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# Connectivity of Local Fusion Graphs for Finite Simple Groups

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**Abstract**

The main result proved here is that for a finite simple group $G$ and a $G$-conjugacy class of involutions $X$ the local fusion graph $\mathcal{F}(G, X)$ is a connected graph.

## 1 Introduction

Studying the action of a group on an appropriate set can reveal much about the group. An excellent illustration of this is Wielandt's proof [39] of Sylow's theorems. If the set carries additional structure preserved by the group action, then even further light may be shed on properties of the group. Examples of this are legion – the set could be a vector space, leading into the realms of representation theory, or some combinatorial object such as a graph. Many of the most interesting combinatorial objects may be defined in terms of certain internal properties of the group. For example buildings (for groups of Lie-type) may be obtained from the $BN$-pair configuration. And their near cousins, the Kac-Moody groups, yield yet more exotic structures, namely twin buildings [37]. While Cayley graphs (see [17]) have the group as their vertex set with adjacency determined by a generating set of the group. Others we mention are the collinearity graphs of group geometries (see [14]) and commuting graphs (see [6]-[11], [13], [23], [24], [36], [42]).

In this paper we investigate local fusion graphs for finite simple groups. We recall the definition of a local fusion graph, this also being a graph given by certain group theoretic data. Let $G$ be a finite group with $X$ a $G$-conjugacy class of involutions. Then the *local fusion graph* $\mathcal{F}(G, X)$ has vertex set $X$ with $x, y \in X$ joined by an edge whenever $x \neq y$ and the order of $xy$ is odd. Such graphs have been investigated in [3], [4], and [19], [20] study graphs of a similar nature. While in [5] local fusion graphs have been deployed in a computational context. Odd order products of involutions also feature frequently in [32]. Suppose that $x = x_1, x_2, \ldots, x_n = y$ is a path in the graph $\mathcal{F}(G, X)$, and let $k_i$ be the order of $x_i x_{i+1} (1 \leq i \leq n-1)$. Then the element $g_i = (x_i x_{i+1})^{(k_i - 1)/2}$ conjugates $x_i$ to $x_{i+1}$. Hence $g = g_1 g_2 \ldots g_{n-1}$ conjugates $x$ to $y$, and this is why such graphs are so-named.

The main theorem of this paper is as follows.

**Theorem 1.** *Suppose that $G$ is a finite simple group with $X$ a $G$-conjugacy class of involutions. Then $\mathcal{F}(G,X)$ is connected.*

We do not have to look very far to find examples of disconnected local fusion graphs. Suppose $H$ is a finite group and $Y$ an $H$-conjugacy class of involutions, and take $G$ to be the wreath product $H \wr \mathbb{Z}_2$. Let $H_1 \times H_2$ be the base group of $G$ with $H_i \cong H$ and $Y_i$ the $H_i$-conjugacy class of involutions corresponding to $Y$. Then $X = Y_1 \cup Y_2$ is a $G$-conjugacy class of involutions and in $\mathcal{F}(G,X)$ there are no edges between vertices in $Y_1$ and vertices in $Y_2$. Closer to Theorem 1, consider $G \cong \mathrm{Aut}(\mathrm{Sym}(6))$, where $\mathrm{Sym}(6)$ is the symmetric group of degree 6. Then $G$ contains a subgroup $H$ isomorphic to $\mathrm{Sym}(6)$ which contains the $H$-conjugacy classes $(1,2)^H$ and $(1,2)(3,4)(5,6)^H$. These two $H$-conjugacy classes fuse in $G$ and will be the two connected components of $\mathcal{F}(G,X)$ where $X = (1,2)^G$.

When $G$ is an abelian simple group, Theorem 1 trivially holds. So we only need consider $G$ a non abelian simple group and shall rely upon the Classification of the Finite Simple Groups, in as much as we will check through the list of non-abelian simple groups. We now discuss the organization of this paper and mention a few salient features of the various intermediate results. Our brief Section 2 establishes notation and assembles a number of results we shall need. Then Section 3 focusses upon the conjugacy classes of involutions in groups of Lie-type, giving representatives of these classes in addition to various of their properties, while Section 4 concerns the generation and subgroup structure of groups of Lie-type. The proof of Theorem 1 commences in Section 5 where we investigate the case when $G$ is a simple group of Lie-type of even characteristic. In fact, along the way, we also show connectivity of local fusion graphs for groups such as $SL_n(q)$ and $GL_n(q)$ ($q$ even) (see Theorem 31). For these even characteristic groups, our method for proving the local fusion graph $\mathcal{F}(G,X)$ is connected proceeds along the following lines. From Section 3 we first select a suitable representative involution $t \in X$. Let $Y$ be the connected component of $t$ in $\mathcal{F}(G,X)$. Next we seek a generating set $\mathcal{S}$ of $G$ for which we are able to show that for every $g \in \mathcal{S}$ we have $g \in \mathrm{Stab}_G(Y)$. Thus $G = \mathrm{Stab}_G(Y)$ whence, as $G$ acts vertex-transitively on $\mathcal{F}(G,X)$, $Y = X$ and so $\mathcal{F}(G,X)$ is connected. In Section 6 we move on to consider the simple groups of Lie-type of odd characteristic. For these groups the direction we take is heavily influenced by Theorem 48. This theorem asserts that if $X \cap T \neq \emptyset$ for $T$ a maximal split torus of $G$, then $\mathcal{F}(G,X)$ is connected. Thus much of our attention is directed to showing that $X \cap T \neq \emptyset$. There are, however, instances when $X \cap T = \emptyset$ (this occurs in some projective orthogonal groups, for example) and then alternative arguments must be employed. Section 7, by deploying Theorem 48, shows connectedness of the local fusion graphs of the linear groups of odd characteristic. Also in this section, in Lemma 54, we consider a graph very similar to a local fusion graph for $GL_2(q)$ and $GU_2(q)$ - this result is needed in Theorems 58, 65 and 75. Our next section looks at the odd characteristic symplectic groups, the main result being Theorem 58. Most of the attention in this theorem is focussed on the involution class in $PSp_{2m}(q)$ which is the image of a conjugacy class of elements of order 4 in $Sp_{2m}(q)$. In Section 9 we examine the unitary groups, and here we see different arguments are required depending on whether $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$. The former case is quickly wrapped up in Theorem 62. For the latter case we need, in Lemma 64, to deal separately with groups of the form $PSU_{2m}(3)$ before we can achieve our goals in Theorem 65 and Corollary 66. The local fusion graphs of the orthogonal groups occupy Section 10. Not surprisingly, we subdivide according to whether we have minus-type, plus-type or odd dimension. First in Corollary 69, Theorem 70 and Theorem 71 we examine $\Omega^{\pm}_{2m}(q)$, $SO^{\pm}_{2m}$, $\Omega_{2m+1}(q)$ and $SO_{2m+1}(q)$. Then we move onto the projective orthogonal groups of even dimension. Again we see a subdivision depending on the congruence of $q$ modulo 4. The case $q \equiv 1 \mod 4$ is resolved fairly quickly in Lemma 72 and Theorem 73, while $q \equiv 3 \mod 4$ is more involved - see Lemma 74 and Theorem 75. At the end of Section 10, courtesy of Theorem 48, the local fusion graphs of exceptional and twisted Lie-type groups are quickly shown to be connected. In Section 11 we rapidly deal with the sporadic simple groups and, as the alternating groups of degree at least 5 are covered by Theorem 9, the proof of Theorem 1 is complete.

# 2 Preliminary Results and Notation

Suppose that $G$ is a finite group and $X$ is a $G$-conjugacy class of involutions. Let $x, y \in \mathcal{F}(G, X)$ and $i \in \mathbb{N} \cup \{0\}$. We shall use $d(x, y)$ to denote the distance between $x$ and $y$ in $\mathcal{F}(G, X)$, and $\mathrm{Diam}(\mathcal{F}(G, X))$ to denote the diameter of $\mathcal{F}(G, X)$. The *i-th disc* of $\mathcal{F}(G, X)$, $\Delta_i(x)$, is defined by

$$\Delta_i(x) = \{y \in X \mid d(x, y) = i\}.$$

Thus $\Delta_0(x) = \{x\}$, while $\Delta_1(x)$ consists of all the neighbours of $x$ in $\mathcal{F}(G, X)$. It is possible to extract $|\Delta_1(x)|$ from the complex irreducible characters of $G$. We briefly recap the relevant theory underpinning this approach.

Let $G$ be a finite group, with conjugacy classes $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ and corresponding class sums $K_1, \ldots, K_\ell$ in the group algebra $\mathbb{C}G$. Let $a_{ijk}$, the so-called structure constants, be defined by

$$K_i K_j = \sum_{k=1}^{\ell} a_{ijk} K_k.$$

Letting $g_k \in \mathcal{K}_k$ for $k = 1, \ldots, \ell$ we have that

$$a_{ijk} = \frac{|\mathcal{K}_i||\mathcal{K}_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)},$$

where $\mathrm{Irr}(G)$ denotes the set of complex irreducible characters of $G$. See Theorem 4.2.12 of [27] for a proof of this formula. Thus the $a_{ijk}$'s may be computed from the character table of $G$. Using the above notation we have the following lemma.

**Lemma 2.** *Suppose $G$ is a finite group with $X$ a $G$-conjugacy class of involutions. Assume that $X = \mathcal{K}_i$. Then for $x \in X$ we have*

$$|\Delta_1(x)| = \sum_j a_{jii},$$

*where the sum is over all $j$ such that the conjugacy class $\mathcal{K}_j$ contains elements of odd order (excluding the conjugacy class of the identity element).*

*Proof.* Let $x \in X$. Then $a_{jii}$ is the number of pairs $(z, y)$ where $z \in \mathcal{K}_j$ and $y \in \mathcal{K}_i = X$ are such that $zy = x$. So, letting $\mathcal{K}_j$ run over all $G$-conjugacy classes of non-trivial odd order elements, $\sum_j a_{jii}$ is the number of $y \in X$ such that $xy$ has odd order, whence the lemma holds. $\square$

**Lemma 3.** *Suppose that $G$ is a finite group with $X$ a $G$-conjugacy class of involutions. Let $x \in X$. If $|\Delta_1(x)| > |X|/2$, then $\mathcal{F}(G, X)$ is connected and* $\mathrm{Diam}(\mathcal{F}(G, X)) \leq 2$.

*Proof.* Since $|\Delta_1(x)| > |X|/2$, the regularity of $\mathcal{F}(G, X)$ implies connectedness. Suppose there exists $y \in X$ such that $d(x, y) = 3$. Then $\Delta_1(x) \cap \Delta_1(y) = \emptyset$, since otherwise $d(x, y) \leq 2$. Therefore

$$|\Delta_1(x)| \leq |X| - |\Delta_1(y)| = |X| - |\Delta_1(x)|$$

by regularity. Hence $|\Delta_1(x)| \leq |X|/2$, a contradiction. Thus the diameter of $\mathcal{F}(G, X)$ is at most 2. $\square$

We shall also need some results regarding local fusion graphs of direct products and normal subgroups, along with an observation regarding involution centralizers.

**Lemma 4.** *Suppose $G = G_1 \times G_2$ and $t$ is an involution in $G$. Let $\pi_i : G \to G_i$ $(i = 1, 2)$ be the projection maps. Set $X = t^G$ and $X_i = \pi_i(t)^{G_i}$. If $\mathcal{F}(G_i, X_i)$ is connected for $i = 1, 2$, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Note that $X = \{(x_1, x_2) \mid x_i \in X_i\}$. Also if $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, with $x \neq y$, then $x$ and $y$ are adjacent in $\mathcal{F}(G, X)$ if, and only if, $x_i$ and $y_i$ are adjacent in $\mathcal{F}(G_i, X_i)$ for $i = 1, 2$, or for $i \in \{1, 2\}$ $x_i = y_i$ with $x_{3-i}$ and $y_{3-i}$ adjacent in $\mathcal{F}(G, X_{3-i})$. Hence the lemma holds. $\square$

**Lemma 5.** *If all the local fusion graphs of a group $G$ are connected, then the local fusion graphs of its normal subgroups are also connected.*

*Proof.* Let $N$ be a normal subgroup of $G$ and $X$ an $N$-conjugacy class of involutions. We claim that $X$ is also a $G$-conjugacy class. Let $Y$ be the involution $G$-conjugacy class containing $X$. Since $N \trianglelefteq G$, $Y \subseteq N$. If $y_1, y_2 \in Y$ with $y_1$ and $y_2$ adjacent in $\mathcal{F}(G, Y)$, then $y_1$ and $y_2$ are conjugate in $\langle y_1, y_2 \rangle$. Because $\langle y_1, y_2 \rangle \leq \langle Y \rangle \leq N$ and $\mathcal{F}(G, Y)$ is connected by assumption, it follows that $X = Y$. Hence $\mathcal{F}(N, X)$ is connected, as desired. $\square$

**Lemma 6.** *Suppose $G$ is a finite group containing an involution $t$. Set $X = t^G$. If $t \notin O_2(G)$ and $C_G(t)$ is a maximal subgroup of $G$, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $Y$ denote the connected component of $\mathcal{F}(G, X)$ which contains $t$. Since $C_G(t)$ is a maximal subgroup of $G$, $G$ acts primitively on the vertices of $\mathcal{F}(G, X)$. Hence, as $Y$ is a block of imprimitivity, either $|Y| = 1$ or $Y = X$. Now $t \notin O_2(G)$ implies, by the Baer-Suzuki theorem (Theorem 8.2 of [27]), that there exists $x \in X$ such that $\langle t, x\rangle$ is not a 2-group. Therefore $t$ inverts a non-trivial subgroup of odd order, whence $|Y| \neq 1$. Thus $Y = X$ and the lemma holds. □

**Lemma 7.** *Let $\phi$ be an involutary automorphism of $H$, a finite group of odd order, and set $I = \{h \in H \mid h^\phi = h^{-1}\}$. Then $H = C_H(\phi)I$.*

*Proof.* See Lemma 10.4.1(i) in [27]. □

**Lemma 8.** *Suppose $G$ is a finite group containing an involution $t$. Set $X = t^G$, and let $Y$ be the connected component of $\mathcal{F}(G, X)$ which contains $t$. Set $M = \mathrm{Stab}_G(Y)$.*

*(i) For all $y \in Y$, $C_G(y) \leq M$, and in particular $\langle Y\rangle \leq M$.*

*(ii) $Y = t^M$.*

*(iii) Let $y \in Y$. If $H$ is a subgroup of $G$ of odd order which is normalized by $y$, then $H \leq M$.*

*Proof.* The proof of (i) is clear. Since $Y$ is, by definition, $M$-invariant under conjugation, $t^M \subseteq Y$. For $y_1, y_2 \in Y$ which are adjacent in $\mathcal{F}(G, X)$, $y_1$ and $y_2$ are conjugate in $\langle y_1, y_2\rangle$. So, as $\langle y_1, y_2\rangle \leq M$ by (i), $y_1$ and $y_2$ are $M$-conjugate. Hence $t^M = Y$, proving (ii). For the final part, by Lemma 7, $H = C_H(y)I$ where $I = \{h \in H \mid h^y = h^{-1}\}$. For $h \in I$, $\langle y, y^h\rangle$ is a dihedral group of order $2n$, where $n$ is the order of $h$. Note that $y^h$ is $G$-conjugate to $y$ and so $y^h \in X$. Also $y$ and $y^h$ are adjacent in $\mathcal{F}(G, X)$ and thus $y^h \in Y$. By (i), $\langle y, y^h\rangle \leq M$ and hence $h \in M$. Therefore $H = C_H(y)I \leq M$, so proving (iii). □

**Theorem 9.** *Suppose that $G = \mathrm{Sym}(n)$ with $n \geq 5$ and $X$ is a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected with*

$$\mathrm{Diam}(\mathcal{F}(G, X)) = 2.$$

*Proof.* See Theorem 1.1 of [4].

□

In dealing with the classical groups we shall often exploit the geometric structures afforded by their natural modules. We now recall some standard concepts and terminology in this area, using notation from [35]. Let $V$ be a vector space over a field $k$, and assume that $\dim V \geq 3$. Recall that the symplectic, unitary and orthogonal classical groups may be defined as isometry groups of alternating, unitary and orthogonal forms, respectively.

**Definition 10.** *Let $V$ be a vector space equipped with an alternating, symmetric or unitary form $\beta$.*

*(i) We say a non-zero vector $u \in V$ is* isotropic *if $\beta(u,u) = 0$.*

*(ii) A subspace $W \subseteq V$ is called* totally isotropic *if $W \subseteq W^{\perp}$.*

*(iii) A pair of vectors $(u,v)$ such that $u$ and $v$ are isotropic and $\beta(u,v) = 1$ is called a* hyperbolic pair*. The line $\langle u,v \rangle$ in the projective geometry $\mathcal{P}(V)$ is called a* hyperbolic line*.*

*(iv) A subspace $W \subseteq V$ is* non-degenerate *if $W \cap W^{\perp} = 0$.*

*(v) If $V = U \oplus W$ and $\beta(u,w) = 0$ for all $u \in U$ and $w \in W$, we say that $V$ is the* orthogonal *direct sum of $U$ and $W$, and write $V = U \perp W$.*

*Now suppose that $V$ is equipped with a quadratic form $Q$.*

*(i) A non-zero vector $v \in V$ is called* singular *if $Q(u) = 0$.*

*(ii) A subspace $W \subseteq V$ is called* totally singular *if $Q(w) = 0$ for all $w \in W$.*

The following lemma allows us in many situations to choose a 'nice' basis for our vector space $V$.

**Lemma 11.** *If $U$ and $W$ are totally isotropic (respectively totally singular) subspaces of $V$ such that $U^{\perp} \cap W = \{0\}$, then there is a totally isotropic (respectively totally singular) subspace $U'$ containing $W$ such that $V = U^{\perp} \oplus U'$. Moreover, for each basis $u_1, u_2, \ldots, u_k$ of $U$, there is a unique basis $u'_1, u'_2, \ldots, u'_k$ of $U'$ such that $\langle u_1, u'_1 \rangle, \langle u_2, u'_2 \rangle, \ldots, \langle u_k, u'_k \rangle$ are mutually orthogonal hyperbolic pairs.*

*Proof.* This may be found in Lemma 7.5 of [35]. □

A *flag* of a projective geometry $\mathcal{P}(V)$ is a chain of distinct subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_k,$$

and a flag is called *proper* if neither 0 nor $V$ occurs in the chain. The *type* of such a flag is the set $\{d_1, \ldots, d_k\}$, where $d_i = \dim(V_i)$ for $i = 1, \ldots, k$. If $\dim(V) = m$, a *maximal* flag is one of type $\{1, 2, \ldots, m-1\}$. In a polar geometry $(\mathcal{P}, \pi)$ (as defined in [35]), the flags are defined to be those flags of $\mathcal{P}(V)$ which are fixed by $\pi$. These can be identified with the flags of totally isotropic subspaces.

We can now state the following important result, which gives a geometric characterisation of the Borel subgroups of the majority of finite classical groups.

**Theorem 12.** *Suppose $G$ is a classical group which acts on a $GF(q)$-vector space $V$. If $G$ is an orthogonal group assume that $q$ is odd and $G$ is not of plus type. Then the Borel subgroups of $G$ are precisely the stabilizers of maximal flags in a suitable polar geometry $(\mathcal{P}(V), \pi)$.*

The Borel subgroups of some of the orthogonal groups not included in Theorem 12 will be described when required in Section 6.

