotopy equivalence $\Omega(X \times Y) \xrightarrow{\simeq} \Omega X \times \Omega Y$ given by $\gamma \mapsto (\pi_X \circ \gamma, \pi_Y \circ \gamma)$, we see that $\beta \circ \alpha$ is mapped to (α, β).

Thus by Proposition 2.18, the fibration

$$\Omega(\Sigma\Omega X \wedge \Omega Y) \longrightarrow \Omega(X \vee Y) \xrightarrow{\Omega i} \Omega(X \times Y)$$

splits giving the desired homotopy equivalence. □

We conclude the chapter by stating a splitting for the suspension of the half-smash product $(\Sigma X) \rtimes Y$. Recall that the (right) half-smash product $A \rtimes B$ is defined as $(A \times B)/B$.

Proposition 2.32. *For spaces X, Y there is a homotopy equivalence*

$$(\Sigma X) \rtimes Y \simeq \Sigma X \vee (\Sigma X \wedge Y).$$

Proof. See [22], Lemma 9. □

Chapter 3

Polyhedral Products

Polyhedral products arose as a natural generalization of the moment angle complex introduced in 1991 by Davis and Januszkiewicz [8]. In this influential paper, which provided the impetus for the birth of toric topology as a field of mathematics, they introduced a class of $2n$-dimensional manifolds admitting an action of the torus $T^n = (S^1)^n$, called quasitoric manifolds. Quasitoric manifolds serve as a purely topological analogue of the algebraic non-singular projective toric varieties of algebraic geometry.

In the course of their research into the cohomology of these manifolds, Davis and Januszkiewicz were led to the construction of two very important spaces.

Firstly, for any given simple polytope P^n with m facets, they constructed a T^m-manifold $\mathcal{Z}_P$ called the *moment-angle complex* associated to P. The explicit construction can be found in [8] or [4]. In the same paper, Davis and Januskiewicz also went on to generalize their construction of the moment angle complex to work not only for polytopes, but for all simplicial complexes. In this more general context, the space $\mathcal{Z}_K$ associated to a simplicial complex K, is no longer necessarily a manifold.

Davis and Januszkiewicz were led to the construction of a second space DJ_K, for any simplicial complex K, which is very closely related to $\mathcal{Z}_K$. It turns out that the integral cohomology ring $H^*(DJ_K;\mathbb{Z})$ is isomorphic to the Stanley-Reisner ring $\mathbb{Z}[K]$, (for the definition, see for example [4] Chapter 3). Davis and Januszkiewicz went on to show that the integral cohomology ring of a given quasitoric manifold over the polytope P is a certain quotient of $H^*(DJ_{K_P};\mathbb{Z})$, where K_P is the boundary of the polar dual of P.

Although the original motivations behind their construction were merely as tools for understanding the cohomology of quasitoric manifolds, the moment angle complex and it's counterpart DJ_K have since become objects of great interest in their own right, providing interesting interconnections between fields as diverse as combinatorics and subspace arrangements, to robotics and configuration spaces.

In their survey [4], Buchstaber and Panov gave an alternative construction of the spaces $\mathcal{Z}_K$ and DJ_K as unions of certain product spaces indexed by the faces of K. Their construction is much more intuitive than that of Davis and Januszkiewicz and is the construction we shall work with throughout this thesis. As observed by Neil Strickland, Buchstaber and Panov's construction of the spaces $\mathcal{Z}_K, DJ_K$ has a very natural generalization to a class of spaces which we call polyhedral products. We shall spend this chapter looking at polyhedral products and some of the useful properties they possess.

3.1 Simplicial complexes

Before jumping into the construction of polyhedral products, we take the chance here to set some notation and recall some basic definitions related to simplicial complexes. The *standard n-simplex* is the set

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i\}.$$

A co-ordinate hyperplane in $\mathbb{R}^{n+1}$ is a set of the form

$$\{\{t_0, \ldots, t_n\} \subseteq \mathbb{R}^{n+1} \mid t_{i_1} = t_{i_k} = 0, \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, n\}\}$$

Definition 3.1. A subset $\sigma \in \Delta^n$ is called a face if it is given by the intersection of Δ^n with a collection of co-ordinate hyperplanes in $\mathbb{R}^{n+1}$.

Definition 3.2. A *geometric simplicial complex* K is a union of sets, called simplices, each homeomorphic to a standard simplex, such that the following conditions are satisfied:

1. If σ is a simplex in K, then each of its faces is also a simplex of K,
2. The intersection of any two simplices in K is a face of each simplex.

Definition 3.3. An *abstract simplicial complex* on a set S is a collection K of subsets of S such that for each $\sigma \in K$, all subsets of σ belong to K, including the empty set $\emptyset$.

The one element subsets are called *vertices* and if K contains all possible one element subsets we say that K is a simplicial complex on the vertex set S. In the case that S is a finite set of cardinality n, it is convenient to fix an ordering of S and identify it with the set of natural numbers $[n] = \{1, \ldots, n\}$.

We shall use Δ^n synonomously, to denote the standard n-simplex, the geometic simplicial complex which is the union of the standard n-simplex together with all of its faces, and the abstract simplicial complex which is the power set of $[n + 1]$. In this way, a geometric simplicial complex may be viewed as an abstract simplicial complex, and vice versa, and from now on we simply use the term *simplicial complex* to cover both definitions.

A subset of a simplicial complex K which is also a simplicial complex is called a *subcomplex* of K. Let L be a simplicial complex on n vertices and let $K \subseteq L$

be a subcomplex on k vertices. Suppose the vertices of L are ordered such that the first k vertices are the vertices of K. Then K is called a *full subcomplex* of L if $K = \{\sigma \cap \{1, \ldots, k\} \mid \sigma \in L\}$. A face $\sigma \in K$ is called *maximal* if there is no face $\tau \in K$ such that $\sigma \subset \tau$. When listing the faces of a simplicial complex, it suffices to list only the maximal faces.

Definition 3.4. Let K, L be simplicial complexes on $[n]$ and $[m]$ respectively. A map $f\colon K \longrightarrow L$ is called a *simplicial map* if it sends $[n]$ to $[m]$ and if $\sigma = (i_1, \ldots, i_k)$ is a simplex of K, then $f(\sigma) = (f(i_1), \ldots, f(i_k))$.

Some special examples of simplicial complexes

Here we introduce some constructions of simplicial complexes which we shall make use of in this thesis. Let K be a simplicial complex on $[n]$ and let $\sigma \in K$ be a face.

The *star* of σ is the subcomplex of K, denoted $\mathrm{star}_K(\sigma)$, consisting of those faces $\tau \in K$ such that $\sigma \cup \tau$ is a face of K.

The *link* of σ is the subcomplex of K, denoted $\mathrm{link}_K(\sigma)$, consisting of those faces $\tau \in K$ such that $\sigma \cup \tau$ is a face of K, and $\sigma \cap \tau = \emptyset$.

The *restriction* of K to a subset $S \subseteq [n]$ is the subcomplex $\mathrm{res}_K(S) = \{\tau \cap S \mid \tau \in K\}$. In the case that $S = \{1, \ldots, i\}$ for $i \leq n$, we shall write $\mathrm{res}_K(i)$.

If L is another simplicial complex on $[m]$. Identify the vertices of K with $\{1, \ldots, n\} \subseteq [n+m]$ and the vertices of L with $\{n+1, \ldots, n+m\} \subseteq [n+m]$. The *join* of K and L, is the simplicial complex on $[n+m]$, with faces $\sigma \cup \tau$ for $\sigma \in K$ and $\tau \in L$.

Example 3.5. Let K be the simplicial complex $\{\{1,2\}, \{1,3\}, \{2,3,4\}\}$, and let $L = \{5\}$. Then

- $\mathrm{link}_K(\{1\}) = \{\{2\}, \{3\}\}$ and $\mathrm{link}_K(\{4\}) = \{\{2,3\}\}$.

- $\text{star}_K(\{1\}) = \{\{1,2\},\{1,3\}\}$ and $\text{star}_K(\{4\}) = \{\{2,3,4\}\}$.
- $\text{res}_K(3) = \{\{1,2\},\{1,3\},\{2,3\}\}$.
- $K * L = \{\{1,2,5\},\{1,3,5\},\{2,3,4,5\}\}$.

3.2 The moment angle complex $\mathcal{Z}_K$, and DJ_K

We shall not use the original construction of the spaces $\mathcal{Z}_K, DJ_K$ due to Davis and Januszkiewicz in this thesis. From a homotopy-theoretic point of view, the following construction due to Buchstaber and Panov [4] is much more effective. Let D^2 denote the 2-dimensional disk $\{x \in \mathbb{C} \mid |x| \leq 1\}$ and let T denote the subspace $S^1 = \{x \in \mathbb{C} \mid |x| = 1\}$. For a group G, let BG denote the classifying space of G.

Definition 3.6. Let K be a simplicial complex on n vertices. For each face $\sigma \in K$, let $B_\sigma = \{(x_1,\ldots,x_n) \in (D^2)^n \mid x_i \in S^1 \text{ if } i \notin \sigma\}$ and let $BT_\sigma = \{(x_1,\ldots,x_n) \in (BT)^n \mid x_i = * \text{ if } i \notin \sigma\}$. The *moment angle complex* associated to K is defined to be the union $\mathcal{Z}_K = \cup_{\sigma\in K} B_\sigma$, and the space $DJ_K = \cup_{\sigma\in K} BT_\sigma$.

Example 3.7.

1. If K is the full simplex Δ^{n-1}, then $\mathcal{Z}_K = (D^2)^n$ and $DJ_K = BT^n$.
2. For $K = \partial\Delta^{n-1}$, we get $\mathcal{Z}_K = (D^2\times\ldots\times D^2\times S^1)\cup\ldots\cup(S^1\times D^2\times\ldots\times D^2) = \partial D^{2n} = S^{2n-1}$.

An interesting result that Buchstaber and Panov were able to derive is that the homotopy fibre of the inclusion map $DJ_K \hookrightarrow BT^n$ is homotopy equivalent to $\mathcal{Z}_K$. Thus there is a homotopy fibration

$$\mathcal{Z}_K \longrightarrow DJ_K \longrightarrow BT^n. \tag{3.1}$$

3.3 Polyhedral products

A very straightforward generalization of the moment-angle complexes, due to Neil Strickland, leads to an interesting class of spaces called polyhedral products. These polyhedral products form much of the focus of our thesis.

Definition 3.8. Let K be a simplicial complex on the vertex set $[n]$, and for $i = 1, \ldots, n$, let (X_i, A_i) be a pair of topological spaces such that A_i is a subspace of X_i. Denote by $(\underline{X}, \underline{A})$, the sequence of pairs $\{(X_i, A_i)\}_{i=1}^n$. Now for each face $\sigma \in K$ define $(\underline{X}, \underline{A})^\sigma$ to be the subspace $\prod_{i=1}^n Y_i \subseteq \prod_{i=1}^n X_i$ where

$$Y_i = \begin{cases} X_i, \text{if } i \in \sigma, \\ A_i, \text{if } i \notin \sigma. \end{cases}$$

The colimit $\cup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma$ over all faces in K is called the *polyhedral product* determined by the sequence $(\underline{X}, \underline{A})$ and the simplical complex K, and is denoted by $(\underline{X}, \underline{A})^K$. In the case that $X_i = X$ and $A_i = A$ for all $i = 1, \ldots, n$ we remove the underlines and write simply $(X, A)^K$.

Example 3.9. 1. The polyhedral product $(D^2, S^1)^K$ is precisely the moment angle complex $\mathcal{Z}_K$.

2. $(BT, *)^K = DJ_K$.

3. Let K_k^n be the k-skeleton of the the simplex Δ^{n-1} for $0 \leq k \leq n-1$. Then $(\underline{X}, *)^{K_k^n}$ is the space T_k^n as defined in equation (2.9). In particular, $(\underline{X}, *)^{K_{n-1}^n} = X_1 \vee \ldots \vee X_n$ and $(\underline{X}, *)^{K_1^n}$ is the fat wedge $FW(\underline{X})$.

A slight generalization of Definition 3.8 can be made to allow for simplicial complexes whose set of vertices V is a proper subset of $[n]$. An element $i \in [n]$ which is not a vertex is called a *ghost vertex* of K and the resulting polyhedral product is homeomorphic to $(\underline{X},\underline{A})^{\mathrm{res}_K(V)} \times \prod_{i\notin V} A_i$. Usually we shall assume that K has no ghost vertices, but they will make a one off appearance in Chapter 6 in which we review the work of Félix and Tanré [11].

Maps between polyhedral products

There are two main ways to construct maps between polyhedral products. Firstly, a map $f\colon (\underline{X},\underline{A}) \longrightarrow (\underline{Y},\underline{B})$ is a set $\{f_i\}$ of maps of pairs $f_i\colon (X_i,A_i) \longrightarrow (Y_i,B_i)$. Given such a map, the product map $\prod_i f_i\colon \prod_i X_i \longrightarrow \prod_i Y_i$ restricts to a map $(\underline{X},\underline{A})^K \longrightarrow (\underline{Y},\underline{B})^K$.

Let $f\colon K \longrightarrow L$ be a simplicial map which is the inclusion of a full subcomplex. Then for any face $\sigma \in K$, $(\underline{X},\underline{A})^\sigma$ includes into $(\underline{X},\underline{A})^L$. Passing to the colimit gives an induced map $(\underline{X},\underline{A})^K \longrightarrow (\underline{X},\underline{A})^L$. We shall make use of the following useful result, taken from [10], later in the thesis.

Proposition 3.10 (See [10]). *Let L be a simplicial complex on $[m]$ and suppose K is a full subcomplex of L on $n \leq m$ vertices. Then the induced map $f\colon (\underline{X},\underline{A})^K \longrightarrow (\underline{X},\underline{A})^L$ has a strict left inverse.*

Proof. Relabelling the vertices of L if necessary, we may assume that K is a simplicial complex on the vertex set $[n] \subseteq [m]$. By definition, f is the restriction of the inclusion map $\prod_{i=1}^n X_i \hookrightarrow \prod_{i=1}^m X_i$. On the other hand, the projection map $\prod_{i=1}^m X_i \longrightarrow \prod_{i=1}^n X_i$ restricts to a surjective map $l\colon (\underline{X},\underline{A})^L \longrightarrow (\underline{X},\underline{A})^K$ and $l \circ f$ is clearly the identity map on $(\underline{X},\underline{A})^K$. □

3.4 Some useful properties of polyhedral products

We record some useful properties of polyhedral products in this section which crop up again and again in our main results.

Proposition 3.11. *Let K, L be simplicial complexes on vertices $\{1, \ldots, n\}$ and $\{n+1, \ldots, m\}$ respectively. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{n+m}$ be any sequence of pairs of spaces, and let $(\underline{X}, \underline{A})_n = \{(X_i, A_i)\}_{i=1}^{n}$ and $(\underline{X}, \underline{A})_m = \{(X_i, A_i)\}_{i=n+1}^{m}$. Then there is a homeomorphism*

$$(\underline{X}, \underline{A})^{K*L} \cong (\underline{X}, \underline{A})_n^K \times (\underline{X}, \underline{A})_m^L .$$

Proof. Note that $(\underline{X}, \underline{A})_n^K$ is a subspace of $\prod_{i=1}^n X_i$, $(\underline{X}, \underline{A})_m^L$ is a subspace of $\prod_{i=n+1}^{n+m} X_i$ and $(\underline{X}, \underline{A})^{K*L}$ is a subspace of $\prod_{i=1}^{n+m} X_i$. The desired homeomorphism is given by the obvious restriction of the homeomorphism

$$\prod_{i=1}^{n+m} X_i \xrightarrow{\cong} \left(\prod_{i=1}^{n} X_i\right) \times \left(\prod_{i=n+1}^{n+m} X_i\right).$$

□

Proposition 3.12 (See Denham and Suciu [10])**.** *Let $p_i \colon (E_i, E_i') \longrightarrow (B_i, B_i')$ be a map of pairs for $i = 1, \ldots n$ such that $p_i \colon E_i \longrightarrow B_i$ has homotopy fibre F_i and $p_i|_{E_i'} \colon E_i' \longrightarrow B_i'$ has homotopy fibre F_i'. If either*

1. *$F_i = F_i'$ for all $i = 1, \ldots, n$,*

2. *$B_i = B_i'$ for all $i = 1, \ldots, n$,*

then the product fibration $\prod_{i=1}^{n} F_i \longrightarrow \prod_{i=1}^{n} E_i \longrightarrow \prod_{i=1}^{n} B_i$ *restricts to a homotopy fibration*

$$(\underline{F}, \underline{F'})^K \longrightarrow (\underline{E}, \underline{E'})^K \longrightarrow (\underline{B}, \underline{B'})^K .$$

Proof. We include a proof using Mather's Cube Lemma in the case that $B_i = B_i'$ for all $i = 1, \ldots, n$. In this case $(\underline{B}, \underline{B'})^\sigma = \prod_{i=1}^{n} B_i$ for any simplex $\sigma \in K$. Now, the polyhedral product $(\underline{E}, \underline{E'})^K$ is the colimit $\cup_{\sigma \in K} (\underline{E}, \underline{E'})^\sigma$. The map of pairs induces a product map $(\underline{E}, \underline{E'})^\sigma \longrightarrow (\underline{B}, \underline{B'})^\sigma$, for each $\sigma \in K$, and the homotopy fibre is given by $(\underline{F}, \underline{F'})^\sigma$. Mather's Cube Lemma now shows that the map $(\underline{E}, \underline{E'})^K \longrightarrow (\underline{B}, \underline{B'})^K$ has homotopy fibre $(\underline{F}, \underline{F'})^K$. □

A generalization of Buchstaber and Panov's fibration

Using Proposition 3.12 we are able to construct a certain homotopy fibration which generalizes the classical fibration (3.1) of Buchstaber and Panov. Let K be a simplicial complex on n vertices and let $\underline{X} = \{X_1, \ldots, X_n\}$ be a sequence of spaces. The fact that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathrm{Cone}\,\Omega X_i & \longrightarrow & X_i & \xrightarrow{1} & X_i \\ \uparrow & & \uparrow & & \| \\ \Omega X_i & \longrightarrow & * & \longrightarrow & X_i \end{array}$$

for each $i = 1, \ldots, n$, in which the horizontal rows are homotopy fibrations, shows that there is an induced map of pairs $(\underline{X}, *) \longrightarrow (\underline{X}, \underline{X})$, and that there is a homotopy fibration

$$(\underline{\mathrm{Cone}\,\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, *)^K \longrightarrow \prod_{i=1}^{n} X_i. \tag{3.2}$$

Putting $X_i = BT$ for all i, recovers the the homotopy fibration (3.1). We record here the result that this homotopy fibration splits after looping. The proof generalizes that of Proposition 2.31 and can also be found in [15] in the special case that $K = \partial\Delta^{n-1}$.

