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2012

MIMS EPrint: **2012.109**

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GEOMETRIC STRUCTURE AND THE LOCAL LANGLANDS CONJECTURE

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ABSTRACT. We prove that a strengthened form of the local Langlands conjecture is valid throughout the principal series of any connected split reductive p-adic group. The method of proof is to establish the presence of a very simple geometric structure, in both the smooth dual and the Langlands parameters. We prove that this geometric structure is present, in the same way, for the general linear group, including all of its inner forms. With these results as evidence, we give a detailed formulation of a general geometric structure conjecture.

CONTENTS

1. Introduction

Let $\mathcal{G}$ be a connected reductive p-adic group. The smooth dual of $\mathcal{G}$ — denoted $\mathbf{Irr}(\mathcal{G})$ — is the set of equivalence classes of smooth irreducible representations of $\mathcal{G}$. In this paper we state a conjecture based on [5, 6, 7, 8] which asserts that a very simple geometric structure is present in $\mathbf{Irr}(\mathcal{G})$. A first feature of our conjecture is that it provides a guide to determining $\mathbf{Irr}(\mathcal{G})$. A second feature of the conjecture is that it connects very closely to the local Langlands conjecture.

For any connected reductive p-adic group $\mathcal{G}$, validity of the conjecture gives an explicit description of Bernstein's infinitesimal character [15] and of the intersections of L-packets with Bernstein components in $\mathbf{Irr}(\mathcal{G})$.

The conjecture can be stated at four levels:

- K-theory of C^*-algebras
- Periodic cyclic homology of finite type algebras
- Geometric equivalence of finite type algebras
- Representation theory

At the level of K-theory, the conjecture interacts with the Baum–Connes conjecture [13]. BC has been proved for reductive p-adic groups by V. Lafforgue [44]. The conjecture of this paper ABPS can be viewed as a "lifting" of BC from K-theory to representation theory. In this paper ABPS will be stated at the level of representation theory.

The overall point of view of the paper is as follows. Denote the Langlands dual group of the p-adic group $\mathcal{G}$ by $^L\mathcal{G}$. If $\mathcal{G}$ is split, then $^L\mathcal{G}$ is a connected reductive complex algebraic group. A Langlands parameter is a homomorphism of topological groups

$$\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathcal{G}$$

which is required to satisfy some conditions. Here $\mathbf{W}_F$ is the Weil group of the p-adic field F over which $\mathcal{G}$ is defined. Let {Langlands parameters} $/^L\mathcal{G}$ be the set of all the Langlands parameters for $\mathcal{G}$ modulo the action of $^L\mathcal{G}$. The local Langlands correspondence asserts that there is a surjective finite-to-one map

$$\mathbf{Irr}(G) \longrightarrow \{\text{Langlands parameters}\}/^L\mathcal{G},$$

which is natural in various ways. A more subtle version [3, 65] conjectures that one can naturally enhance the Langlands parameters with irreducible representations of certain finite groups, such that the map becomes bijective.

The aim of the paper is to introduce into this context a countable disjoint union of complex affine varieties, denoted {Extended quotients}, such that there is a commutative triangle of maps

{Extended quotients}

$\mathbf{Irr}(\mathcal{G})$ $\longrightarrow$ {Langlands parameters}$/^L\mathcal{G}$

in which the left slanted arrow is bijective and the other two arrows are surjective and finite to one.

The key point is that in practice {Extended quotients} is much more easily calculated than either $\mathbf{Irr}(\mathcal{G})$ or the $^L\mathcal{G}$-conjugacy classes of Langlands parameters. In examples, bijectivity of the left slanted map is proved by using results on the representation theory of affine Hecke algebras. The right slanted map is defined and studied by using an appropriate generalization of the Springer correspondence.

The paper is divided into three parts:

- Part 1: Statement of the conjecture
- Part 2: Examples
- Part 3: Principal series of connected split reductive p-adic groups

We should emphasize that the conjecture in Part 1 of this article is a *strengthening* of the geometric conjecture formulated in [5, 6, 7, 8]. In [5, 6, 7, 8] the local Langlands correspondence was not part of the conjecture. Now, by contrast, the local Langlands correspondence is locked into our conjecture.