We end this section with a few general comments about notation. Almost all our group-theoretic notation is standard as laid out, for example, in [1], [27] and [34]. For describing specific group structures we follow the Atlas [18], with some exceptions which we now note. So we use $\mathrm{Sym}(n)$ (as we already have) and $\mathrm{Alt}(n)$ to denote the symmetric and alternating groups of degree $n$. The dihedral group of order $n$ is denoted $\mathrm{Dih}(n)$. While $\Omega_n^{\pm}(q)$ will denote the derived subgroup of $SO_n^{\pm}(q)$, and $P\Omega_n^{\pm}(q)$ will denote the simple orthogonal groups (rather than $O_n^{\pm}(q)$ as in [18]). The symbol $\sim$ will be used to indicate that two groups have the same shape and $\circ$ is used to denote a central product construction.

# 3 Involutions in Groups of Lie-type

The majority of the material in this section concerns groups of Lie-type defined over fields of even characteristic, and comes almost exclusively from the paper by Aschbacher and Seitz [2]. Let $V$ be an $n$-dimensional vector space over $GF(q)$, where $q$ is even. Suppose we have an involution $t \in SL(V)$. The *rank* of $t$ is defined to be the dimension of the commutator space $[V,t]$ of $t$. The following result is well known.

**Lemma 13.** *Let $x, y \in SL(V)$ be involutions. Then $x$ and $y$ are conjugate in $SL(V)$ if and only if they have the same rank.*

Now fix an ordered basis for $V$. Given an integer $\ell$ such that $1 \leq \ell \leq n/2$, we define the involution $j_\ell$ of $SL(V)$ to be

$$j_\ell = \begin{pmatrix} I_\ell & & \\ & I_{n-2\ell} & \\ I_\ell & & I_\ell \end{pmatrix},$$

where $I_m$ is the $m \times m$ identity matrix, and $I_0$ is defined to be the 'empty' matrix. Then $j_\ell$ has rank $\ell$, and is referred to as the *Suzuki form* of its involution conjugacy class.

**Lemma 14.** *The involutions $j_\ell$, where $1 \leq \ell \leq n/2$, form a complete set of representatives for the conjugacy classes of involutions in $SL(V)$.*

*Proof.* This follows as there are exactly $\lfloor n/2 \rfloor$ possibilities for the rank of an involution in $SL(V)$. $\square$

We shall also require a further set of representatives for involution conjugacy classes in $SL(V)$.

**Lemma 15.** *Let $H = SL(V)$ and, for $1 \leq \ell \leq \lfloor n/2 \rfloor$, define*

$$x_\ell = \begin{pmatrix} B_\ell & \\ & I_{n-2\ell} \end{pmatrix},$$

*where $B_\ell$ has $\ell$ $2 \times 2$ blocks*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*along its main diagonal, and zeroes elsewhere. Then*

$$\mathcal{I} = \{x_\ell \mid 1 \le \ell \le \lfloor n/2 \rfloor\}$$

*is a complete set of representatives for the conjugacy classes of involutions in $H$.*

*Proof.* Since the elements of $\mathcal{I}$ have distinct ranks, the result follows from Lemma 13. □

Now suppose that $V$ is a symplectic space of dimension $n = 2m$, with symplectic form $\beta$. If $t$ is an involution in $Sp(V)$, we define

$$V(t) = \{v \in V \mid \beta(v, v^t) = 0\}.$$

Also define $E_{2k}$ to be the $2k \times 2k$ matrix with 1 in the $(2i, 2i-1)$ and $(2i-1, 2i)$ positions and zeroes elsewhere ($i = 1, \ldots, k$). We shall make extensive use of the next result later in Section 5.

**Theorem 16.** *Let $t$ be an involution in $Sp(V)$ of rank $\ell$. Then there exists a basis $\{e_1, \ldots, e_n\}$ of $V$ such that $\beta$ has Gram matrix*

$$J = \begin{pmatrix} & & F \\ & E_{n-2\ell} & \\ F & & \end{pmatrix}$$

*in which $t = j_\ell$ and exactly one of the following holds:*

*(i) $\ell$ is even, $V = V(t)$ and $F = E_\ell$;*

*(ii) $\ell$ is odd, $V(t) = \langle e_i \mid i \ne n - \ell + 1\rangle$, $V(t)^\perp = \langle e_1 \rangle$,*

$$[V(t), t]^\perp = \langle e_i \mid 1 \le i \le n - \ell + 1\rangle,$$

$$[V(t), t] = \langle e_i \mid 1 \le i \le \ell\rangle,$$

$$F = \begin{pmatrix} 1 & \\ & E_{\ell-1} \end{pmatrix}.$$

*(iii) $\ell$ is even, $V(t) = \langle e_i \mid 1 \le i < n\rangle$, $V(t)^\perp = \langle e_1 \rangle$,*

$$[V(t), t]^\perp = \langle e_i \mid 1 \le i \le n - \ell + 1\rangle,$$

$$[V(t), t] = \langle e_i \mid 1 \le i < l\rangle,$$

$$F = \begin{pmatrix} & & 1 \\ & E_{\ell-2} & \\ 1 & & 1 \end{pmatrix}.$$

*Proof.* See 7.6 of [2]. □

An involution $t \in Sp(V)$ is said to be in *symplectic Suzuki form* if the basis for $V$ is chosen as in Theorem 16. We also denote by $a_\ell$, $b_\ell$ and $c_\ell$ the Suzuki forms in parts (i), (ii) and (iii) of Theorem 16.

**Theorem 17.** *Let $t$ and $s$ be involutions in $Sp(V)$. Then the following are equivalent:*

*(i) $t$ is conjugate to $s$ in $Sp(V)$;*

*(ii) $t$ and $s$ have the same symplectic Suzuki form; and*

*(iii) $t$ and $s$ have the same rank $\ell$, and if $\ell$ is even then $V(t)$ and $V(s)$ have the same dimension.*

*Proof.* This follows from Theorem 16 and the fact that $Sp(V)$ is transitive on the set of ordered symplectic bases of $V$ (see [35]). □

We now drop our requirement that $q$ be even. To help determine conjugacy in unitary groups there is the following result:

**Theorem 18.** *Elements of $GU_n(q)$ are conjugate in $GU_n(q)$ if, and only if, they are conjugate in $GL_n(q^2)$.*

*Proof.* This is proved by G. E. Wall in [38]. □

**Theorem 19.** *Suppose that $G$ is a finite group of Lie-type defined over a field of odd characteristic, with $t \in G$ an involution. If $G = GL_n(q)$, $GU_n(q)$ or $Sp_n(q)$, or $G$ is a simple exceptional or twisted group of Lie-type, then $t$ lies in a maximal split torus of $G$.*

*Proof.* If $G$ is a classical group acting naturally on a vector space $V$, the diagonal subgroup forms a maximal split torus of $G$. Suppose then that $G = GL_n(q) = GL(V)$. Since $t$ is an involution, its only eigenvalues are $\pm 1$. As $q$ is odd, $\pm 1 \in GF(q)$, so $t$ is $G$-conjugate to a diagonal element, and thus lies in a maximal split torus of $G$. Now let $G = GU_n(q) = GU_n(V)$. If we canonically embed $G$ into $GL_n(q^2)$, then by Theorem 18 $G$ has at most the number of involution conjugacy classes as $GL_n(q^2)$. As diagonal involutions of $GL_n(q^2)$ will also lie in $G$, $t$ must be conjugate to a diagonal element, as required. If $G = Sp_n(q)$, then it is straighforward to show that $t$ must stabilize a maximal isotropic flag of $V$, and so lies in a Borel subgroup of $G$ by Theorem 12, and consequently lies in a maximal split torus of $G$.

If $G$ is a simple exceptional group of Lie-type, we refer to the representations given in Chapter 4 of [40], in which it is straighforward to construct involutions of each $G$-conjugacy class which lie in a maximal split torus. □

# 4 Generation and Subgroups of Lie-type Groups

First we identify generating sets in the matrix groups $SL(V)$, $Sp(V)$ and $SU(V)$ which we will employ in the proof of Theorem 1. For the next three results we use $E_{ij}$ to denote a square matrix of an appropriate size whose $(i,j)^{\text{th}}$ entry is 1 and the remaining entries are 0.

**Theorem 20.** *If $G = SL_n(q) \cong SL(V)$, then $G$ is generated by the set*

$$\mathcal{A} = \{I + \lambda E_{ij} \mid i \neq j, \lambda \in GF(q)\}.$$

*Moreover, if $n \geq 4$, then when $i$ is odd and $j = i+1$, and when $i$ is even and $j = i-1$, we may exclude the corresponding matrices from $\mathcal{A}$ and the resulting set $\mathcal{A}'$ still generates $G$.*

*Proof.* It is well known that the set $\mathcal{A}$ generates $G$ (see [34], for example). Now let $I + \lambda E_{ij} \in \mathcal{A}$ be such that $i$ is odd and $j = i+1$. Then an easy matrix calculation shows that, when $i \neq k \neq j$,

$$I + \lambda E_{ij} = (I + E_{ik})(I + \lambda E_{kj})(I - E_{ik})(I - \lambda E_{kj}),$$

and due to the restrictions on $i, j$, and our assumption that $n \geq 4$, it is possible to choose $k$ such that all the matrices on the right hand side of this equation lie in $\mathcal{A}'$. A similar equality holds when $i$ is even and $j = i-1$, so proving the theorem. $\square$

**Theorem 21.** *Let $G = Sp_{2n}(q) \cong Sp(V)$, where the symplectic form $\beta$ on $V$ has Gram matrix*

$$J = \left(\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right).$$

*Then $G$ is generated by the matrices*

$$\left(\begin{array}{c|c} I_n & \lambda E_{ii} \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \lambda E_{ii} & I_n \end{array}\right),$$

$$\left(\begin{array}{c|c} I_n & \lambda(E_{ij} + E_{ji}) \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \lambda(E_{ij} + E_{ji}) & I_n \end{array}\right), \lambda \in GF(q).$$

*Proof.* Looking in Section 2.2 of [31], we see that $G$ is certainly generated by the matrices given above along with those of the form

$$\left(\begin{array}{c|c} I_n + \lambda E_{ij} & \\ \hline & I_n - \lambda E_{ji} \end{array}\right).$$

However, a straightforward matrix calculation shows that, when $\lambda \neq 0$,

$$\left(\begin{array}{c|c} I_n + \lambda E_{ij} & \\ \hline & I_n - \lambda E_{ji} \end{array}\right) = \left(\begin{array}{c|c} I_n & \\ \hline E_{ij} + E_{ji} & I_n \end{array}\right)^{gh}$$

where

$$g = \left(\begin{array}{c|c} I_n & -\lambda E_{ii} \\ \hline & I_n \end{array}\right)$$

and

$$h = \left(\begin{array}{c|c} I_n & \\ \hline \lambda^{-1}E_{ii} & I_n \end{array}\right),$$

where $i < j$. A similar relation holds when $i > j$, which completes the proof. $\square$

**Theorem 22.** *Let $G = SU_{2n}(q) \cong SU(V)$ where $q$ is even, and the unitary form $\beta$ on $V$ has Gram matrix*

$$J = \left(\begin{array}{c|c} & I_n \\ \hline I_n & \end{array}\right).$$

*Denote by $\tau$ the involutary automorphism of $GF(q^2)$ associated to $\beta$. Then $G$ is generated by the matrices*

$$\left(\begin{array}{c|c} I_n & \mu E_{ii} \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \mu E_{ii} & I_n \end{array}\right),$$

$$\left(\begin{array}{c|c} I_n & \lambda E_{ij} + \lambda^\tau E_{ji} \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \lambda E_{ij} + \lambda^\tau E_{ji} & I_n \end{array}\right),$$

*where $\mu, \lambda \in GF(q^2)$, and $\mu + \mu^\tau = 0$.*

*Proof.* Write $G_0$ for the subgroup of $G$ which is generated by the matrices in the statement of the result, along with those of the form

$$\left(\begin{array}{c|c} I_n + \lambda E_{ij} & \\ \hline & I_n + \lambda^\tau E_{ji} \end{array}\right).$$

We first show that $G = G_0$. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ be the basis for $V$, where $\langle e_i, f_i \rangle$ are hyperbolic lines for $i = 1, \ldots, n$. From Corollary 10.10 of [35], $G$ is generated by *unitary transvections*. These unitary transvections correspond to the isotropic vectors in $V$. Given an isotropic vector $u \in V$, and a scalar $a \in GF(q^2)$ which satisfies $a + a^\tau = 0$, the unitary transvection $t_{u,a}$ is the linear transformation given by

$$t_{u,a}(v) = v + a\beta(v, u)u$$

for all $v \in V$. If we can show that $G_0$ contains all unitary transvections, the claim will follow. Certainly $G_0$ contains the transvections which correspond to the isotropic vector $e_1$. Thus, if we can show $G_0$ acts transitively on the isotropic vectors in $V$, then it will follow that $G = G_0$.

Let

$$v = a_1e_1 + \cdots + a_ne_n + b_1f_n + \cdots + b_nf_n$$

be isotropic, where $a_i, b_i \in GF(q^2)$ are not all zero. We show that we can find a matrix in $G_0$ which sends $e_1$ to $v$. Certainly a matrix in $G_0$ with its first column equal to

$$(a_1, \dots, a_n, b_1, \dots, b_n)$$

will suffice. Firstly, note that $G_0$ contains elements

$$\left(\begin{array}{c|c} A & \\ \hline & A^* \end{array}\right),$$

where $A$ runs over all elements of $SL_n(q^2)$, and $A^*$ depends on our choice of $A$. Suppose all the $a_i$ are 0 in our expression for $v$. Then pick $A$ with first column equal to

$$(b_1, \dots, b_n),$$

and note that

$$\left(\begin{array}{c|c} 0 & A^* \\ \hline A & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right) \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A^* \end{array}\right).$$

Since, as $q$ is even,

$$\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right) = \left(\begin{array}{c|c} I_n & I_n \\ \hline 0 & I_n \end{array}\right) \left(\begin{array}{c|c} I_n & 0 \\ \hline I_n & I_n \end{array}\right) \left(\begin{array}{c|c} I_n & I_n \\ \hline 0 & I_n \end{array}\right)$$

it now follows that

$$\left(\begin{array}{c|c} 0 & A^* \\ \hline A & 0 \end{array}\right) \in G_0.$$

Now suppose we have $a_i \neq 0$ for some $i$. Let $A$ be chosen with first column equal to

$$(a_1, \dots, a_n).$$

Let $B$ be the $n \times n$ matrix with $i$-th row and column as follows:

$$\left(\begin{array}{ccccc} & & b_1 a_i^{-1} & & \\ & & \vdots & & \\ (b_1 a_i^{-1})^\tau & \cdots & x & \cdots & (b_n a_i^{-1})^\tau \\ & & \vdots & & \\ & & b_n a_i^{-1} & & \end{array}\right),$$

where $x$ is given by

$$(b_1 a_i^{-1})^\tau a_1 + \cdots + x a_i + \cdots + (b_n a_i^{-1})^\tau a_n = b_i.$$

We require that $x + x^\tau = 0$, to yield

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline B & I_n \end{array}\right) \in G_0.$$

But a straightforward check shows that

$$x = a_i^{-1} (a_i^{-1})^\tau \left[ a_i^\tau b_i + a_1 b_1^\tau + \cdots + a_{i-1} b_{i-1}^\tau + a_{i+1} b_{i+1}^\tau + \cdots + a_n b_n^\tau \right],$$

and since $v$ is isotropic, we have

$$a_1 b_1^\tau + \cdots a_n b_n^\tau + a_1^\tau b_1 + \cdots + a_n^\tau b_n = 0,$$

from whence a rearrangement of terms demonstrates $x + x^\tau = 0$ as desired. Now the first column of

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline B & I_n \end{array}\right) \left(\begin{array}{c|c} A & \\ \hline & A^* \end{array}\right) \in G_0$$

is equal to

$$(a_1, \ldots, a_n, b_1, \ldots, b_n).$$

This proves that $G_0 = G$. To complete the proof, we note that a straightforward matrix calculation shows that

$$\left(\begin{array}{c|c} I_n + \lambda E_{ij} & \\ \hline & I_n + \lambda^\tau E_{ji} \end{array}\right) = \left(\begin{array}{c|c} I_n & \\ \hline \lambda E_{ij} + \lambda^\tau E_{ji} & I_n \end{array}\right)^{gh},$$

where

$$g = \left(\begin{array}{c|c} I_n & E_{ii} \\ \hline & I_n \end{array}\right)$$

and

$$h = \left(\begin{array}{c|c} I_n & \\ \hline E_{ii} & I_n \end{array}\right),$$

so matrices of this form are not required in our generating set, whence the result holds. □

When investigating the local fusion graphs of finite groups of Lie-type, we shall require some knowledge of the maximal subgroup structure of such groups. We first have the following result concerning the centralizers of field automorphisms.

**Theorem 23.** *Let $G = G(q)$ be a finite, simple group of Lie-type, defined over the field $GF(q)$. Suppose we may write $q = q_0^r$ where $r$ is prime. Then $G(q_0)$ is a maximal subgroup of $G$, where $G(q_0)$ denotes the finite group of the same type defined over the field $GF(q_0)$.*

*Proof.* This is an immediate consequence of Theorem 1 in [15]. □

When dealing with classical groups, more detailed information regarding maximal subgroups will be required. In [30], Kleidman and Liebeck determine the maximal subgroup structure of the finite classical groups with dimension at least 13, while the lower dimensional cases are covered in [29]. We shall need but a fraction of the information contained in this source, and include only the relevant results here.

