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A continuum model of a channelized, free-surface granular flow is developed to calculate the rate at which it expands into an initially grain-free region when lateral constraints are removed. The spreading is driven by cross-stream pressure gradients and resisted by basal drag. The boundary between the granular vacuum and the flowing grains is elucidated both in the near and far fields.

1 INTRODUCTION

Whenever a confined granular material flows into an unbounded region, there is a well defined boundary to the flowing grains, as the material moves into initially grain-free domain. For example, such *granular vacua* are found in the lee of obstacles placed in free-surface chute flows; particles are deflected around the stationary obstacle and do not infill the region immediately downstream of it, forming a grain-free region in its wake. Infact such deflection patterns often underlie the design of many avalanche defence barriers (Gray et al. 2003). Also large-scale natural rockfalls and avalanches are often initiated in gulleys from which the flows may spread laterally when they are no longer confined. In this paper we examine the generic processes by which free-surface granular flows expand into grain-free regions by studying in detail the lateral spreading of a channelised flow when the boundaries that confine it are removed.

We analyse rapid, shallow flows for which the steady lateral, cross-slope spreading is driven by the generation of normal stresses and resisted by basal drag. Using a shallow layer continuum model we calculate the shape of the boundary between the flowing grains and the grain-free region. First we show that close to the point at which the flow becomes unconfined, it spreads laterally at a constant rate that is determined only by the upstream Froude number. This expansion is analogous to the Prandtl-Meyer expansion of a high speed flow of gas into a vacuum (Chapman 2000). Further downslope, however, the rate of lateral spreading enters a different dynamical regime and becomes dependent upon the gravitational acceleration and basal resistance.

In this paper we formulate a shallow-layer model of rapid granular flows (§2) and then analyse the expansion into the grain-free region in the near (§3.1) and far field (§3.2). Finally we draw some brief conclusions (§4).

2 SHALLOW LAYER MODELS

The granular material flows steadily down a rigid plane, inclined at an angle θ to the the horizontal. We align the coordinate axes such that the z-axis is perpendicular to the plane, the x-axis is parallel to the direction of steepest descent and the y-axis is perpendicular to these two (see figures 1,2). The flow is laterally constrained for $x < 0$, but is unconstrained for $x > 0$ and we analyse its expansion into the granular vacuum. The depth and bulk density of the flowing layer of grains are denoted by h and ρ and thus the steady-state conservation of mass is given by

$$\frac{\partial}{\partial x}\int_0^h \rho u \, \mathrm{d}z + \frac{\partial}{\partial y}\int_0^h \rho v \, \mathrm{d}z = 0, \qquad (1)$$

where u & v are the components of the velocity field along the x & y axes, respectively.

The depth of the flow is assumed to be much less than the lengthscales over which the flow varies in the plane. This implies that vertical accelerations are negligible and the vertical momentum equation is considerably simplified. Denoting the pressure tensor by p_{ij}, this implies that to leading order

$$p_{zz} = \int_z^h \rho g \cos\theta \, \mathrm{d}\eta, \quad (2)$$

where it has been further assumed that the upper surface of the flow is stress-free. The steady momentum equations parallel with the plane are then given by

$$\frac{\partial}{\partial x}\int_0^h \rho u^2 \, \mathrm{d}z + \frac{\partial}{\partial y}\int_0^h \rho uv \, \mathrm{d}z = -\frac{\partial}{\partial x}\int_0^h p_{xx} \, \mathrm{d}z - \frac{\partial}{\partial y}\int_0^h p_{xy} \, \mathrm{d}z - \tau_x + \int_0^h \rho g \sin\theta \, \mathrm{d}z, \quad (3)$$

$$\frac{\partial}{\partial x}\int_0^h \rho uv \, \mathrm{d}z + \frac{\partial}{\partial y}\int_0^h \rho v^2 \, \mathrm{d}z = -\frac{\partial}{\partial x}\int_0^h p_{xy} \, \mathrm{d}z - \frac{\partial}{\partial y}\int_0^h p_{yy} \, \mathrm{d}z - \tau_y. \quad (4)$$

In these expressions the basal shear stress has components τ_x and τ_y, which are equal to p_{xz} and p_{yz} evaluated at the boundary $z = 0$.