Proposition 3.13. *The inclusion map* $i\colon (\underline{X},*)^K \longrightarrow \prod_{i=1}^n X_i$ *admits a right homotopy inverse after looping.*

Proof. Denote the inclusion $\bigvee_{i=1}^n X_i \longrightarrow (\underline{X},*)^K$ by j and let α_k denote the inclusion $X_k \hookrightarrow \bigvee_{i=1}^n X_i$. Let μ denote the loop multiplication on $\Omega\left(\bigvee_{i=1}^n X_i\right)$. Then the following composite gives the desired right homotopy inverse,

$$r\colon \prod_{i=1}^n \Omega X_i \xrightarrow{\prod \Omega\alpha_i} \prod_{i=1}^n \Omega\left(\bigvee_{i=1}^n X_i\right) \xrightarrow{\mu} \Omega\left(\bigvee_{i=1}^n X_i\right) \xrightarrow{\Omega j} \Omega\left(\underline{X},*\right)^K.$$

To see that this is the case, an easy generalization of Corollary 2.31 shows that the composite

$$\prod_{i=1}^n \Omega X_i \xrightarrow{r} \Omega\left(\underline{X},*\right)^K \xrightarrow{\Omega i} \Omega\Big(\prod_{i=1}^n X_i\Big) \xrightarrow{\simeq} \prod_{i=1}^n \Omega X_i$$

is homotopic to the identity. Restricting to $\Omega\left(\underline{X},*\right)^K$ gives the result. □

Corollary 3.14. *Let K be a simplicial complex on n vertices and let $\underline{X} = \{X_1,\ldots,X_n\}$ be a sequence of spaces. Then there is a homotopy equivalence*

$$\Omega\left(\underline{X},*\right)^K \simeq \Omega\left(\underline{\mathrm{Cone}\,\Omega X}, \underline{\Omega X}\right)^K \times \prod_{i=1}^n \Omega X_i.$$

Proof. By Proposition 3.13, the homotopy fibration

$$\Omega\left(\underline{\mathrm{Cone}\,\Omega X}, \underline{\Omega X}\right)^K \longrightarrow \Omega\left(\underline{X},*\right)^K \longrightarrow \prod_{i=1}^n \Omega X_i,$$

obtained by looping (3.2), splits giving the desired homotopy equivalence. □

Chapter 4

Homotopy Exponents

Serre [28], famously showed that the homotopy groups of a simply connected finite CW-complex are finitely generated Abelian groups. In particular $\pi_i(X)$ is isomorphic to a direct sum $F \oplus T$, where $F = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ is the free part, and T, the torsion part, is a direct sum of a finite number of cyclic groups. This is not true for non-simply-connected finite complexes. For example $\pi_2(S^1 \vee S^2)$ is generated by the Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ where t is a choice of a generator for $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$. Due to the intractability of the calculation of homotopy groups in general, even for spaces as fundamentally simple as the sphere, homotopy theory inevitably branched off into two distinct fields: rational homotopy theory and primary homotopy theory. We discuss these two fields next.

Rational homotopy theory

It is an over simplification to say so in general, but in many important examples at least, it tends to be the torsion which makes the problem of computing homotopy groups so difficult. In many cases, the free part can often be much simpler to calculate. In rational homotopy theory, the main objects of study are the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$ of a space X. Tensoring $\pi_*(X)$ with the rationals kills all of the torsion and in doing so isolates the information about the free part. Interestingly, $\pi_*(\Omega X) \otimes \mathbb{Q}$ comes equipped with a Lie algebra structure with the bracket given by the Samelson product. Various techniques involving commutative differential graded algebras, and the Sullivan minimal model for example, have been successfully developed over time in order to gain insight into spaces after rationalization. A nice result is the Dichotomy Theorem which shows that the rational homotopy of simply-connected CW-complexes can behave in one of two ways. Either $\pi_*(X) \otimes \mathbb{Q}$ is finite dimenional as a vector space over $\mathbb{Q}$ in which case X is called *elliptic*. Or else the sequence $\sum_{i=0}^{n} \dim(\pi_i(X) \otimes \mathbb{Q})$ grows exponentially with n. Such spaces are called *hyperbolic*. A few important examples of elliptic spaces are the following:

Example 4.1.

1. Serre [27] showed that the sphere S^n is elliptic. Specifically, he showed that $\pi_n(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$, with generator the identity map, and if $n = 2k$ then additionally $\pi_{4k-1}(S^{2k}) \otimes \mathbb{Q} \cong \mathbb{Q}$, generated by the Whitehead product of the identity $1 : S^{2k} \longrightarrow S^{2k}$ with itself. All other rational homotopy groups of the n-sphere are trivial. Thus $\pi_*(S^n) \otimes \mathbb{Q}$ is either 1-dimensional, over $\mathbb{Q}$, when n is odd, or 2-dimensional when n is even.

2. A product of finitely many elliptic spaces is elliptic. This is immediate since $\pi_*(X \times Y) \otimes \mathbb{Q} \cong (\pi_*(X) \otimes \mathbb{Q}) \oplus (\pi_*(Y) \otimes \mathbb{Q})$.

3. The wedge of two spheres $S^n \vee S^k$ is hyperbolic. To see this, apply the Hilton-Milnor Theorem to see that $\Omega(S^n \vee S^k)$ is homotopy equivalent to a product of infinitely many looped spheres. Iterated use of the Hilton-Milnor Theorem shows similarly that the wedge sum of more than two spheres is hyperbolic.

4. The Moore space $P^m(p^r)$ is a rationally trivial space, that is, $\pi_*(P^m(p^r)) \otimes$

$\mathbb{Q} = 0$. Thus $P^m(p^r)$ is clearly an elliptic space.

Primary homotopy theory

In primary homotopy theory, it is the torsion in $\pi_*(X)$ that is of interest. Usually it is too difficult to calculate the torsion subgroup of $\pi_*(X)$ completely. Even for spaces as simple as the sphere S^n, which as we saw in example (4.1), is completely understood from the perspective of rational homotopy theory, has highly complex torsion. The calculation of the integral homotopy groups of spheres remains one of the most important and long-standing unsolved problems in homotopy theory. Taking a more qualitative approach in our study of homotopy groups can often be fruitful in gaining new insight. One such approach is encapsulated in the notion of homotopy exponents, which forms the main subject matter of this chapter of the thesis.

Definition 4.2. Let G be an Abelian group. We say that $n \in \mathbb{N}$ is an exponent for G if $ng = 0$ for all $g \in G$. If G has an exponent, we denote the least such by $\exp(G)$.

Definition 4.3. Let X be a simply connected space and let p be a prime. We say that p^r is the *homotopy exponent* for X at p, (or the p-primary *homotopy exponent* of X), if $\exp(T_p) = p^r$ where T_p is the the p-torsion subgroup of $\pi_*(X)$. We write $\exp_p(X)$ for $\exp(T_p)$.

We shall often abuse notation by omitting the prime p and writing simply $\exp(X)$, when it is safe that no confusion can arise, or in the case that the choice of a specific prime p is unimportant.

Example 4.4. Suppose X is a simply connected space such that

$$\pi_*(X) = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^4 \oplus \mathbb{Z}/5^2 \oplus \mathbb{Z}/11^6.$$

Then $\exp_2(X) = 2^4$, $\exp_5(X) = 5^2$ and $\exp_{11}(X) = 11^6$. On the other hand, suppose there is a space Y such that

$$\pi_*(Y) = \mathbb{Z} \oplus \left(\bigoplus_{k=1}^{\infty} \mathbb{Z}/2^k \right) \oplus \mathbb{Z}/3.$$

Then the 2-primary homotopy exponent of Y does not exist, but $\exp_3(Y) = 3$.

Often the exact value of the homotopy exponent $\exp_p(X)$ at a particular prime may be too difficult to calculate. Sometimes looping a space can unlock new information about the homotopy exponents of a space since the isomorphism $\pi_*(\Omega X) \cong \pi_{*+1}(X)$ implies that $\exp_p(\Omega X) = \exp_p(X)$ for all primes p. In cases where the homotopy exponent refuses to be pinned down precisely, it is desirable to determine good upper and lower bounds. One of the simplest methods we have for calculating upper bounds for the homotopy exponents of a space X is to construct a homotopy fibration $Y \longrightarrow X \longrightarrow Z$ for spaces Y, Z whose homotopy exponents, or upper bounds thereof, are already known.

Proposition 4.5. *Let* $Y \xrightarrow{i} X \xrightarrow{f} Z$ *be a homotopy fibration and* p *a prime. Then* $\exp_p(X) \leq \exp_p Y \cdot \exp_p Z$.

Proof. The result follows from the long exact sequence of homotopy groups associated to the fibration. Suppose $\alpha \in \pi_*(X)$ and suppose $\exp_p(Y) = p^s$ and $\exp_p(B) = p^t$. Then $f_*(p^t \cdot \alpha) = p^t \cdot f_*(\alpha) = 0$. Thus by exactness, there exists $\beta \in \pi_*(B)$ such that $i_*(\beta) = p^t \cdot \alpha$. As a result we have $0 = p^s \cdot i_*(\beta) = (p^s p^t) \cdot \alpha$. □

The upper bound given in Proposition 4.5 is generally quite a rough estimate. In the case of the trivial fibration $X \longrightarrow X \times Y \longrightarrow Y$ for example, it gives the upper bound $\exp_p(X \times Y) \leq \exp_p(X) \cdot \exp_p(Y)$. However, for products, the homotopy exponent can be determined precisely in terms of the exponents of it's factors.

Proposition 4.6. *For spaces* X, Y *we have* $\exp_p(X \times Y) = \max\{\exp_p(X), \exp_p(Y)\}$.

Proof. Suppose $\exp_p(X) = p^s$ and $\exp_p(Y) = p^t$. Since there is an isomorphism $\pi_*(X \times Y) \cong \pi_*(X) \oplus \pi_*(Y)$, it is trivial that for $\alpha \in \pi_*(X)$ and $\beta \in \pi_*(Y)$ we have $p^k(\alpha, \beta) = 0$ if and only if $p^r \cdot \alpha = p^r \cdot \beta = 0$. □

Before proceeding to look at the homotopy exponents of some fundamental spaces, we discuss the related ideas of H-exponents and co-H-exponents.

H-exponents and co-H-exponents

Throughout this section, we use the notation $X_{(p)}$ to denote the localization of X at a prime p.

Suppose X is an H-group, for example a loop space, with multiplication $\mu\colon X \times X \longrightarrow X$. Then the set $[X, X]$ of homotopy classes of self maps of X has the structure of a group with addition given by

$$f + g\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} X \times X \xrightarrow{\mu} X.$$

The p^r *power map* on X, denoted $p^r\colon X \longrightarrow X$, is the map $p^r(1)$ where 1 is the identity map on X.

Definition 4.7. Let X be an H-group and let p be a prime. Then p^r is called an *H-exponent* for X if $p^r(1) = 0$ in $[X_{(p)}, X_{(p)}]$. We shall call the least such p^r, *the H-exponent for X.*

A useful result is the following:

Proposition 4.8. *Suppose X is an H-group and suppose that p^r is an H-exponent for X. Then* $\exp_p(X) \leq p^r$.

Proof. Since p^r is an H-exponent for X, then by definition, the p^r power map on $X_{(p)}$ is null-homotopic. But the p^r power map induces multiplication by p^r on $\pi_*(X_{(p)})$, and hence $p^r\pi_*(X_{(p)}) = 0$. □

Similarly, suppose X is a co-H-group, for example a suspension, with co-multiplication $\psi\colon X \longrightarrow X \vee X$. Then the set $[X, X]$ of homotopy classes of self maps of X has the structure of a group with addition given by

$$f + g\colon X \xrightarrow{\psi} X \vee X \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

For a prime p, the map $p^r(1)\colon X \longrightarrow X$ is called the p^r-degree map, and is denoted $[p^r]$.

Definition 4.9. Let X be a co-H-group and and let p be a prime. Then p^r is called the co-H-exponent of X at p if $[p^r]\colon X_{(p)} \longrightarrow X_{(p)}$ is null-homotopic.

4.1 Homotopy exponents of spheres and Moore spaces

Inspired by a conjecture of Barratt, which we shall discuss in Section 4.2.2, Cohen, Moore and Neisendorfer began searching in the 70's for elements of order p^{r+1} in the homotopy groups of $P^m(p^r)$ with $m \geq 3$ and p an odd prime. They believed such elements could be detected by certain Bockstein maps. Their work in [7] and [6] led them to the following homotopy exponent results for spheres and Moore spaces.

Theorem 4.10 (Cohen, Moore, Neisendorfer)**.** *For p an odd prime and $n \geq 1$, $exp_p(S^{2n+1}) = p^n$.*

Theorem 4.11 (Cohen, Moore, Neisendorfer). *For p an odd prime, $m \geq 3$ and $r \geq 1$, $\exp_p(P^m(p^r)) = p^{r+1}$. In fact, the p^{r+1} power map on $\Omega^2\Sigma^2 P^m(p^r)$ is nullhomotopic.*

In the next section we show that Cohen, Moore and Neisendorfers exponent result for Moore spaces can be used to deduce the homotopy exponents of wedges of Moore spaces.

4.1.1 Wedges of Moore spaces

It is interesting to note that the p-primary homotopy exponent of $P^m(p^r)$ is unaffected by the dimension m. This is in marked contrast to the behaviour of the spheres where the homotopy exponent of S^{2n+1} increases exponentially as n increases. It is this interesting fact which belies the reason, as we see in this section, why a wedge of at least two spheres has no homotopy exponent at any prime, yet a wedge of at least two mod p^r Moore spaces has a finite homotopy exponent at p. We prove this result in this section. First of all we will need to show that the p^r degree map $[p^r]\colon P^m(p^r) \longrightarrow P^m(p^r)$ is null-homotopic, and to show this we need the following lemma.

Lemma 4.12. *The degree map is natural with respect to co-H-maps. That is, if $f\colon X \longrightarrow Y$ is a co-H-map then $[k] \circ f$ is homotopic to $f \circ [k]$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\psi_X} & \bigvee_k X & \xrightarrow{\nabla} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle \vee f}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow[\psi_Y]{} & \bigvee_k Y & \xrightarrow[\nabla]{} & Y. \end{array}$$

The right hand square is clearly homotopy commutative, and since f is a co-H-map, the left hand square also homotopy commutes. Since the composite $\nabla \circ \psi_X$

is precisely the degree k map on X, and $\nabla \circ \psi_Y$ is the degree k map on Y, the result follows. $\square$

Consider the homotopy cofibration sequence

$$S^{m-1} \xrightarrow{[p^r]} S^{m-1} \xrightarrow{i} P^m(p^r) \xrightarrow{\text{pinch}} S^m$$

which defines $P^m(p^r)$. For $m \geq 2$, we have that $[p^r]\colon S^m \longrightarrow S^m$ is homotopic to $S^1 \wedge S^{m-1} \xrightarrow{1\wedge[p^r]} S^1 \wedge S^{m-1}$ which is by definition the co-H-map $\Sigma[p^r]$. If $m \geq 3$, then similarly, i is homotopic to $S^1 \wedge S^{m-2} \xrightarrow{1\wedge i'} S^1 \wedge P^{m-1}(p^r)$, where i' is the inclusion of the bottom cell, which by definition is the co-H-map $\Sigma i'$. Additionally, for $m \geq 3$ the pinch map $P^m(p^r) \longrightarrow S^m$ is homotopic to the co-H-map $\Sigma(\text{pinch})\colon S^1 \wedge P^{m-1}(p^r) \xrightarrow{1\wedge\text{pinch}} S^1 \wedge S^{m-1}$. This observation, combined with Lemma 4.12 leads to the following co-H-exponent result:

Proposition 4.13. *For $m \geq 3$ and $p \neq 2$, the Moore space $P^m(p^r)$ has co-H-exponent p^r.*

Proof. Since $m \geq 3$, each of the maps in the homotopy cofibration sequence $S^{m-1} \xrightarrow{[p^r]} S^{m-1} \xrightarrow{i} P^m(p^r) \xrightarrow{\text{pinch}} S^m$ are co-H-maps. Thus by Lemma 4.12, we obtain a homotopy commutative diagram

$$\begin{array}{ccccccc}
S^{m-1} & \xrightarrow{[p^r]} & S^{m-1} & \xrightarrow{i} & P^m(p^r) & \xrightarrow{\text{pinch}} & S^m \\
\downarrow{\scriptstyle [p^r]} & & \downarrow{\scriptstyle [p^r]} & & \downarrow{\scriptstyle [p^r]} & & \downarrow{\scriptstyle [p^r]} \\
S^{m-1} & \xrightarrow[{[p^r]}]{} & S^{m-1} & \longrightarrow & P^m(p^r) & \longrightarrow & S^m.
\end{array}$$

The restriction of $[p^r]$ to the bottom cell of $P^m(p^r)$ is given by the composite $S^{m-1} \xrightarrow{i} P^m(p^r) \xrightarrow{[p^r]} P^m(p^r)$ which is homotopic to $[p^r] \circ i$ and therefore trivial. Consequently there is a lift of the pinch map $P^m(p^r) \longrightarrow S^m$ to a map $l\colon S^m \longrightarrow$

$P^m(p^r)$, giving a factorization of the p^r degree map on $P^m(p^r)$

$$\begin{array}{ccc} P^m(p^r) & \xrightarrow{\text{pinch}} & S^m \\ {\scriptstyle [p^r]}\downarrow & \swarrow {\scriptstyle l} & \\ P^m(p^r). & & \end{array}$$

Since $p \neq 2$, $\pi_m(P^m(p^r)) = 0$. See for example [21]. Thus l is null-homotopic and consequently so too is the degree map $[p^r]\colon P^m(p^r) \longrightarrow P^m(p^r)$.