**Theorem 24.** *Let $H = Sp_n(q)$ and $\overline{H} = PSp_n(q)$, where $n \geq 4$. Suppose $M = \mathrm{Stab}_H(W)$, where $W \subseteq V$ is a non-degenerate subspace of even dimension*

*$m$, with $2 \leq m < n/2$. Then $M$ and $\overline{M}$ are maximal subgroups of $H$ and $\overline{H}$ respectively. Moreover,*

$$M \cong Sp_m(q) \times Sp_{n-m}(q)$$

*and*

$$\overline{M} \cong Sp_m(q) \circ Sp_{n-m}(q).$$

*Proof.* See Proposition 4.1.3 of [30]. □

**Theorem 25.** *Let $H = SU_n(q)$ and $\overline{H} = PSU_n(q)$, where $n \geq 3$. Suppose $M = \mathrm{Stab}_H(W)$, where $W \subseteq V$ is a non-degenerate subspace of dimension $m < n/2$. Then $M$ and $\overline{M}$ are maximal subgroups of $H$ and $\overline{H}$ respectively. Moreover,*

$$SU_m \times SU_{n-m}(q) \leq M \leq GU_m \times GU_{n-m}(q).$$

*Proof.* See Proposition 4.1.4 of [30]. □

**Theorem 26.** *Let $H = SU_n(q)$ and $\overline{H} = PSU_n(q)$, where $n \geq 3$ and $q$ is odd. Suppose that $M$ is the stabilizer of a decomposition*

$$V = W_1 \perp W_2 \perp \ldots \perp W_k$$

*where $\mathrm{dim} W_i = m < n$, for $1 \leq i \leq k$. Then $M$ and $\overline{M}$ are maximal subgroups of $H$ and $\overline{H}$ respectively, and*

$$SU_m(q) \wr \mathrm{Sym}(k) \leq M \leq GU_m(q) \wr \mathrm{Sym}(k).$$

*Proof.* See Proposition 4.2.9 of [30]. □

**Theorem 27.** *Let $H = SO_n^{\epsilon}(q)$ and $\overline{H} = PSO_n^{\epsilon}(q)$, where $q$ is odd, $n \geq 8$ and $\epsilon = \pm$. Suppose $M = \mathrm{Stab}_H(W)$, where $W \subseteq V$ is a totally singular subspace of dimension $m < n/2 - 1$. Then $M$ and $\overline{M}$ are maximal subgroups of $H$ and $\overline{H}$ respectively. Moreover,*

$$M \sim [q^a] : (GL_m(q) \times SO_{n-2m}^{\epsilon}(q))$$

*and*

$$\overline{M} \sim [q^a] : (GL_m(q) \circ SO_{n-2m}^{\epsilon}(q)),$$

*where $a = mn - \frac{m}{2}(3m+1)$.*

*Proof.* See Proposition 4.1.20 of [30]. □

**Theorem 28.** *Let $H = \Omega_n^+(q)$, where $q$ is even and $n \equiv 0 \bmod 4$. Suppose $M = \mathrm{Stab}_H(W \oplus W')$, where $V = W \oplus W'$ is a decomposition of $V$ into totally singular subspaces of dimension $n/2$. Then $M$ is a maximal subgroup of $H$. Moreover,*

$$M \sim GL_{n/2}(q).2,$$

*where the outer automorphism of $GL_{n/2}(q)$ is an involution $\sigma$ which has the effect of interchanging $W$ and $W'$.*

*Proof.* See Proposition 4.1.20 of [30]. $\square$

**Theorem 29.** *Let $H = SO_n^\epsilon(q)$ and $\overline{H} = PSO_n^\epsilon(q)$, where $n \geq 8$ and $\epsilon = \pm$. Suppose $M = \mathrm{Stab}_H(W)$, where $W \subseteq V$ is a non-degenerate subspace of dimension $m$ and type $\eta = \pm$. In addition, assume that $m < n/2$ and that when $q \leq 3$ we have $(\eta, m) \neq (+, 2)$. Then $M$ and $\overline{M}$ are maximal subgroups of $H$ and $\overline{H}$ respectively. Moreover,*

*when $\epsilon = +$ we have*

$$SO_m^\eta(q) \times SO_{n-m}^\eta(q) \leq M \leq GO_m^\eta(q) \times GO_{n-m}^\eta(q); \text{ and}$$

*when $\epsilon = -$ we have*

$$SO_m^\eta(q) \times SO_{n-m}^{-\eta}(q) \leq M \leq GO_m^\eta(q) \times GO_{n-m}^{-\eta}(q).$$

*Proof.* See Proposition 4.1.6 of [30]. $\square$

# 5 The Even Characteristic Case

Throughout this section we assume $q$ is even, and start by examining the local fusion graphs of linear groups.

## 5.1 Linear Groups

Let $H = SL_n(q) \cong SL(V)$ with $n \geq 2$. First we have a lemma concerning our set of involution representatives $\mathcal{I}$ from Lemma 15.

**Lemma 30.** *Let $x_i \in \mathcal{I}$ be a representative of an $H$-conjugacy class $X$, and let $x_i'$ be equal to $x_i$ but with at least one $2 \times 2$ block $B_i$ transposed. Then $x_i$ and $x_i'$ are adjacent in $\mathcal{F}(H, X)$.*

*Proof.* Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

it is easy to see that there exists an element of $H$, built up of suitable blocks, which conjugates $x_i$ to $x_i'$. Furthermore, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

which has order 3, and it follows that $x_i x_i'$ also has order 3. Hence $x_i$ and $x_i'$ are adjacent in $\mathcal{F}(H, X)$. $\square$

**Theorem 31.** *If $H = SL_n(q)$ or $GL_n(q)$, and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

*Proof.* Since $q$ is even, any involution of $GL_n(q)$ has determinant 1, so lies in $SL_n(q)$. Moreover, it is clear that no involution classes of $SL_n(q)$ fuse in $GL_n(q)$. Hence it suffices to prove the result for $H = SL_n(q)$. By Lemma 15 we have $x_k \in X$ for some $k$. Denote by $Y$ the connected component of $\mathcal{F}(H, X)$ which contains $x_k$. From Lemma 8(i), for any $y \in Y$, $C_H(y) \leq \mathrm{Stab}_G(Y)$. By Theorem 20, the set

$$\mathcal{A} = \{I_n + \lambda E_{ij} \mid i \neq j, \lambda \in GF(q)\}$$

generates $H$ (where $\{E_{ij}\}$ are elementary $n \times n$ matrices). We claim that if $a \in \mathcal{A}$, then $a \in C_H(y)$ for some $y \in Y$. Indeed, let $a = I_n + \lambda E_{ij}$. It is easy to check that $a$ centralizes $x_k$ if and only if the $i$-th column and $j$-th row of $x_k$ contain exactly one non-zero entry each. If this is not the case, by transposing a suitable $2 \times 2$ block $B_i$, we obtain an element $x_k'$ which is centralized by $a$. Moreover, by Lemma 32, $x_k'$ is adjacent to $x_k$ in $\mathcal{F}(H, X)$, so is certainly in $Y$. Hence

$$\mathcal{A} \subseteq \langle C_H(y) \mid y \in Y \rangle \leq \mathrm{Stab}_H(Y).$$

Since $\mathcal{A}$ generates $H$, we have that $H = \mathrm{Stab}_H(Y)$. But $H$ acts transitively on $X$, implying that $Y = X$, so $\mathcal{F}(H, X)$ is connected. $\square$

In preparation for dealing with the projective groups $PSL_n(q)$, we require a lemma regarding $GL_2(q)$.

**Lemma 32.** *Let $K = GL_2(q) \cong GL(V)$, and suppose $g \in K$ is an element of even order such that $g^2 \in Z(K)$. Then $g$ is $K$-conjugate to an element $y$ such that the product $yy^T$ has odd order (where $y^T$ denotes the transpose of $y$).*

*Proof.* Observe that

$$Z = Z(K) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in GF(q)^* \right\},$$

and hence $|Z| = q - 1$, which is odd. Let $SL_2(q) \cong H \leq K$, and note that $Z \cap H = 1$. We therefore have $K = ZH$. Furthermore, we know that any element of even order in $H$ must be an involution. Hence if $x \in K$ has even order, then $x = zy$ for some $z \in Z$ and $y \in H$ an involution. Thus $x^2 \in Z$, so

$$x^2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

for some $\lambda \in GF(q)^*$. Thus the minimal polynomial of $x$ is $\chi^2 + \lambda^2$.

Now suppose $g \in K$ is such that $g^2 \in Z$. Then by the above argument $g$ has minimal polynomial $\chi^2 + \alpha$ for some $\alpha \in GF(q)^*$. Choose $v \in V$ such that $v^g = \mu w$, where $w \notin \langle v \rangle$ and $\mu^2 = \alpha$ (this must be possible since otherwise $g$ would be diagonalizable, contradicting our assumption that $g$ has even order). Suppose the scalar $\mu$ has multiplicative order $k$ in $GF(q)^*$. Then by choosing the ordered basis $\mathcal{B} = \{v + w, \mu^{k-1} v\}$ for $V$, we see that $g$ has representation

$$g_{\mathcal{B}} = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}.$$

But now

$$g_{\mathcal{B}} g_{\mathcal{B}}^T = \begin{pmatrix} \mu^2+1 & \mu \\ \mu & \mu^2 \end{pmatrix},$$

which has minimal polynomial $\chi^2 + \chi + \mu^4$. Hence $g_{\mathcal{B}} g_{\mathcal{B}}^T$ cannot have even order. Since $K$ acts transitively on ordered bases of $V$, $g$ and $y = g_{\mathcal{B}}$ must be $K$-conjugate. $\square$

**Lemma 33.** *Let $H = SL_n(q) \cong SL(V)$, suppose $g \in H$ is such that $g^2 \in Z(H)$. Then $g$ is $H$-conjugate to an element*

$$y = \begin{pmatrix} A & \\ & \lambda I_{n-2k} \end{pmatrix},$$

*where $A$ is a $2k \times 2k$ matrix consisting of $k$ $2 \times 2$ blocks*

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

*along its main diagonal and zeroes elsewhere, and $\lambda \in GF(q)^*$.*

*Furthermore, if $y^*$ is equal to $y$ but with one or more $2 \times 2$ block on the diagonal of $A$ transposed, then $y$ and $y^*$ are $H$-conjugate and $yy^*$ has odd order.*

*Proof.* Put $G = GL_n(q)$. Write $Z = Z(G)$ and note that $|Z| = q - 1$, which is odd. As $g^2 \in Z(H) \leq Z$, $g$ must have minimal polynomial $\chi^2 + \alpha$ for some $\alpha \in GF(q)^*$. By choosing a basis for $V$ in a similar manner to that in the proof of Lemma 32, we see that $g$ is $G$-conjugate to an element $y$ as in the statement of the lemma, where $\lambda^2 = \alpha$. Suppose $h \in G$ is such that $g^h = y$, and that $\det(h) = \omega$. Then set

$$c = \begin{pmatrix} \sqrt{\omega^{-1}} & & & & \\ & \sqrt{\omega^{-1}} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then $c \in C_G(y)$, and $\det(c) = \omega^{-1}$, so $\det(hc) = 1$. Thus $hc \in H$, and hence $g$ and $y$ are $H$-conjugate.

Now let $y^*$ equal $y$ but with one or more blocks of $A$ transposed. Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix},$$

it follows that $y$ and $y^*$ are $H$-conjugate. Furthermore, using Lemma 32 and the fact that $|Z|$ is odd, we see that $yy^*$ is essentially a direct sum of $2 \times 2$ blocks, each of which has odd order. Hence $yy^*$ also has odd order. $\square$

**Theorem 34.** *If $G = PSL_n(q)$ and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $H = SL_n(q)$. So $G = \overline{H} = H/Z(H)$. Choose $t \in X$, and let $Y$ be the connected component of $\mathcal{F}(G, X)$ containing $t$. Then $t = \overline{g}$ where $g \in H$ is such that $g^2 \in Z(H)$. By Lemma 33 $g$ is $H$-conjugate to an element

$$y = \begin{pmatrix} A & \\ & \lambda I_{n-2k} \end{pmatrix}.$$

Therefore $t$ and $\overline{y}$ are $G$-conjugate, so without loss of generality we let $t = \overline{y}$. From Theorem 20

$$\mathcal{A} = \{I_n + \lambda E_{ij} \mid i \neq j, \lambda \in GF(q)\}$$

generates $H$. If $a \in \mathcal{A}$, then $a$ centralizes either $y$ or $y^*$, where $y^*$ is obtained from $y$ by transposing an appropriate $2 \times 2$ block of $y$. Hence each $\overline{a} \in \overline{\mathcal{A}}$ centralizes either $\overline{y}$ or $\overline{y^*}$. By Lemma 33 $\overline{y}\overline{y^*}$ has odd order and so $\overline{y}$ and $\overline{y^*}$ are adjacent in $\mathcal{F}(G, X)$. Therefore $G = \langle \overline{\mathcal{A}} \rangle \leq \mathrm{Stab}_G(Y)$, and Theorem 34 follows. □

## 5.2 Symplectic Groups

Next we address the case of projective symplectic groups. Since $q$ is even, $PSp_{2n}(q) \cong Sp_{2n}(q)$, so it suffices to prove the following theorem.

**Theorem 35.** *If $G = Sp_{2n}(q)$, with $X$ a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* By Theorem 21, $G$ is generated by the set $\mathcal{E}$ consisting of matrices of the form

$$\left(\begin{array}{c|c} I_n & \lambda E_{ii} \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \lambda E_{ii} & I_n \end{array}\right),$$

$$\left(\begin{array}{c|c} I_n & \lambda(E_{ij} + E_{ji}) \\ \hline & I_n \end{array}\right), \left(\begin{array}{c|c} I_n & \\ \hline \lambda(E_{ij} + E_{ji}) & I_n \end{array}\right).$$

Let $t \in X$, with $Y$ the relevant connected component of $\mathcal{F}(G, X)$. By 2.1.16 of [31], $t$ is $G$-conjugate to an element

$$y = \left(\begin{array}{c|c} I_n & A \\ \hline & I_n \end{array}\right),$$

where

$$A = \left(\begin{array}{c|c} A' & \\ \hline & 0 \end{array}\right)$$

and $A'$ is either invertible and diagonal, or has blocks

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

along its main diagonal. In particular, note every row (and column) of $A'$ contains one nonzero entry. Without loss of generality let $t = y$. Note that

$t^T \in G$, and that $t^T \in X$ since $t^T$ has the same symplectic Suzuki form as $y$, as in Theorem 17.

We claim that $t$ and $t^T$ are adjacent in $\mathcal{F}(G,X)$. Define elements $a_i \in GF(q)$, $1 \leq i \leq n$, by setting $a_i$ to be the nonzero entry in the $i$-th row of $A'$ if it exists, and zero otherwise. Note that at least one of the $a_i$ must be nonzero. Now define an element $x$ of $GL_{2n}(q)$ as follows:

$$x = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_n \end{pmatrix},$$

where

$$B_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}.$$

Consider the product $xx^T$ as a direct sum of elements in $GL_2(q)$. Each $2 \times 2$ block is either $I_2$, or has minimal polynomial $\chi^2 + a_i^2\chi + 1$ where $a_i \neq 0$. We may therefore apply the argument of Lemma 32 to see that $xx^T$ has odd order.

Suppose we interchange two rows of $t$, and then interchange the corresponding columns. This can be achieved by conjugating $t$ by an invertible matrix $r$, where $r^{-1} = r^T$ (indeed, $r$ can be identified with an element of the symmetric group $\mathrm{Sym}(2n)$). Furthermore, note that $t$ can be transformed into $x$ by a series of these operations. Hence there exists an invertible matrix $s$, with $s^{-1} = s^T$, such that $s^T ts = x$. Now considering the product $tt^T$, we have (where $o$ denotes order)

$$\begin{aligned} o(tt^T) &= o(s^{-1}tt^Ts) = o(s^{-1}tss^{-1}t^Ts) \\ &= o(s^Ttss^Tt^Ts) = o(s^Tts(s^Tts)^T) \\ &= o(xx^T). \end{aligned}$$

Thus $tt^T$ has odd order.

But now let $e \in \mathcal{E}$ be a generator of $G$. By observing the configuration of any element of $\mathcal{E}$, and of $t$, it is easily seen that either $e \in C_G(t)$, or $e \in C_G(t^T)$. Since both $t$ and $t^T$ lie in $Y$, we have that $e \in \mathrm{Stab}_G(Y)$ for all $e \in \mathcal{E}$. But $\langle \mathcal{E} \rangle = G$, so $G = \mathrm{Stab}_G(Y)$. Hence $\mathcal{F}(G,X)$ is connected. □

As an immediate corollary, we have the following result concerning orthogonal groups of odd dimension.

**Corollary 36.** *Let $G = GO_n(q)$, where $n$ is odd. For $X$ a $G$-conjugacy class of involutions, $\mathcal{F}(G,X)$ is connected.*

*Proof.* Write $n = 2m+1$. Then since $q$ is even, $GO_n(q) \cong Sp_{2m}(q)$ (see, for example, Theorem 11.9 of [35]), and the result follows from Theorem 35. □

### 5.3 Unitary Groups

We now move on to address the case of unitary groups. To begin with, we concentrate on the case when the dimension is even. Let $H = SU_{2m}(q) \cong SU(V)$, where the unitary form $\beta$ on $V$ has Gram matrix

$$J = \left(\begin{array}{c|c} & I_m \\ \hline I_m & \end{array}\right).$$

Denote by $\tau$ the involutary automorphism of $GF(q^2)$ associated to $\beta$.

**Theorem 37.** *If $H = SU_{2m}(q) \cong SU(V)$ and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

*Proof.* By Lemma 13 and Theorem 18, involutions $x, y \in H$ are $H$-conjugate if and only if they have the same rank. Therefore we may index the $H$-conjugacy classes of involutions $\{X_i\}$, $1 \leq i \leq m$, where $i$ is the rank, and for the class $X_i$ choose representative

$$x_i = \left(\begin{array}{c|c} I_m & B_i \\ \hline 0 & I_m \end{array}\right)$$

where

$$B_i = \left(\begin{array}{c|c} I_i & \\ \hline & 0 \end{array}\right).$$

Denote by $\mathcal{A}$ the set of matrices of the form

$$\left(\begin{array}{c|c} I_m & \mu E_{ii} \\ \hline & I_m \end{array}\right), \left(\begin{array}{c|c} I_m & \\ \hline \mu E_{ii} & I_m \end{array}\right),$$

$$\left(\begin{array}{c|c} I_m & \lambda E_{ij} + \lambda^\tau E_{ji} \\ \hline & I_m \end{array}\right), \left(\begin{array}{c|c} I_m & \\ \hline \lambda E_{ij} + \lambda^\tau E_{ji} & I_n \end{array}\right),$$

where $\mu, \lambda \in GF(q^2)$, and $\mu + \mu^\tau = 0$. By Theorem 22 we have $H = \langle \mathcal{A} \rangle$. Notice that for every $a \in \mathcal{A}$, either $a \in C_H(x_i)$ or $a \in C_H(x_i^T)$. Since $x_i$, $x_i^T$ are $H$-conjugate, and $x_i x_i^T$ has odd order (as in the proof of Theorem 35), the result follows. $\square$

Now suppose $H = SU_{2m+1}(q)$, and choose a basis so that the unitary form has Gram matrix

$$J = \left(\begin{array}{c|c|c} & I_m & \\ \hline I_m & & \\ \hline & & 1 \end{array}\right).$$

**Theorem 38.** *If $H = SU_{2m+1}(q)$, and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

*Proof.* Let $\mathcal{A}'$ be our set of generators from the even dimension case, considered as elements of $H$ in the obvious way. Then $\mathcal{A}'$ generates a subgroup $K \leq H$, where $K \cong SU_{2m}(q)$. Notice that $K$ stabilizes a non-degenerate $2m$-space $W$

of $V$. By Theorem 25, $K$ lies in a unique maximal subgroup $M \leq H$, where $M = \mathrm{Stab}_H(W)$, and

$$SU_{2m}(q) \times SU_1(q) \leq M \leq GU_{2m}(q) \times GU_1(q).$$

Let $\alpha \in GF(q^2)$ be such that $\alpha + \alpha^\tau = 1$, and define

$$y = \left(\begin{array}{c|c|c} I_m & \alpha E_{11} & E'_{11} \\ \hline & I_m & \\ \hline & E''_{11} & 1 \end{array}\right) \in SU_{2m+1}(q),$$

where $E_{11}$, $E'_{11}$ and $E''_{11}$ are elementary $m \times m$, $m \times 1$ and $1 \times m$ matrices respectively, each with 1 in position $(1, 1)$. This element does not stabilize $W$, so by the maximality of $M$ we have $H = \langle K, y \rangle$.

As in the $2m$-dimensional case there are $n$ conjugacy classes of involutions which we may index by rank. For the class $X_i$ choose representative

$$x_i = \left(\begin{array}{c|c|c} I_m & B_i & \\ \hline & I_m & \\ \hline & & 1 \end{array}\right)$$

where

$$B_i = \left(\begin{array}{c|c} I_i & \\ \hline & 0 \end{array}\right).$$

We have that $x_i \in K$ for all $i$, and so if $Y$ is the connected component of $\mathcal{F}(H, X_i)$ which contains $x_i$, then by Theorem 37 we have $K \leq \mathrm{Stab}_H(Y)$. But also note that $y \in C_H(x_i)$ for all $i$. Hence $H = \langle K, y \rangle \leq \mathrm{Stab}_H(Y)$, as required. $\square$

The case of the projective unitary groups now follows very quickly, using a result of Dye [22].