Let $\mathfrak{s}$ be a point in the Bernstein spectrum of $\mathcal{G}$, let $\mathbf{Irr}^\mathfrak{s}(\mathcal{G})$ be the part of $\mathbf{Irr}(G)$ belonging to $\mathfrak{s}$ and let {Langlands parameters}$^\mathfrak{s}$ be the set of Langlands parameters whose L-packets contain elements of $\mathbf{Irr}^\mathfrak{s}(G)$. Furthermore let $T^\mathfrak{s}$ and $W^\mathfrak{s}$ be the complex torus and finite group assigned by Bernstein to $\mathfrak{s}$, and let $(T^\mathfrak{s}//W^\mathfrak{s})_2$ is the extended quotient (of the second kind) for the action of $W^\mathfrak{s}$ on $T^\mathfrak{s}$. According to our conjecture, the local Langlands correspondence, restricted to the objects attached to $\mathfrak{s}$, *factors through* the extended quotient $(T^\mathfrak{s}//W^\mathfrak{s})_2$. In this precise sense, our conjecture reveals a geometric structure latent in the local Langlands conjecture.

The essence of our conjecture is:

Conjecture 1.1. *Let $\mathcal{G}$ be a connected reductive p-adic group, and let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be any Bernstein component in $\mathbf{Irr}(\mathcal{G})$. Then there is a commutative triangle*

$$
\begin{array}{ccc}
 & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 & \\
 & \swarrow \qquad \searrow & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & \longrightarrow & \{\text{Langlands parameters}\}^{\mathfrak{s}}/^{L}\mathcal{G}
\end{array}
$$

in which the horizontal arrow is the map of the local Langlands conjecture, and the two slanted arrows are canonically defined maps with the left slanted arrow bijective and the right slanted arrow surjective and finite-to-one.

In Part 2, we give an account of the general linear group $\mathrm{GL}_n(F)$ and its inner forms $\mathrm{GL}_m(D)$. Here, D is an F-division algebra of dimension d^2 over its centre F and $n = md$. Except for $\mathrm{GL}_n(F)$, these groups are non-split. The main result of Part 2 is:

Theorem 1.2. *Conjecture 1.1 is valid for $\mathcal{G} = \mathrm{GL}_m(D)$. In this case, for each Bernstein component $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \subset \mathbf{Irr}(\mathcal{G})$, all three maps in the commutative triangle are bijective.*

Calculations involving two other examples — $\mathcal{G} = \mathrm{Sp}_{2n}(F)$ and $\mathcal{G} = G_2(F)$ — are also given in Part 2.

In Part 3 we assume that the local field F satisfies a mild restriction on its residual characteristic, depending on $\mathcal{G}$. For the principal series we then prove that the conjectured geometric structure (i.e. extended quotients) is present in the enhanced Langlands parameters. In this case the enhanced Langlands parameters reduce to simpler data, which we call Kazhdan–Lusztig–Reeder parameters.

More precisely, the main result in Part 3, namely Theorem 24.1, is:

Theorem 1.3. *Let $\mathcal{G}$ be a connected split reductive p-adic group. Assume that the residual characteristic of the local field F is not a torsion prime for $\mathcal{G}$. Let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be a Bernstein component in the principal series of $\mathcal{G}$. Then Conjecture 1.1 is valid for $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ i.e. there is a commutative triangle of natural bijections*

$$
\begin{array}{ccc}
 & (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 & \\
 & \swarrow \qquad \searrow & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & \longrightarrow & \{\text{KLR parameters}\}^{\mathfrak{s}}/^{L}\mathcal{G}^{\mathfrak{s}}
\end{array}
$$

In this triangle $\{\text{KLR parameters}\}^{\mathfrak{s}}/^{L}\mathcal{G}^{\mathfrak{s}}$ is the set of Kazhdan–Lusztig–Reeder parameters for the Bernstein component $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ (modulo conjugation by the appropriate subgroup of the Langlands dual group $^{L}\mathcal{G}$).