□

Lemma 4.14. *Let p be an odd prime and let $n, m \geq 2$ and $r \geq 1$. Then there is a homotopy equivalence $P^m(p^r) \wedge P^n(p^r) \simeq P^{m+n}(p^r) \vee P^{m+n-1}(p^r)$.*

Proof. The degree map $[p^r]\colon P^3(p^r) \longrightarrow P^3(p^r)$ can be defined by $[p^r] \wedge 1\colon S^1 \wedge P^2(p^r) \longrightarrow S^1 \wedge P^2(p^r)$. since there is a homotopy cofibration sequence $S^1 \xrightarrow{[p^r]} S^1 \longrightarrow P^2(p^r)$, then taking the smash product with $P^2(p^r)$ gives a homotopy cofibration sequence

$$S^1 \wedge P^2(p^r) \overset{[p^r]\wedge 1 = [p^r]}{\longrightarrow} S^1 \wedge P^2(p^r) \longrightarrow P^2(p^r) \wedge P^2(p^r).$$

On the other hand, the map $[p^r]\colon P^3(p^r) \longrightarrow P^3(p^r)$ is nullhomotopic by Proposition 4.13 and so by Proposition 2.19, we see that

$$P^2(p^r) \wedge P^2(p^r) \simeq P^3(p^r) \vee P^4(p^r).$$

By suspending, we may obtain the required decomposition for any $n, m \geq 2$. □

A nice corollary of Lemma 4.14 which we shall also make use of in Chapter 6 is the following:

Corollary 4.15. *Let p be an odd prime and let $m, n \geq 3$. Then the p-primary homotopy exponent of the wedge of Moore spaces $P^m(p^r) \vee P^n(p^r)$ is equal to p^{r+1}.*

Proof. By Corollary 2.31 there is a homotopy decomposition

$$\Omega(P^m(p^r) \vee P^n(p^r)) \simeq \Omega P^m(p^r) \times \Omega P^n(p^r) \times \Omega(\Sigma\Omega P^m(p^r) \wedge \Omega P^n(p^r)).$$

Consider the final factor $\Omega(\Sigma\Omega P^m(p^r) \wedge \Omega P^n(p^r))$. By iterated use of the James splitting, there is a sequence of homotopy equivalences:

$$\begin{aligned}
\Sigma\Omega P^m(p^r) \wedge \Omega P^n(p^r) &\simeq \Sigma\left(\bigvee_{k=1}^{\infty}(P^{m-1}(p^r))^{\wedge k}\right) \wedge \Omega P^n(p^r) \\
&\simeq \left(\bigvee_{j=1}^{\infty}(P^{m-1}(p^r))^{\wedge j}\right) \wedge \Sigma\Omega P^n(p^r) \\
&\simeq \left(\bigvee_{j=1}^{\infty}(P^{m-1}(p^r))^{\wedge j}\right) \wedge \Sigma\left(\bigvee_{k=1}^{\infty}(P^{n-1}(p^r))^{\wedge k}\right) \\
&\simeq \bigvee_{j,k=1}^{\infty} \Sigma(P^{m-1}(p^r))^{\wedge j} \wedge (P^{n-1}(p^r))^{\wedge k}
\end{aligned}$$

By iterating Proposition 4.14 the space $P^{m-1}(p^r))^{\wedge j} \wedge (P^{n-1}(p^r))^{\wedge k}$ decomposes as wedge of mod p^r Moore spaces, and suspending just raises the dimension of the wedge summands. Hence the factor $\Omega(\Sigma\Omega P^m(p^r) \wedge \Omega P^n(p^r))$ is actually homotopy equivalent to ΩX where X is a wedge of mod p^r Moore spaces of dimension at least 3. By iterated application of Corollary 2.31 we obtain a decomposition $\Omega X \simeq \Omega A \times \Omega B$, where A is a product of mod p^r Moore spaces and B is a wedge of mod p^r Moore spaces. By induction, it follows that $\Omega(P^m(p^r) \vee P^n(p^r))$ decomposes upto homotopy as a product of mod p^r Moore spaces, and since the homotopy exponent of a product is equal to the greatest homotopy exponent of its factors, we see that $\exp_p(P^m(p^r) \vee P^n(p^r)) = p^{r+1}$. □

4.2 Two conjectures of Moore and Barratt

To close this chapter we discuss two conjectures related to homotopy exponents.

4.2.1 Moore's Conjecture

On the face of it, despite sharing the common goal of bettering our understanding of the integral homotopy groups, the rational homotopy groups of a space appear to have little in common with it's primary homotopy. But during the academic year 1977-78, John Moore proposed the following remarkable conjecture which suggests that for simply connected finite CW-complexes, the two are intimately intertwined.

Conjecture 4.16 (Moore's Conjecture)**.** *Let p be a prime and X be a simply connected finite CW-complex. Then X is elliptic if and only if X has a finite homotopy exponent at p.*

It is interesting to note that a nice consequence of this conjecture, should it be true, is that since the statement is independent of the prime p, then the existence of the homotopy exponent at a particular prime is equivalent to the existence of the homotopy exponent at all primes.

Not many examples of spaces are known for which Moore's Conjecture has been verified. Two important examples which do fit the bill are the sphere S^n for $n \geq 1$, and the Moore space $P^m(p^r)$ for all p, r except when $p = 2$ and $r = 1$. Spheres and Moore spaces are elliptic spaces, and the results of Cohen, Moore and Neisendorfer show that their homotopy exponents are known to exist at all primes, except in the case of $P^m(2)$.

4.2.2 Barratt's Conjecture

Barratt initiated the study of suspension spaces with finite co-H exponents, which he termed finite characteristic, in [3]. The $[p^r]$ degree map on a suspension space ΣX is by definition homotopic to the composite

$$\Sigma X \xrightarrow{\psi} \bigvee_{p^r} \Sigma X \xrightarrow{\nabla} \Sigma X$$

where ψ is the coproduct and ∇ is the fold map. Barratt was able to find bounds on the order of elements in $\pi_*(\Sigma X)$ for suspensions ΣX with finite co-H exponent p^r. He did this by applying the Hilton-Milnor Theorem to decompose $\Omega(\bigvee_{p^r} \Sigma X)$ as a weak product which has p^r factors $\Omega\Sigma X$ and other terms of the form $\Omega\Sigma X^{\wedge r}$ for $r \geq 2$.

If X is $n-1$ connected and ΣX has characteristic p^r, then $X^{\wedge r}$ is $rn-1$ connected and the co-H-exponent of $\Sigma X^{\wedge r}$ divides p^r. It follows that after looping the null homotopic degree map $[p^r]$, there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega[p^r]\colon \Omega\Sigma X & \xrightarrow{\Omega\psi} & \Omega(\bigvee_{p^r} \Sigma X) & \xrightarrow{\Omega\nabla} & \Sigma X \\ & \searrow & \downarrow \simeq & \nearrow & \\ & & \prod_\alpha \Omega\Sigma X^{\wedge n_\alpha} & & \end{array}$$

which factors the nullhomotopic map $\Omega[p^r]$ via $\prod_\alpha \Omega\Sigma X^{\wedge n_\alpha}$. Barratt was able to deduce that multiplication by p^r annihilates the p-torsion in $\pi_q(\Omega\Sigma X)$ for $q \leq 2n-1$, and by an inductive argument found that multiplication by p^{sr} annihilates the p-torsion in $\pi_q(\Omega\Sigma X)$ for $q \leq 2^s n$. In particular, the homotopy groups $\pi_q(\Sigma X)$ have a finite exponent for each q, but Barratt's bound is dependent on q.

Barratt conjectured that if X is itself a suspension ΣY, with co-H-exponent

p^r, then $\pi_*(\Sigma X)$ is annihilated by p^{r+1}.

Conjecture 4.17 (Barratt's Conjecture)**.** *a) (Weak Form) Suppose that* ΣY *has co-*H*-exponent* p^r*. Then* $\exp_p(\Sigma^2 Y) \leq p^{r+1}$*.*

b) (Strong Form) Suppose that ΣY *has co-*H*-exponent* p^r*. Then* p^{r+1} *is an* H*-exponent for* $\Omega^2\Sigma^2 Y$*.*

Notice that if $\Omega\Sigma^2 Y$ has H-exponent p^{r+1} then it follows that $\exp_p(\Sigma^2 Y) \leq p^{r+1}$, so the strong form of Barratt's Conjecture implies the weak form. The standard example of a space satisfying Barratt's Conjecture is the Moore space $P^{m+1}(p^r) \simeq \Sigma P^m(p^r)$ for $m \geq 3$. We saw in 4.13 that $\Sigma P^m(p^r)$ has co-H-exponent p^r. The fact that $\Omega^2\Sigma^2 P^m(p^r)$ has H-exponent p^{r+1} is precisely the result of Cohen Moore and Neisendorfer from Theorem 4.11.

Chapter 5

Polyhedral Products For n-gons

The motivating questions behind the work in this thesis are the following:

Question 1: Are there conditions on $(\underline{X}, \underline{A})$ and K which guarantee that $(\underline{X}, \underline{A})^K$ is elliptic/hyperbolic?

Question 2: Are there conditions on $(\underline{X}, \underline{A})$ and K which guarantee the existence, (or not), of the homotopy exponent of $(\underline{X}, \underline{A})^K$ at a given prime p? And if the homotopy exponent exists, can we get an upper bound?

Question 3: Can we add to the existing evidence in support of Moore's Conjecture, by exhibiting some sub-collection of polyhedral products which are either elliptic and admit a homotopy exponent at every prime, or else hyperbolic and do not admit a homotopy exponent at any prime?

Background - What is already known

Suppose we fix the pair (D^2, S^1). One of the first non-trivial contributions in the direction of Question 1 came in 2004 when Grbić and Theriault [13] showed that if K is a disjoint union of n vertices, then $(D^2, S^1)^K$ is homotopy equivalent to a wedge of spheres $\bigvee_{j=2}^{n} \binom{n}{j}(j-1)S^{j+1}$. In particular, $(D^2, S^1)^K$ is elliptic when $n = 2$ and hyperbolic and when $n \geq 3$.

In 2007, Grbić and Theriault [14] extended their previous result to show that $(D^2, S^1)^K$ is homotopy equivalent to a wedge of spheres whenever K belongs to a class of simplicial complexes called shifted complexes.

Definition 5.1. A simplicial complex K is called *shifted* if there exists an ordering of the vertices of K such that whenever $\sigma \in K$, and v, w are vertices of K such that $v < w$ and $w \in \sigma$, then $(\sigma \backslash w) \cup v \in K$.

Example 5.2.

1. The k-dimensional skeleton of the simplex Δ^n is shifted for $0 \leq k \leq n-1$.
2. The simplicial complex $\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ is shifted.

In fact Grbić and Theriault were able to further extend the family of simplicial complexes for which $(D^2, S^1)^K$ has the homotopy type of a wedge of spheres beyond the class of shifted complexes by including also disjoint unions of shifted complexes, and complexes $K_1 \cup_\sigma K_2$ obtained from gluing shifted complexes along a common face, ([14]).

In 2008, Géry Debongnie [9] gave a complete and very aesthetically pleasing answer to Question 1 in the case of the pair (D^2, S^1) by showing that there is a very simple condition on K which determines whether $(D^2, S^1)^K$ is elliptic or hyperbolic.

Theorem 5.3 (Debongnie)**.** *The following statements are equivalent:*

a) $(D^2, S^1)^K$ *is elliptic,*

b) K *is a join of simplices and boundaries of simplices,*

c) $(D^2, S^1)^K$ *is homotopy equivalent to a product of spheres.*

Proof. See [9]. Note that the implications b) $\Rightarrow$ c) is trivial by Proposition 3.11, and also the implication c) $\Rightarrow$ a) is trivial since spheres are elliptic. □

In fact, Debongnie's result was stated in the language of complements of coordinate subspaces but his result is equivalent to that stated above. The methods employed by Debongnie were therefore very combinatorial, using various rational models and techniques of rational homotopy theory, which are very particular to the polyhedral product $(D^2, S^1)^K$ but are not very adaptable to general polyhedral products. It would be desirable to obtain a purely homotopy theoretic proof of Theorem 5.3 with the hope that the techniques used may be adaptable to the case of pairs (X, A) other than (D^2, S^1).

For example, we conjecture that a very similar result to Theorem 5.3 should hold for the pair (D^m, S^{m-1}) for $m \geq 2$.

Conjecture 5.4. *The following statements are equivalent:*

a) $(D^m, S^{m-1})^K$ *is elliptic,*

b) K *is a join of simplices and boundaries of simplices,*

c) $(D^m, S^{m-1})^K$ *is homotopy equivalent to a product of spheres.*

A proof of Conjecture 5.4 would would in turn have immediate implications for polyhedral products constructed from the pair $(P^m(p^r), S^{m-1})$, where the inclusion of S^{m-1} in $P^m(p^r)$ is the inclusion of the bottom cell. Because there is a rational homotopy equivalence of pairs $(P^m(p^r), S^{m-1}) \simeq_{\mathbb{Q}} (D^m, S^{m-1})$, and the polyhedral product functor preserves rational homotopy equivalences, then $(P^m(p^r), S^{m-1})^K \simeq_{\mathbb{Q}} (D^m, S^{m-1})^K$. Thus we have

Proposition 5.5. *If Conjecture 5.4 holds then the polyhedral product* $(P^m(p^r), S^{m-1})^K$ *is elliptic if and only if* K *is a join of simplices and boundaries of simplices.*

Proof. Since there is a rational homotopy equivalence $(P^m(p^r), S^{m-1})^K \simeq_{\mathbb{Q}} (D^m, S^{m-1})^K$

it follows that there is an isomorphism of the rational homotopy groups

$$\pi_*(\left(P^m(p^r), S^{m-1}\right)^K) \otimes \mathbb{Q} \cong \pi_*(\left(D^m, S^{m-1}\right)^K) \otimes \mathbb{Q}.$$

Thus $(P^m(p^r), S^{m-1})^K$ is elliptic if and only if $(D^m, S^{m-1})^K$ is. □

Polyhedral products associated to n-gons

In this thesis, the main family of polyhedral products we study are those of the form $(\underline{X}, \underline{A})^K$, where K is an n-gon.

Definition 5.6. For a positive integer $n \geq 3$, the n-gon is the 1-dimensional simplicial complex on n vertices consisting of the faces $\{1, 2\}, \ldots, \{n-1, n\}, \{1, n\}$.

In the case of the pair (D^2, S^1), the polyhedral products for n-gons have a very nice form. For example, the 3-gon is the simplicial complex $\partial\Delta^2$ and we have seen that $(D^2, S^1)^{\partial\Delta^2} \simeq S^5$. Also, the 4-gon is the join $\partial\Delta^1 * \partial\Delta^1$ and $(D^2, S^1)^{\partial\Delta^1 * \partial\Delta^1} \simeq S^3 \times S^3$ by Proposition 3.11. In particular, the n-gon for $n \leq 4$ is a join of boundaries of simplices and the associated polyhedral products are elliptic, as stated in Theorem 5.3. For $n \geq 5$, the n-gon is not a join of simplices and boundaries of simplices and so Proposition 5.3 tells us that $(D^2, S^1)^K$ is hyperbolic in these cases.

In this chapter we study a certain homotopy fibration over $(D^2, S^1)^K$, which exists when K is an n-gon, with the intention of understanding more about its rational homotopy groups. We show that $\pi_*((D^2, S^1)^K) \otimes \mathbb{Q}$ has a basis realized by an infinite set of iterated Samelson products.

It is worth noting that for $n \geq 4$ the n-gon is not a shifted complex, and consequently the corresponding polyhedral product for the pair (D^2, S^1) is not covered by the result of Grbić and Theriault.

Proposition 5.7. *For $n \geq 4$, the n-gon is not shifted.*

Proof. Let $n \geq 4$ and let K be the n-gon. Every vertex in K is connected to precisely two other vertices in K. Fix an ordering on the vertices. Now there are precisely two integers p, q with $1 < p < q \leq n$ such that $\{1, p\}$ and $\{1, q\}$ belong to K. Furthermore, since $n \neq 3$, there exists an integer s with $1 < s \leq n$ with $s \neq p, q$ such that $\{s, p\} \in K$. But $\{1, s\}$ is not a face in K, so K is not shifted. □

It is therefore not unreasonable to suspect that for $n \geq 5$, the polyhedral product $(D^2, S^1)^K$ despite being hyperbolic, is not in fact homotopy equivalent to a wedge of spheres. In fact, this is indeed the case. In 1979, Mcgavran showed that for $n \geq 4$, $(D^2, S^1)^K$ is homotopy equivalent to a connected sum of various products of spheres. We shall make use of this result in our calculations.

Theorem 5.8 (Mcgavran [20])**.** *Let K be the n-gon for $n \geq 4$. Then there is a homotopy equivalence*

$$\left(D^2, S^1\right)^K \simeq \#_{j=1}^{n-3}\left(\#_{j\binom{n-2}{j+1}}(S^{j+2} \times S^{n-j})\right).$$

5.1 $\left(D^2, S^1\right)^K$ when K is the 5-gon

We begin our work by considering the polyhedral product $(D^2, S^1)^K$ in the case that K is the 5-gon. By Theorem 5.8, $(D^2, S^1)^K$ is homotopy equivalent to the connected sum $\#_5(S^3 \times S^4)$. With the obvious cell structure, the 4-skeleton is given by the wedge $\bigvee_5(S^3 \vee S^4)$. Let i denote the inclusion $\bigvee_5(S^3 \vee S^4) \longrightarrow \#_5(S^3 \times S^4)$.

Our aim in this section is to show that there is a basis for the rational homotopy of $(D^2, S^1)^K$ realized by an infinite set of iterated Samelson products. The method of attack we adopt is to determine the homotopy type of the fibre $\mathcal{F}_5$ in the homotopy fibration

$$\mathcal{F}_5 \longrightarrow \bigvee_5 (S^3 \vee S^4) \overset{i}{\hookrightarrow} \#_5(S^3 \times S^4). \tag{5.1}$$

In particular, we prove that $\mathcal{F}_5$ is homotopy equivalent to a wedge sum of infinitely many spheres. Moreover, we prove that the homotopy fibration sequence splits after looping, and by considering the homotopy equivalence given by the Hilton-Milnor Theorem, we are able to conclude $\pi_*(\#_5(S^3 \times S^4)) \otimes \mathbb{Q}$ is isomorphic to an infinite dimensional vector sub space of $\pi_*(\bigvee_5(S^3 \vee S^4))$. In section 5.3 we generalize our results to n-gons where $n \geq 5$.

5.2 A simplified version of the problem

In order to keep the calculations tidy, we shall in fact study a simplified but analogous version of the problem outlined above, and show that the same techniques applied in the simplified version carry over to the main problem.

Instead of studying the fibre of the inclusion $\bigvee_5(S^3 \vee S^4) \xrightarrow{i} \#_5(S^3 \times S^4)$, we shall instead determine the homotopy fibre $\mathcal{F}_5^2$ in the homotopy fibration

$$\mathcal{F}_5^2 \longrightarrow \bigvee_2 (S^3 \vee S^4) \hookrightarrow \#_2(S^3 \times S^4).$$

Let $i_1, i_2 \colon S^3 \longrightarrow \bigvee_2(S^3 \vee S^4)$ be the inclusions of the first and second S^3 summand respectively, and let $j_1, j_2 \colon S^4 \longrightarrow \bigvee_2(S^3 \vee S^4)$ be the inclusions of the first and second S^4 summand. For the sake of brevity we write $2(S^3 \vee S^4)$ for the wedge sum $\bigvee_2(S^3 \vee S^4)$. Forming the Whitehead products $w_1 = [i_1, j_1] \colon S^6 \longrightarrow 2(S^3 \vee S^4)$, and $w_2 = [i_2, j_2] \colon S^6 \longrightarrow 2(S^3 \vee S^4)$ and taking their sum, we obtain a map

$$w\colon S^6 \xrightarrow{\psi} S^6 \vee S^6 \xrightarrow{w_1 \vee w_2} 2(S^3 \vee S^4) \vee 2(S^3 \vee S^4) \xrightarrow{\nabla} 2(S^3 \vee S^4),$$

where ψ is the standard co-H-space structure on S^6. The following proposition shows that the homotopy cofibre of w is the connected sum $\#_2(S^3 \times S^4.)$

Proposition 5.9. *In the following homotopy pushout*

$$\begin{array}{ccc} S^6 & \xrightarrow{w} & 2(S^3 \vee S^4) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q, \end{array} \tag{5.2}$$

Q is homotopy equivalent to the connected sum $(S^3 \times S^4)\#(S^3 \times S^4)$.