To close this mathematical model of the motion we make a series of further assumptions. First, we assume that the bulk density of the flowing material is uniform. Although granular materials must dilate in order to flow, once mobilised, their bulk density varies only negligibly (Savage & Hutter (1989), Pouliquen & Forterre (2002), Gray et al. (2003)). Thus, we treat the material as incompressible. In addition, we assume that the velocity fields, u and v are vertically uniform; such an assumption underlies most depth-averaged, hydraulic models of river, estuarine, debris and avalanche flows. Current understanding of the wide range of dynamical behaviours that may be exhibited by granular flows remains incomplete and there is no widely accepted constitutive law for the internal stresses developed by these flows. However, we may simplify the depth-averaged equations by invoking two further assumptions. First we assume that the ratio of the shear to normal stresses is small ($p_{xx}, p_{yy} \gg p_{xy}$) and that to leading order the normal stresses are equal, $p_{xx} = p_{yy} = p_{zz}$ (Gray et al. 2003). This diverges from the approach of Savage & Hutter (1989) who introduced an earth pressure coefficient that measures the ratio p_{xx}/p_{zz} and is different from unity. However, normal stress differences have been found to be small in simulations Ertas et al. (2001) and both Pouliquen & Forterre (2002) and Gray et al. (2003) suggest that for chute flows they should be neglected. Thus we find that the steady governing equations are given by

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \quad (5)$$

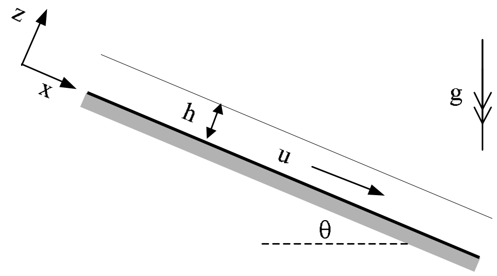

Figure 1. Side of the flow.

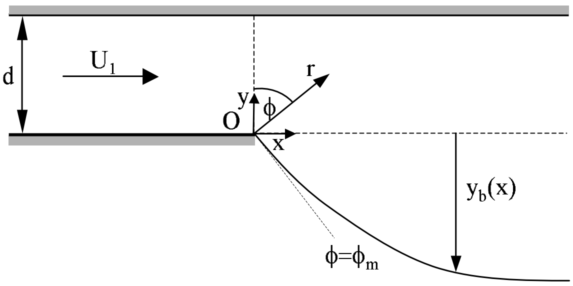

Figure 2. Plan view of the flow and its expansion.

$$\frac{\partial}{\partial x}\left(u^2 h\right) + \frac{\partial}{\partial y}(uvh) + \frac{\partial}{\partial x}\left(\frac{g\cos\theta h^2}{2}\right) = -\frac{\tau_x}{\rho} + gh\sin\theta, \quad (6)$$

$$\frac{\partial}{\partial x}(uvh) + \frac{\partial}{\partial y}\left(v^2 h\right) + \frac{\partial}{\partial y}\left(\frac{g\cos\theta h^2}{2}\right) = -\frac{\tau_y}{\rho}. \quad (7)$$

With the exception of the drag and downslope acceleration terms on the right-hand side of (6) & (7), these equations are equivalent to those that express mass and momentum conservation for the steady flow of a polytropic gas with index 2 (Chapman 2000). Furthermore they are identical to the shallow-water equations used to model hydraulic phenomena (Whitham 1974).

We consider a uniform channelized flow upstream of the expansion, that is purely downslope ($\mathbf{u} = (U_1, 0)$) and of depth h_1. A constant flow depth and velocity are possible if the basal drag, τ_x is equal to the downslope gravitational force ($\rho g h_1 \sin\theta$) and recent laboratory experiments have found that steady uniform flows can exist for a range of chute inclinations (see, for example, Pouliquen & Forterre (2002)). We introduce dimensionless variables by scaling velocities by U_1, lengths by h_1, stresses by ρU_1^2 and henceforth will assume that variables are dimensionless, unless indicated otherwise. An important dimensionless parameter is the upstream Froude number, F, which is given by

$$F^2 = \frac{U_1^2}{g h_1 \cos\theta}. \quad (8)$$

This is equivalent to the Mach number in gas flows and measures the speed of the flow relative to the speed of the small amplitude surface waves. Generally for snow avalanches, $F \gg 1$; for example, Issler (2003) suggests that for dry-snow avalanches, F lies between 5 and 10.

3 EXPANSION INTO A GRANULAR VACUUM

3.1 *Local Analysis*

Sufficiently close to the point at which the flow is no longer laterally constrained $(x = 0)$, the flow expands and its motion is independent of the gravitational acceleration down the slope and the effects of basal drag (see figure 2). Thus the local behaviour may be examined by analysing the governing equations (5)-(7) in the absence of 'source' terms; in other words the right-hand sides of (6) and (7) may be neglected. The behaviour then becomes akin to Prandtl-Meyer expansions in gas dynamics (Chapman 2000).