The construction of the bottom horizontal map in the triangle generalizes results of Reeder [54]. Reeder requires that $\mathcal{G}$ shall have connected centre; we have removed this restriction. Therefore, our result applies to the principal series of $\mathrm{SL}_n(F)$.

Additional points in Part 3 are — labelling by unipotent classes, correcting cocharacters, and proof of the L-packet conjecture stated in [8].

Finally, the appendix defines *geometric equivalence.* This is an equivalence relation on finite type algebras which is a weakening of Morita equivalence. Geometric equivalence underlies and is the foundation of Conjecture 1.1.

Acknowledgements. Thanks to Mark Reeder for drawing our attention to the article of Kato [41]. We thank Joseph Bernstein, David Kazhdan, and David Vogan for enlightening comments and discussions.

Part 1. **Statement of the conjecture**

2. Extended quotient

Let Γ be a finite group acting on a complex affine variety X as automorphisms of the affine variety

$$\Gamma \times X \to X.$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x:

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x. The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$. This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}$. $\widetilde{X}$ is an affine variety and is a subvariety of $\Gamma \times X$. The group Γ acts on $\widetilde{X}$:

$$\Gamma \times \widetilde{X} \to \widetilde{X}$$
$$\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha x), \qquad \alpha \in \Gamma, \quad (\gamma, x) \in \widetilde{X}.$$

The extended quotient, denoted $X//\Gamma$, is $\widetilde{X}/\Gamma$. Thus the extended quotient $X//\Gamma$ is the usual quotient for the action of Γ on $\widetilde{X}$. The projection $\widetilde{X} \to X$, $(\gamma, x) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to give a morphism of affine varieties

$$\rho \colon X//\Gamma \to X/\Gamma.$$

This map will be referred to as the projection of the extended quotient onto the ordinary quotient.

The inclusion

$$X \hookrightarrow \widetilde{X}$$
$$x \mapsto (e, x) \qquad e = \text{identity element of } \Gamma$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion of affine varieties $X/\Gamma \hookrightarrow X//\Gamma$. This will be referred to as the inclusion of the ordinary quotient in the extended quotient. We will denote $X//\Gamma$ with X/Γ removed by $X//\Gamma - X/\Gamma$.

3. Bernstein spectrum

We recall some well-known parts of Bernstein's work on p-adic groups, which can be found for example in [15, 55].

With $\mathcal{G}$ fixed, a cuspidal pair is a pair $(\mathcal{M}, \sigma)$ where $\mathcal{M}$ is a Levi factor of a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ and σ is an irreducible supercuspidal representation of $\mathcal{M}$. Here supercuspidal means that the support of any matrix coefficient of such a representation is compact modulo the centre of the group. Pairs $(\mathcal{M}, \sigma)$ and $(\mathcal{M}, \sigma')$ with σ isomorphic to σ' are considered equal. The group $\mathcal{G}$ acts on the space of cuspidal pairs by conjugation:

$$g \cdot (\mathcal{M}, \sigma) = (g\mathcal{M}g^{-1}, \sigma \circ \mathrm{Ad}_g^{-1}).$$

We denote the space of $\mathcal{G}$-conjugacy classes by $\Omega(\mathcal{G})$. We can inflate σ to an irreducible smooth $\mathcal{P}$-representation. Normalized smooth induction then produces a $\mathcal{G}$-representation $I_{\mathcal{P}}^{\mathcal{G}}(\sigma)$.

For any irreducible smooth $\mathcal{G}$-representation π there is a cuspidal pair $(\mathcal{M}, \sigma)$, unique up to conjugation, such that π is a subquotient of $I_{\mathcal{P}}^{\mathcal{G}}(\sigma)$. (The collection of irreducible subquotients of the latter representation does not depend on the choice of $\mathcal{P}$.) The $\mathcal{G}$-conjugacy class of (M, σ) is called the *cuspidal support* of π. We write the cuspidal support map as