Proof. The Whitehead product $[i_1, j_1]$ is the attaching map of the top cell in $(S^3 \times S^4) \vee S^3 \vee S^4$. Likewise, the Whitehead product $[i_2, j_2]$ is the attaching map of the top cell in $S^3 \vee S^4 \vee (S^3 \times S^4)$. Denote the two hemispheres of S^6 by S^6_+ and S^6_-. The map w is by definition homtotopic to the composite

$$S^6 \xrightarrow{\psi} S^6 \vee S^6 \xrightarrow{[i_1,j_1] \vee [i_2,j_2]} (S^3 \vee S^4) \vee (S^3 \vee S^4)$$

where ψ is the standard coproduct on the co-H-space S^6. That is, ψ restricted to S^6_+ is homotopic to the composite $S^6_+ \xrightarrow{pinch} S^6 \hookrightarrow S^6 \vee S^6$ where the first map pinches the boundary of the hemisphere to a point, and the second map is the inclusion of the first wedge summand. Similarly, the restriction of ψ to S^6_- is homotopic to the composite $S^6_- \xrightarrow{pinch} S^6 \hookrightarrow S^6 \vee S^6$ where the first map again pinches the boundary of the hemisphere to a point and the second map this time is the inclusion of the second wedge summand.

Since the Whitehead product $[i_1, j_1]$ is the attaching map of the top cell in in

$(S^3 \times S^4) \vee S^3 \vee S^4$, there is a cofibration $S^6 \xrightarrow{[i_1,j_1]} 2(S^3 \vee S^4) \longrightarrow (S^3 \times S^4) \vee S^3 \vee S^4$. By restricting $[i_1, j_1]$ to the hemisphere S^6_+ we obtain the cofibration $S^6_+ \longrightarrow 2(S^3 \vee S^4) \longrightarrow ((S^3 \times S^4)\backslash D^7) \vee S^3 \vee S^4$. Similarly, the cofibre of the restriction of $[i_2, j_2]$ to the hemisphere S^6_- is homotopy equivalent to $S^3 \vee S^4 \vee ((S^3 \times S^4)\backslash D^7)$. Now, gluing S^6_+ and S^6_- back along the common boundary we see that the homotopy cofibre of w is homotopy equivalent to two copies of $(S^3 \times S^4)\backslash D^7$ identified along the common boundary. □

Notation 5.10. For the remainder of this chapter, let C denote the connected sum $(S^3 \times S^4)\#(S^3 \times S^4)$.

In the next section, we consider a certain map χ which we shall call the collapse map. Let X, Y be manifolds of the same dimension n. By definition, the connected sum of X, Y is the adjunction space obtained by removing the interior of a disc from each of X, Y and identifying $X\backslash D^n$ with $Y\backslash D^n$ along the common boundary sphere S^{n-1}. The quotient map which collapses this copy of S^{n-1} to a point defines a map $\chi\colon X\#Y \longrightarrow X \vee Y$.

Definition 5.11. We call the map $\chi\colon X\#Y \longrightarrow X \vee Y$ the collapse map.

For the sake of completeness, we include a direct proof that C is hyperbolic.

Proposition 5.12. *The connected sum C is a hyperbolic space.*

Proof. To see that C is hyperbolic, consider the map $\alpha\colon S^3 \vee S^3 \longrightarrow C$ defined as the composite $S^3 \vee S^3 \hookrightarrow 2(S^3 \vee S^4) \longrightarrow C$, and the map $\beta\colon C \longrightarrow S^3 \vee S^3$ defined by the composite $C \xrightarrow{\chi} (S^3 \times S^4) \vee (S^3 \times S^4) \xrightarrow{\pi \vee \pi} S^3 \vee S^3$, where χ is the collapse map and π is the projection $S^3 \times S^4 \longrightarrow S^3$. Clearly the composite

$$S^3 \vee S^3 \xrightarrow{\alpha} C \xrightarrow{\beta} S^3 \vee S^3$$

is the identity map, and thus α is a right homotopy inverse for β. Let G be the homotopy fibre of $\Omega\beta$. Then there is a homotopy decomposition $\Omega C \simeq G \times \Omega(S^3 \vee S^3)$, and therefore an isomorphism of rational homotopy groups

$$\pi_*(\Omega C) \otimes \mathbb{Q} \cong (\pi_*(G) \otimes \mathbb{Q}) \oplus \pi_*(\Omega(S^3 \vee S^3) \otimes \mathbb{Q}).$$

Since $\pi_*(\Omega(S^3 \vee S^3) \otimes \mathbb{Q})$ is a free Lie algebra on two generators, it follows that $\pi_*(\Omega C) \otimes \mathbb{Q}$ is infinite dimensional over $\mathbb{Q}$. That is, C is hyperbolic. $\square$

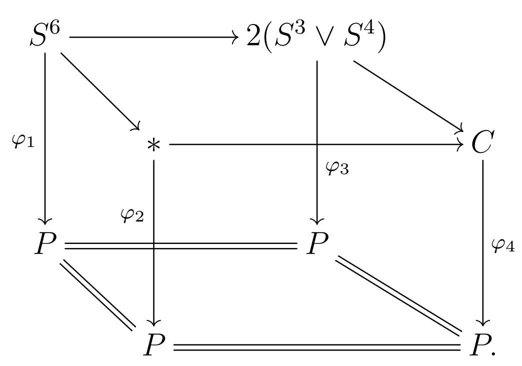
5.2.1 Construction of a certain cofibration

The collapse map $\chi\colon \#_2(S^3 \times S^4) \longrightarrow (S^3 \times S^4) \vee (S^3 \times S^4)$ induces maps from each corner of the homotopy pushout (5.2) into $(S^3 \times S^4) \vee (S^3 \times S^4)$, by composition.

Notation 5.13. For the remainder of this chapter, let P denote the space $(S^3 \times S^4) \vee (S^3 \times S^4)$, denote χ by φ_4, and let

$$S^6 \xrightarrow{\varphi_1} P, \qquad * \xrightarrow{\varphi_2} P, \qquad 2(S^3 \vee S^4) \xrightarrow{\varphi_3} P, \tag{5.3}$$

denote the maps induced by φ_4 and homotopy pushout (5.2), giving the homotopy commutative cube



Proposition 5.14. *There is a homotopy pushout*

$$\begin{array}{ccc} S^6 \times \Omega P & \xrightarrow{f} & N \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \Omega P & \longrightarrow & M \end{array} \tag{5.4}$$

where

1. *N is the homotopy fibre of φ_3,*
2. *M is the homotopy fibre of φ_4,*
3. *π is the projection map,*
4. *the restriction of f to ΩP is null-homotopic,*
5. *and f is homotopic to the composite*

$$S^6 \times \Omega P \xrightarrow{(f|_{S^6})\times 1} N \times \Omega P \xrightarrow{\quad\theta\quad} N$$

where $f|_{S^6}$ is the restriction of f to S^6 and θ is the homotopy action of ΩP on N, defined via the homotopy fibration $N \longrightarrow 2(S^3 \vee S^4) \xrightarrow{\varphi_3} P$.

Before proving Proposition 5.14 we need a lemma.

Lemma 5.15. *Given spaces A, B, C and D, the inclusion map $i\colon A \vee B \vee C \vee D \longrightarrow (A \times B) \vee (C \times D)$ has a right homotopy inverse after looping.*

Proof. Recall from Proposition 2.30 that for spaces X, Y there is a homotopy fibration $\Omega X * \Omega Y \longrightarrow X \vee Y \xrightarrow{j} X \times Y$, where j is the inclusion. Moreover, by Corollary 2.31, there is a right homotopy inverse $r\colon \Omega(X \times Y) \longrightarrow \Omega(X \vee Y)$ of the map Ωj, and hence a homotopy equivalence $\Omega(X\vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$.

Now for spaces A, B, C, D, there are homotopy equivalences

$$\Omega(A \vee B \vee C \vee D) \simeq \Omega(A \vee B) \times \Omega(C \vee D) \times \Omega\Sigma(\Omega(A \vee B) \wedge \Omega(C \vee D))$$

and

$$\Omega((A \times B) \vee (C \times D)) \simeq \Omega(A \times B) \times \Omega(C \times D) \times \Omega\Sigma(\Omega(A \times B) \wedge \Omega(C \times D)).$$

Furthermore, the map Ωi decomposes as the product of the three maps

$$\begin{aligned} \Omega i_1 \colon \Omega(A \vee B) &\longrightarrow \Omega(A \times B) \\ \Omega i_2 \colon \Omega(C \vee D) &\longrightarrow \Omega(C \times D) \\ \Omega\Sigma(i_1 \wedge i_2) \colon \Omega\Sigma(\Omega(A \vee B) \wedge \Omega(C \vee D)) &\longrightarrow \Omega\Sigma(\Omega(A \times B) \wedge \Omega(C \times D)). \end{aligned}$$

where i_1, i_2 are the inclusion maps. Clearly $\Omega i_1, \Omega i_2$ admit right homotopy inverses $r_1 \colon \Omega(A \times B) \longrightarrow \Omega(A \vee B)$ and $r_2 \colon \Omega(C \times D) \longrightarrow \Omega(C \vee D)$ respectively. To see that the third map admits a right homotopy inverse, first notice that the product of the composite maps

$$\Omega(A \times B) \xrightarrow{r_1} \Omega(A \vee B) \xrightarrow{\Omega i_1} \Omega(A \times B)$$
$$\Omega(C \times D) \xrightarrow{r_2} \Omega(C \vee D) \xrightarrow{\Omega i_2} \Omega(C \times D)$$

is homotopic to $(r_1 \circ \Omega i_1) \times (r_2 \circ \Omega i_2)$ which is the identity. Therefore there is an induced homotopy commutative diagram

$$\begin{array}{ccc} \Omega(A \times B) \wedge \Omega(C \times D) & \xrightarrow{r_1 \wedge r_2} & \Omega(A \vee B) \wedge \Omega(C \vee D) \\ \downarrow{\scriptstyle (r_1 \circ \Omega i_1) \wedge (r_2 \circ \Omega i_2)} & & \downarrow{\scriptstyle i_1 \wedge i_2} \\ \Omega(A \times B) \wedge \Omega(C \times D) & = & \Omega(A \times B) \wedge \Omega(C \times D) \end{array} \tag{5.5}$$

and thus $r_1 \wedge r_2$ is a right homotopy inverse of $i_1 \wedge i_2$.

Secondly, applying the functor $\Omega\Sigma$ to all maps in diagram (5.5), we see that $\Omega\Sigma(i_1 \wedge i_2) \circ \Omega\Sigma(r_1 \wedge r_2) \simeq \Omega\Sigma((i_1 \wedge i_2) \circ (r_1 \wedge r_2))$, which of course is the identity. Thus the product of these three right homotopy inverses $r_1 \times r_2 \times \Omega\Sigma(r_1 \wedge r_2)$ is a right homotopy inverse for Ωi. □

In particular, Lemma 5.15 shows that the map φ_3 has a right homotopy inverse after looping, a fact we shall use now as we go back and prove Proposition 5.14.

Proof of Proposition 5.14. Taking homotopy fibres of the maps $\varphi_1, \ldots, \varphi_4$, in (5.3), we obtain four homotopy fibrations:

$$F \longrightarrow S^6 \xrightarrow{\varphi_1} P, \tag{5.6}$$

$$G \longrightarrow * \xrightarrow{\varphi_2} P, \tag{5.7}$$

$$N \longrightarrow 2(S^3 \vee S^4) \xrightarrow{\varphi_3} P, \tag{5.8}$$

$$M \longrightarrow C \xrightarrow{\varphi_4} P. \tag{5.9}$$

The map φ_1 is by definition homotopic to the composite $S^6 \longrightarrow * \xrightarrow{\varphi_2} P$ and is therefore null-homotopic. So we can express φ_1 as the product $S^6 \times * \xrightarrow{* \times *} * \times P$ from which it follows that F is homotopy equivalent to the product of the individual homotopy fibres $S^6 \times \Omega P$. The homotopy fibration (5.7) is simply the path-loop fibration and thus G is homotopy equivalent to ΩP. As for the homotopy types of N and M, we leave these undetermined for now.

By mapping each corner of homotopy pushout (5.2) into P, we see from Mather's cube lemma, (Lemma 2.20), that there is a homotopy pushout of the fibres

$$\begin{array}{ccc} S^6 \times \Omega P & \xrightarrow{f} & N \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \Omega P & \longrightarrow & M \end{array}$$

for some maps π and f. This proves parts (1) - (2). Next we prove part (3).

Since φ_1 is homotopic to the composite $S^6 \longrightarrow * \xrightarrow{\varphi_2} P$, there is an induced morphism of homotopy fibrations

$$\begin{array}{ccccc} S^6 \times \Omega P & \longrightarrow & S^6 & \xrightarrow{\varphi_1} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \| \\ \Omega P & \longrightarrow & * & \xrightarrow[\varphi_2]{} & P \end{array}$$

which defines π. Since the map on the common base space P, is the identity, the left hand square is a homotopy pullback diagram. By example 2.2 π is homotopic to the projection map.

Now we prove part (4). Since φ_1 is homotopic to the composite $S^6 \xrightarrow{w} 2(S^3 \vee S^4) \xrightarrow{\varphi_3} P$, there is a morphism of homotopy fibration sequences

$$\begin{array}{ccccccc} \Omega P & \longrightarrow & S^6 \times \Omega P & \longrightarrow & S^6 & \xrightarrow{\varphi_1} & P \\ \| & & {\scriptstyle f}\downarrow & & {\scriptstyle w}\downarrow & & \| \\ \Omega P & \longrightarrow & N & \longrightarrow & 2(S^3 \vee S^4) & \xrightarrow[\varphi_3]{} & P. \end{array} \tag{5.10}$$

By Lemma 5.15, the map φ_3 has a right homotopy inverse after looping and so it follows that the map $\Omega P \longrightarrow N$ is null-homotopic. Furthermore, since φ_1 is null-homotopic, the map $\Omega P \longrightarrow S^6 \times \Omega P$ is homotopic to the inclusion of the second factor and so the square on the far left of the diagram shows precisely that the restriction of f to ΩP is null-homotopic.

Now we prove part (5). Consider again the morphism of homotopy fibrations in diagram (5.10). Associated to the homotopy fibration on the bottom row there is a homotopy action $\theta\colon N\times\Omega P \longrightarrow \Omega P$, and similarly, there is a homotopy action $\theta'\colon (S^6\times\Omega P)\times\Omega P \longrightarrow S^6\times\Omega P$ associated to the homotopy fibration in the top row. By the naturality property of homotopy actions there is a homotopy commutative diagram

$$\begin{array}{ccc} (S^6\times\Omega P)\times\Omega P & \xrightarrow{\theta'} & S^6\times\Omega P \\ \downarrow{\scriptstyle f\times 1} & & \downarrow{\scriptstyle f} \\ N\times\Omega P & \xrightarrow[\theta]{} & N. \end{array}$$

Since the homotopy fibration $S^6\times\Omega P \longrightarrow S^6\times * \longrightarrow *\times P$ on the top row of diagram (5.10) is a product of homotopy fibrations, the associated homotopy action θ' is homotopic to $\theta''\times\mu$ where $\theta''\colon S^6 \longrightarrow S^6$ is the homotopy action associated to the trivial fibration $S^6 \longrightarrow S^6 \longrightarrow *$ and $\mu\colon \Omega P\times\Omega P \longrightarrow \Omega P$ is the homotopy action associated to the path-loop fibration $\Omega P \longrightarrow * \longrightarrow P$. It follows directly from the definition of the homotopy action that θ'' is the identity map and μ is the loop multiplication.

Let i denote the map $S^6\times *\times\Omega P \xrightarrow{1\times *\times 1} S^6\times\Omega P\times\Omega P$. Since the restriction of μ to $*\times\Omega P$ is homotopic to the identity, then the composite

$$S^6\times\Omega P \xrightarrow{i} S^6\times\Omega P\times\Omega P \xrightarrow{\theta'} S^6\times\Omega P.$$

is homotopic to the identity. So we now see that

$$\begin{aligned} f &\simeq f\circ(\theta'\circ i) \\ &\simeq (\theta\circ(f\times 1))\circ i \\ &\simeq \theta\circ(f|_{S^6}\times 1) \end{aligned}$$

as required. □

The homotopy type of N

We will need to identify the map $N \longrightarrow M$ appearing in diagram (5.4). Our first step is to determine the homotopy type of N and show that it is homotopy equivalent to a wedge of infinitely many spheres $\bigvee_{\alpha\in I} S^{n_\alpha}$ by appealing to the Hilton-Milnor Theorem. We will need the following lemma which determines the homotopy type of the homotopy fibre of the pinch map $A \vee B \longrightarrow B$ which collapses the wedge summand A to a point.

Proposition 5.16. *The homotopy fibre of the pinch map $A \vee B \longrightarrow B$ is homotopy equivalent to $A \rtimes \Omega B$.*

Proof. See [5], Proposition 2.3. □

The homotopy type of N follows as a special case of the following more general proposition. For spaces $X_1, \ldots, X_n$, and a subset $\sigma = \{i_1, \ldots, i_k\}$ of $[n]$, let $X_{i_1 \ldots i_k}$ denote the smash product $X_{i_1} \wedge \ldots \wedge X_{i_k}$.

Proposition 5.17. *The homotopy fibre of the inclusion*

$$X_1 \vee X_2 \vee X_3 \vee X_4 \longrightarrow (X_1 \times X_2) \vee (X_3 \times X_4)$$

is homotopy equivalent to $\overline{G} \rtimes \Omega G$ where

a) $G \simeq \Sigma\Omega(X_1 \times X_2) \wedge \Omega(X_3 \times X_4)$,

b) and $\overline{G} \simeq \Sigma(\Omega X)_{12} \vee \Sigma(\Omega X)_{34} \vee \Sigma(\Omega X)_{123} \vee \Sigma(\Omega X)_{124} \vee \Sigma(\Omega X)_{134} \vee \Sigma(\Omega X)_{234} \vee 2\Sigma(\Omega X)_{1234}$.

Proof. Let H be the homotopy fibre of the inclusion $\bigvee_{i=1}^{4} X_i \longrightarrow (X_1 \times X_2) \vee$

$(X_3 \times X_4)$. Then there is a homotopy commutative diagram

$$
\begin{array}{ccccc}
H & \longrightarrow & F & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow \\
H & \longrightarrow & X_1 \vee X_2 \vee X_3 \vee X_4 & \longrightarrow & (X_1 \times X_2) \vee (X_3 \times X_4) \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & X_1 \times X_2 \times X_3 \times X_4 & = & X_1 \times X_2 \times X_3 \times X_4
\end{array}
\tag{5.11}
$$

where all rows and columns are homotopy fibrations. By Proposition 2.30 there is a homotopy equivalence $G \simeq \Sigma\Omega(X_1 \times X_2) \wedge \Omega(X_3 \times X_4)$. Then using the decomposition of a product after suspending $\Sigma A \times B \simeq \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$ in Proposition 2.21 we obtain a sequence of homotopy equivalences

$$
\begin{aligned}
G &\simeq \Sigma(\Omega X_1 \times \Omega X_2) \wedge (\Omega X_3 \times \Omega X_4) \\
&\simeq \Sigma(\Omega X_1 \vee \Omega X_2 \vee (\Omega X)_{12}) \wedge (\Omega X_3 \vee \Omega X_4 \vee (\Omega X)_{34}) \\
&\simeq \Sigma(\Omega X)_{13} \vee \Sigma(\Omega X)_{14} \vee \Sigma(\Omega X)_{23} \vee \Sigma(\Omega X)_{24} \\
&\quad \ldots \Sigma(\Omega X)_{123} \vee \Sigma(\Omega X)_{124} \vee \Sigma(\Omega X)_{134} \vee \Sigma(\Omega X)_{234} \\
&\quad \ldots \Sigma(\Omega X)_{1234}.
\end{aligned}
$$

In particular, G is the wedge sum of the domains of the following iterated universal Whitehead products:

1. $[\zeta_j, \zeta_k]$ for all pairs $1 \leq j < k \leq 4$ except $[\zeta_1, \zeta_2]$ and $[\zeta_3, \zeta_4]$. Call this collection of Whitehead products W_2.