To analyse the flow we adopt plane polar coordinates, (r, ϕ) in the vicinity of the origin as shown in figure 2. Furthermore we seek solution for u, v and h that are dependent only upon the polar angle ϕ. This ansatz implies that the flow is irrotational. Denoting $u = U \sin\phi + V \cos\phi$ and $v = U \cos\phi - V \sin\phi$, where $U(\phi)$ and $V(\phi)$ are velocities in the radial and angular directions, respectively, we find that the governing equations are given by

$$Uh + \frac{\mathrm{d}}{\mathrm{d}\phi}(Vh) = 0, \quad (9)$$

$$-V^2 + V\frac{\mathrm{d}U}{\mathrm{d}\phi} = 0, \quad (10)$$

$$V\frac{\mathrm{d}V}{\mathrm{d}\phi} + UV + \frac{\mathrm{d}h}{\mathrm{d}\phi} = 0. \quad (11)$$

These equations represent conservation of mass, radial and angular momentum, respectively. We observe from (10) that $V = \mathrm{d}U/\mathrm{d}\phi$ and from (9) & (11) that $V = \sqrt{h}$. Thus substituting these into (9), we find that

$$\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2} + \frac{1}{3}U = 0, \quad (12)$$

with solutions $U = R\sin(\phi/\sqrt{3} + \alpha)$ and $V = R/\sqrt{3}\cos(\phi/\sqrt{3} + \alpha)$, where R and α are constants to be evaluated.

This solution must be matched to the upstream uniform flow. We introduce an angle ϕ_0, such that for $\phi < \phi_0$ the flow is the upstream uniform flow, but for $\phi > \phi_0$ the flow is given by the solution above. Also the upstream Froude number may be related to the angle ϕ_0 and is given by $\cos\phi_0 = F^{-1/2}$. Thus we may solve these simultaneous equations to find that

$$\alpha = \tan^{-1}\left[\left(\frac{F^2-1}{3}\right)^{\frac{1}{2}}\right] - \frac{1}{\sqrt{3}}\tan^{-1}\left[\left(F^2-1\right)^{\frac{1}{2}}\right]. \quad (13)$$

Finally we may calculate the maximum expansion that occurs when the flow becomes laterally unconfined. This is given by the streamline on which the height and angular velocity vanish, which may be evaluated as

$$\phi_m = \sqrt{3}\left(\frac{\pi}{2} - \alpha\right). \quad (14)$$

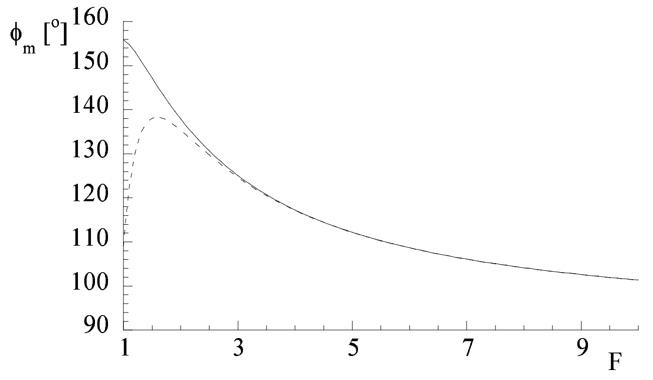

Figure 3. The angle at which the flow initially expands into the granular vacuum, ϕ_m, as a function of the upstream Froude number, F. Exact (——) and asymptotic (– – –) evaluations of ϕ_m.

We plot the dependence of ϕ_m on the Froude number, F, in figure 3 and note that it attains a maximum value of $\pi\sqrt{3}/2$ when $F = 1$. This implies that that flows that are critical expand at the greatest rate into granular vacua. Furthermore when $F \gg 1$,

$$\phi_m = \frac{\pi}{2} + \frac{2}{F} - \frac{5}{3F^3} + O\left(\frac{1}{F^5}\right), \quad (15)$$

and as can be noted from figure 3, this asymptotic approximation works very well when $F \geq 3$.

3.2 *Far-field analysis*

Far from the point at which the flow is first laterally unconstrained, the motion once again become affected by the downslope gravitational acceleration and the basal resistance. The motion differs from the uniform flow that occurs on the confined upper part of the chute $(x < 0)$, because it is still able to spread laterally. However the rate at which the granular material spreads into the particle-free region is strongly modified and the solution of the preceding subsection is no longer appropriate; in particular the boundary of the expansion is no longer at a constant angle, ϕ_m, to the direction of steepest descent but is curved.

In this subsection we analyse the expansion far from the point at which it becomes unconfined and the asymptotic solution developed below is based on the downslope component of the velocity field far exceeding the lateral component; *i.e.* $|v/u| \equiv \epsilon \ll 1$. The detailed structure of the motion depends strongly on the relationship between the basal drag and the flow speed and depth. In this paper we analyse one representation of the basal shear stress, namely Coulomb drag,

$$\boldsymbol{\tau} = \mu \rho g h \cos\theta \frac{\mathbf{u}}{|\mathbf{u}|}, \tag{16}$$

where $\mu \equiv \tan\delta$ and δ is the friction angle (Savage & Hutter 1989)[1]. Using this model, a steady flow is only possible when the inclination of the chute matches the basal friction angle ($\theta = \delta$).