$$\mathbf{Sc} : \mathbf{Irr}(\mathcal{G}) \to \Omega(\mathcal{G}).$$

For any unramified character ν of $\mathcal{M}$, $(\mathcal{M}, \sigma \otimes \nu)$ is again a cuspidal pair. Two cuspidal pairs $(\mathcal{M}, \sigma)$ and $(\mathcal{M}', \sigma')$ are said to be inertially equivalent, written $(\mathcal{M}, \sigma) \sim (\mathcal{M}', \sigma')$, if there exists an unramified character $\nu : \mathcal{M} \to \mathbb{C}^{\times}$ and an element $g \in \mathcal{G}$ such that

$$g \cdot (\mathcal{M}, \psi \otimes \sigma) = (\mathcal{M}', \sigma').$$

The Bernstein spectrum of $\mathcal{G}$, denoted $\mathfrak{B}(\mathcal{G})$, is the set of inertial equivalence classes of cuspidal pairs. It is a countable set, infinite unless $\mathcal{G} = 1$. Let $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathcal{G}} \in \mathfrak{B}(\mathcal{G})$ be the inertial equivalence class of $(\mathcal{M}, \sigma)$ and let $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ be the subset of $\mathbf{Irr}(\mathcal{G})$ of representations that have cuspidal support in $\mathfrak{s}$. Then $\mathbf{Irr}(\mathcal{G})$ is the disjoint union of the Bernstein components $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$:

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G})} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

The space $X_{\mathrm{unr}}(\mathcal{M})$ of unramified characters of $\mathcal{M}$ is a natural way a complex algebraic torus. Put

$$(1) \qquad \qquad \mathrm{Stab}(\sigma) = \{\nu \in X_{\mathrm{unr}}(\mathcal{M}) \mid \sigma \otimes \nu \cong \sigma\}.$$

This is known to be a finite group, so $X_{\mathrm{unr}}(M)/\mathrm{Stab}(\sigma)$ is again a complex algebraic torus. The map

$$(2) \qquad X_{\mathrm{unr}}(\mathcal{M})/\mathrm{Stab}(\sigma) \to \mathbf{Irr}(\mathcal{M})^{[\mathcal{M}, \sigma]_{\mathcal{M}}}, \ \nu \mapsto \sigma \otimes \nu$$

is bijective and thus provides $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M}, \sigma]_{\mathcal{M}}}$ with the structure of an algebraic torus. This structure is canonical, in the sense that it does not depend on the choice of σ in $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M}, \sigma]_{\mathcal{M}}}$.

The Weyl group of $(\mathcal{G}, \mathcal{M})$ is defined as

$$W(\mathcal{G}, \mathcal{M}) := N_{\mathcal{G}}(\mathcal{M})/\mathcal{M}.$$

It is a finite group which generalizes the notion of the Weyl group associated to a maximal torus. The Weyl group of $(\mathcal{G}, \mathcal{M})$ acts naturally on $\mathbf{Irr}(\mathcal{M})$, via the conjugation action on $\mathcal{M}$. The subgroup

$$(3) \qquad W^{\mathfrak{s}} := \{w \in W(\mathcal{G}, \mathcal{M}) \mid w \text{ stabilizes } [\mathcal{M}, \sigma]_{\mathcal{M}}\}.$$

acts on $\mathbf{Irr}(\mathcal{M})^{[\mathcal{M}, \sigma]_{\mathcal{M}}}$. We define

$$(4) \qquad T^{\mathfrak{s}} := \mathbf{Irr}(\mathcal{M})^{[\mathcal{M}, \sigma]_{\mathcal{M}}}$$

with the structure (2) as algebraic torus and the $W^{\mathfrak{s}}$-action (3). We note that the $W^{\mathfrak{s}}$-action is literally by automorphisms of the algebraic variety $T^{\mathfrak{s}}$, via (2) they need not become group automorphisms. Two elements of $T^{\mathfrak{s}}$ are $\mathcal{G}$-conjugate if and only if they are in the same $W^{\mathfrak{s}}$-orbit.

An inertially equivalent cuspidal pair $(\mathcal{M}', \sigma')$ would yield a torus $T'^{\mathfrak{s}}$ which is isomorphic to $T^{\mathfrak{s}}$ via conjugation in $\mathcal{G}$. Such an isomorphism $T^{\mathfrak{s}} \cong T'^{\mathfrak{s}}$ is unique up to the action of $W^{\mathfrak{s}}$.