2. $[\zeta_j, [\zeta_k, \zeta_l]]$ where $1 \leq j < k < l \leq 4$, and $[\zeta_k, \zeta_l]$ belongs to W_2. Call this collection L_3.

3. $[[\zeta_j, \zeta_k], \zeta_l]$ where $1 \leq j < k < l \leq 4$, and $[\zeta_j, \zeta_k]$ belongs to W_2. Call this collection R_3 and let $W_3 = L_3 \cup R_3$.

4. $[\zeta_j,[\zeta_k,[\zeta_l,\zeta_m]]]$ and $[\zeta_j,[[\zeta_k,\zeta_l],\zeta_m]]$ where $(j,k,l,m)=(1,2,3,4)$ and $[\zeta_k,[\zeta_l,\zeta_m]]$ and $[[\zeta_k,\zeta_l],\zeta_m]$ belong to W_3.

5. $[[\zeta_j,[\zeta_k,\zeta_l]],\zeta_m]$ and $[[[\zeta_j,\zeta_k],\zeta_l],\zeta_m]$ where $(j,k,l,m)=(1,2,3,4)$ and $[\zeta_j,[\zeta_k,\zeta_l]]$ and $[[\zeta_j,\zeta_k],\zeta_l]$ belong to W_3.

On the other hand, Porter's Theorem tells us that there is a homotopy equivalence

$$\begin{aligned} F &\simeq \Sigma(\Omega X)_{12}\vee\Sigma(\Omega X)_{13}\vee\Sigma(\Omega X)_{14}\vee\Sigma(\Omega X)_{23}\vee\Sigma(\Omega X)_{24}\vee\Sigma(\Omega X)_{34} \\ &\quad \ldots 2(\Sigma(\Omega X)_{123}\vee\Sigma(\Omega X)_{124}\vee\Sigma(\Omega X)_{134}\vee\Sigma(\Omega X)_{234}) \\ &\quad \ldots 3\Sigma(\Omega X)_{1234}. \end{aligned}$$

which is the wedge sum of the domains of the set of all the iterated universal Whitehead products as constructed in W together with those new Whitehead products obtained by adding $[\zeta_1,\zeta_2]$ and $[\zeta_3,\zeta_4]$ to the list W_2.

So G is the wedge sum of those iterated universal Whitehead products in F which are not generated by $[\zeta_1,\zeta_2]$ and $[\zeta_3,\zeta_4]$, namely $F\simeq G\vee\overline{G}$ where

$$\begin{aligned} \overline{G} &\simeq \Sigma(\Omega X)_{12}\vee\Sigma(\Omega X)_{34}\vee\Sigma(\Omega X)_{123}\vee\Sigma(\Omega X)_{124} \\ &\quad \ldots\vee\Sigma(\Omega X)_{134}\vee\Sigma(\Omega X)_{234}\vee 2\Sigma(\Omega X)_{1234} \end{aligned}$$

and the map $F\longrightarrow G$ in diagram 5.11 is homotopic to the pinch map $G\vee\overline{G}\longrightarrow G$. Thus by Proposition 5.16, there is a homotopy equivalence $H\simeq\overline{G}\rtimes\Omega G$. □

As an immediate corollary of proposition 5.17 we obtain the homotopy type of N

Corollary 5.18. *There is a homotopy equivalence*

$$N \simeq \overline{G} \rtimes \Omega(\Sigma\Omega(S^3 \times S^4) \wedge \Omega(S^3 \times S^4))$$

where $\overline{G} \simeq (\Sigma\Omega S^3 \wedge \Omega S^4) \vee (\Sigma\Omega S^3 \wedge \Omega S^4) \vee (\Sigma\Omega S^3 \wedge \Omega S^4 \wedge \Omega S^3) \vee (\Sigma\Omega S^3 \wedge \Omega S^4 \wedge \Omega S^4) \vee (\Sigma\Omega S^3 \wedge \Omega S^3 \wedge \Omega S^4) \vee (\Sigma\Omega S^4 \wedge \Omega S^3 \wedge \Omega S^4) \vee (2\Sigma\Omega S^3 \wedge \Omega S^4 \wedge \Omega S^3 \wedge \Omega S^4)$.

Proof. The result follows from Proposition 5.17 by putting $X_1 = X_3 = S^3$ and $X_2 = X_4 = S^4$. □

In order to further decompose N into a wedge of spheres, the following two lemmas will be useful.

Lemma 5.19. *Suppose that* $\overline{X} = \Sigma X$ *and* $\overline{Y} = \Sigma Y$ *are suspension spaces. Then* $\Sigma\Omega\overline{X} \wedge \Omega\overline{Y}$ *is homotopy equivalent to the wedge* $\bigvee_{j,k=1}^{\infty} \Sigma X^{\wedge j} \wedge Y^{\wedge k}$.

Proof. The result follows from iterated use of the James splitting. In particular, applying the James splitting there is a homotopy equivalence

$$\Sigma\Omega\Sigma X \wedge \Omega\Sigma Y \simeq (\Sigma \bigvee_{j=1}^{\infty} X^{\wedge j}) \wedge \Omega\Sigma Y.$$

Taking the suspension through the smash product, this is homotopy equivalent to $(\bigvee_{j=1}^{\infty} X^{\wedge j}) \wedge \Sigma\Omega\Sigma Y$ which by another application of the James splitting decomposes as $(\bigvee_{j=1}^{\infty} X^{\wedge j}) \wedge (\Sigma \bigvee_{k=1}^{\infty} Y^{\wedge k})$. Since the smash product distributes over the wedge, this gives the required result. □

Lemma 5.20. *Let* $n, m, k, l \geq 2$. *Then* $(S^n \vee S^m) \rtimes \Omega(S^k \vee S^l)$ *is homotopy equivalent to a wedge of spheres.*

Proof. Since $(S^n \vee S^m)$ is a suspension space, then by Proposition 2.32, there is a homotopy equivalence

$$\begin{aligned}(S^n \vee S^m) \rtimes \Omega(S^k \vee S^l) &\simeq S^n \vee S^m \vee ((S^n \vee S^m) \wedge \Omega(S^k \vee S^l)) \\ &\simeq S^n \vee S^m \vee (S^n \wedge \Omega(S^k \vee S^l)) \dots \\ &\quad \dots \vee (S^m \wedge \Omega(S^k \vee S^l))\end{aligned} \tag{5.12}$$

By the Hilton-Milnor Theorem $\Omega(S^k \vee S^l)$ is homotopy equivalent to the product $\prod_B \Omega(\Sigma(S^k)^{\wedge n_1} \wedge (S^l)^{\wedge n_2})$ indexed by a basis B for the free Lie algebra on 2 elements. Notice that $\Sigma(S^k)^{\wedge n_1} \wedge (S^l)^{\wedge n_2} \simeq S^{kn_1+ln_2+1}$. It follows that the wedge summand $S^n \wedge \Omega(S^k \vee S^l)$ in equation (5.12) is homotopy equivalent to $\Sigma^n \prod_B \Omega S^{kn_1+ln_2+1}$. Proposition 2.21 says that there is a decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma X \wedge Y$ for arbitrary spaces X, Y. Iterating this decomposition shows that $\Sigma^n \prod_B \Omega S^{kn_1+ln_2+1}$ is homotopy equivalent to a wedge of spaces of the form $\Sigma^n \Omega X_1 \wedge \dots \wedge \Omega X_t$ where each X_i is a sphere, and iterated use of the James splitting, as in Lemma 5.19 shows that each of these summands is a wedge of spheres.

The same argument shows that the summand $S^m \wedge \Omega(S^k \vee S^l)$ in equation (5.12) is also homotopy equivalent to a wedge of spheres, and this gives the desired result. □

We therefore obtain the following corollary.

Corollary 5.21. *N has the homotopy type of an infinite wedge of spheres.*

Proof. By Corollary 5.18, N is homotopy equivalent to

$$\overline{G} \rtimes \Omega(\Sigma\Omega(S^3 \times S^4) \wedge \Omega(S^3 \times S^4))$$

where $\overline{G}$ is a wedge sum of spaces of the form $\Sigma\Omega X_1 \wedge \dots \wedge \Omega X_k$, where each X_i

is a sphere. By Lemma 5.19 it follows that $\overline{G}$ decomposes as a wedge of spheres.

On the other hand, by Proposition 2.21, there is a homotopy decomposition

$$\begin{aligned}\Sigma\Omega(S_1^3 \times S_2^4) \wedge \Omega(S_3^3 \times S_4^4) &\simeq (\Sigma\Omega S_1^3 \vee \Sigma\Omega S_2^4 \vee \Sigma(\Omega S_1^3 \wedge \Omega S_2^4)) \\ &\quad \wedge(\Omega S_3^3 \vee \Omega S_4^4 \vee (\Omega S_3^3 \wedge \Omega S_4^4)).\end{aligned}$$

Since the smash product distributes over the wedge, this is homotopy equivalent to a wedge of spaces of the form $\Sigma\Omega X_1 \wedge \ldots \wedge \Omega X_k$ where each X_i is a sphere. By Lemma 5.19 each of these wedge summands is homotopy equivalent to a wedge of spheres.

It follows that $N \simeq \overline{G} \rtimes \Omega G$, where $\overline{G}$ and G are both wedges of spheres. By generalizing the argument in the proof of Lemma 5.20 we obtain the desired result that N is homotopy equivalent to a wedge of infinitely many spheres.

□

5.2.2 An extension of $f\colon S^6 \times \Omega P \longrightarrow N$

Suppose $f\colon X \times Y \longrightarrow Z$ is a map such that the restriction of f to Y is null-homotopic. Then since there is a homotopy cofibration $Y \hookrightarrow X \times Y \longrightarrow X \rtimes Y$, there is an extension of f to a map $\overline{f}\colon X \rtimes Y \longrightarrow Z$.

In particular, the map $f\colon S^6 \times \Omega P \longrightarrow N$ from Proposition 5.14 has an extension $\overline{f}\colon S^6 \rtimes \Omega P \longrightarrow N$.

Proposition 5.22. *The sequence*

$$S^6 \rtimes \Omega P \xrightarrow{\overline{f}} N \longrightarrow M.$$

is a homotopy cofibration.

Proof. Consider the homotopy pushout from Proposition 5.14

$$\begin{array}{ccc} S^6 \times \Omega P & \xrightarrow{f} & N \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \Omega P & \longrightarrow & M. \end{array}$$

Since π is the the projection, it has a right homotopy inverse, given by the inclusion $i\colon \Omega P \longrightarrow S^6 \times \Omega P$, we see that the map $\Omega P \longrightarrow M$ factors as $\Omega P \xrightarrow{f|_{\Omega P}} N \longrightarrow M$. But the restriction of f to ΩP is trivial by Proposition 5.14 and hence the map $\Omega P \longrightarrow M$ is null-homotopic. Furthermore, the extension $\overline{f}\colon S^6 \rtimes \Omega P \longrightarrow N$, of f, induces a monomorphism in homology. To see this, notice that since S^6 is a suspension, there is a splitting $S^6 \rtimes \Omega P \simeq S^6 \vee (S^6 \wedge \Omega P)$ by Proposition 2.32. It is enough to check that $\overline{f}$ induces a monomorphism on each wedge summand.

It follows from the homotopy decomposition of N found in Proposition 5.18 and Proposition 5.21 that N is a wedge of spheres of dimension at least 6, and in particular, N is 5-connected. It therefore follows from the Hurewicz Theorem, that the Hurewicz homomorphism gives an isomorphism $\pi_6(S^6) \longrightarrow \pi_6(N)$. In particular, the homomorphism induced in homology by the restriction of $\overline{f}$ to S^6, induces an isomorphism onto it's image.

For the second summand $S^6 \wedge \Omega P$, recall that in Proposition 5.14 we found that f is homotopic to the composite

$$S^6 \times \Omega P \xrightarrow{(f|_{S^6})\times 1} N \times \Omega P \xrightarrow{\ \theta\ } N$$

In particular, the restriction of $\overline{f}$ to $S^6 \wedge \Omega P$ is defined via the homotopy action

θ, associated to the fibration $N \longrightarrow 2(S^3 \vee S^4) \xrightarrow{\varphi_3} P$. That is,

$$\overline{f}|_{(S^6\wedge\Omega P)} \simeq S^6 \wedge \Omega P \xrightarrow{(f|_{S^6})\wedge 1} N \wedge \Omega P \xrightarrow{\theta} N.$$

The map $S^6 \wedge \Omega P \xrightarrow{(f|_{S^6})\wedge 1} N \wedge \Omega P$ induces a monomorphism in homology since $f|_{S^6}$ does, and therefore so too does $\overline{f}|_{(S^6\wedge\Omega P)}$.

Since $\overline{f}$ induces a monomorphism in homology we may therefore pinch out the factor ΩP in the homotopy pushout diagram to obtain a new homotopy pushout

$$\begin{array}{ccc} S^6 \rtimes \Omega P & \xrightarrow{\overline{f}} & N \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ * & \longrightarrow & M. \end{array}$$

In particular, $S^6 \rtimes \Omega P \xrightarrow{\overline{f}} N \longrightarrow M$ is a homotopy cofibration. □

5.2.3 The homotopy type of $\mathcal{F}_5^2$

We are now ready to return to our main aim for this section, the determination of the homotopy type of $\mathcal{F}_5^2$, and of a basis for $\pi_*(\Omega C) \otimes \mathbb{Q}$.

Proposition 5.23. *The homotopy fibre $\mathcal{F}_5^2$ is homotopy equivalent to $(S^6 \rtimes \Omega P) \rtimes \Omega M$.*

Proof. Recall that from the proof of Proposition 5.14, there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_5^2 & = & \mathcal{F}_5^2 & & \\ \downarrow & & \downarrow & & \\ N & \longrightarrow & 2(S^3 \vee S^4) & \xrightarrow{\varphi_3} & P \\ \downarrow & & \downarrow{\scriptstyle i} & & \| \\ M & \longrightarrow & C & \xrightarrow{\varphi_4} & P \end{array} \qquad (5.13)$$

in which all rows and columns are homotopy fibrations. In particular, $\mathcal{F}_5^2$ is homotopy equivalent to the homotopy fibre of the map $N \longrightarrow M$.

By Proposition 5.22 there is a homotopy cofibration $S^6 \rtimes \Omega P \xrightarrow{\overline{f}} N \longrightarrow M$, where $\overline{f}$ induces a monomorphism in homology. By Proposition 5.21, N is homotopy equivalent to a wedge of spheres, and by a similar argument, it is clear to see that so is $S^6 \rtimes \Omega P$. It follows that $\overline{f}$ is the inclusion of a wedge summand and consequently, there is a homotopy equivalence $N \simeq (S^6 \rtimes \Omega P) \vee M$, and the map $N \longrightarrow M$ is homotopic to the pinch map $(S^6 \rtimes \Omega P) \vee M \longrightarrow M$.

Finally, Proposition 5.16 shows that there is a homotopy equivalence $\mathcal{F}_5^2 \simeq (S^6 \rtimes \Omega P) \rtimes \Omega M$. □

Corollary 5.24. *$\mathcal{F}_5^2$ is homotopy equivalent to a wedge of infinitely many spheres.*

Proof. This follows from Proposition Lemma 5.20 since $S^6 \rtimes \Omega P$ and M are both homotopy equivalent to a wedge of infinitely many spheres. □

We are now ready to state our main theorem of this section.

Theorem 5.25. *The rational homotopy groups $\pi_*(\Omega C) \otimes \mathbb{Q}$ are generated by an infinite set of Samelson products.*

Proof. By loooping all spaces and maps in diagram 5.13 there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega\mathcal{F}_5^2 & = & \Omega\mathcal{F}_5^2 & & \\ \downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle \Omega j} & & \\ \Omega N & \xrightarrow{\Omega g} & \Omega 2(S^3 \vee S^4) & \xrightarrow{\Omega\varphi_3} & \Omega P \\ \downarrow & & \downarrow & & \| \\ \Omega M & \longrightarrow & \Omega C & \xrightarrow[\Omega\varphi_4]{} & \Omega P. \end{array} \tag{5.14}$$

in which all rows and columns are homotopy fibrations. Since $\Omega\varphi_3$ admits a right homotopy inverse, it follows by Proposition 2.18, that Ωg admits a left homotopy inverse g'. Similarly, the map $\Omega M \longrightarrow \Omega N$ has a right homotopy inverse given by looping the inclusion of the wedge summand M into N. Hence Ωf has a left homotopy inverse f'. It follows that the composite $\Omega 2(S^3 \vee S^4) \xrightarrow{f'} \Omega N \xrightarrow{g'} \Omega \mathcal{F}_5^2$ is a left homotopy inverse for Ωj and consequently, Proposition 2.18 implies there are homotopy equivalences

$$\Omega 2(S^3 \vee S^4) \simeq \Omega \mathcal{F}_5^2 \times \Omega C \tag{5.15}$$

$$\Omega 2(S^3 \vee S^4) \simeq \Omega N \times \Omega P \tag{5.16}$$

$$\Omega N \simeq \Omega \mathcal{F}_5^2 \times \Omega M \tag{5.17}$$

Let $S(n_1, n_2, n_3, n_4)$ denote the smash product $(S^3)^{n_1} \wedge (S^4)^{\wedge n_2} \wedge (S^3)^{\wedge n_3} \wedge (S^4)^{\wedge n_4}$. Then by the Hilton-Milnor Theorem, $\Omega 2(S^3 \vee S^4) \simeq \prod_{b\in B} \Omega\Sigma S(n_1, n_2, n_3, n_4)$ where B is a basis for the free Lie algebra on 4 elements, and $\pi_*(\Omega 2(S^3 \vee S^4)) \otimes \mathbb{Q}$ is generated by the set of iterated Samelson products

$$S(n_1, n_2, n_3, n_4) \xrightarrow{E} \Omega\Sigma S(n_1, n_2, n_3, n_4) \xrightarrow{\Omega w_b} \Omega 2(S^3 \vee S^4)$$

where $E\colon X \longrightarrow \Omega\Sigma X$ is the adjoint of the identity on ΣX and w_b is the Whitehead product indexed by b.

Combining decompositions (5.16) and (5.17), there is a homotopy equivalence

$$\Omega(2(S^3 \vee S^4)) \simeq \Omega \mathcal{F}_5^2 \times \Omega M \times \Omega P.$$

It is clear that under the Hilton-Milnor equivalence, ΩM and ΩP correspond to subproducts $\prod_{b\in B'\subset B}\Omega\Sigma S(n_1,n_2,n_3,n_4)$ and $\prod_{b\in B''\subset B}\Omega\Sigma S(n_1,n_2,n_3,n_4)$ respectively, for disjoint B' and B''. It follows that $\Omega\mathcal{F}_5^2$ corresponds to the subproduct

$$\prod_{b\in\overline{B}\subset B}\Omega\Sigma S(n_1,n_2,n_3,n_4)$$

where $\overline{B}=B\backslash(B'\cup B'')$. Finally, decomposition (5.15) confirms that ΩC corresponds under the Hilton-Milnor equivalence to the subproduct $\prod_{b\in B\backslash\overline{B}}\Omega\Sigma S(n_1,n_2,n_3,n_4)$ and so $\pi_*(\Omega C)\otimes\mathbb{Q}$ is generated by those Samelson products in $\pi_*(\Omega 2(S^3\vee S^4))\otimes\mathbb{Q}$ which do not belong to $\pi_*(\Omega\mathcal{F}_5^2)\otimes\mathbb{Q}$. $\square$

5.3 Generalization to n-gons for $n\geq 5$

In this section we show how to generalize our result, proved in Theorem 5.25, which shows that the rational homotopy groups of the looped connected sum ΩC has an infinite basis given by iterated Samelson products, to $(D^2,S^1)^K$ where K is an n-gon and $n\geq 5$.