**Theorem 39.** *If $G = PSU_n(q)$ and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $H = SU_n(q)$ and $G = H/Z(H)$. By Theorem 3 of [22] the conjugacy classes of involutions of $G$ are in one to one correspondence with those of $H$. The result now follows from Theorems 37 and 38. $\square$

## 5.4 Orthogonal Groups

Recall that the orthogonal group $GO(V)$ preserves a nonsingular quadratic form $Q : V \to GF(q)$, and that $Q$ defines a non-degenerate, symmetric, bilinear form $\beta$ on $V$ which $GO(V)$ also preserves. By Corollary 36 we need only consider the case where $n = 2m$. Since $\beta$ must be an alternating form, $GO_{2m}(q) \leq Sp_{2m}(q)$; in particular, every orthogonal group in even characteristic must have trivial centre (when $m \geq 3$).

**Theorem 40.** *If $G = \Omega_{2m}^-(q) \cong \Omega^-(V)$, where $m \geq 3$, and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* When $m = 3$ we have $\Omega_{2m}^-(q) \cong PSU_4(q)$ (see Proposition 2.9.1 of [30], for example) and so the result follows by Theorem 39. Therefore we may assume that $m \geq 4$. Let $t \in X$, with $Y$ the connected component of $\mathcal{F}(G, X)$ which contains $t$, and let $r$ be the dimension of the fixed space $W$ of $t$ in $V$. From [21] we have that $m \leq r \leq 2m - 1$. As the Witt index of $G$ is equal to $m - 1$, there must exist $w \in W$ such that $Q(w) \neq 0$. Therefore $t$ fixes a non-singular 1-space of $V$.

Let $M = \mathrm{Stab}_G(\langle w \rangle)$. By Theorem 29 we have that $M$ is a maximal subgroup of $G$, and

$$\Omega_{2m-1}(q) \times \Omega_1(q) \leq M \leq GO_{2m-1}(q) \times GO_1(q).$$

Since the elements of $G$ have determinant 1, and

$$GO_{2m-1}(q) \cong Sp_{2m-2}(q),$$

we deduce that $M \cong Sp_{2m-2}(q)$. Now Theorem 35 tells us that $M \leq \mathrm{Stab}_G(Y)$. We wish to show that $t$ fixes another non-singular 1-space of $V$. If $r \geq m + 1$, then this is clear, so suppose $r = m$. Then there must exist $u \in W$ such that $Q(u) = 0$. Now $t$ fixes $\lambda u + w$, where $\lambda \in GF(q)$, and since

$$Q(\lambda u + w) = \lambda^2 Q(u) + \lambda\beta(u, w) + Q(w) = \lambda\beta(u, w) + Q(w)$$

we can choose $\lambda \neq 0$ so that $\lambda u + w$ is non-singular. Hence $t$ does indeed fix another non-singular 1-space of $V$, and $t$ therefore lies in another maximal subgroup, $\tilde{M}$ say, distinct from $M$ but also isomorphic to $Sp_{2m-2}(q)$. Thus $\tilde{M} \leq \mathrm{Stab}_G(Y)$. But $M$ is maximal, so $\langle M, \tilde{M} \rangle = G$, and we have $G = \mathrm{Stab}_G(Y)$. It follows that $\mathcal{F}(G, X)$ is connected. $\square$

**Theorem 41.** *Let $G = \Omega_{2m}^+(q) \cong \Omega^+(V)$, where $m \geq 4$ is odd, and let $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected.*

*Proof.* We use the notation from the previous proof. In this case the Witt index of $G$ is equal to $m$. However, using [21] we have $m+1 \leq r \leq 2m-1$. By arguing as in the proof of Theorem 40, we see that $t$ fixes two distinct non-singular 1-spaces of $V$, and thus lies in distinct maximal subgroups $M$ and $M'$, both of which are isomorphic to $Sp_{2m-2}(q)$. As previously, the result follows. $\square$

**Theorem 42.** *Let $G = \Omega_{2m}^+(q) \cong \Omega^+(V)$, where $m \geq 2$ is even, and let $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected.*

*Proof.* When $m = 2$, note that $\Omega_4^+(q) \cong PSL_2(q) \times PSL_2(q)$ (see Proposition 2.9.1 of [30]), and so by Lemma 4 and Theorem 34 the theorem holds in this case. We may therefore assume that $m \geq 4$. Let $t \in X$. If $\mathrm{Rank}(t) < m$, then $t$ stabilizes a non-singular 1-space of $V$, and we may argue as in the proof of Theorem 40. However, we must be careful when $\mathrm{Rank}(t) = m$. Since $G \leq$

$Sp_{2m}(q)$, we make use of Theorem 16. Adopting its notation, we see that in this case $t$ is either of type $a_m$ or $c_m$. Firstly, without loss let $t = a_m$, and let $V$ have basis $\{v_1, \ldots, v_{2m}\}$. Then

$$t = \left( \begin{array}{c|c} I_m & 0 \\ \hline I_m & I_m \end{array} \right)$$

and the symplectic form on $V$ has Gram matrix

$$J = \left( \begin{array}{c|c} 0 & F \\ \hline F & 0 \end{array} \right)$$

where $F$ is the $m \times m$ matrix with 1 in the $(2i, 2i-1)$ and $(2i-1, 2i)$ positions and 0 elsewhere $(1 \leq i \leq m/2)$. Furthermore, by 8.2 of [2] we have that $Q(v_i) = 0$ for $1 \leq i \leq 2m$.

Let $O = \{1 \leq i \leq 2m \mid i \text{ odd}\}$, and $E = \{1 \leq j \leq 2m \mid j \text{ even}\}$. Let $W_1 = \langle v_i \mid i \in O \rangle$, and $W_2 = \langle v_j \mid j \in E \rangle$. Certainly it is the case that $V = W_1 \oplus W_2$, and both $W_1$ and $W_2$ are totally singular. Furthermore, $t$ leaves both $W_1$ and $W_2$ invariant. Thus $t \in \mathrm{Stab}_G(W_1 \oplus W_2) = M$. By Theorem 28 we have that $M \cong H : 2$, where $H \cong GL_m(q)$ leaves both $W_1$ and $W_2$ invariant, and the outer automorphism of $GL_m(q)$ is an involution $\sigma$ which has the effect of interchanging $W_1$ and $W_2$. Thus $t \in H$. Now Theorem 31 implies that $H \leq \mathrm{Stab}_G(Y)$. Also, since $\sigma$ interchanges $W_1$ and $W_2$, the dimension of the fixed spaces of $t$ and $t^\sigma$ are the same. Theorem 16 now tells us that $t$ and $t^\sigma$ are $H$-conjugate, which yields $\sigma \in \mathrm{Stab}_G(Y)$, and so $M \leq \mathrm{Stab}_G(Y)$.

We now partition $\{1, \ldots, 2m\}$ into four subsets, by defining

$$\begin{aligned} I_1 &= \{1, \ldots, m/2\}, \\ I_2 &= \{m/2 + 1, \ldots, m\}, \\ I_3 &= \{m + 1, \ldots, 3m/2\}, \\ I_4 &= \{3m/2 + 1, \ldots, 2m\}. \end{aligned}$$

Note that this is possible since $m$ is even. Now define

$$U_1 = \langle v_i, v_j, v_k, v_\ell \mid i \in I_1 \cap O, j \in I_3 \cap O, k \in I_2 \cap E, \ell \in I_4 \cap E \rangle$$

and

$$U_2 = \langle v_i, v_j, v_k, v_\ell \mid i \in I_1 \cap E, j \in I_3 \cap E, k \in I_2 \cap O, \ell \in I_4 \cap O \rangle.$$

Once more we have that $V = U_1 \oplus U_2$, both $U_1$ and $U_2$ are totally singular, $t(U_1) = U_1$ and $t(U_2) = U_2$. Hence $t \in \tilde{M}$, where $\tilde{M} \cong M$ and $\tilde{M} \neq M$. Arguing as above we see that $\tilde{M} \leq \mathrm{Stab}_G(Y)$, and so $\langle M, \tilde{M} \rangle \leq \mathrm{Stab}_G(Y)$. But $M$ is maximal (as is $\tilde{M}$), so the result follows.

The other possibility is that $t$ is of type $c_m$. Then

$$t = \left( \begin{array}{c|c} I_m & 0 \\ \hline I_m & I_m \end{array} \right)$$

and the symplectic form on $V$ has Gram matrix

$$J = \left( \begin{array}{c|c} 0 & F \\ \hline F & 0 \end{array} \right),$$

where

$$F = \left( \begin{array}{c|c|c} & & 1 \\ \hline & E_{m-2} & \\ \hline 1 & & 1 \end{array} \right)$$

and $E_{m-2}$ is the $(m-2) \times (m-2)$ matrix with 1 in the $(2i, 2i-1)$ and $(2i-1, 2i)$ positions and 0 elsewhere $(1 \leq i \leq (m-2)/2)$. We can see from this description that $t$ stabilizes a non-degenerate 4-space, namely

$$U = \langle v_1, v_m, v_{m+1}, v_{2m} \rangle.$$

Thus, using Theorem 29 with further details gleaned from the proof of Proposition 4.1.6 in [30], $t$ lies in a maximal subgroup

$$M \sim (L_1 \times L_2) : 2 \sim (\Omega_4^+(q) \times \Omega_{2m-4}^+(q)) : 2,$$

where the outer automorphism is $r = r_1 r_2$, a product of reflections in nonsingular vectors in $U$ and $U^\perp$, respectively. We wish to show that without loss of generality $t$ can be chosen to lie in the base group $L = L_1 \times L_2$.

Let $h_1 \in L_1$, $h_2 \in L_2$ be involutions of type $c_2$ and $c_{m-2}$ repectively, and let $s = h_1 h_2$. Then $s$ is an involution of $G$ of rank $2 + (m-2) = m$. However, we do not yet know the type of $s$. Recall from Section 3 that

$$V(s) = \{v \in V \mid \beta(v, v^s) = 0\}.$$

This is a subspace of $V$ of codimension 0 or 1. Since, for example, $h_2$ is of type $c_{m-2}$, we see from Theorem 16 that $V(h_2)$ has codimension 1. Now, since $V = U \oplus U^\perp$, it is easy to see that $V(s)$ must also have codimension 1. But now using Theorem 16 once more we see that this implies that $s$ is $G$-conjugate to an involution of type $c_m$, so is therefore conjugate to $t$. Thus without loss we may take $t = s \in L$.

By Lemma 4, Theorem 34 (as $\Omega_4^+(q) \cong PSL_2(q) \times PSL_2(q)$) and induction we have that $L \leq \mathrm{Stab}_G(Y)$. We next show that $r \in \mathrm{Stab}_G(Y)$, whence $M \leq \mathrm{Stab}_G(Y)$. It suffices to show that $r$ does not fuse $t^L$ with another $L$-conjugacy class. As $r$ leaves $U$ and $U^\perp$ invariant, the problem reduces to showing nonfusion in the relevant classes of $L_1$ and $L_2$.

Suppose $r_2$ is the reflection in some non-singular vector $v \in U^\perp$, with $Q(v) = \delta \in GF(q)$. Since $q$ is even, every element of $GF(q)^*$ is a square, so we may choose $\alpha \in GF(q)$ such that $\alpha^2 = \delta^{-1}$. Now $Q(\alpha v) = \alpha^2 \delta = 1$, so without loss we may assume that $Q(v) = 1$. By definition we have

$$w r_2 = w - \frac{\beta(w, v)}{Q(v)} v,$$

for all $w \in U^{\perp}$, which in this case simplifies to

$$wr_2 = w + \beta(w, v)v.$$

We observe from this that $r_2$ fixes $U$ pointwise. Similarly, $r_1$ will fix $U^{\perp}$ pointwise. Thus when considering the subgroup $L_i$ ($i \in \{1, 2\}$) we need only consider the effect of conjugation by $r_i$.

We wish to show that $t^r = h_1^{r_1} h_2^{r_2}$ is such that $h_1^{r_1}$ and $h_2^{r_2}$ have types $c_2$ and $c_{m-2}$ respectively. As the rank is invariant under conjugation, the only possibilities for fusion in, say $L_2$, are that $h_2^{r_2} = h_2'$ where $h_2'$ has type $a_{m-2}$ or $c_{m-2}$. But if $h_2'$ has type $a_{m-2}$, then $h_2'$ must stabilize a decomposition $U^{\perp} = W_1 \oplus W_2$, where both $W_1$ and $W_2$ are totally singular subspaces of dimension $m-2$, and so $h_2$ must stabilize the decomposition $W_1 r_2 \oplus W_2 r_2$. Since elements in $G$ preserve $Q$, $W_1 r_2$ and $W_2 r_2$ must also be totally singular. But this shows that $h_2$ has type $a_{m-2}$, a contradiction. Thus $h_2'$ must have type $c_{m-2}$. Applying the same reasoning to the subgroup $L_1$ we see that $h_1^{r_1}$ has type $c_2$. Hence $r$ does not fuse the relevant classes of $L$. Thus $M \leq \mathrm{Stab}_G(Y)$.

Now note that $t$ stabilizes another non-degenerate 4-space, for example

$$\tilde{U} = \langle v_1, v_m, v_{m+1}, v_{2m-1} \rangle,$$

and thus lies in a maximal subgroup $\tilde{M}$ such that $\tilde{M} \neq M$. We may apply the same argument as above to show that $\tilde{M} \leq \mathrm{Stab}_G(Y)$, and so $G = \langle M, \tilde{M} \rangle \leq \mathrm{Stab}_G(Y)$. Hence $\mathcal{F}(G, X)$ is connected. □

## 5.5 Exceptional and Twisted Groups

To conclude our investigations into the local fusion graphs of finite groups of Lie-type of even characteristic, we must consider the exceptional and twisted groups of Lie-type in even characteristic. The Suzuki groups are straightfoward to deal with.

**Theorem 43.** *If $G = Suz(q)$ with $X$ a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected, and* $\mathrm{Diam}(\mathcal{F}(G, X)) = 2$.

*Proof.* From [33] we have that $G$ contains just a single class of involutions, and for $x, y \in X$ either $x$ and $y$ commute, or the product $xy$ has odd order. Furthermore, if $x \in P$ where $P \in \mathrm{Syl}_2(G)$, then $C_G(x) \leq P$. Also, the number of involutions which lie in $P$ is $q-1$. We therefore have that $|X \setminus \Delta_1(x)| = q-1$. Since $|X| > 2(q-1)$, the result follows by Lemma 3. □

Table 1 lists the number of involution classes of the remaining exceptional and twisted groups. This data is taken from [2].

Recall from Theorem 23 if $G(q^r)$ is a finite group of Lie-type defined over the field $GF(q^r)$ where $r$ is prime, then $G(q)$, the group of the same type defined over $GF(q)$, is a maximal subgroup of $G(q^r)$. We shall need the following two lemmas regarding involutions and their centralizers.

Table 1: Involutions in Exceptional and Twisted Groups - $q$ even

| Group | Number of Involution Classes |
|---|---|
| $G_2(q)$ | 2 |
| ${}^3D_4(q)$ | 2 |
| $F_4(q)$ | 4 |
| ${}^2F_4(2)'$ | 2 |
| ${}^2F_4(q)$ | 2 |
| $E_6(q)$ | 3 |
| ${}^2E_6(q)$ | 3 |
| $E_7(q)$ | 5 |
| $E_8(q)$ | 4 |

**Lemma 44.** *Let $G = G(2^r)$ be an exceptional or twisted group of Lie-type, with $X$ a $G$-conjugacy class of involutions. Then there exists $t \in X$ such that $t \in H$, where $H$ is the subgroup of $G$ naturally isomorphic to $G(2)$.*

*Proof.* In [2], representatives for every involution class in each exceptional or twisted group are given, in terms of products of involutions from commuting root groups. Since each of these involutions is defined using only the base field $GF(2)$, these representatives lie in $H$. □

**Lemma 45.** *Let $G = G(q^r)$ be an exceptional or twisted group of Lie-type, and suppose that $r$ is prime. Write $H$ for the subgroup of $G$ which is naturally isomorphic to $G(q)$. If $t$ is an involution of $H$, then $C_G(t) \not\leq H$.*

*Proof.* Again we refer to the representatives given in [2] in terms of root groups. Each of these root groups is isomorphic to the additive group of the field $GF(q^r)$, and contains elements which do not lie in $H$. Since the root groups are abelian, and pairwise commute, such elements lie in $C_G(t)$. □

**Theorem 46.** *Let $G$ be a finite, simple, exceptional or twisted group of Lie-type. If $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $Y$ be the connected component of $t$, $t \in X$. First suppose that $q = 2$. In this case we can verify the result computationally, using MAGMA. For $G_2(2)'$, $G_2(2)$, ${}^3D_4(2)$, $F_4(2)$, ${}^2F_4(2)'$, ${}^2F_4(2)$, $E_6(2)$ and ${}^2E_6(2)$ the complex character tables are known, and so we can use Lemma 2 to calculate $|\Delta_1(t)|$ for each local fusion graph $\mathcal{F}(G, X)$. The results are recorded in Table 2, along with the value $f$ which is defined to be the floor of $|X|/|\Delta_1(t)|$. If $\mathcal{F}(G, X)$ has $m$ connected components, then $m \leq f$ and there exists a homomorphism from $G$ to a subgroup of $\mathrm{Sym}(m)$. By Table 2 we must have $m = 1$. Within each group, the involution classes can be distinguished by the dimension of the fixed spaces of their elements, and we use Table 1 to ensure we have dealt with every involution class.

For $E_7(2)$ and $E_8(2)$ we do not have access to their character tables. However, representations of both groups are available from the online ATLAS [41], where we are given matrices $a, b$ such that $G = \langle a, b \rangle$. By using the MAGMA command `Random` to find elements of $G$ with even order, and then taking an appropriate power, we can find involutions from each conjugacy class of $G$, again distinguishing them using fixed space dimension and Table 1. Now, for each representative involution $t \in G$, we use random conjugation to find elements $x_1, x_2, x_3, x_4 \in X = t^G$ such that $tx_i$ has odd order for $i = 1, \ldots, 4$, $x_1^a = x_3$ and $x_2^b = x_4$. Hence $a, b \in \mathrm{Stab}_G(Y)$, and since $G = \langle a, b \rangle$ we have that $\mathcal{F}(G, X)$ is connected. Representatives for suitable elements to use in this process are available from the authors on request.