The essence of the asymptotic analysis is that since the lateral velocity is much smaller than the downslope velocity, the leading order terms in the momentum equations are considerably simplified; the downslope equation reduces to a balance between gravitational acceleration and the lateral pressure gradient is balanced by weak basal drag ($\rho g \cos\theta \partial h/\partial y \sim \mu \rho g \cos\theta v/u$). However the volume flux of particles remains constant and is given by

$$\int_{-d}^{y_b} uh \, \mathrm{d}y = Q, \tag{17}$$

where Q is the dimensionless volume flux and $y_b(x)$ is the expanding edge of the granular material. By balancing the terms in these equation, we find the following distinguished scaling

$$u = U_0 + \ldots, \qquad v = \epsilon V_0(X, Y) + \ldots$$

$$\text{and} \qquad h = \epsilon^{1/2} H_0(X, Y), \tag{18}$$

where $X = \epsilon^{3/2} x$ and $Y = \epsilon^{1/2} y$. Then the leading order equations are given by

$$U_0 \frac{\partial H}{\partial X} + \frac{\partial}{\partial Y}(H_0 V_0) = 0 \quad \& \quad \frac{\partial H_0}{\partial Y} = -\frac{\mu V_0}{U_0}, \tag{19}$$

and U_0 is a constant determined by upstream conditions. Combing these solutions Thus we find that the height field satisfies the following partial differential equation,

$$\frac{\partial H}{\partial X} = \frac{1}{\mu} \frac{\partial}{\partial Y} \left(H \frac{\partial H}{\partial Y} \right), \tag{20}$$

together with the volume flux constraint

$$\int_{-\epsilon^{1/2} d}^{Y_b(X)} H \, \mathrm{d}Y = Q/U_0. \tag{21}$$

[1]We note that an analogous analysis may be carried out with other friction laws.

This equation exhibits a similarity solution that provide the intermediate asymptotics for the solution (Barenblatt 1996) and is given by

$$H = \frac{\mu}{6x^{1/3}} \left[\left(\frac{9Q}{U_0 \mu} \right)^{2/3} - \frac{(y + \epsilon^{1/2} d)^2}{x^{2/3}} \right]. \tag{22}$$

Thus sufficient far downstream so that the solution has converged to the similarity solution, the leading order position of vacuum boundary in the original variables is given by

$$y_b = [9Qx/(U_0 \mu)]^{1/3}. \tag{23}$$

4 CONCLUSIONS

We have developed a continuum model of the rapid, shallow flow of a granular material down an inclined chute when the flow is no longer laterally confined. Close to the position at which the flow becomes unconfined, the rate of expansion depends only upon the upstream Froude number, but further downslope it emerges as a balance between a lateral pressure gradient and basal drag. This motion is of fundamental interest because the rate of spreading reveals some of the consequences of different mechanisms of stress generation; indeed different representations of the basal drag will lead to different predictions. It is also of direct practical application in the design of effective barriers to defend against avalanche damage. Work is currently underway to compare our calculations with laboratory experiments and numerical computations of the flow.

REFERENCES

Barenblatt, G. I. 1996. *Scaling, self-similarity and intermediate asymptotics*. Cambridge Univ. Press.

Chapman, C. J. 2000. *High speed flow*. Cambridge Univ. Press. pp. 258.

Ertas, D., Grest, G. S., Halsey, T. C., Levine, D., & Silbert, L. E. 2001. Gravity-driven dense granular flows. *Europhysics Letters* 56: 214–220.

Gray, J. M. N. T., Tai, Y. C., & Noelle, S. 2003. Shock waves, dead zones and particle-free regions in rapid granular free-surface flows. *J. Fluid Mech.* 491: 161–181.

Issler, D. 2003. Experimental information on the dynamics of dry-snow avalanches. In K. Hutter & N. Kirchner (eds), *Dynamic response of granular and porous materials under large and catastrophic deformations*. Springer (Berlin). 109–160.

Pouliquen, O. & Forterre, Y. 2002. Friction law for dense granular flows: application to the motion of a mass down a rough inclined plane. *J. Fluid Mech.* 453: 133–151.

Savage, S. B. & Hutter, K. 1989. The motion of a finite mass of granular material down a rough incline. *J. Fluid Mech.* 199: 177–215.

Whitham, G. B. 1974. *Linear and nonlinear waves*. Wiley. pp. 636.