Fix $n\geq 5$ and let K be the n-gon. Recall that by Theorem 5.8 there is a homotopy equivalence

$$\left(D^2,S^1\right)^K\simeq \#_{j=1}^{n-3}\#_{\binom{n-2}{j+1}}(S^{j+2}\times S^{n-j}). \tag{5.18}$$

Denote this connected sum by C and let W be the wedge sum $\bigvee_{j=1}^{n-3}\binom{n-2}{j+1}(S^{j+2}\vee S^{n-j})$. Let $\mathcal{F}_n$ denote the homotopy fibre of the inclusion $W\longrightarrow C$. We claim that all the results of Section 5.1 have analogues in the current setting.

To begin with, for fixed j, set an ordering of the $\binom{n-2}{j+1}$ copies of $S^{j+2}\vee S^{n-j}$. Then for $j<j'$ set $(S^{j+2}\vee S^{n-j})<(S^{j'+2}\vee S^{n-j'})$. This defines an ordering of the $\Sigma_{j=1}^{n-3}\binom{n-2}{j+1}$ wedge sums of pairs of spheres appearing in W.

For $k = 1, \ldots, \Sigma_{j=1}^{n-3}\binom{n-2}{j+1}$ let

$$i_k \colon S^{j+2} \hookrightarrow (S^{j+2} \vee S^{n-j})_k \hookrightarrow W$$
$$j_k \colon S^{n-j} \hookrightarrow (S^{j+2} \vee S^{n-j})_k \hookrightarrow W$$

be the the inclusion maps into W and let w be the sum of the Whitehead products

$$\left(\Sigma_{k=1}^{\Sigma_{j=1}^{n-3}\binom{n-2}{j+1}}[i_k, j_k]\right) : S^{n+1} \longrightarrow W.$$

This map is the attaching map of the top cell in C and so there is a homotopy pushout diagram

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{w} & W \\ \downarrow & & \downarrow \\ * & \longrightarrow & C. \end{array} \tag{5.19}$$

Now let $P = \bigvee_{j=1}^{n-3}\binom{n-2}{j+1}(S^{j+2} \times S^{n-j})$. The collapse map $\chi \colon C \longrightarrow P$ induces maps from each corner of the homotopy pushout (5.19) into P, and so Mather's Cube Lemma yields a homotopy pushout of homotopy fibres

$$\begin{array}{ccc} S^{n+1} \times \Omega P & \xrightarrow{f} & N \\ \pi\downarrow & & \downarrow \\ \Omega P & \longrightarrow & M. \end{array}$$

By analogy with the results of Section 5.2, we may there is a homotopy cofibration $S^{n+1} \rtimes \Omega P \xrightarrow{\bar{f}} N \longrightarrow M$ where $N \simeq (S^{n+1} \rtimes \Omega P) \vee M$ is an infinite wedge of spheres, M is the wedge of an infinite subcollection of the spheres in N, and the map $N \longrightarrow M$ is homotopic to the pinch map. Therefore $\mathcal{F}_n$ is homotopy equivalent to $(S^{n+1} \rtimes \Omega P) \longrightarrow \Omega M$, and by analogy with Theorem 5.25 we are able to state our main result for this chapter:

Theorem 5.26. *If K is an n-gon with $n \geq 5$, then $(D^2, S^1)^K$ is hyperbolic and $\pi_*(\Omega\,(D^2, S^1)^K) \otimes \mathbb{Q}$ is generated by an infinite set of iterated Samelson products.*

Chapter 6

Homotopy exponents for $(\mathrm{Cone}\,\Omega P^m(p^r), \Omega P^m(p^r))^K$

Throughout this chapter, let P denote a Moore space $P^m(p^r)$, where p is an odd prime and $m \geq 3$. As we saw in Chapter 3, Cohen, Moore and Neisendorfer found that $\exp_p(P) = p^{r+1}$. In this chapter we study the homotopy exponent of the polyhedral product $(\mathrm{Cone}\,\Omega P, \Omega P)^K$, and show that when K is an n-gon, the value of the exponent is also p^{r+1}. For general K, we do not definitively obtain the value of the exponent, but we believe it is also p^{r+1}. In fact we show that under the assumption that Barratt's Conjecture is true, then this is indeed the case.

Note that by Corollary 3.14 there is a homotopy decomposition

$$\Omega\,(P,*)^K \simeq \Omega\,(\mathrm{Cone}\,\Omega P, \Omega P)^K \times \prod_{i=1}^{n} \Omega P_i.$$

and so our results are also valid for $(P,*)^K$ by Proposition 4.6.

Before getting to our main results, we first need to review some work of Félix and Tanré [11].

6.1 Review of the work of Félix and Tanré

In [11], Félix and Tanré studied the rational homotopy theory of polyhedral products. One of the constructions arising in their work is the existence of a particular sequence of fibrations, for any polyhedral product $(\underline{X}, \underline{A})^K$, which can be viewed as a decomposition of the fibration $F \longrightarrow (\underline{X}, \underline{A})^K \longrightarrow \prod_{i=1}^n X_i$.

Theorem 6.1 ([11], Theorem 2)**.** *Let K be a simplicial complex on n vertices. For $i = 1, \ldots, n$, suppose that A_i is a subspace of X_i and define*

$$\begin{aligned} K_i &= res_K(i) * \{i+1, \ldots, n\}, \\ Y_i &= (\underline{X}, \underline{A})^{link_{res_K(i)}(\{i\})}, \\ Z_i &= (\underline{X}, \underline{A})^{res_K(i-1)}. \end{aligned}$$

Let F_i' denote the homotopy fibre of $Y_i \longrightarrow Z_i$ and F_i'' denote the homotopy fibre of the inclusion $A_i \longrightarrow X_i$. Then there is a sequence of homotopy fibrations

$$\begin{array}{ccccc} F_2 & \longrightarrow & (\underline{X}, \underline{A})^{K_2} & \longrightarrow & \prod_{i=1}^n X_i, \\ F_3 & \longrightarrow & (\underline{X}, \underline{A})^{K_3} & \longrightarrow & (\underline{X}, \underline{A})^{K_2} \\ & & \vdots & & \\ F_n & \longrightarrow & (\underline{X}, \underline{A})^{K} & \longrightarrow & (\underline{X}, \underline{A})^{K_{n-1}}. \end{array}$$

where F_i is homotopy equivalent to $F_i' * F_i''$.

Proof. Any simplicial complex can be expressed as a homotopy pushout

$$\begin{array}{ccc} \mathrm{link}_K(\{n\}) & \longrightarrow & \mathrm{star}_K(\{n\}) \\ \downarrow & & \downarrow \\ \mathrm{res}_K(n-1) & \longrightarrow & K. \end{array}$$

Notice that $\mathrm{star}_K(\{n\})$ is the join $\mathrm{link}_K(\{n\}) * \{n\}$. Since $(\underline{X}, \underline{A})^{K*L} = (\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^L$, then $(\underline{X}, \underline{A})^{\mathrm{star}_K(\{n\})} = (\underline{X}, \underline{A})^{\mathrm{link}_K(\{n\})} \times X_n$.

By considering $\mathrm{link}_K(\{n\})$ as a simplicial complex on $\{1, \ldots, n-1\}$ it can be expressed as a join $\mathrm{link}_K(\{n\}) * \emptyset$, and similarly $\mathrm{res}_K(n-1)$ can be expressed as $\mathrm{res}_K(n-1) * \emptyset$. Thus there is an induced homotopy pushout

$$\begin{array}{ccc} Y_n \times A_n & \longrightarrow & Y_n \times X_n \\ \downarrow & & \downarrow \\ Z_n \times A_n & \longrightarrow & (\underline{X}, \underline{A})^K . \end{array} \tag{6.1}$$

where $Y_n = (\underline{X}, \underline{A})^{\mathrm{link}_K(\{n\})}$ and $Z_n = (\underline{X}, \underline{A})^{\mathrm{res}_K(n-1)}$.

Now set $K_{n-1} = \mathrm{res}_K(n-1) * \{n\}$ so that $(\underline{X}, \underline{A})^{K_{n-1}} = Z_n \times X_n$. The inclusion of K in K_{n-1} induces a map $(\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})^{K_{n-1}}$. Denote the homotopy fibre of this map by F_n. By mapping the four corners of the homotopy pushout in (6.1) into $Z_n \times X_n$ and taking homotopy fibres, we obtain by Mather's Cube Lemma a homotopy pushout

$$\begin{array}{ccc} F'_n \times F''_n & \xrightarrow{\pi_1} & F'_n \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ F''_n & \longrightarrow & F_n. \end{array}$$

where F'_n is the homotopy fibre of the map $Y_n \longrightarrow Z_n$ and F''_n is the homotopy fibre of the inclusion $A_n \longrightarrow X_n$, and the maps π_1, π_2 are the projection maps. It follows by Proposition 2.29 that F_n is homotopy equivalent to the join $F'_n * F''_n$.

For the inductive step, let $K_i = \mathrm{res}_K(i) * \{i+1, \ldots, n\}$. Then

$$\begin{aligned} (\underline{X}, \underline{A})^{K_{i+1}} &= (\underline{X}, \underline{A})^{\mathrm{res}_K(i+1)} \times X_{i+2} \times \ldots \times X_n, \\ (\underline{X}, \underline{A})^{K_i} &= (\underline{X}, \underline{A})^{\mathrm{res}_K(i)} \times X_{i+1} \times X_{i+2} \ldots \times X_n, \end{aligned}$$

and the homotopy fibre F_{i+1} of the map $(\underline{X}, \underline{A})^{K_{i+1}} \longrightarrow (\underline{X}, \underline{A})^{K_i}$ is homotopy equivalent to the homotopy fibre of

$$(\underline{X}, \underline{A})^{\mathrm{res}_K(i+1)} \longrightarrow (\underline{X}, \underline{A})^{\mathrm{res}_K(i) \times X_{i+1}} \tag{6.2}$$

But since $\mathrm{res}_K(i)$ can be expressed as the restriction of $\mathrm{res}_K(i+1)$ to the first i vertices, then the same argument as before shows that F_{i+1} is homotopy equivalent to the join $F'_{i+1} * F''_{i+1}$ where F'_{i+1} is the homotopy fibre of $Y_{i+1} \longrightarrow Z_{i+1}$ and F''_{i+1} is the homotopy fibre of $A_{i+1} \longrightarrow X_{i+1}$. □

One of the main results of [11] shows that for most choices of X and K, the polyhedral product $(\mathrm{Cone}\,\Omega X, \Omega X)^K$ is hyperbolic.

Proposition 6.2 ([11], Corollary 2)**.** *Suppose X is a CW-complex, and is nilpotent of finite type, and let K be a simplicial complex on n vertices. If*

1. *K is not the full simplex Δ^{n-1} and*
2. *$\tilde{H}^*(X; \mathbb{Q}) \neq 0$ and*
3. *and if the rational cohomology algebra $H^*(X; \mathbb{Q})$ is not a polynomial algebra $\mathbb{Q}[\alpha]$ on one generator α of even degree,*

then $(\mathrm{Cone}\,\Omega X, \Omega X)^K$ is hyperbolic.

We omit the defintion of nilpotent space here, but we note that the class of nilpotent spaces of finite type includes simply-connected spaces of finite type.

Notice that in the case $X = BT$ we have $(\mathrm{Cone}\,\Omega X, \Omega X)^K = (D^2, S^1)^K$, and since $H^*(BT; \mathbb{Q}) \cong \mathbb{Q}[\alpha]$ where α is of degree 2, then Proposition 6.2 is not applicable and thus does not contradict Debongnie's result stated in Theorem 5.3. Another family of polyhedral products which is not covered by the hypotheses of Proposition 6.2 is that of $(\mathrm{Cone}\,\Omega P, \Omega P)^K$, since $\tilde{H}^*(P; \mathbb{Q}) = 0$. In the next section, we study the homotopy exponents of precisely this family.

6.2 Homotopy exponents for n-gons

First of all we consider the special case where K is an n-gon and we determine the value of the p-primary homotopy exponent for $(\text{Cone}\,\Omega P^m(p^r), \Omega P^m(p^r))^K$. We see that the value of the exponent is independent of both n and m, and is in fact equal to p^{r+1}. For $n = 3, 4$, the result is straightforward:

Proposition 6.3. *Let K be the 3-gon. Then* $\exp_p((\text{Cone}\,\Omega P, \Omega P)^K) \leq p^{r+1}$.

Proof. By equation (3.2), there is a homotopy fibration

$$(\text{Cone}\,\Omega P, \Omega P)^K \longrightarrow (P, *)^K \longrightarrow \prod_{i=1}^{3} P_i,$$

and since $K = \partial\Delta^2$, we have that $(P, *)^K = T_1^3$ in the notation of equation (2.9). Thus Porter's Theorem shows that $(\text{Cone}\,\Omega P, \Omega P)^K \simeq \Sigma^2 \Omega P_1 \wedge \Omega P_2 \wedge \Omega P_3$. By appealing to the James' splitting together with Lemma 4.14, this is homotopy equivalent to a wedge of mod p^r Moore spaces and hence $\exp_p((\text{Cone}\,\Omega P, \Omega P)^K) \leq p^{r+1}$ by Proposition 4.15. □

Proposition 6.4. *Let K be the 4-gon. Then* $\exp_p((\text{Cone}\,\Omega P, \Omega P)^K) \leq p^{r+1}$.

Proof. By Corollary 3.14 there is a homotopy decomposition

$$\Omega\,(P, *)^K \simeq \Omega\,(\text{Cone}\,\Omega P, \Omega P)^K \times \prod_{i=1}^{4} \Omega P_i.$$

Since K is the join $\partial\Delta^1 * \partial\Delta^1$, there is a homeomorphism $(P, *)^K \simeq (P_1 \vee P_2) \times (P_3 \vee P_4)$ by Proposition 3.11. Thus $\exp_p((P, *)^K)$ is equal to

$$\max\{\exp_p(P_1 \vee P_2), \exp_p(P_3 \vee P_4)\} = p^{r+1}.$$

□

The 5-gon is the first non-trivial case, which we consider in Section 6.2.1. We then consider the case of $n \geq 5$ in Section 6.2.2.

6.2.1 The 5-gon

To keep our proofs tidy we shall initially work in the more general setting, working with $(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^K$ for an arbitrary choice of spaces X_i. Once we have laid some necessary groundwork we shall then specialize to the case $X_i = P$ for all i in order to obtain our homotopy exponent results for $(\mathrm{Cone}\,\Omega P, \Omega P)^K$.

First of all we consider the 5-gon and subsequently generalize our results to all n-gons for $n \geq 5$. Recall that by Theorem 6.1 there is a sequence of four homotopy fibrations:

$$\begin{array}{ccccc} F_2 & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_2} & \longrightarrow & \prod_{i=1}^5 \mathrm{Cone}\,\Omega X_i, \\ F_3 & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_3} & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_2} \\ F_4 & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_4} & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_3} \\ F_5 & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K} & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{K_4}. \end{array} \tag{6.3}$$

where

$$\begin{aligned} K_2 &= \mathrm{res}_K(2) * \{3,4,5\} = \{1,2\} * \{3,4,5\} \\ K_3 &= \mathrm{res}_K(3) * \{4,5\} = \{\{1,2\},\{2,3\}\} * \{4,5\} \\ K_4 &= \mathrm{res}_K(4) * \{5\} = \{\{1,2\},\{2,3\},\{3,4\}\} * \{5\} \\ K_5 &= \mathrm{res}_K(5) = K \end{aligned}$$

and $F_i \simeq F_i' * F_i''$ for each i. In this case F_i'' is the homotopy fibre of the inclusion $\Omega X_i \longrightarrow \mathrm{Cone}\,\Omega X_i$ and therefore F_i is homotopy equivalent to $F_i' * \Omega X_i$. We wish to determine the homotopy type of each of the F_i, and as we have just reasoned, to do so, it is enough to calculate the homotopy type of the F_i'.

Proposition 6.5. *The homotopy fibre F_2' is contractible.*

Proof. Recall that F_i' is the homotopy fibre of the inclusion map $Y_i \longrightarrow Z_i$ where $Y_i = (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{\mathrm{link}_{\mathrm{res}_K(i)}(\{i\})}$ and $Z_i = (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{\mathrm{res}_K(i-1)}$. When $i = 2$ we have $\mathrm{link}_{\mathrm{res}_K(2)}(\{2\}) = \{1\} = \mathrm{res}_K(1)$ and hence the map $Y_2 \longrightarrow Z_2$ is the identity map on $\mathrm{Cone}\,\Omega X_1$. Therfore F_2' is contractible. □

Proposition 6.6. *There is a homotopy equivalece $F_3' \simeq \Omega X_1$.*

Proof. We have $\mathrm{res}_K(3) = \{\{1,2\},\{2,3\}\}$ and thus $\mathrm{link}_{\mathrm{res}_K(3)}(\{3\})$ is the simplicial complex with a single 0-simplex $\{2\}$. Taking into account the ghost vertex $\{1\}$ we see that $Y_i \simeq \Omega X_1 \times \mathrm{Cone}\,\Omega X_2$. On the other hand, $\mathrm{res}_K(2)$ is the full 1-simplex $\{1,2\}$ and so $Z_3 = \mathrm{Cone}\,\Omega X_1 \times \mathrm{Cone}\,\Omega X_2$. Thus the map $Y_3 \longrightarrow Z_3$ is homotopic to the trivial map $\Omega X_1 \longrightarrow *$ and hence $F_3' \simeq \Omega X_1$. □

For the calculation of the homotopy type of F_4' and F_5', the following Lemma will be needed.

Lemma 6.7. *The homotopy fibre of the inclusion map $A \hookrightarrow A \vee B$ is homotopy equivalent to $\Omega B \times \Omega(\Sigma\Omega A \wedge \Omega B)$.*

Proof. Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
G & \longrightarrow & \Omega B & \xrightarrow{j} & \Sigma\Omega A \wedge \Omega B \\
\| & & \downarrow & & \downarrow \\
G & \longrightarrow & A & \longrightarrow & A \vee B \\
 & & \downarrow & & \downarrow \\
 & & A \times B & = & A \times B
\end{array}$$

in which all rows and columns are homotopy fibrations. The map j is null-homotopic by Proposition 2.29, and hence $G \simeq \Omega B \times \Omega(\Sigma\Omega A \wedge \Omega B)$. □

Now we may calculate the homotopy type of F'_4.