Now suppose $G = G(2^r)$ where $r \geq 2$, and write $r = r_1 r_2 \cdots r_k$ as a product of primes. We proceed by induction on $k$. If $k = 1$ then by Lemma 44 we may choose $t \in X \cap H$, where $H \cong G(2)$. By the treatment of the cases above we have that $H \leq \mathrm{Stab}_G(Y)$. By Lemma 8, $C_G(t) \leq \mathrm{Stab}_G(Y)$. Lemma 45 implies that $C_G(t) \not\leq H$ and since, by Theorem 23, $H$ is a maximal subgroup of $G$, $\langle H, C_G(t) \rangle = G$. So $\mathcal{F}(G, X)$ is connected. If $k \geq 2$, then let $H \leq G$ be such that $H \cong G(2^{r_1 \cdots r_{k-1}})$. Using induction, Lemmas 44 and 45 we again have that $\langle H, C_G(t) \rangle \leq \mathrm{Stab}_G(Y)$, and since $r_k$ is prime the maximality of $H$ yields $G = \mathrm{Stab}_G(Y)$, as required. $\square$

Table 2: First disc size for some exceptional groups

| Group | Class | Class size | $\lvert\Delta_1(t)\rvert$ | $f$ |
|---|---|---|---|---|
| $G_2(2)'$ | 2A | 165 | 80 | 2 |
| ${}^3D_4(2)$ | 2A | 819 | 512 | 1 |
| ${}^3D_4(2)$ | 2B | 68796 | 30080 | 2 |
| $F_4(2)$ | 2A | 69615 | 32768 | 2 |
| $F_4(2)$ | 2B | 69615 | 32768 | 2 |
| $F_4(2)$ | 2C | 4385745 | 1114112 | 3 |
| $F_4(2)$ | 2D | 350859600 | 76808192 | 4 |
| ${}^2F_4(2)'$ | 2A | 11700 | 4352 | 2 |
| ${}^2F_4(2)'$ | 2B | 1755 | 1024 | 1 |
| ${}^2F_4(2)$ | 2A | 1755 | 1024 | 1 |
| ${}^2F_4(2)$ | 2B | 11700 | 4352 | 2 |
| $E_6(2)$ | 2A | 5081895 | 2097152 | 2 |
| $E_6(2)$ | 2B | 8822169720 | 958660608 | 9 |
| $E_6(2)$ | 2C | 1587990549600 | 267829772288 | 5 |
| ${}^2E_6(2)$ | 2A | 3968055 | 2097152 | 1 |
| ${}^2E_6(2)$ | 2B | 3142699560 | 1226571776 | 2 |
| ${}^2E_6(2)$ | 2C | 1319933815200 | 398070775808 | 3 |

# 6 The Odd Characteristic Case

We are now ready to tackle the local fusion graphs of finite groups of Lie-type when the defining characteristic is odd. Here is the main result we prove in the Sections 7-10.

**Theorem 47.** *Let $G$ be a finite, simple group of Lie-type defined over a field of odd characteristic. If $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G,X)$ is connected.*

To begin, we state a result which will be the major tool in dealing with the odd characteristic case.

**Theorem 48.** *Suppose $G$ is a finite group with a split $BN$-pair and $G$-conjugacy class of involutions $X$, and suppose $B = UT$ where $U$ is the unipotent radical of $B$ and $T$ is a maximal split torus. If $X \cap T \neq \emptyset$, then $\mathcal{F}(G,X)$ is connected.*

In view of this result, our motivation for treating the odd and even characteristic cases separately becomes clear. For suppose $G$ is a finite group of Lie-type defined over a field of even characteristic. Then the involutions of $G$ are not semisimple elements. However, the torus $T$ consists only of semisimple elements, so the requirement that $X \cap T \neq \emptyset$ is never satisfied. Thus the theorem is of no use to us in even characteristic.

In odd characteristic, the immediate question is: when does $X \cap T \neq \emptyset$? Recall that the maximal split tori of a finite group of Lie-type are $G$-conjugate. Moreover, for a classical subgroup of $GL(V)$, the subgroup of diagonal matrices forms a maximal split torus $T$. Thus for finite classical groups the question reduces to determining when an involution can be diagonalized (within $G$) with respect to a certain basis.

First, however, let us prove Theorem 48.

*Proof of Theorem 48.* Let $t \in X \cap T$, $Y$ be the connected component of $\mathcal{F}(G,X)$ containing $t$, and set $M = \mathrm{Stab}_G(Y)$. By [16], for example, we may consider the Borel subgroup *opposite* to $B$, which can be written $B^- = HU^-$. Since $X \cap H \neq \emptyset$, the defining characteristic must be odd, and hence $U$ and $U^-$ are of odd order. Without loss of generality, as $X \cap H \neq \emptyset$, we may suppose $t \in H$. Thus $t$ normalizes both $U$ and $U^-$, so $\langle U, U^- \rangle \leq M$ by Lemma 8(iii). But $\langle U, U^- \rangle = G$ (see, for example, [16] once again), so $M = G$ and hence $\mathcal{F}(G,X)$ is connected. □

We now move on to study the various families of groups of Lie-type in detail - for Sections 7-10 $q$ is assumed to be odd.

# 7 Linear Groups, odd characteristic

For this section we suppose $H = SL_n(q)$, and let $H$ act on $V$ in the natural way. We may take for a Borel subgroup $B$ the group of upper uni-triangular matrices.

Then $U$ is the group of upper uni-triangular matrices, while $T$ is the diagonal subgroup. It follows from Theorem 19 that $X \cap T \neq \emptyset$ holds for all $H$-conjugacy classes $X$ of involutions. Thus we may immediately apply Theorem 48 to deduce the following:

**Theorem 49.** *If $H = SL_n(q)$ or $GL_n(q)$ and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

Since Theorem 47 concerns simple groups, we must consider the projective groups $PSL_n(q)$. We first consider the case were $n$ is odd.

**Theorem 50.** *If $G = PSL_{2m+1}(q)$ and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected,*

*Proof.* Since the centre of $SL_{2m+1}(q)$ has odd order, the local fusion graphs of $G$ are isomorphic to those of $SL_{2m+1}(q)$, and the result follows immediately from Theorem 49. □

For the next two results we require the following information on maximal subgroups of $PSL_2(q)$.

**Theorem 51.** *Let $G = PSL_2(q)$ where $q \geq 11$ is odd. If $M$ is a maximal subgroup of $G$, then $M$ is isomorphic to one of the following groups:*

*(i) the dihedral group* $\mathrm{Dih}(q + 1)$*, which is the centralizer of an involution of $G$;*

*(ii) the dihedral group* $\mathrm{Dih}(q - 1)$*;*

*(iii) a group of order $q(q - 1)$;*

*(iv)* $\mathrm{Alt}(4)$, $\mathrm{Sym}(4)$ *or* $\mathrm{Alt}(5)$*;*

*(v) $PSL_2(r)$ or $PGL_2(r)$, where $r^m = q$.*

*Proof.* See Theorem 6.25 of [34]. □

We now deal with the groups $PSL_n(q)$ when $n$ is even.

**Theorem 52.** *Suppose $G \cong PSL_{2m}(q)$, and $X$ is a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected, unless $m = 1$ and $q = 3$, in which case $\mathcal{F}(G, X)$ is totally disconnected.*

*Proof.* Write $H = SL_{2m}(q)$, with $G = \overline{H}$, and let $\bar{t} \in X$, with $Y$ the connected component of $\mathcal{F}(G, X)$ which contains $\bar{t}$. When $m = 1$ and $q = 3$ we have $PSL_2(3) \cong \mathrm{Alt}(4)$, which has a totally disconnected local fusion graph. For $PSL_2(5)$, $PSL_2(7)$ and $PSL_2(9)$ it is easy to check that $\mathcal{F}(G, X)$ is connected. When $m = 1$ and $q \geq 11$, Theorem 51 implies (as there is only one involution class) that $C_G(t)$ is a maximal subgroup. Therefore $\mathcal{F}(G, X)$ is connected by Lemma 6. So we may now assume $m \geq 2$. If $t$ has any eigenvalues, then $GF(q)$

must contain a suitable root of unity, so $t$ may be diagonalized in $H$, whence Theorem 48 yields the result. Therefore we may assume that

$$t = \begin{pmatrix} 0 & \alpha_1 & & & & & \\ \alpha_2 & 0 & & & & & \\ & & 0 & \alpha_3 & & & \\ & & \alpha_4 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \alpha_{2m-1} \\ & & & & & \alpha_{2m} & 0 \end{pmatrix},$$

where for each $1 \leq i \leq m$ we have $\alpha_{2i-1}\alpha_{2i} = \omega \neq 1$. Denote by $L$ the inverse image of $\mathrm{Stab}_G(Y)$ in $H$. Our aim is to show that $a \in L$ for all $a \in \mathcal{A}'$, where $\mathcal{A}'$ is the generating set for $H$ given in Theorem 20 (whence the requirement that $m \geq 2$).

Notice that for any $a = I + \lambda E_{ij} \in \mathcal{A}'$, the entry $(i, j)$ does not coincide with the position of any of the $\alpha_i$ in $t$. We can easily check that $t^a = t + s$, where $s$ is an upper triangular nilpotent matrix with nonzero entries which do not coincide with the positions of any $\alpha_i$ in $t$. As a consequence, $t^a t = \omega I_{2m} r$, where $r$ is unipotent, and hence has odd order. Thus $\overline{t^a t}$ has odd order, from which we deduce that $a \in L$. Since the image of $\mathcal{A}'$ generates $G$, the result follows. □

**Theorem 53.** *Suppose $G \cong PGL_{2m}(q)$, and $X$ is a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected, unless $m = 1$ and $q = 3$, in which case $\mathcal{F}(G, X)$ has two local fusion graphs, one of which is totally disconnected.*

*Proof.* Write $H = GL_{2m}(q)$, with $G = \overline{H}$, and let $\bar{t} \in X$, with $Y$ the connected component of $\mathcal{F}(G, X)$ which contains $\bar{t}$. Also let $L \leq G$ be the subgroup which is naturally isomorphic to $PSL_{2m}(q)$, so $[G : L] = (q - 1, 2m)$. When $m = 1$ and $q = 3$ we have $PGL_2(3) \cong \mathrm{Sym}(4)$, which has one totally disconnected local fusion graph, the other being connected. It is straightforward to check that the local fusion graphs of $PGL_2(5)$, $PGL_2(7)$ and $PGL_2(9)$ are connected, so assume $m = 1$ and $q \geq 11$. Then $G$ has two involution classes. One corresponds to an involution class of $H$, whence the result follows by Theorem 31, while the other lies in the normal subgroup of $G$ which is isomorphic to $PSL_2(q)$. In this case the result follows by Theorem 52.

Now assume that $m \geq 2$. We may assume $t$ has the same form as in the proof of Theorem 52, and so $\overline{\mathcal{A}'} \leq \mathrm{Stab}_G(Y)$, whence $L \leq \mathrm{Stab}_G(Y)$. Write $t_1$ for the first $2 \times 2$ block $\begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}$ of $t$, and consider $t_1$ as an element of $K = GL_2(q)$. Then

$$C_K(t_1) = \left\{ \begin{pmatrix} a & b \\ b\alpha_1^{-1}\alpha_2 & a \end{pmatrix} \mid a, b \in GF(q), a^2 - \alpha_1^{-1}\alpha_2 b^2 \neq 0 \right\}.$$

Write $C$ for $C_K(t_1)$ considered as a subgroup of $H$ in the natural way. Then $\overline{C} \leq \mathrm{Stab}_G(Y)$. Since $C$ contains elements with determinant any square $\gamma \in GF(q)^*$

(and such elements cannot lie in $L$ unless $\gamma = 1$), we have that $\langle \overline{C}, L\rangle$ has index at most 2 in $G$. However, by using the description of $C$ given above, we see that there exist elements of $C$ with non-square determinant, and so $\langle \overline{C}, L\rangle = G$. Thus $G = \mathrm{Stab}_G(Y)$, as required. $\square$

Now we define a graph closely related to a local fusion graph, to be used in subsequent sections. Let $H$ be a finite group, and suppose $X$ is an $H$-conjugacy class of elements which square to $z \in Z(H)$, where $z \neq 1$. We define the graph $\mathcal{D}(H, X)$ to have $X$ as its vertex set, with $x, y \in X$ adjacent in $\mathcal{D}(H, X)$ if, and only if, $x \neq y$ and $xy = zw$, where $w$ is some element of $H$ of odd order.

**Lemma 54.** *Suppose $H = GL_2(q)$ or $GU_2(q)$ where $q \neq 3$, and that $X$ is an $H$-conjugacy class of elements which square to $\lambda I_2$.*

*(i) If $\lambda = -1$, then $\mathcal{D}(H, X)$ is connected.*

*(ii) If $q \equiv 3$ mod 4, then $\mathcal{D}(H, X)$ is connected.*

*Proof.* First we prove the lemma for $H = GL_2(q)$. When $q < 11$ we may verify the result using Magma, so assume $q \geq 11$. Let $t \in H$ be such that $t^2 = \lambda I_2$, and without loss of generality take

$$t = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

We write $Z = Z(H)$, $C = C_H(t)$, $N = N_H(\langle t\rangle)$, and $L$ for the subgroup of $H$ which is naturally isomorphic to $SL_2(q)$. Notice that

$$C = \left\{ \begin{pmatrix} a & b \\ \lambda b & a \end{pmatrix} \mid a, b \in GF(q), a^2 - \lambda b^2 \neq 0 \right\}.$$

Our first claim is that $N$ is a maximal subgroup of $H$. Indeed, it is certainly the case that $\overline{N} \cap \overline{L} = C_{\overline{L}}(\bar{t})$, which is a maximal subgroup of $\overline{L}$ isomorphic to $\mathrm{Dih}(q + 1)$, by Theorem 51. Since $[\overline{H} : \overline{L}] = 2$, any maximal subgroup of $\overline{H}$ must be one of the subgroups listed in Theorem 51, an extension of one of these subgroups by a group of order 2, or $\overline{L}$ itself. Using the description of $C$ given above, we see there exist elements of $C$ (and so $N$) which have non-square determinant, and so lie outside $N_L(\langle t\rangle)Z$. Therefore $C_{\overline{L}}(\bar{t})$ is a proper subgroup of $C_{\overline{H}}(\bar{t})$, so we must have $\overline{N} \sim C_{\overline{L}}(\bar{t}).2$ and $\overline{N}$ is a maximal subgroup of $\overline{H}$. Since $Z \leq N$, this implies that $N$ is a maximal subgroup of $H$.

Next, we show that $N$ is the unique maximal subgroup of $H$ which contains $C$. Since $t^2 \in Z$, we have $[N : C] = 2$, and consequently $|\overline{C}| = q + 1$. Suppose $M$ is such that $C < M < H$. Then $\overline{M}$ is a maximal subgroup of $\overline{H}$ with order divisible by $q + 1$. Since $C$ contains elements with determinant not equal to 1, it cannot be that $M \cong SL_2(q)$, and now an examination of the orders of the groups in Theorem 51 shows that the only possibility is $\overline{M} = \overline{N}$, whence $M = N$.

Now suppose that $\lambda = -1$, and notice that in this case $t \in L$, so $X \subseteq L$. If there exists $h \in H \setminus N$ such that $t$ and $t^h$ are adjacent in $\mathcal{D}(H, X)$, then

$h \in \mathrm{Stab}_H(Y)$, where $Y$ is the connected component of $\mathcal{D}(H, X)$ which contains $t$. Moreover, since $h \notin N$, $\langle C, h\rangle = H$. Hence it suffices to show the existence of such an $h$. Note that if $n \in N \backslash C$, then $t^n = -I_2 t$ and so $tt^n = I_2$. Consequently $t$ and $t^n$ are not adjacent in $\mathcal{D}(G, X)$. Thus, if we can show the existence of some $x \in X$ such that $x$ is adjacent to $t$, then choosing $h$ so that $t^h = x$ will satisfy our requirements.

If for all $x, y \in X$ it is the case that $\langle x, y\rangle$ is a 2-group, then the Baer-Suzuki Theorem (Theorem 8.2 of [27]) implies that $X \subseteq O_2(H)$, a contradiction. So there must exist $x \in X$ such that $tx$ has order $2^k m$, where $m > 1$ has odd order. If $k = 0$, then $t(-I_2 x)$ has order $2m$, and since $x$ and $-I_2 x$ are conjugate in $L$ we see that $-I_2 x$ is adjacent to $t$ in $\mathcal{D}(H, X)$. If $k = 1$, then $(tx)^m$ has order 2, which is sufficient since the only involution in $L$ is $-I_2$. Finally, suppose that $k \geq 2$. Then $(tx)^{2^{k-1}m}$ has order 2, so must be equal to $-I_2$. Using the facts that $t^{-1} = -I_2 t$, $x^{-1} = -I_2 x$ and $2^{k-1}m$ is even, we have

$$\begin{aligned} -I_2 &= (tx)^{2^{k-1}m} \\ &= txtx \cdots tx \\ &= (-I_2 t)(-I_2 x) \cdots (-I_2 t)(-I_2 x) tx \cdots tx \\ &= (-I_2 t) t^{xt \cdots x} \\ &= t(-I_2 t^{xt \cdots x}). \end{aligned}$$

Since $t^{xt\cdots x}$ and $-I_2 t^{xt\cdots x}$ are $L$-conjugate we again have an edge in $\mathcal{D}(H, X)$, as required, completing the proof of (i).

To prove (ii), suppose that $q \equiv 3 \bmod 4$, and let $\omega \in GF(q)$ be an element of maximal (multiplicative) odd order. Since $q \equiv 3 \bmod 4$, this order must be greater than 1. Now define

$$y = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}.$$

Then we may easily check that $t^y$ is adjacent to $t$ in $\mathcal{D}(H, X)$, and so $y \in \mathrm{Stab}_H(Y)$, where $Y$ is the connected component of $\mathcal{D}(H, X)$ which contains $t$. But clearly we also have $C \leq \mathrm{Stab}_H(Y)$. Furthermore, $y \notin N$, and since $N$ is the unique maximal subgroup of $H$ which contains $C$, it must be that $\langle C, y\rangle = H$. Thus $H = \mathrm{Stab}_H(Y)$, as required.

Now suppose that $H = GU_2(q)$, and note that $PGU_2(q) \cong PGL_2(q)$ and $SU_2(q) \cong SL_2(q)$. Since, using the notation above, we may easily check that $C$ contains elements which have non-square determinant, and so lie outside $N_L(\langle t\rangle)Z$, the proof is almost identical to that for $GL_2(q)$. $\square$

# 8 Symplectic Groups, odd characteristic

Let $H = Sp_{2m}(q)$ act on $V$. Recall that $H$ preserves a non-degenerate, alternating bilinear form $\beta$ on $V$. From Theorem 19 we have that $X \cap T \neq \emptyset$ for each $G$-conjugacy class of involutions $X$, so Theorem 48 gives the following result.

**Theorem 55.** *If $H = Sp_{2n}(q)$, and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

We now consider the local fusion graphs of the projective groups $PSp_{2m}(q)$. In view of Theorem 55, when proving Theorem 58 we need only be concerned with involution classes of $PSp_{2m}(q)$ which arise from elements of $Sp_{2m}(q)$ which square to $-I_{2m}$. Fortunately, there is only ever one such class.