Proposition 6.8. *There is a homotopy equivalence*

$$F'_4 \simeq \Omega X_1 \times \Omega X_2 \times \Omega(\Omega X_1 * \Omega X_3).$$

Proof. Let $L = \mathrm{link}_{\mathrm{res}_K(4)}(\{4\})$ and $R = \mathrm{res}_K(3)$. Then L is the simplicial complex with a single vertex $\{3\}$ and ghost vertices $\{1\}, \{2\}$, and R is the simplicial complex $\{\{1\}, \{3\}\} * \{2\}$. Let L' be the simplicial complex $\{3\}$ obtained from L by forgetting the ghost vertices. Then $(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^L = (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{L'} \times \Omega X_1 \times \Omega X_2$ and $(\underline{X}, *)^L = (\underline{X}, *)^{L'} \times * \times *$.

There is a homotopy commutative diagram

$$\begin{array}{ccccc}
F'_4 & \longrightarrow & (\Omega X_1 \times \Omega X_2) \times (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{L'} & \longrightarrow & (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{R} \\
\| & & \downarrow & & \downarrow \\
F'_4 & \longrightarrow & (\underline{X}, *)^{L'} & \xrightarrow{\quad f \quad} & (\underline{X}, *)^{R} \\
 & & \downarrow & & \downarrow \\
 & & \prod_{i=1}^{3} X_i & = & \prod_{i=1}^{3} X_i
\end{array}$$

in which all rows and columns are homotopy fibrations. In particular, F'_4 is homotopy equivalent to the map $f\colon (\underline{X}, *)^{L'} \longrightarrow (\underline{X}, *)^{R}$ induced by the inclusion of L' into R. But f is homotopic to the inclusion map $X_3 \hookrightarrow (X_1 \vee X_3) \times X_2$. And hence $F'_4 \simeq G \times \Omega X_2$ where G is the homotopy fibre of $X_1 \longrightarrow (X_1 \vee X_3)$. By Lemma 6.7, G is homotopy equivalent to $\Omega X_1 \times \Omega(\Sigma\Omega X_1 \wedge \Omega X_3)$. ◻

Before calculating the homotopy type of F'_5 we state a further lemma, proven by Grbić and Theriault in [14], which is very useful for calculating the homotopy type of $(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^K$ where K is obtained by gluing two sub-complexes along a common face.

Lemma 6.9 ([14], Theorem 10.2)**.** *Let K be a simplicial complex on n vertices and suppose there are sub-complexes K_1, K_2 and a face $\sigma \in K$ such that $K = K_1 \cup_\sigma K_2$. List the vertices of K_1 as $\{1, \ldots, l, \ldots, m\}$ and the vertices of K_2 as $\{l+1, \ldots, m, \ldots, n\}$, with $\{l+1, \ldots, m\}$ being the vertices of the common face σ. Let*

a) $M = \prod_{i=1}^{l} \Omega X_i$, and $N = \prod_{i=m+1}^{n} \Omega X_i$,

b) $A_1 = (\mathrm{Cone}\,\Omega \underline{X}, \Omega \underline{X})^{K_1}$, and $A_2 = (\mathrm{Cone}\,\Omega \underline{X}, \Omega \underline{X})^{K_2}$.

Then there is a homotopy equivalence

$$(\mathrm{Cone}\,\Omega X, \Omega X)^K \simeq (\Sigma M \wedge N) \vee (M \ltimes A_2) \vee (A_1 \rtimes N).$$

Proof. See [14]. □

Now we calculate the homotopy type of F_5'.

Proposition 6.10. *There is a homotopy equivalence*

$$F_5' \simeq (\Omega X_2 \times \Omega X_3) \times \Omega B \times \Omega(\Sigma \Omega A \wedge \Omega B)$$

where $A = \Sigma \Omega X_1 \wedge \Omega X_4$ and

$$B = (\Sigma \Omega X_1 \wedge \Omega X_3) \vee (\Sigma \Omega X_2 \wedge \Omega X_4) \vee (\Sigma \Omega X_1 \wedge \Omega X_2 \wedge \Omega X_4) \vee (\Sigma \Omega X_1 \wedge \Omega X_3 \wedge \Omega X_4).$$

Proof. Let $L = \mathrm{link}_{\mathrm{res}_K(5)}(\{5\})$ and $R = \mathrm{res}_K(4)$. Then L is the simplicial complex $\{\{1\},\{4\}\}$ and ghost vertices $\{2\},\{3\}$, and R is the simplicial complex $\{\{1,2\},\{2,3\},\{3,4\}\}$.

Let L' be the simplicial complex $\{\{1\},\{4\}\}$ obtained from L by forgetting the ghost vertices. Then $(\mathrm{Cone}\,\Omega \underline{X}, \Omega \underline{X})^L = (\mathrm{Cone}\,\Omega \underline{X}, \Omega \underline{X})^{L'} \times \Omega X_2 \times \Omega X_3$ and

$\left(\underline{X}, *\right)^L = \left(\underline{X}, *\right)^{L'} \times * \times *.$

Now, the homotopy commutative diagram

$$\begin{array}{ccccc}
F_5' & \longrightarrow & \left(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X}\right)^L \times (\Omega X_2 \times \Omega X_3) & \longrightarrow & \left(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X}\right)^R \\
\Vert & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} \\
F_5' & \longrightarrow & \left(\underline{X}, *\right)^L & \longrightarrow & \left(\underline{X}, *\right)^R \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} \\
 & & \prod_{i=1}^4 X_i & = & \prod_{i=1}^4 X_i
\end{array}$$

in which all rows and columns are homotopy fibrations shows that the homotopy fibre of the map $\left(\underline{X}, *\right)^L \longrightarrow \left(\underline{X}, *\right)^R$ induced by the inclusion $L \longrightarrow R$ is also homotopy equivalent to F_5'. Thus there is a second homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \Omega\left(\underline{X}, *\right)^R & \xrightarrow{\Omega i} & \prod_{i=1}^4 \Omega X_i \\
 & & \downarrow & & \downarrow \\
G & \xrightarrow{\psi} & F_5' & \longrightarrow & \Omega X_2 \times \Omega X_3 \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X}\right)^{L'} & \longrightarrow & (X, *)^{L'} & \longrightarrow & X_1 \times X_4 \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X}\right)^R & \longrightarrow & (X, *)^R & \xrightarrow{i} & \prod_{i=1}^4 X_i
\end{array} \qquad (6.4)$$

in which all rows and columns are homotopy fibrations. By Proposition 3.13, Ωi admits a right homotopy inverse $r\colon \prod_{i=1}^4 \Omega X_i \longrightarrow \Omega\left(\underline{X}, *\right)^R$. Let r' be the restriction of r to $\Omega X_2 \times \Omega X_3$ and let θ be the homotopy action associated to the middle vertical fibration in diagram (6.4). Then it follows that

$$(\Omega X_2 \times \Omega X_3) \times G \xrightarrow{r' \times \psi} \Omega\left(\underline{X}, *\right)^R \times F_5' \xrightarrow{\theta} F_5'$$

is a homotopy equivalence.

To determine the homotopy type of G, consider the homotopy fibration

$$G \longrightarrow (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{L'} \longrightarrow (\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{R}. \tag{6.5}$$

Clearly, $(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{L'}$ is homotopy equivalent to $\Sigma\Omega X_1 \wedge \Omega X_4$. Now since $R = R_1 \cup_\sigma R_2$ where R_1 is the simplicial complex $\{\{1,2\},\{2,3\}\}$, R_2 is the simplicial complex $\{3,4\}$ and $\sigma = \{3\}$, then by Lemma 6.9, there is a homotopy equivalence

$$(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{R} \simeq (\Sigma M \wedge N) \vee (M \ltimes A_2) \vee (A_1 \rtimes N).$$

where

$$\begin{aligned}
\Sigma M \wedge N &\simeq \Sigma(\Omega X_1 \times \Omega X_2) \wedge \Omega X_4 \\
&\simeq (\Sigma\Omega X_1 \vee \Sigma\Omega X_2 \vee \Sigma\Omega X_1 \wedge \Omega X_2) \wedge \Omega X_4 \\
&\simeq (\Sigma\Omega X_1 \wedge \Omega X_4) \vee (\Sigma\Omega X_2 \wedge \Omega X_4) \vee (\Sigma\Omega X_1 \wedge \Omega X_2 \wedge \Omega X_4),
\end{aligned}$$

$$\begin{aligned}
M \rtimes A_2 &\simeq (\Omega X_1 \times \Omega X_2) \rtimes (\mathrm{Cone}\,\Omega X_3 \times \mathrm{Cone}\,\Omega X_4) \\
&\simeq *,
\end{aligned}$$

$$\begin{aligned}
A_1 \rtimes N &\simeq (\Sigma\Omega X_1 \wedge \Omega X_3) \rtimes \Omega X_4 \\
&\simeq (\Sigma\Omega X_1 \wedge \Omega X_3) \vee (\Sigma\Omega X_1 \wedge \Omega X_3 \wedge \Omega X_4).
\end{aligned}$$

Thus the fibration in equation (6.5) can be written

$$G \longrightarrow A \hookrightarrow A \vee B \tag{6.6}$$

where $A = \Sigma\Omega X_1 \wedge \Omega X_4$ and $B = (\Sigma\Omega X_1 \wedge \Omega X_3) \vee (\Sigma\Omega X_2 \wedge \Omega X_4) \vee (\Sigma\Omega X_1 \wedge \Omega X_2 \wedge \Omega X_4) \vee (\Sigma\Omega X_1 \wedge \Omega X_3 \wedge \Omega X_4)$.

Finally, by Lemma 6.7 we see that G is homotopy equivalent to $\Omega B \times \Omega(\Sigma\Omega A \wedge \Omega B)$. □

The Exponent Bound

Before proceeding to calculate the p-primary homotopy exponent for the polyhedral product $(\mathrm{Cone}\,\Omega P, \Omega P)^K$ in the case that K is the 5-gon, we first need a couple of lemmas.

Lemma 6.11. *Let K be a simplicial complex on n vertices, and let K_i be defined as in Proposition 6.1, and suppose that $(\underline{X}, \underline{A})$ is a sequence of pairs such that X_i is a contractible space for each $i = 1, \ldots, n$. Then, for $i = 2, \ldots, n$ the map $f_i\colon (\underline{X}, \underline{A})^{K_i} \longrightarrow (\underline{X}, \underline{A})^{K_{i-1}}$ induced by the inclusion $K_i \longrightarrow K_{i-1}$, admits a right homotopy inverse.*

Proof. Recall that for $1 \leq l \leq n$, K_l is defined as the simplicial complex $\mathrm{res}_K(l) * \{l+1, \ldots, n\}$. By Proposition 3.11 $(\underline{X}, \underline{A})^{K_l}$ is homeomorphic to the product $(\underline{X}, \underline{A})^{\mathrm{res}_K(l)} \times \prod_{j=l+1}^{n} X_j$, and it follows that f_i is homotopic to the map

$$(\underline{X}, \underline{A})^{\mathrm{res}_K(i)} \times Q \xrightarrow{\hat{f}_i \times 1} \left((\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)} \times X_i\right) \times Q$$

where $Q = \prod_{j=i+1}^{n} X_j$ and $\hat{f}_i$ is the restriction of f_i. Note that since Q is contractible f_i is actually homotopic to $\hat{f}_i$.

Finally, since X_i is contractible, the projection $(\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)} \times X_i \longrightarrow (\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)}$ is a homotopy equivalence and so f_i may be expressed as the composite

$$(\underline{X}, \underline{A})^{\mathrm{res}_K(i)} \xrightarrow{\hat{f}_i} (\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)} \times X_i \xrightarrow{\simeq} (\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)}$$

Since we have simply restricted f_i to the first i co-ordinates and then projected to the first $i-1$ co-ordinates, this composite is clearly homotopic to the map $(\underline{X}, \underline{A})^{\mathrm{res}_K(i)} \longrightarrow (\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)}$ induced by the projection $\prod_{j=1}^{i} X_j \longrightarrow\!\!\!\!\rightarrow \prod_{j=1}^{i-1} X_j$. It follows that the map $(\underline{X}, \underline{A})^{\mathrm{res}_K(i-1)} \longrightarrow (\underline{X}, \underline{A})^{\mathrm{res}_K(i)}$ induced by the inclusion $\mathrm{res}_K(i-1) \longrightarrow \mathrm{res}_K(i)$ is a right homotopy inverse for f_i. □

Lemma 6.12. *Suppose Y is a space of the form $\Sigma\Omega P^m(p^r) \wedge \Omega(\prod_i A_i)$ where each A_i is either a product of mod p^r Moore spaces, or a wedge sum of mod p^r Moore spaces. Then Y is homotopy equivalent to a wedge of mod p^r Moore spaces*

Proof. The proof follows by iterated use of the James splitting and the decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ of Proposition 2.21. □

Now we turn our attention to the polyhedral product $(\mathrm{Cone}\,\Omega P, \Omega P)^K$ where K is the 5-gon, and use the results of this chapter to obtain an upper bound for its homotopy exponent.

Theorem 6.13. *Let K be the 5-gon. Then*

$$\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}.$$

Proof. From Equation (6.3) we have a sequence of homotopy fibrations

$$\begin{array}{ccccc}
F_2 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} & \xrightarrow{f_2} & \prod_{i=1}^{5} \mathrm{Cone}\,\Omega P_i, \\
F_3 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_3} & \xrightarrow{f_3} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} \\
F_4 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_4} & \xrightarrow{f_4} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_3} \\
F_5 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K} & \xrightarrow{f_5} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_4}.
\end{array}$$

By Proposition 6.6, F_3 is homotopy equivalent to $\Sigma\Omega P_1 \wedge \Omega P_3$ which by Lemma 6.12 is homotopy equivalent to a wedge of mod p^r Moore spaces. The homotopy exponent of a wedge of Moore spaces was calculated in Proposition 4.15 and we see that $\exp_p(F_3) = p^{r+1}$. In fact, since $\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_2}$ is contractible, we see that there is a homotopy equivalence $\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_3} \simeq F_3$ and so too $\exp_p(\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_3}) = p^{r+1}$.

Now, by checking the homotopy decomposition of F_4' given in Proposition 6.8, Lemma 6.12 shows that F_4 is homotopy equivalent to a wedge of Moore spaces and so $\exp_p(F_4) = p^{r+1}$. Since f_4 admits a right homotopy inverse by Lemma 6.11, there is a homotopy decomposition $\Omega\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_4} \simeq \Omega F_4 \times \Omega\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_3}$. Thus, comparing exponents we have $\exp_p(\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_4}) = \max\{\exp_p(F_4), \exp_p(\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_3})\} = p^{r+1}$.

Similarly, Proposition 6.10, and Lemma 6.12 together show that F_5 is homotopy equivalent to a wedge of mod p^r Moore spaces. Thus $\exp_p(F_5) = p^{r+1}$. Since f_5 admits a right homotopy inverse, there is a homotopy decomposition $\Omega\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K} \simeq \Omega F_5 \times \Omega\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_4}$ and therefore comparing exponents as before, gives the result $\exp_p(\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K}) = p^{r+1}$. □

6.2.2 Generalization to $n \geq 5$

In this section we generalize Theorem 6.13 to show that the p-primary homotopy exponent for $\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K}$ where K is an n-gon for $n \geq 5$, is also bounded above by p^{r+1}.

In the proof of Theorem 6.13 we could actually have obtained the result by considering only the final fibration in the sequence $F_5 \longrightarrow \left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K} \xrightarrow{f_5} \left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_4}$ and showing directly that $\exp_p(\left(\mathrm{Cone}\,\Omega P, \Omega P\right)^{K_4}) = p^{r+1}$. We shall follow this approach in this section. Specifically, fix $n \geq 5$ and let K be the

n-gon. Then we have a sequence of homotopy fibrations

$$\begin{array}{ccccc} F_2 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} & \xrightarrow{f_2} & \prod_{i=1}^n \mathrm{Cone}\,\Omega X_i, \\ F_3 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_3} & \xrightarrow{f_3} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} \\ & & \vdots & & \\ F_n & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K} & \xrightarrow{f_n} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}. \end{array}$$

We consider the fibration $F_n \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^K \xrightarrow{f_{n-1}} (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$ and show that $(\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$ has the homotopy type of a wedge of Moore spaces. Recall that K_{n-1} is the join $\mathrm{res}_K(n-1) * \{n\}$, where $\mathrm{res}_K(n-1) = \{\{1,2\}, \ldots, \{n-2, n-1\}\}$. This restricted complex has the nice property that it is built inductively by gluing 1-simplices, one at a time, along a common 0-simplex. The following useful lemma shows that there is an analogous iterative construction of $(\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$.

Lemma 6.14. *Let $n \geq 3$ and let $A(n)$ denote the simplicial complex*

$$A(n) = \{\{1,2\}, \ldots, \{n-1, n\}\}.$$

Let $\underline{X} = \{X_i\}_{i=1}^n$ be a sequence of n arbitrary spaces. Then there is a homotopy equivalence

$$(\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{A(n)} \simeq (\Sigma M \wedge N) \vee ((\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^{A(n-1)} \rtimes N)$$

where $M = \prod_{i=1}^{n-2} \Omega X_i$ and $N = \Omega X_n$.

Proof. The simplicial complex $A(n)$ is obtained by gluing $A(n-1)$ and the simplicial complex $L = \{n-1, n\}$ along the common vertex $\{n-1\}$. It follows from

Lemma 6.9 that there is a homotopy equivalence

$$(\mathrm{Cone}\,\Omega\underline{X},\Omega\underline{X})^{A(n)} \simeq (\Sigma M\wedge N)\vee(M\ltimes(\mathrm{Cone}\,\Omega\underline{X},\Omega\underline{X})^L)\vee((\mathrm{Cone}\,\Omega\underline{X},\Omega\underline{X})^{A(n-1)}\rtimes N).$$

But since $(\mathrm{Cone}\,\Omega\underline{X},\Omega\underline{X})^L = \mathrm{Cone}\,\Omega X_{n-1}\times\mathrm{Cone}\,\Omega X_n$ is contractible, the result follows. □

Proposition 6.15. *Let $n \geq 4$ and let $A(n)$ be the simplicial complex of Lemma 6.14. Then $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n)}$ has the homotopy type of a wedge of mod p^r Moore spaces.*

Proof. We proceed by induction. For the case $n = 4$, refer to the proof of Proposition 6.10. There it was shown that for an arbitrary space X, $(\mathrm{Cone}\,\Omega X,\Omega X)^{A(4)}$ decomposes as a wedge sum of spaces of the form $\Sigma(\Omega X)^{\wedge k}$ with $k \leq 3$. When X is a Moore space P, it is seen by iterating the James splitting that each of these wedge summands splits as a wedge of mod p^r Moore spaces.

Now suppose that $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}$ has the homotopy type of a wedge of mod p^r Moore spaces. Then Lemma 6.14 shows that $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n)} \simeq (\Sigma M\wedge N)\vee(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}\rtimes\Omega P$ where $M = (\Omega P)^{n-2}$ and $N = \Omega P$. The summand $\Sigma M\wedge N$ splits as a wedge of mod p^r Moore spaces by Lemma 6.12. Furthermore, $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}$ is a suspension space and so there is a homotopy splitting of the wedge summand

$$(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}\rtimes\Omega P \simeq (\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}\vee(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}\wedge\Omega P$$

by Proposition 2.32. The induction hypothesis tells us that the polyhedral product $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}$ decomposes as a wedge of mod p^r Moore spaces and therefore so does the summand $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n-1)}\wedge\Omega P$. In other words $(\mathrm{Cone}\,\Omega P,\Omega P)^{A(n)}$ decomposes as a wedge of mod p^r Moore spaces, as required.

□

Now we can calculate the homotopy type of F'_n where $F_n \simeq F'_n * F''_n$ is the homotopy fibre in the homotopy fibration $F_n \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^K \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$, as in the notation of Theorem 6.1.

Proposition 6.16. *There is a homotopy equivalence* $F'_n \simeq \Omega A_1 \times \Omega A_2$ *where* A_1, A_2 *are both homotopy equivalent to a wedge of mod* p^r *Moore spaces.*

Proof. By definition, F'_n is the homotopy fibre of the map $(\mathrm{Cone}\,\Omega P, \Omega P)^L \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^R$ where $L = \mathrm{link}_K(\{n\})$ and $R = \mathrm{res}_K(n-1) = A(n-1)$. Note that L is the simplicial complex $\{\{1\}, \{n-1\}\}$ with ghost vertices $\{2\}, \ldots, \{n-2\}$. Let L' be the simplicial complex $\{\{1\}, \{n-1\}\}$ obtained from L by forgetting the ghost vertices. Then $(\mathrm{Cone}\,\Omega P, \Omega P)^{L'} \simeq \Sigma\Omega P_1 \wedge \Omega P_{n-1}$ and $(\mathrm{Cone}\,\Omega P, \Omega P)^L \simeq (\mathrm{Cone}\,\Omega P, \Omega P)^{L'} \times \prod_{i=2}^{n-2} \Omega P_i$.

Similarly to the proof of Proposition 6.10, there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \Omega\,(P,*)^R & \xrightarrow{\Omega i} & \prod_{i=1}^{n-1} \Omega P_i \\
 & & \downarrow & & \downarrow \\
G & \xrightarrow{\psi} & F'_n & \longrightarrow & \prod_{i=2}^{n-2} \Omega P_i \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{Cone}\,\Omega P, \Omega P)^{L'} & \longrightarrow & (P,*)^{L'} & \longrightarrow & P_1 \times P_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{Cone}\,\Omega P, \Omega P)^R & \longrightarrow & (P,*)^R & \xrightarrow{i} & \prod_{i=1}^{n-1} P_i
\end{array}$$

in which all rows and columns are homotopy fibrations. By Proposition 3.13 the map Ωi has a right homotopy inverse $r\colon \prod_{i=1}^{n-1} \Omega P_i \longrightarrow \Omega\,(P,*)^R$, and so it follows that there is a homotopy equivalence

$$\left(\prod_{i=2}^{n-2} \Omega P_i\right) \times G \xrightarrow{r' \times \psi} \Omega\,(P,*)^R \times F'_n \xrightarrow{\theta} F'_n$$

where r' is the restriction of r to $\prod_{i=2}^{n-2}\Omega P_i$ and θ is the homotopy action associated with the middle vertical fibration.

It remains to calculate the homotopy fibre G of the inclusion map

$$f\colon\ (\text{Cone}\,\Omega P,\Omega P)^{L'} \longrightarrow (\text{Cone}\,\Omega P,\Omega P)^{R}$$

induced by the inclusion of simplicial complexes $L' \longrightarrow R$. By Lemma 6.14, $(\text{Cone}\,\Omega P,\Omega P)^R \simeq (\Sigma M\wedge N)\vee B$ where $B=(\text{Cone}\,\Omega P,\Omega P)^{A(n-2)}\rtimes N$, M is the product $\prod_{i=1}^{n-3}\Omega P_i$, and $N=\Omega P_{n-1}$. By Proposition 2.21 there is a homotopy decomposition $\Sigma(Y_1\times Y_2)\simeq \Sigma Y_1\vee\Sigma Y_2\vee\Sigma Y_1\wedge Y_2$ for arbitrary spaces Y_1,Y_2. Iterating this decomposition, we see that

$$\begin{aligned}\Sigma M\wedge N &= \Sigma(\Omega P_1\times\ldots\times\Omega P_{n-3})\wedge\Omega P_{n-1}\\ &\simeq (\Sigma\Omega P_1\wedge\Omega P_{n-1})\vee C\end{aligned}$$

where C is a wedge of spaces of the form $\Sigma\Omega P_{i_1}\wedge\ldots\wedge\Omega P_{i_k}$ with $1\le i_1<\ldots< i_k\le n-1$.

The map f is homotopy equivalent to the inclusion of the wedge summand $\Sigma\Omega P_1\wedge\Omega P_{n-1}\hookrightarrow(\Sigma\Omega P_1\wedge\Omega P_{n-1})\vee D$ where $D=C\vee B$. Thus by Lemma 6.7 the homotopy fibre G is homotopy equivalent to

$$\Omega D\times\Omega(\Sigma\Omega(\Sigma\Omega P_1\wedge\Omega P_{n-1})\wedge\Omega D).$$

Let $A_1=D$ and $A_2=\Sigma\Omega(\Sigma\Omega P_1\wedge\Omega P_{n-1})\wedge\Omega D$. A_1 is clearly homotopy equivalent to a wedge of Moore spaces since both B and C are. To see that A_2 is homotopy equivalent to a wedge of mod p^r Moore spaces, notice first that the space $Z=\Sigma\Omega P_1\wedge\Omega P_{n-1}$ decomposes as a wedge of Moore spaces by Lemma 6.12. Therefore

$A_2 \simeq \Sigma\Omega Z \wedge \Omega D$ where both Z, D are homotopy equivalent to a wedge of mod p^r Moore spaces. Thus iterated use of the James splitting shows that A_2 is in fact homotopy equivalent to a wedge of mod p^r Moore spaces. □

Theorem 6.17. *Let $n \geq 3$ and let K be the n-gon. Then*

$$\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}.$$

Proof. The cases $n < 5$ were proven independently in Proposition 6.3 and Proposition 6.4. For $n \geq 5$, consider the homotopy fibration

$$F_n \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^K \xrightarrow{f_n} (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}.$$

Since f_n admits a right homotopy inverse by Proposition 6.11, there is a homotopy equivalence $\Omega\,(\mathrm{Cone}\,\Omega P, \Omega P)^K \simeq \Omega F_n \times \Omega\,(\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$ and hence

$$\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = \max\{\exp_p(F_n), \exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}})\}. \tag{6.7}$$

Now, the fibre $F_n \simeq \Sigma F'_n \wedge F''_n$ is by Proposition 6.16 homotopy equivalent to $\Sigma(\Omega A_1 \times \Omega A_2) \wedge \Omega P$ where A_1, A_2 both decompose as a wedge of mod p^r Moore spaces. By Lemma 6.12 it follows that F_n is also homotopy equivalent to a wedge of Moore spaces. Moreover, by Proposition 6.15, $(\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}$ is homotopy equivalent to a wedge of mod p^r Moore spaces. Therefore $\exp_p(F_n) = \exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}) = p^{r+1}$ and as a consequence, equation (6.7) shows that $\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}$, as required. □

Finally, we conclude this section with a couple of observations. Throughout this chapter, we we were concerned with polyhedral products $(\mathrm{Cone}\,\Omega \underline{X}, \Omega \underline{X})^K$ where $X_i = P = P^m(p^r)$ for all i, and $m \geq 3$ is fixed. However, the results we have proven will work equally well in the case that $X_i = P^{m_i}(p^r)$, where $m_i \geq 3$ for each i. Since the homotopy exponent of $P^m(p^r)$ is independent of the dimension, the methods used in this chapter can be applied equally well to show that the homotopy exponent is also p^{r+1} in this more general case.

Moreover, our methods can be further extended to sequences $\underline{X} = \{X_1, \ldots, X_n\}$ where X_i is a mod p^{r_i} Moore space of dimension $m_i \geq 3$, with $r_i \geq 1$. There is only one aspect of our methods in this chapter which needs some extra care in this situation. Throughout this chapter we have made extensive use of Proposition 4.14 which says that a smash product $P^m(p^r) \wedge P^k(p^r)$ of mod p^r Moore spaces decomposes as a wedge sum $P^{m+k}(p^r) \vee P^{m+n-1}(p^r)$. In fact this is a special case of a more general result which says that for $t \geq s$ there is a homotopy equivalence $P^m(p^s) \wedge P^k(p^t) \simeq P^{m+k}(p^t) \vee P^{m+k-1}(p^t)$.

These observations mean that we can immediately extend Theorem 6.17 to the following more general result:

Theorem 6.18. *Let $n \geq 3$ and let K be the n-gon. For $i = 1, \ldots, n$ let $X_i = P^{m_i}(p^{r_i})$ where $m_i \geq 3$ and $r_i \geq 1$. Then*

$$\exp_p((\mathrm{Cone}\,\Omega\underline{X}, \Omega\underline{X})^K) = p^{R+1}.$$

where $R = \max\{r_i\}_{i=1}^n$.

6.3 An application of Barratt's Conjecture

As in the previous section we use P to denote a mod p^r Moore space $P^m(p^r)$ where $r \geq 1$, but this time we must take $m \geq 4$. In this concluding section of the thesis we show that if Barratt's Conjecture is true, then we may actually determine the p-primary homotopy exponent of the polyhedral product $(\mathrm{Cone}\,\Omega P, \Omega P)^K$, for any simplicial complex K. In fact the value of the homotopy exponent does not depend on K and is given by p^{r+1} in all cases.

In section 6.2.2, where K was taken to be an n-gon, we were able to prove Proposition 6.12 which said that for certain spaces X, the join $\Sigma\Omega P^m(p^r) \wedge X$ always decomposes as a wedge of mod p^r Moore spaces. As it turned out, we were able to show that each of the homotopy fibres F_i' for $i = 2, \ldots, n$ always have this property, and as such, we saw that for each $i = 2, \ldots, n$, the fibre $F_i \simeq \Sigma F_i' \wedge F_i''$ decomposes as a wedge of Moore spaces.

When we turn our consideration to the case of arbitrary K, it is not clear whether the corresponding statement is true about the F_i. If we assume the truth of Barratt's Conjecture, then we have the following lemma which provides an analogous result to Proposition 6.12.

Lemma 6.19. *Let X be a space and suppose $m \geq 4$. If Barratt's Conjecture holds, then*

$$\exp_p(\Sigma\Omega P^m(p^r) \wedge X) \leq p^{r+1}.$$

Proof. By the James splitting, together with Proposition 4.14 we have

$$\begin{aligned} \Sigma\Omega P^m(p^r) \wedge X &\simeq (\Sigma\Omega\Sigma P^{m-1}(p^r)) \wedge X \\ &\simeq (\vee_\alpha P^{n_\alpha}(p^r)) \wedge X \\ &\simeq \Sigma(\vee_\alpha P^{n_\alpha - 1}(p^r)) \wedge X. \end{aligned}$$

Since $\vee_\alpha P^{n_\alpha-1}(p^r)$ has co-H-exponent p^r, then so does $(\vee_\alpha P^{n_\alpha-1}(p^r)) \wedge X$. Thus, if Barratt's conjecture holds then it follows that $\exp_p(\Sigma(\vee_\alpha P^{n_\alpha-1}(p^r)) \wedge X) \leq p^{r+1}$. But since $\Sigma(\vee_\alpha P^{n_\alpha-1}(p^r)) \wedge X \simeq \Sigma\Omega P^m(p^r) \wedge X$, the result follows. □

If K is the full simplex Δ^{n-1} on n vertices then $(\mathrm{Cone}\,\Omega P, \Omega P)^K = \prod_{i_1}^n \mathrm{Cone}\,\Omega P_i$ is contractible. For allother simplicial complexes we have the following result:

Theorem 6.20. *Let K be a simplicial complex on $n \geq 1$ vertices, $K \neq \Delta^{n-1}$, and let $P = P^m(p^r)$ where $m \geq 4$ and $r \geq 1$. If Barratt's Conjecture holds, then*

$$\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}.$$

Proof. The proof is similar to that of Theorem 6.13. Consider the sequence of homotopy fibrations

$$\begin{array}{ccccc} F_2 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} & \xrightarrow{f_2} & \prod_{i=1}^n \mathrm{Cone}\,\Omega P_i, \\ F_3 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_3} & \xrightarrow{f_3} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} \\ & & \vdots & & \\ F_n & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K} & \xrightarrow{f_n} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{n-1}}. \end{array}$$

where $K_i = \mathrm{res}_K(i) * \{i+1, \ldots, n\}$, as defined in Theorem 6.1. Since $K \neq \Delta^{n-1}$, then K contains at least one missing face, that is, there exists a subset $\sigma \subseteq [n]$ such that $\sigma \notin K$ but all proper subsets of σ belong to K. Let τ be a missing face of least dimension, $k-1$ say, and relabel the vertices of K if necessary so that $\tau = \{1, \ldots, k\}$.

Now, for $i = 2, \ldots, k-1$ we have $\mathrm{res}_K(i) = \Delta^{i-1}$ and it follows that each of the spaces in the fibrations

$$\begin{array}{ccccc} F_2 & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_2} & \xrightarrow{f_2} & \prod_{i=1}^n \mathrm{Cone}\,\Omega P_i, \\ & & \vdots & & \\ F_{k-1} & \longrightarrow & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{k-1}} & \xrightarrow{f_{k-1}} & (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{k-2}}. \end{array}$$

is contractible. The first non-trivial fibration in the sequence is

$$F_k \longrightarrow (\mathrm{Cone}\,\Omega P, \Omega P)^{K_k} \xrightarrow{f_k} (\mathrm{Cone}\,\Omega P, \Omega P)^{K_{k-1}}.$$

Since $K_k = \partial\Delta^{k-1} * \{k+1,\ldots,n\}$ then $(\mathrm{Cone}\,\Omega P,\Omega P)^{K_k}$ is homotopy equivalent to $(\mathrm{Cone}\,\Omega P,\Omega P)^{\partial\Delta^{k-1}}$. To determine the homotopy type of $(\mathrm{Cone}\,\Omega P,\Omega P)^{\partial\Delta^{k-1}}$, notice that equation (3.2) gives a fibration

$$(\mathrm{Cone}\,\Omega P,\Omega P)^{\partial\Delta^{k-1}} \longrightarrow (P,*)^{\partial\Delta^{k-1}} \longrightarrow \prod_{i=1}^{k} P_i.$$

The space $(P,*)^{\partial\Delta^{k-1}}$ is the fat wedge, which is by definition the space T_1^k in the notation of equation (2.9). Thus Porter's Theorem shows that $(\mathrm{Cone}\,\Omega P,\Omega P)^{K_k}$ is homotopy equivalent to a wedge of spaces of the form $\Sigma^{k-1}\Omega P_{i_1}\wedge\ldots\wedge\Omega P_{i_j}$, and therefore by the James splitting, is homotopy equivalent to a wedge of Moore spaces. In particular, $\exp_p((\mathrm{Cone}\,\Omega P,\Omega P)^{K_k}) = p^{r+1}$. This concludes the base step of our induction.

Now, let $k \le l < n$ and suppose that $\exp_p((\mathrm{Cone}\,\Omega P,\Omega P)^{K_l}) \le p^{r+1}$. By Theorem 6.1, $F_{l+1}\simeq \Sigma F'_{l+1}\wedge F''_{l+1}$ where $F''_{l+1}\simeq \Omega P_{l+1}$. Thus by Lemma 6.19, under our assumption that Barratt's Conjecture holds, we have $\exp_p(F_{l+1}) \le p^{r+1}$.

Now, by Proposition, 6.11, f_{l+1} has a right homotopy inverse and so there is a homotopy decomposition $\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_{l+1}} \simeq \Omega F_{l+1}\times\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_l}$. It follows that $\exp_p((\mathrm{Cone}\,\Omega P,\Omega P)^{K_{l+1}}) \le p^{r+1}$.

By induction, we obtain the upper bound $\exp_p((\mathrm{Cone}\,\Omega P,\Omega P)^{K}) \le p^{r+1}$. Next we show that p^{r+1} also bounds below. By Proposition 6.11, f_i has a right homotopy inverse for each $k\le i\le n$. Thus there are homotopy decompositions

$$\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_i} \simeq \Omega F_i\times\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_{i-1}}$$

for $k\le i\le n$. It follows that $\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K}\simeq\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_k}\times Z$ for some space Z. In particular, $\pi_*(\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K})$ contains $\pi_*(\Omega\,(\mathrm{Cone}\,\Omega P,\Omega P)^{K_k})$

as a direct summand, and therefore contains elements of order p^{r+1} since $(\mathrm{Cone}\,\Omega P, \Omega P)^{K_k}$ is a wedge of Moore spaces. this proves the result.

□

We conclude by observing that since Moore spaces are rationally contractible, then so is $(\mathrm{Cone}\,\Omega P, \Omega P)^K$. That is the rational equivalence $P^m(p^r) \simeq_{\mathbb{Q}} *$ induces a rational equivalence $(\mathrm{Cone}\,\Omega P, \Omega P)^K \simeq_{\mathbb{Q}} (*, *)^K$. It follows that $(\mathrm{Cone}\,\Omega P, \Omega P)^K$ has trivial rational homotopy groups and is therefore elliptic. Therefore Theorem 6.18 provides a family of spaces satisfying Moore's Conjecture, that we believe were not known previously. If Barratt's Conjecture is true then Theorem 6.20 extends this family.

Chapter 7

Further Work

We end the thesis by outlining some possibilities for further work which would build on the results of this thesis.

Conjecture A. *The following statements are equivalent:*

a) $(D^m, S^{m-1})^K$ *is elliptic,*

b) K *is a join of simplices and boundaries of simplices,*

c) $(D^m, S^{m-1})^K$ *is homotopy equivalent to a product of spheres.*

A proof of Conjecture A would extend Debongnie's result, (Theorem 5.3), which classifies those simplicial complexes for which $(D^2, S^1)^K$ is elliptic, to the pair (D^m, S^{m-1}) and consequently to $(P^m(p^r), S^{m-1})$ for $m \geq 2$. It should be noted that during the final stages of writing this thesis, Bahri, Bendersky, Cohen and Gitler ([2]) have found a geometric proof of Debongnie's result, by showing that when K is not a join of simplices and boundaries of simplices, then a pair of intersecting missing faces in K can be chosen which realize a rational wedge of spheres $S^n \vee S^k$ as a retract of $(D^2, S^1)^K$. This of course implies that the rational homotopy groups of $(D^2, S^1)^K$ contains a free Lie algebra on two generators, yielding the hyperbolicity result. We have not proven it here due to time constraints, but we believe that the same methods should work for the pair (D^m, S^{m-1}) and therefore confirm Conjecture A.

Let p be an odd prime, and let $P = P^m(p^r)$ for $m \geq 3$ and $r \geq 1$.

Conjecture B. *If K is any simplicial complex other than a full simplex, then* $\exp_p((\mathrm{Cone}\,\Omega P, \Omega P)^K) = p^{r+1}$.

We proved in Theorem 6.20, that this result holds if we assume that Barratt's Conjecture is true. The proof that p^{r+1} is a lower bound did not depend on Barratt's Conjecture and so this is holds in general. To prove Conjecture B, it is enough to show that p^{r+1} is an upper bound.

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