**Lemma 56.** *Let $H = Sp_{2m}(q)$. Then there is a unique $H$-conjugacy class of elements which square to $-I_{2m} \in Z(H)$.*

*Proof.* This is contained in the proof of Lemma 11.52 in [35]. □

**Lemma 57.** *Suppose $H = Sp_{2m}(q)$, and that $x \in H$ is an element which squares to $-I_{2m}$. Then $x \in L$, where $L \leq H$ and $L \cong Sp_2(q^m)$.*

*Proof.* Without loss of generality suppose that $V$ has symplectic basis

$$\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$$

with the sympletic form $\beta$ on $V$ having Gram matrix

$$J = \left( \begin{array}{c|c} & I_n \\ \hline -I_n & \end{array} \right),$$

and suppose that

$$x = \left( \begin{array}{c|c} & A_m \\ \hline -A_m & \end{array} \right)$$

where $A_m$ is the $m \times m$ matrix with 1 in positions $(1, m), (2, m-1), \ldots, (m, 1)$ and zeroes elsewhere. Notice that there exists an $GF(q)$-vector space isomorphism $\phi$ between $V$ and $\tilde{V}$, where $\tilde{V}$ is a 2-dimensional $GF(q^m)$-vector space. For example, if $\tilde{V}$ has basis $\{\tilde{e_1}, \ldots, \tilde{e_m}, \tilde{f_1}, \ldots, \tilde{f_m}\}$ as an $GF(q)$-vector space, we can simply set $e_i\phi = \tilde{e_i}$ and $f_i\phi = \tilde{f_i}$ for $1 \leq i \leq m$, and extend $GF(q)$-linearly. We can also endow $\tilde{V}$ with a symplectic form $\tilde{\beta}$, where $\tilde{\beta}$ has Gram matrix

$$\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Clearly $H$ acts on $\tilde{V}$ via the isomorphism $\phi$, and one can show (see page 111 of [30], for example) that the subgroup of $H$ consisting of elements which preserve $\tilde{\beta}$ is in fact isomorphic to $Sp_2(q^m)$. Moreover, it is straightforward to check by explicit calculation that $x$ also preserves $\tilde{\beta}$, so lies in this subgroup. □

**Theorem 58.** *Let $G = PSp_{2m}(q)$, and $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected, unless $m = 1$ and $q = 3$, where $\mathcal{F}(G, X)$ is totally disconnected.*

*Proof.* We prove the result by induction on $m$. First note that if $q \geq 5$ then $PSp_2(q) \cong PSL_2(q)$, and the result holds by Theorem 52. If $q = 3$, then $PSp_2(q) \cong \mathrm{Alt}(4)$, which has a totally disconnected local fusion graph. However, we may easily check using MAGMA that $PSp_4(3)$ and $PSp_6(3)$ have connected local fusion graphs.

Let $G = \overline{H}$, where $H = Sp_{2m}(q) = Sp(V)$, and suppose $V$ has basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ so that the sympletic form $\beta$ on $V$ has Gram matrix

$$J = \left(\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array}\right).$$

Set

$$t = \left(\begin{array}{c|c} & A_m \\ \hline -A_m & \end{array}\right)$$

where $A_m$ is the $m \times m$ matrix with 1 in positions $(1, m), (2, m-1), \ldots, (m, 1)$ and zeroes elsewhere. Then $t \in H$, and $t^2 = -I_{2m} \in Z(H)$, so using Lemma 56 we may take $\bar{t}$ as our representative for the relevant involution class of $G$. We may write $t = t_1 t_2$, where

$$t_1 = \left(\begin{array}{c|cc|c} I_{m-k} & & & \\ \hline & & A_k & \\ & -A_k & & \\ \hline & & & I_{m-k} \end{array}\right)$$

and

$$t_2 = \left(\begin{array}{c|cc|c} & & & A_{m-k} \\ \hline & I_k & & \\ & & I_k & \\ \hline -A_{m-k} & & & \end{array}\right),$$

where $1 \leq k \leq m-1$. It is clear that $t_1$ and $t_2$ commute. Notice that $t$ stabilizes a non-degenerate subspace of $V$ of dimension $2k$, namely

$$W = \langle e_{2m-k+1}, \ldots, e_{2m}, f_{2m}, \ldots, f_{2m-k+1} \rangle,$$

and so by Theorem 24 lies in a subgroup $M \leq H$, where

$$M = M_1 \times M_2 \cong Sp_{2k}(q) \times Sp_{2m-2k}(q).$$

Furthermore, if $k \neq m$, then $M$ is a maximal subgroup of $H$ and $\overline{M}$ is a maximal subgroup of $G$.

Our aim is to show that $\overline{M} \leq \mathrm{Stab}_G(Y)$, where $Y$ is the connected component of $\mathcal{F}(G, X)$ which contains $\bar{t}$. Denote by $X_i$ the $\overline{M_i}$-conjugacy class of involutions which contains $\overline{t_i}$, for $i = 1, 2$. If $q \geq 5$, then by induction $\mathcal{F}(\overline{M_1}, X_1)$ and $\mathcal{F}(\overline{M_2}, X_2)$ are connected. If $q = 3$ then, since we have checked cases when $m = 2$ and $m = 3$, we may assume that $m \geq 4$. We can now choose $k \notin \{1, m-1\}$ so that neither $\overline{M_1}$ nor $\overline{M_2}$ is isomorphic to $PSp_2(3)$. This allows us to use induction in this case also.

Let $\overline{x} \in \tilde{X}$ have preimage $x \in M$, where $\tilde{X}$ denotes the $\overline{M}$-conjugacy class of $\overline{t}$, and write $x = x_1 x_2$, where $x_1 \in M_1$ and $x_2 \in M_2$. By induction there exists a path

$$t_1 = x_1^{(0)} \to x_1^{(1)} \to x_1^{(2)} \to \cdots \to x_1^{(\ell)} = x_1$$

of elements in $x_1^{M_1}$ such that $x_1^{(i)} x_1^{(i+1)} = y_1^{(i)} z_1^{(i)}$, where $y_1^{(i)}$ has odd order and $z_1^{(i)} \in Z(M_1)$, for $0 \leq i \leq \ell - 1$. This path induces in a natural way a path in $\mathcal{F}(G, X)$ from $\overline{t_1 t_2}$ to either $\overline{x_1 t_2}$ or $\overline{x_1(-I_{M_2} t_2)}$. For suppose first that $z_1^{(i)} = -I_{M_1} \in Z(M_1)$ for some $i$. Then

$$(x_1^{(i)} t_2)(x_1^{(i+1)} t_2) = (x_1^{(i)} x_1^{(i+1)}) t_2^2 = y_1^{(i)}(-I_{M_1})(-I_{M_2}).$$

Since $(-I_{M_1})(-I_{M_2}) = -I_{2m} \in Z(H)$, we have that $\overline{x_1^{(i)} t_2}$ and $\overline{x_1^{(i+1)} t_2}$ are adjacent in $\mathcal{F}(G, X)$. Now suppose that $z_1^{(j)} = I_{M_1}$ for some $j$. Then

$$(x_1^{(j)} t_2)(x^{(j+1)}(-I_{M_2} t_2) = x_1^{(j)} x_1^{(j+1)}(-I_{M_2} t_2^2) = y_1^{(j)}$$

has odd order. Moreoever, $(-I_{M_2} t_2)^2 = -I_{M_2}$, so by Lemma 56 we deduce that $t_2$ and $-I_{M_2} t_2$ are $M_2$-conjugate. Hence we have a path from $\overline{t_1 t_2}$ to either $\overline{x_1 t_2}$ or $\overline{x_1(-I_{M_2} t_2)}$ in $\mathcal{F}(G, X)$. A similar argument now allows us to find a path in $\mathcal{F}(G, X)$ from this element to one of $\overline{x_1 x_2}$, $\overline{-I_{2m}(x_1 x_2)}$, $\overline{(-I_{M_1} x_1) x_2}$ or $\overline{x_1(-I_{M_2} x_2)}$. In the former two cases we are done, since both $x_1 x_2$ and $-I_{2m}(x_1 x_2)$ have image $\overline{x}$ in $G$. So without loss of generality suppose we are in one of the latter two cases, say with a path from $\overline{t_1 t_2}$ to $\overline{(-I_{M_1} x_1) x_2}$. Note that we may decompose $W$ into an orthogonal sum of non-degenerate symplectic 2-spaces, each of which is left invariant by both $x_1$ and $-I_{M_1} x_1$, and as such both elements lie in a subgroup of $M_1$ which is isomorphic to a direct product of copies of $Sp_2(q)$. Now the isomorphism $Sp_2(q) \cong SL_2(q)$ allows us to use Lemma 54(i) to see that there exists a path in the graph $\mathcal{D}(M_1, x_1^{M_1})$ from $x_1$ to $-I_{M_1} x_1$, which in turn induces a path from $\overline{(-I_{M_1} x_1) x_2}$ to $\overline{x_1 x_2}$ in $\mathcal{F}(G, X)$. As this method allows us to construct a path between $\overline{t}$ and an arbitrary $\overline{x}$ in $\mathcal{F}(\overline{M}, \tilde{X})$, we deduce that $\overline{M} \leq \mathrm{Stab}_G(Y)$.

Now denote by $M^*$ the stabilizer in $G$ of the decomposition $V = W \oplus W^{\perp}$. If $k \neq m$, then $M = M^*$, while if $k = m$ then $M$ is a subgroup of index 2 in $M^*$. In the former case, by Theorem 24, $\overline{M^*}$ is a maximal subgroup of $G$, while in the latter case reference to Table 3.5C of [30] tells us that $\overline{M^*}$ is again a maximal subgroup of $G$, and is the unique maximal subgroup of $G$ which contains $\overline{M}$. However, by Lemma 57 we have that $\overline{t} \in \overline{L}$, where $\overline{L} \cong PSp_2(q^m)$. As $PSp_2(q^m) \cong PSL_2(q^m)$, and $m \geq 2$, Theorem 52 tells us that $\overline{L} \leq \mathrm{Stab}_G(Y)$. Since $\overline{L} \not\leq \overline{M^*}$ (by comparing group orders from the formulae given in [30], for example), it must be that $\langle \overline{M}, \overline{L} \rangle = G$, and so $\mathcal{F}(G, X)$ is connected. $\square$

# 9 Unitary Groups, odd characteristic

The unitary groups are next on our agenda. Let $H = GU_n(q) \cong GU(V)$. Recall that this means the entries of matrices in $H$ are taken from $GF(q^2)$.

**Theorem 59.** *If $H = SU_n(q)$ or $GU_n(q)$, and $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* By Theorem 19 the condition $X \cap T \neq \emptyset$ is satisfied for each $G$-conjugacy class of involutions $X$. Thus we may apply Theorem 48 to deduce the result. $\square$

When dealing with the projective unitary groups, we must consider elements of $GU_n(q)$ which square to non-trivial central elements. We first consider how the vector space $V$ can decompose under the action of such an element. So for $H = GU_n(q)$, let $t \in H$ be such that $t^2 \in Z(H)$, say $t^2 = \lambda I_n$. For $v \in V$ the subspace $\langle v, v^t \rangle$ must be either 1 or 2-dimensional. Thus, first taking $v$ to be a non-isotropic vector, then taking a non-isotropic vector in $\langle v, v^t \rangle^\perp$, and so on, we see that $t$ must stabilize a decomposition

$$V = W_1 \perp \ldots \perp W_k \perp U_1 \perp U_{n-2k},$$

where the $W_i$ are non-degenerate 2-spaces and the $U_i$ are non-degenerate 1-spaces. For each 2-space we may choose a basis so that the restriction of $\beta$ to $W_i$ has Gram matrix $J_i = I_2$, and then by taking a suitable basis vector for each $U_i$ we may ensure that $\beta$ has Gram matrix $J = I_n$. With respect to this basis for $V$ we have

$$t = \left( \begin{array}{c|c|c|c} t_1 & & & \\ \hline & \ddots & & \\ \hline & & t_k & \\ \hline & & & \mu I_{n-2k} \end{array} \right),$$

where $t_i \in GU_2(q)$ and $t_i^2 = \lambda I_2$ for each $i$, and $\mu^2 = \lambda$.

**Theorem 60.** *Let $H = GU_{2m+1}(q) \cong GU(V)$, and suppose that $t \in H$ is such that $t^2 \in Z(H)$. Then $t$ lies in a maximal split torus of $H$.*

*Proof.* Suppose $t^2 = \lambda I_{2m+1}$, and choose the basis for $V$ and representative for $t$ in the form described above. Since $V$ has odd dimension $2m+1$, $t$ has at least one eigenvalue $\mu$, and since $t$ must preserve the unitary form $\beta$ we deduce that $\mu\mu^\tau = 1$, where $\tau$ is the involutory automorphism of $GF(q^2)$ associated to $\beta$. Since $t^2 = \lambda I_{2m+1}$, $t$ has characteristic polynomial

$$(\chi - \mu)^{2m+1-2k}(\chi^2 - \lambda)^k,$$

and since $\mu^2 = \lambda$ this splits into linear factors

$$(\chi - \mu)^{2m+1-k}(\chi + \mu)^k.$$

Hence $t$ is conjugate in $GL_{2m+1}(q^2)$ to a diagonal element. But since $\mu\mu^\tau = 1$, such a diagonal element will also preserve $\beta$, so will lie in $H$, and by Theorem 18 will be $H$-conjugate to $t$. Thus without loss of generality we may choose $t$ to be diagonal, and so $t$ lies in a maximal split torus of $H$. $\square$

**Corollary 61.** *If $G = PSU_{2m+1}(q)$ and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* By Theorem 60, any $G$-conjugacy class of involutions intersects non-trivially with a maximal split torus of $G$. We may now apply Theorem 48 to deduce the result. □

We deal with the groups $PSU_{2m}(q)$ in two stages. First, we consider the case when $q \equiv 1 \bmod 4$.

**Theorem 62.** *Let $G = PSU_{2m}(q)$, where $m \geq 2$ and $q \equiv 1$ mod 4. If $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $\beta$ have Gram matrix

$$J = \left(\begin{array}{c|c} & I_m \\ \hline I_m & \end{array}\right).$$

By consulting Table 4.5.1 of [28], we see that $G$ has $\lfloor m/2+1 \rfloor$ conjugacy classes of involutions. For elements of $H = SU_{2m}(q)$ which map canonically into $\lfloor m/2 \rfloor$ of these classes, we may simply take diagonal elements $t_i$, for $1 \leq i \leq \lfloor m/2 \rfloor$, with eigenvalues $\pm 1$. The final $G$-conjugacy class of involutions comes from elements which square to a non-trivial central element in $H$. Since $q \equiv 1$ mod 4, there exists $\omega \in GF(q^2)$ which squares to $-1$. Define

$$t = \begin{pmatrix} \omega & & & & & \\ & \ddots & & & & \\ & & \omega & & & \\ & & & \omega^{-1} & & \\ & & & & \ddots & \\ & & & & & \omega^{-1} \end{pmatrix} \in GL_{2m}(q^2).$$

Notice that $\det(t) = 1$. Furthermore, $t$ will preserve the unitary form $\beta$ if $\omega^{q-1} = 1$. But 4 divides $q - 1$, so this is certainly the case. Hence $t \in H$. Moreover, since $\omega I_{2m}$ does not preserve $\beta$ (as $\omega^{q+1} \neq 1$), so does not lie in $H$, we see that $\bar{t}$ must lie in a different $G$-conjugacy class of involutions to $\overline{t_i}$ for $1 \leq i \leq \lfloor m/2 \rfloor$. Nevertheless, $\bar{t}$ lies in a maximal split torus of $G$, so once again the result follows by Theorem 48.

□

**Lemma 63.** *If $H = SU_4(3)$, and $X$ is a conjugacy class of elements which square to $-I_4$, then $\mathcal{D}(H, X)$ is connected.*

*Proof.* This can be easily verified using Magma. □

We shall see in a moment that our general method for dealing with the groups $PSU_{2m}(q)$ when $q \equiv 3$ mod 4 does not cover the case when $q = 3$. This is due to the fact that $PSU_2(3) \cong PSL_2(3)$, which has a disconnected local fusion graph. Therefore we deal with this case separately.

**Lemma 64.** *If $G = PSU_{2m}(3)$, where $m \geq 2$ and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* We can check, using MAGMA, the cases $PSU_4(3)$, $PSU_6(3)$ and $PSU_8(3)$, so assume that $m \geq 5$. Let $H = SU_{2m}(3)$, and note that $|Z(H)| = 2$ or $4$, depending on whether $m$ is odd or even, respectively.

If $|Z(H)| = 2$, then any $G$-conjugacy class of involutions must either be the image of an $H$-conjugacy class of involutions, or the image of an $H$-conjugacy class of elements which square to $-I_{2m}$. In the former case we may apply Theorem 48, while in the latter case we may argue as in the proof of Theorem 62.

Now suppose that $|Z(H)| = 4$, so $m$ is even. Reference to Table 4.5.1 of [28] tells us that $G$ has either $m/2$ or $m/2 + 1$ conjugacy classes of involutions, and by considering the representatives $t$ and $t_i$ for $1 \leq i \leq m/2$, as given in the proof of Theorem 62, we see that at most one $G$-conjugacy class of involutions is not the image of an $H$-conjugacy class of involutions. Moreover, we can take as our representative for the possible remaining class the involution $\bar{t}$, where

$$t = \left( \begin{array}{c|c} & A_m \\ \hline -A_m & \end{array} \right)$$

and $A_m$ is the $m \times m$ matrix with 1 in positions $(1, m), (2, m-1), \ldots, (m, 1)$ and zeroes elsewhere. Since $m \geq 5$ and $m$ is even, we see that $t$ stabilizes a non-degenerate subspace of $V$ of dimension 6, for example

$$W = \langle e_{2m-2}, \ldots, e_{2m}, f_{2m}, \ldots, f_{2m-2} \rangle,$$

and so lies in a subgroup $M \leq H$, where

$$M = M_1 \times M_2 \cong SU_6(3) \times SU_{2m-6}(3).$$

Since 4 does not divide 6 or $2m-6$, $|Z(M_1)| = |Z(M_2)| = 2$, and so we may argue as above to see that $\overline{M} \leq \mathrm{Stab}_G(Y)$, where $Y$ is the connected component of $\mathcal{F}(G, X)$ which contains $t$. If $6 \neq m$, then by Theorem 25 $M \trianglelefteq K$ where $K$ is a maximal subgroup of $H$, and

$$K \leq GU_6(3) \times GU_{2m-6}(3).$$

By Theorem 18, no $M$-conjugacy classes fuse in $K$, so $\overline{K} \leq \mathrm{Stab}_G(Y)$. But $t$ stabilizes a further non-degenerate 6-space of $V$, say

$$U = \langle e_1, \ldots, e_3, f_3, \ldots, f_1 \rangle,$$

so by the same argument we get $\overline{K_0} \leq \mathrm{Stab}_G(Y)$, where $K_0$ is another maximal subgroup of $H$. The result now follows by the maximality of $\overline{K}$ or $\overline{K_0}$.

Suppose then that $m = 6$, so $H = SU_{12}(3)$. Here $M$ lies in a unique maximal subgroup $K^*$ of $H$, the stabilizer of the decomposition

$$V = W \perp W_0$$

into two non-degenerate 6-spaces. Notice that $t$ also stabilizes the decomposition

$$V = W_1 \perp W_2 \perp W_3,$$

where

$$W_1 = \langle e_1, e_2, f_1, f_2 \rangle,$$
$$W_2 = \langle e_3, e_4, f_3, f_4 \rangle$$

and

$$W_3 = \langle e_5, e_6, f_5, f_6 \rangle.$$

Using Lemma 63, we have that $\overline{L} \leq \mathrm{Stab}_G(Y)$, where

$$L \cong SU_4(3) \times SU_4(3) \times SU_4(3).$$

Since $L$ does not lie in $K^*$, we have that $G = \mathrm{Stab}_G(Y)$ by the maximality of $K^*$. Hence $\mathcal{F}(G, X)$ is connected. $\square$

We now tackle the case when $q \neq 3$.

**Theorem 65.** *Let $G = PGU_{2m}(q)$, where $m \geq 2$ and $q \equiv 3$ mod 4. If $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* If $\bar{t} \in X$ has preimage $t \in H = GU_{2m}(q)$ which is an involution, then we may apply Theorem 59, and indeed if $t$ has any eigenvectors we may argue as in the proof of Theorem 60 to show that $\bar{t}$ lies in a maximal split torus of $G$, and so the result follows by Theorem 48. Assume therefore that $t^2 = \lambda I_{2m}$, $t$ has no eigenvectors, and that $t$ preserves a decomposition

$$V = W_1 \perp W_2 \perp \ldots \perp W_m$$

of $V$ into non-degenerate 2-spaces. For each $W_i$ we may choose a basis $\mathcal{B}_i$ so that the restriction of $\beta$ to $W_i$ has Gram matrix $J_i = I_2$, so $J = I_{2m}$. As $t^2 = \lambda I_{2m}$, $t$ is conjugate in $GL_{2m}(q)$ to an element

$$t' = \begin{pmatrix} 0 & 1 & & & & & \\ \lambda & 0 & & & & & \\ & & 0 & 1 & & & \\ & & \lambda & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & \lambda & 0 \end{pmatrix}.$$

Since $\lambda I_{2m} \in Z(H)$, we must have $\lambda\lambda^\tau = 1$, and using this fact we see that $t'$ also preserves $J$, so lies in $H$. Now Theorem 18 implies that $t$ and $t'$ are $H$-conjugate, so without loss of generality let $t = t'$. We now rearrange our basis vectors for $V$, and write $\tilde{\mathcal{B}}$ for this new basis. If $m$ is even, we do this so that $\beta$ has Gram matrix

$$J_{\text{even}} = \left(\begin{array}{c|c} & I_m \\ \hline I_m & \end{array}\right),$$

while if $m$ is odd we rearrange so that

$$J_{\mathrm{odd}} = \left(\begin{array}{c|c|c} & I_{m-1} & \\ \hline I_{m-1} & & \\ \hline & & I_2 \end{array}\right).$$

Write $\tilde{V}$ for $V$ with respect $\tilde{\mathcal{B}}$. This change of basis will of course give a new representation $t_{\tilde{\mathcal{B}}}$, but notice that $t_{\mathcal{B}}$ given above also preserves both $J_{\mathrm{even}}$ and $J_{\mathrm{odd}}$, so lies in $SU(\tilde{V})$. Since $t_{\tilde{\mathcal{B}}}$ is conjugate in $GL_{2m}(q)$ to $t_{\mathcal{B}}$, Theorem 18 tells us that these elements are conjugate in $SU(\tilde{V})$. Writing $H = SU(\tilde{V})$, we may therefore take $t = t_{\mathcal{B}} \in H$ as our conjugacy class representative.

Since $t$ preserves the decomposition

$$V = W_1 \perp W_2 \perp \ldots \perp W_m,$$

by Theorem 26 $t$ lies in a maximal subgroup $M \leq H$, where

$$M = L \wr K \leq GU_2(q) \wr \mathrm{Sym}(m).$$

Our aim is to show that $\overline{M} \leq \mathrm{Stab}_G(Y)$, where $Y$ is the connected component of $\mathcal{F}(G, X)$ which contains $\overline{t}$. First consider an element of $M$ which lies in the subgroup $K$. Such an element permutes the subspaces $W_1, \ldots, W_m$, but within each $W_i$ acts trivially. Thus, these elements lie in $C_H(t)$, and so $\overline{K} \leq \mathrm{Stab}_G(Y)$. Next, consider

$$L = L_1 \times L_2 \times \cdots \times L_m,$$

and write $t = t_1 \cdots t_m$ where $t_i \in L_i$ for each $i$. Suppose that

$$h = h_1 h_2 \cdots h_m \in L.$$

By Lemma 54, $t_i^{h_i}$ is connected to $t_i$ in the graph $\mathcal{D}(L_i, t_i^{L_i})$, for $1 \leq i \leq m$. Using any suitable paths in these graph, and the fact that the components of $L$ pairwise commute, we may construct a chain of elements

$$t = x^{(0)} \to x^{(1)} \to x^{(2)} \to \cdots \to x^{(\ell)} = t^h$$

in $X$ such that $x^{(j)} x^{(j+1)} = \lambda I_2 y^{(j+1)}$, where $y^{(j+1)}$ is an element of odd order, for $1 \leq j \leq \ell - 1$. Thus $\overline{x^{(j)}}$ and $\overline{x^{(j+1)}}$ are adjacent in $\mathcal{F}(G, X)$ for each $j$, and so $\overline{t}$ is connected to $\overline{t^h}$ in $\mathcal{F}(G, X)$. Thus $\overline{h} \in \mathrm{Stab}_G(Y)$. As $h$ was chosen arbitrarily, we get that $\overline{L} \leq \mathrm{Stab}_G(Y)$, and consequently $\langle \overline{L}, \overline{K} \rangle = \overline{M} \leq \mathrm{Stab}_G(Y)$.

Since $\overline{M}$ is a maximal subgroup of $G$, it now suffices to show the existence of an element $\overline{y} \in \mathrm{Stab}_G(Y) \setminus \overline{M}$. If $\beta$ has Gram matrix $J_{\mathrm{even}}$, then define

$$y = \left(\begin{array}{c|c} I_m & \mu E_{1,1} \\ \hline & I_m \end{array}\right),$$

while if $\beta$ has Gram matrix $J_{\mathrm{odd}}$ let

$$y = \left(\begin{array}{c|c|c} I_{m-1} & \mu E_{1,1} & \\ \hline & I_{m-1} & \\ \hline & & I_2 \end{array}\right),$$

where $\mu \in GF(q^2)$ is such that $\mu + \mu^\tau = 0$. Then $y \in H$, and since $y$ does not preserve the decomposition of $V$ which $t$ preserves, $y \notin M$. But as in the proof of Theorem 52 we may check that

$$tt^y = \lambda I_{2m} r,$$

where $r$ is an upper-triangular unipotent matrix, which consequently must have odd order. Therefore $\overline{y} \in \mathrm{Stab}_G(Y)$, and the proof is complete. □

**Corollary 66.** *If $G = PSU_{2m}(q)$, where $m \geq 2$ and $q \equiv 3$ mod 4, and $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* When $q = 3$ the result follows by Lemma 64, while if $q \neq 3$ we use Theorem 65 along with Lemma 5. □

# 10 Orthogonal Groups, odd characteristic

In this section we investigate the local fusion graphs of orthogonal groups. The first step is to consider the matrix groups. We deal separately with cases of minus-type, plus-type, and odd dimension. As usual we denote the matrix group by $H$, which acts on a vector space $V$ equipped with a symmetric bilinear form $\beta$.

**Lemma 67.** *Let $H = SO_n^\epsilon(q)$, where $\epsilon \in \{+, -\}$, and suppose $x \in H$ is an involution. Then $V = V_+ \perp V_-$, where $x$ acts trivially on $V_+$ and as $-I$ on $V_-$.*

*Proof.* Let $v \in V$ be an arbitrary vector. Since $q$ is odd we may write

$$v = \frac{1}{2}(v + v^x) + \frac{1}{2}(v - v^x),$$

which implies that $V = V_+ \oplus V_-$, where $x$ acts trivially on $V_+$ and as $-I$ on $V_-$. Suppose that $u \in V_+$ and $v \in V_-$. Then

$$\beta(u, v) = \beta(u^x, v^x) = \beta(u, -v) = -\beta(u, v).$$

Thus $\beta(u, v) = 0$ and we have $V = V_+ \perp V_-$. □

**Theorem 68.** *Let $H = \Omega_{2m}^-(q)$ or $SO_{2m}^-(q)$, where $m \geq 2$, and let $x \in H$ be an involution. Then $x$ lies in a maximal split torus of $H$.*

*Proof.* By Lemma 67 we may write $V = V_+ \perp V_-$. Note that this implies both $V_+$ and $V_-$ are non-degenerate. Also, as $\det(x) = 1$ and $x$ acts as $-I$ on $V_-$, we must have that the dimension of $V_-$ is even, say $\dim V_- = 2k$.

Since $V$ has minus-type we have that $V_+$ and $V_-$ have different types. Suppose that $V_-$ has minus-type, so $V_+$ has plus-type. Then $V_+$ and $V_-$ contain maximal isotropic subspaces of dimensions $m - k$ and $k - 2$ respectively. By considering these and using Lemma 11 we may write

$$V_+ = L_1 \perp L_2 \perp \cdots \perp L_{m-k}$$

and

$$V_- = L_{m-k+1} \perp L_{m-k+2} \perp \cdots \perp L_{m-1} \perp W,$$

where the $L_i$ are hyperbolic lines, and $W$ is a 2-space which contains no singular vectors.

For $i = 1, \ldots, m-1$ let $e_i$ be a singular vector in $L_i$. Then the following is an isotropic flag in $V$:

$$\mathcal{F} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{m-1} \rangle.$$

Moreover, since the Witt index of $V$ is $m-1$, this is a maximal isotropic flag. Since $x$ acts trivially on $V_+$ and as $-I$ on $V_-$, $x$ stabilizes $\mathcal{F}$. Now Theorem 12 implies that $x$ lies in Borel subgroup $B$ of $H$, and so $x$ must lie in a maximal split torus of $H$. The argument for when $V_+$ has minus-type and $V_-$ has plus-type is similar. □

**Corollary 69.** *Let $H = \Omega^-_{2m}(q)$ or $SO^-_{2m}(q)$, and let $t \in H$ be an involution, with $X = t^H$. Then $\mathcal{F}(H, X)$ is connected.*

*Proof.* Both $\Omega^-_2(q)$ and $SO^-_2(q)$ are cyclic groups (see Proposition 2.9.1 of [30]), so the result trivially holds here. When $m \geq 2$, then by Theorem 68, $t$ lies in a maximal split torus of $H$. We now apply Theorem 48. □

We now come to the orthogonal groups of plus-type. Recall that Theorem 12, which characterized Borel subgroups in terms of maximal isotropic flags, excluded such orthogonal groups. In fact, the Borel subgroups of orthogonal groups of plus-type arise as stabilizers of maximal flags in the *oriflamme geometry*. Full details may be found in Chapter 11 of [35], but for our purposes, a maximal flag in the oriflamme geometry consists of a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of distinct maximal isotropic flags, where the first $m-1$ subspaces of $\mathcal{F}_1$ coincide with those of $\mathcal{F}_2$.

**Theorem 70.** *Let $H = \Omega^+_{2m}(q)$ or $SO^+_{2m}(q)$, where $m \geq 3$. If $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $t \in X$. By Lemma 67 we may write $V = V_+ \perp V_-$, where $t$ acts trivially on $V_+$ and as $-I$ on $V_-$. If $\dim V_+ = 0$, then $t$ acts as $-I$ on the whole space $V$, implying $t \in Z(H)$, whence the result clearly holds. So assume $\dim V_- = 2k$ where $k < m$. Since $V$ has plus-type we see that the spaces $V_+$ and $V_-$ have the same type. We first consider the case when both $V_+$ and $V_-$ have plus-type. By Lemma 11 we may write

$$V_+ = L_1 \perp L_2 \perp \cdots \perp L_k$$

and

$$V_- = L_{k+1} \perp L_{k+2} \perp \cdots \perp L_m.$$

If we write $L_i = \langle e_i, f_i \rangle$ for $1 \leq i \leq m$, then $t$ stabilizes the isotropic flag

$$\mathcal{F} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{m-1}, e_m \rangle,$$

and also stabilizes the isotropic flag

$$\mathcal{F}' = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{m-1}, f_m \rangle.$$

Since the first $m-1$ subspace of $\mathcal{F}$ and $\mathcal{F}'$ coincide, $t$ stabilizes a maximal flag in the oriflamme geometry, and so lies in a Borel subgroup $B$ of $H$. Since $t$ is semisimple, it must therefore lie in a maximal split torus $T$ of $H$, and the result follows by Theorem 48.

Now suppose that both $V_+$ and $V_-$ are of minus-type. Then by Lemma 11 we may write

$$V_+ = L_1 \perp L_2 \perp \cdots \perp L_{k-1} \perp W_1$$

and

$$V_- = L_{k+1} \perp L_{k+2} \perp \cdots \perp L_{m-1} \perp W_2,$$

where the $L_i$ are hyperbolic lines and $W_1$ and $W_2$ are 2-spaces which contain no singular vectors. Note that $t$ stabilizes the decomposition $V_+ \perp V_-$, so by Theorem 29 lies in a subgroup $M \leq H$ where

$$M \leq GO^-_{2k}(q) \times GO^-_{2(m-k)}(q).$$

Furthermore, $t$ also preserves the decomposition $\tilde{V_+} \perp \tilde{V_-}$, where

$$\tilde{V_+} = L_1 \perp \cdots \perp L_{k-1} \perp L_{k+1} \perp W_1$$

and

$$\tilde{V_-} = L_{k+2} \perp \cdots \perp L_{m-1} \perp W_2,$$

and so, again by Theorem 29, $t$ lies in a subgroup $\tilde{M}$, where

$$\tilde{M} \leq GO^-_{2(k+1)}(q) \times GO^-_{2(m-k-1)}(q).$$

Since $t$ acts trivially on $V_+$, and as $-I$ on each 2-dimensional component of $V_-$, we have that $t$ lies in a normal subgroups $N \trianglelefteq M$ and $\tilde{N} \trianglelefteq \tilde{M}$, where

$$N \leq SO^-_{2k}(q) \times SO^-_{2(m-k)}(q)$$

and

$$\tilde{N} \leq SO^-_{2(k+1)}(q) \times SO^-_{2(m-k-1)}(q).$$

Now using Lemma 4 and Corollary 69 we see that $N, \tilde{N} \leq \mathrm{Stab}_H(Y)$, where $Y$ is the connected component of $\mathcal{F}(H, X)$ which contains $t$. Since $N \trianglelefteq M$, $\tilde{N} \trianglelefteq \tilde{M}$, and no involution conjugacy classes of $N$ or $\tilde{N}$ fuse in $M$ or $\tilde{M}$, respectively, we have that $M, \tilde{M} \leq \mathrm{Stab}_H(Y)$. Thus $\langle M, \tilde{M} \rangle \leq \mathrm{Stab}_H(Y)$. However, since either $k \neq m-k$ or $k+1 \neq m-k-1$, one of $M$ or $\tilde{M}$ must be a maximal subgroup of $H$ by Theorem 29. Hence $\langle M, \tilde{M} \rangle = H$, which completes the proof. □

**Theorem 71.** *Let $H = \Omega_{2m+1}(q)$ or $SO_{2m+1}(q)$, where $m \geq 2$. If $X$ is an $H$-conjugacy class of involutions, then $\mathcal{F}(H, X)$ is connected.*

*Proof.* Let $t \in X$ with $Y$ the connected component of $t$ in $\mathcal{F}(G, X)$. Once again we use Lemma 67 to write $V = V_+ \perp V_-$. Since $t$ has determinant 1, it must be that $V_-$ has even dimension. Suppose that $V_-$ has plus-type. Then using Lemma 11 we have

$$V_+ = L_1 \perp L_2 \perp \cdots \perp L_k \perp W,$$

where $W$ is a nonsingular 1-space, and

$$V_- = L_{k+1} \perp L_{k+2} \perp \cdots \perp L_m.$$

Notice that $t$ stabilizes the maximal totally isotropic space

$$\langle e_1, e_2, \ldots, e_m \rangle,$$

and so lies in a Borel subgroup, and hence maximal torus of $H$. We may now apply Theorem 48 to show that $\mathcal{F}(H, X)$ is connected.

Now suppose that $V_-$ has minus-type, so we may write

$$V_- = L_{k+1} \perp L_{k+2} \perp \cdots \perp L_{m-1} \perp \tilde{W}$$

where $\tilde{W}$ is a 2-space which contains no singular points. By pairing $W$ with any hyperbolic line $L_i$, or with $\tilde{W}$, we see that $t$ stabilizes a non-degenerate 3-space, and acts with determinant 1 on this space. Hence, using Theorem 29, $t$ lies in a maximal subgroup $M \leq H$ such that $L \leq M \leq K$, where

$$L = \Omega_3(q) \times \Omega^-_{2m-2}(q)$$

and

$$K = SO_3(q) \times SO^-_{2m-2}(q).$$

Assume that $q \neq 3$. We see from Proposition 2.9.1 of [30], for example, that $SO_3(q) \cong PGL_2(q)$, which has connected local fusion graphs by Propostion 53. Therefore, by Lemma 4 and Theorem 69, the local fusion graphs of $K$ are connected. Moreover, since $L \trianglelefteq K$ it must be that $M \trianglelefteq K$, and so by Lemma 5 the local fusion graphs of $M$ must be connected. We deduce that $M \leq \mathrm{Stab}_H(Y)$. However, since $m \geq 2$, it can be seen from the decomposition of $V$ that we may choose another non-degenerate 3-space which is stabilized by $t$. An identical argument shows that $\tilde{M} \leq \mathrm{Stab}_H(Y)$, where $\tilde{M}$ is a maximal subgroup of $H$, and $\tilde{M} \neq M$. Thus by the maximality of $M$ or $\tilde{M}$, $\mathcal{F}(H, X)$ is connected.

When $q = 3$, we cannot use the method above since $PGL_2(3)$ has a disconnected local fusion graph. But we can easily check using Magma that the local fusion graphs of $SO_5(3)$ are connected, and so we may assume that $m \geq 3$. We then observe that $t$ stabilizes a non-degenerate 5-space, and argue as above. □

Having dealt with the matrix groups, we now work towards proving that the local fusion graphs of the corresponding projective orthogonal groups are also connected. As was the case for symplectic groups, the involution classes we must deal with arise from elements of the matrix groups which square to $-I$. Also exerting influence here is the congruence of the field. First we have the case in which $q \equiv 1 \bmod 4$.

**Lemma 72.** *Let $H = SO^{\epsilon}_{2m}(q)$, where $m \geq 3$, $\epsilon = \pm$ and $q \equiv 1 \bmod 4$. If $t \in H$ is such that $t^2 = -I_{2m}$, then $H$ must have plus-type, and $t$ lies in a maximal split torus of $H$.*

*Proof.* Since 4 divides $q-1$, there exists $\omega \in GF(q)$ such that $\omega^2 = -1$. For $v \in V$ we may write
$$v = \frac{1}{2}(v + \omega v^t) + \frac{1}{2}(v - \omega v^t).$$
As $t^2 = -I_{2m}$, we have $(v + \omega v^t)^t = -\omega v + v^t$ and $(v - \omega v^t)^t = \omega v + v^t$, and so
$$V = V_\omega \oplus V_{-\omega},$$
where
$$V_\omega = \{v \in V : v^t = \omega v\}$$
and
$$V_{-\omega} = \{v \in V : v^t = -\omega v\}.$$
Suppose $u, v \in V_\omega$. Then
$$\beta(u, v) = \beta(u^t, v^t) = \beta(\omega u, \omega v) = -\beta(u, v),$$
so $\beta(u, v) = 0$ and $V_\omega$ (and $V_{-\omega}$) is a totally singular subspace of dimension $m$. If $H = SO^-_{2m}(q)$, then this contradicts the fact that $V$ has Witt index $m-1$, so we deduce that $H = SO^+_{2m}(q)$.

Using Lemma 11, choose a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ for $V$ so that
$$V_\omega = \langle e_1, \ldots, e_m \rangle,$$
$$V_{-\omega} = \langle f_1, \ldots, f_m \rangle$$
and $\beta(e_i, f_j) = \delta_{ij}$. Then $t$ stabilizes the maximal isotropic flag
$$\mathcal{F} = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_m \rangle,$$
and $t$ also stabilizes the flag
$$\mathcal{F}' = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{m-1}, f_m \rangle.$$
Since these flags coincide in their first $m-1$ subspaces, we have that $t$ stabilizes a maximal flag in the oriflamme geometry, and hence lies in a Borel subgroup, and consequently a maximal split torus of $H$. □

**Theorem 73.** *Let $G = PSO^{\epsilon}_{2m}(q)$ or $P\Omega^{\epsilon}_{2m}(q)$, where $m \geq 3$, $\epsilon = \pm$, and $q \equiv 1 \bmod 4$, and let $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected.*

*Proof.* This follows from Theorems 69, 70 and Lemma 72. □

When $q \equiv 3 \bmod 4$ the situtation is slightly more complicated. The next lemma establishes some details regarding the relevant involution classes.

**Lemma 74.** *Let $H = SO^{\epsilon}_{2m}(q)$, where $m \geq 3$ and $q \equiv 3 \bmod 4$. Suppose there exists $t \in H$ such that $t^2 = -I_{2m}$. Then the following hold:*

*(i) Either $m$ is even and $H$ has plus-type, or $m$ is odd and $H$ has minus-type;*

*(ii) If $t \in H' = \Omega^{\epsilon}_{2m}(q)$, then there exists exactly one further $H'$-conjugacy class of elements which square to $-I_{2m}$. Moreover, these classes do not fuse in $H$.*

*Proof.* Choose a singular vector $v \in V$, and consider $U = \langle v, v^t \rangle$. If $\dim(U) = 1$, then it must be that $v^t = \omega v$, where $\omega^2 = -1$. Thus $\omega$ has multiplicative order 4 in $GF(q)$, a contradiction since $q \equiv 3 \bmod 4$. Therefore $U$ must be a 2-dimensional subspace of $V$. Since $v$ is singular, it is certainly the case that $v^t$ is also singular. We claim that $\beta(v, v^t) = 0$. Indeed, suppose that $\beta(v, v^t) \neq 0$, so that $U$ has plus-type. Then using Theorem 29, $t$ lies in a subgroup $M \leq H$, where

$$M = M_1 \times M_2 \leq GO^{+}_2(q) \times GO^{\epsilon}_{2m-2}(q)$$

Hence we may write $t = t_1 t_2$, where $t_1 \in M_1$ and $t_2 \in M_2$. Theorem 11.4 of [35] tells us that $GO^{+}_2(q) \cong \mathrm{Dih}(2(q-1))$, and since $q \equiv 3 \bmod 4$ this group contains no elements of order 4. But $t_1^2 = -I_{M_1}$, so $t_1$ must have order 4. This is a contradiction. Hence $\beta(v, v^t) = 0$, and $U$ is totally singular.

Now, if possible, choose a singular vector in $U^{\perp} \setminus U$, and proceed as above. Suppose that $\epsilon = +$ and $m$ is even. Following this method to its conclusion shows that $t$ stabilizes a maximal totally singular subspace of $V$ of dimension $m$. By Theorem 11.61 of [35], both $H$ and its derived subgroup $H'$ have exactly two orbits on the set of maximal totally singular subspaces of $V$, and consequently have exactly two conjugacy classes of elements which square to $-I_{2m}$. Now suppose that $m$ is odd. If such a $t$ were to exist in $H$, then by breaking off totally singular 2-spaces stabilized by $t$ as above, $t$ would stabilize a totally singular subspace of dimension $m+1$, contradicting the Witt index of $V$ being $m$.

We now suppose that $H$ has minus-type. If $m$ is even, then the Witt index $m-1$ of $V$ is odd, and if we have $t \in H$ such that $t^2 = -I_{2m}$, then the preceding argument provides a contradiction. So we have that $m$ is odd and $m-1$ is even. Then $t$ stabilizes a maximal totally singular subspace $W \subseteq V$ of dimension $m-1$, and so, by Theorem 27, $t$ lies in a subgroup $K \leq H$, where

$$K \cong C : (K_1 \times K_2) \sim [q^a] : (GL_{m-1}(q) \times SO^{-}_2(q)).$$

Note that the elements of $K_1$ have determinant 1 when considered as elements of $H$. By Lemma 11 we can find hyperbolic lines $\langle e_1, f_1 \rangle, \ldots, \langle e_{m-1}, f_{m-1} \rangle$ so that

$$W = \langle e_1, \ldots, e_{m-1} \rangle,$$

and so $K_1$, and $t$, stabilize the non-degenerate space

$$W \oplus \tilde{W} = \langle e_1, \ldots, e_{m-1} \rangle \oplus \langle f_1, \ldots, f_{m-1} \rangle,$$

which has plus-type. Further details from Proposition 4.1.20 of [30] yield that $K_2 \cong SO^-((W \oplus \tilde{W})^\perp)$, and $K_2$ fixes $W \oplus \tilde{W}$ pointwise.

We may write $t = t_1 t_2$, where $t_1 \in K_1$ and $t_2 \in K_2$. By Proposition 2.9.1 of [30], $K_2 \cong C_{q+1}$, a cyclic group which contains two elements which square to $-I_{2m}$, namely $t_2$ and $(-I_{K_2})t_2$. We claim that $t$ and $t' = t_1(-I_{K_2})t_2$ are not $H$-conjugate. Since $t$ stabilizes $W \oplus \tilde{W}$, using Theorem 29, $t$ lies in a subgroup $L \leq H$, where

$$L = L_1 \times L_2 \leq GO^+_{2m-2}(q) \times GO^-_2(q).$$

As $K_2$ fixes $W \oplus \tilde{W}$ pointwise, we have $K_2 \leq L_2$. Notice that $t'$ also stabilizes the subspaces $W$ and $\tilde{W}$. Therefore any element $h \in H$ such that $t^h = t'$ must stabilize the decomposition $W \oplus \tilde{W}$, and so lie in $L$. We may therefore write $h = h_1 h_2$, where $h_1 \in L_1$ and $h_2 \in L_2$. If $W^h = W$, then $h \in K$, and since $t_2$ and $(-I_{K_2})t_2$ are not conjugate in $K_2$, it cannot be the case that $t^h = t'$. Therefore $h$ (and consequently $h_1$ since $h_2$ fixes $W \oplus \tilde{W}$ pointwise) must interchange the spaces $W$ and $\tilde{W}$.

We aim to show that $h$ lies in the subgroup of $L$ which is isomorphic to $SO^+_{2m-2}(q) \times SO^-_2(q)$. Denote by $r$ the product of the reflections $r_1, \ldots, r_{m-1}$, where $r_i$ swaps $e_i$ and $f_i$. Then $r$ stabilizes $W \oplus W'$, so lies in $L_1$, and any element of $L$ which interchanges $W$ and $\tilde{W}$ can be written as a product of $r$ and an element of $K_1$ (since $K_1$ is isomorphic to $GL(W)$). Therefore write $h_1 = rk_1$ for some $k_1 \in K_1$. Since $r$ is a product of $m-1$ reflections, where $m-1$ is even, $\det(r) = 1$, and, as it is an element of $K_1$, $\det(k_1) = 1$. Hence $\det(h_1) = 1$, and so $h_1$ lies in the subgroup of $L_1$ which is isomorphic to $SO^+_{2m-2}(q)$. This now forces $\det(h_2) = 1$ also, so $h_2$ lies in the subgroup of $L_2$ which is isomorphic to $SO^-_2(q)$, which is $K_2$. However, as already noted, $t_2$ and $(-I_{K_2})t_2$ are not conjugate in $K_2$. This contradiction implies that $t$ and $t'$ cannot be $H$-conjugate. To complete the proof, we note that since $H$ acts transitively on the set of maximal totally isotropic subspaces of $V$ (see [35]), any element of $H$ which squares to $-I_{2m}$ must be $H$-conjugate to either $t$ or $t'$. $\square$

We are now in a position to deal with the remaining projective orthogonal groups.

**Theorem 75.** *Let $G = PSO^\epsilon_{2m}(q)$, where $\epsilon = \pm$, $q \equiv 3 \bmod 4$, and $m \geq 3$, and let $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected.*

*Proof.* Let $H = SO^\epsilon_{2m}(q)$, so that $G = \overline{H}$, and let $\bar{t} \in G$ be an involution. By Theorems 69 and 70, we need only consider the cases where $t \in H$ squares to $-I_{2m}$. We proceed by induction on $m$, and first verify the cases when $m$ is small.

When $q$ is odd, we have the following isomorphisms (see [35], for example):

$$\begin{aligned}
P\Omega^+_4(q) &\cong PSL_2(q) \times PSL_2(q),\\
P\Omega^-_4(q) &\cong PSL_2(q^2),\\
P\Omega^+_6(q) &\cong PSL_4(q),\\
P\Omega^-_6(q) &\cong PSU_4(q),
\end{aligned}$$

from which we deduce that

$$\begin{array}{rcl} PSO_4^+(q) & \trianglelefteq & PGL_2(q) \times PGL_2(q), \\ PSO_4^-(q) & \trianglelefteq & PGL_2(q^2), \\ PSO_6^+(q) & \trianglelefteq & PGL_4(q), \\ PSO_6^-(q) & \trianglelefteq & PGU_4(q). \end{array}$$

The first four automorphisms, along with Lemma 4, Theorems 52, 62 and Corollary 66 show that, with the exception of $P\Omega_4^+(3)$ (since $PSL_2(3)$ has a disconnected local fusion graph), the groups $P\Omega_4^\epsilon(q)$ and $P\Omega_6^\epsilon(q)$ have connected local fusion graphs. By using Lemma 4, Lemma 5, Theorem 53 and Theorem 65, we infer that the latter four families also have connected local fusion graphs, with the exception of $PSO_4^+(3)$. To allow us to include the case where $q = 3$, we also check using MAGMA the case when $G = PSO_8^+(3)$. We assume therefore that $m \geq 4$, and when $q = 3$ additionally assume that $m \geq 5$. The proof of Lemma 74 shows that $t$ stabilizes a totally singular 2-space $W \subseteq V$, and so by Theorem 27 we have $t \in M$, where $M$ is a maximal subgroup of $H$, and

$$M = C : (M_1 \times M_2) \sim [q^a] : (GL_2(q) \times SO_{2m-4}^\epsilon(q)).$$

We may write $t = t_1 t_2$, where $t_1 \in M_1$ and $t_2 \in M_2$. Since the subgroup $C$ has odd order and is normalized by $t$, Lemma 8(iii) implies that $C \leq \mathrm{Stab}_G(Y)$. Let $\overline{x}$ be any element of the $\overline{M}$-conjugacy class which contains $\overline{t}$, and write $x = x_1 x_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. By induction there exists a path

$$t_2 \to x_2^{(1)} \to x_2^{(2)} \to \cdots \to x_2^{(k)} = x_2$$

of elements of $t_2^{M_2}$ such that $x_2^{(i)} x_2^{(i+1)} = \pm I_{M_2} y^{(i)}$, where $y^{(i)}$ has odd order, for each $i$. Thus for each $i$ we have that the images of either $t_1 x_2^{(i)} t_1 x_2^{(i+1)}$ or $t_1 x_2^{(i)} (-I_{M_1}) t_1 x_2^{(i+1)}$ have odd order in $G$. Hence there exists a path in $\mathcal{F}(G, X)$ from $\overline{t}$ to either $\overline{t_1 x_2}$ or $\overline{(-I_{M_1}) t_1 x_2}$. Since, by Theorem 53, the local fusion graphs of $PGL_2(q)$ are connected (for $q \neq 3$), we may extend this path to either $\overline{x_1 x_2}$ or $\overline{(-I_{M_1}) x_1 x_2}$. In the former case we are done, while in the latter we use Lemma 54 to get a path from $\overline{(-I_{M_1}) x_1 x_2}$ to $\overline{x_1 x_2}$.

We have therefore shown that $\overline{M} \leq \mathrm{Stab}_G(Y)$. However, it is certainly the case that $t$ stabilizes a totally singular 2-space of $V$ which is distinct from $W$, and an identical argument shows that the corresponding maximal subgroup $\overline{M_0} \leq \mathrm{Stab}_G(Y)$. The maximality of either $\overline{M}$ or $\overline{M_0}$ now yields the result. □

**Corollary 76.** *Let $G = P\Omega_{2m}^\epsilon(q)$, where $\epsilon = \pm$, $q \equiv 3 \bmod 4$, and $m \geq 3$, and let $X$ be a $G$-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected.*

*Proof.* We have that $P\Omega_{2m}^\epsilon(q)$ is a subgroup of index at most 2 in $PSO_{2m}^\epsilon(q)$, so is a normal subgroup. If $t \in X$, then the cases when $t$ lies in a maximal split torus of $G$ are covered by Theorems 69 and 70. The remaining cases follow from Lemma 5 and Theorem 75. □

We conclude this section by quickly dealing with the exceptional and twisted groups of Lie-type defined over fields of odd characteristic. Let $G$ be such a group, with $X$ a $G$-conjugacy class of involutions, and $T$ a maximal split torus of $G$. By Theorem 19 the condition $X \cap T \neq \emptyset$ is satisfied, and so we may apply Theorem 48 to deduce the following result.

**Theorem 77.** *Let $G$ be a simple exceptional or twisted group of Lie-type. If $X$ is a $G$-conjugacy class of involutions, then $\mathcal{F}(G, X)$ is connected.*

# 11 The Sporadic Case

This, our final section, looks at the 26 sporadic simple groups, before we complete the proof of Theorem 1. Our approach here is computational, making use of the complex character tables of the sporadic groups stored in Magma [12] or GAP [25]. We summarise results in Table 3; we use $f$ to denote the floor of $|X|/|\Delta_1(t)|$.

**Theorem 78.** *Let $G$ be a sporadic finite simple group, with $X$ a $G$-conjugacy class of involutions. Then the local fusion graph $\mathcal{F}(G, X)$ is connected.*

*Proof.* Suppose $\mathcal{F}(G, X)$ is not connected and has $m$ connected components. Let $t \in X$. Note that $m \leq f$. Then since $G$ acts on the connected components of $\mathcal{F}(G, X)$, $G$ must be isomorphic to a subgroup of $\mathrm{Sym}(m)$. Surveying the final column of Table 3 yields a contradiction in all cases. Thus Theorem 78 holds. □

*Proof of Theorem 1.* We may now prove Theorem 1, making use of the Classification of Finite Simple Groups. When $G$ is the cyclic group of order 2 the result is trivial, while if $G = \mathrm{Alt}(n)$, $n \geq 5$, then the result holds by Theorem 9. The cases where $G$ is a sporadic simple group are covered by Theorem 78.

Suppose that $q$ is a 2-power. For the groups $PSL_n(q)$, $n \geq 2$, except $PSL_2(2)$ we have Theorem 34, while the groups $Sp_{2m}(q)$, $m \geq 2$, except $Sp_4(2)$ are covered by Theorem 35. If $G = PSU_n(q)$, $n \geq 3$, except $PSU_3(2)$ we use Theorem 39, and if $G = \Omega^+_{2m}(q)$ or $\Omega^-_{2m}(q)$, $m \geq 4$, we use Theorems 40, 41 and 42. The result for the exceptional and twisted groups of Lie-type in even characteristic follows from Theorem 46.

Now suppose that $q$ is a power of an odd prime. When $G = PSL_n(q)$, $n \geq 2$, except $PSL_2(3)$ the result holds by Theorems 50 and 52. For $G = PSp_{2m}(q)$, $m \geq 2$, we use Theorem 58, while for $G = PSU_n(q)$ we use Corollary 61, Theorem 62 and Corollary 66. If $G = P\Omega^+_n(q)$ or $P\Omega^-_n(q)$, $n \geq 5$ the result follows from Theorems 71, 73 and Corollary 76. For the exceptional and twisted groups of Lie-type in odd characteristic we use Theorem 77. □

Table 3: First disc size for sporadic groups

| Group | Class | Class size | $\vert\Delta_1(t)\vert$ | $f$ |
|---|---|---|---|---|
| $M_{11}$ | 2A | 165 | 80 | 2 |
| $M_{12}$ | 2A | 396 | 180 | 2 |
| $M_{12}$ | 2B | 495 | 176 | 2 |
| $M_{22}$ | 2A | 1155 | 576 | 2 |
| $M_{23}$ | 2A | 3795 | 1344 | 2 |
| $M_{24}$ | 2A | 11385 | 2816 | 4 |
| $M_{24}$ | 2B | 31878 | 10880 | 2 |
| $Co_3$ | 2A | 170775 | 59264 | 2 |
| $Co_3$ | 2B | 2608200 | 904112 | 2 |
| $Co_2$ | 2A | 56925 | 14336 | 3 |
| $Co_2$ | 2B | 1024650 | 379904 | 2 |
| $Co_2$ | 2C | 28690200 | 5084672 | 5 |
| $Co_1$ | 2A | 46621575 | 13451264 | 3 |
| $Co_1$ | 2B | 2065694400 | 902774912 | 2 |
| $Co_1$ | 2C | 10680579000 | 3014586368 | 3 |
| $J_2$ | 2A | 315 | 224 | 1 |
| $J_2$ | 2B | 2520 | 1212 | 2 |
| $HS$ | 2A | 5775 | 2304 | 2 |
| $HS$ | 2B | 15400 | 7152 | 2 |
| $McL$ | 2A | 22275 | 10304 | 2 |
| $Suz$ | 2A | 135135 | 69632 | 1 |
| $Suz$ | 2B | 2779920 | 1454432 | 1 |
| $Fi_{22}$ | 2A | 3510 | 2816 | 1 |
| $Fi_{22}$ | 2B | 1216215 | 484352 | 2 |
| $Fi_{22}$ | 2C | 36468450 | 12015872 | 3 |
| $Fi_{23}$ | 2A | 31671 | 28160 | 1 |
| $Fi_{23}$ | 2B | 55582605 | 15234560 | 3 |
| $Fi_{23}$ | 2C | 12839581755 | 3308650496 | 3 |
| $Fi_{24}'$ | 2A | 4860485028 | 1504701440 | 3 |
| $Fi_{24}'$ | 2B | 7819305288795 | 3351534645248 | 2 |
| $J_1$ | 2A | 1463 | 1072 | 1 |
| $J_3$ | 2A | 26163 | 16832 | 1 |
| $J_4$ | 2A | 3980549947 | 1112555520 | 3 |
| $J_4$ | 2B | 47766599364 | 26545360896 | 1 |
| $Ru$ | 2A | 593775 | 149504 | 3 |
| $Ru$ | 2B | 1252800 | 570752 | 2 |
| $Ly$ | 2A | 1296826875 | 659509424 | 1 |
| $O'N$ | 2A | 2857239 | 1079168 | 2 |
| $He$ | 2A | 24990 | 4992 | 5 |
| $He$ | 2B | 187425 | 119552 | 1 |
| $Th$ | 2A | 976841775 | 377298944 | 2 |
| $HN$ | 2A | 1539000 | 391424 | 3 |
| $HN$ | 2B | 74064375 | 26906624 | 2 |
| $\mathbb{B}$ | 2A | 13571955000 | 2370830336 | 5 |
| $\mathbb{B}$ | 2B | 11707448673375 | 4010408935424 | 2 |
| $\mathbb{B}$ | 2C | 156849238149120000 | 56546114902065210 | 2 |
| $\mathbb{B}$ | 2D | 355438141723665100 | 94228887171498040 | 3 |
| $\mathbb{M}$ | 2A | 97239461142009240000 | 30528114911948600000 | 3 |
| $\mathbb{M}$ | 2B | 579174806851198200000000000 | 148632542921011000000000000 | 3 |

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