The element of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ associated to any $\pi \in \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is called its *infinitesimal central character*, denoted $\pi^{\mathfrak{s}}(\pi)$. Another result of Bernstein is the existence of a unital finite type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$-algebra $\mathcal{H}^{\mathfrak{s}}$, whose irreducible modules are in natural bijection with $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. The construction is such that $\mathcal{H}^{\mathfrak{s}}$ has centre $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$ and that $\pi^{\mathfrak{s}}(\pi)$ is precisely the central character of the corresponding $\mathcal{H}^{\mathfrak{s}}$-module.

Since $\mathbf{Irr}(\mathcal{H}^{\mathfrak{s}})$ is in bijection with the collection of primitive ideals of $\mathcal{H}^{\mathfrak{s}}$, we can endow it with the Jacobson topology. By transferring this topology to $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$, we make the latter into a (nonseparated) algebraic variety. (In fact this topology agrees with the topology on $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ considered as a subspace of $\mathbf{Irr}(\mathcal{G})$, endowed with the Jacobson topology from the Hecke algebra of $\mathcal{G}$.)

<u>Summary</u>: For each Bernstein component $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ there are:

(1) A finite group $W^{\mathfrak{s}}$ acting on a complex torus $T^{\mathfrak{s}}$;
(2) A subset $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ of $\mathbf{Irr}(\mathcal{G})$;
(3) A morphism of algebraic varieties

$$\pi^{\mathfrak{s}} \colon \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}};$$

(4) A unital finite-type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$-algebra $\mathcal{H}^{\mathfrak{s}}$ with

$$\mathbf{Irr}(\mathcal{H}^{\mathfrak{s}}) = \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

4. Approximate statement of the conjecture

As above, $\mathcal{G}$ is a connected reductive p-adic group and $\mathfrak{s}$ is a point in the Bernstein spectrum of $\mathcal{G}$.

Consider the two maps indicated by vertical arrows:

$$
\begin{array}{ccc}
T^{\mathfrak{s}}//W^{\mathfrak{s}} & & \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \\
\big\downarrow{\scriptstyle \rho^{\mathfrak{s}}} & & \big\downarrow{\scriptstyle \pi^{\mathfrak{s}}} \\
T^{\mathfrak{s}}/W^{\mathfrak{s}} & & T^{\mathfrak{s}}/W^{\mathfrak{s}}
\end{array}
$$

Here $\pi^{\mathfrak{s}}$ is the infinitesimal character and $\rho^{\mathfrak{s}}$ is the projection of the extended quotient on the ordinary quotient. In practice $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\rho^{\mathfrak{s}}$ are much easier to calculate than $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ and $\pi^{\mathfrak{s}}$.

An approximate statement of the conjecture is:

$$\pi^{\mathfrak{s}} \colon \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}} \quad \text{and} \quad \rho^{\mathfrak{s}} \colon T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}} \text{ are almost the same.}$$

The precise statement of the conjecture — in particular, precise meaning of "are almost the same" — is given in section 7 below.

$\pi^{\mathfrak{s}}$ and $\rho^{\mathfrak{s}}$ are both surjective finite-to-one maps and morphisms of algebraic varieties. For $x \in T^{\mathfrak{s}}/W^{\mathfrak{s}}$, denote by $\#(x, \rho^{\mathfrak{s}})$, $\#(x, \pi^{\mathfrak{s}})$ the number of points in the pre-image of x using $\rho^{\mathfrak{s}}$, $\pi^{\mathfrak{s}}$. The numbers $\#(x, \pi^{\mathfrak{s}})$ are of interest in describing exactly what happens when $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is constructed by parabolic induction.

Within $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ there are algebraic sub-varieties $R(\rho^{\mathfrak{s}})$, $R(\pi^{\mathfrak{s}})$ defined by

$$R(\rho^{\mathfrak{s}}) := \{x \in T^{\mathfrak{s}}/W^{\mathfrak{s}} \mid \#(x, \rho^{\mathfrak{s}}) > 1\}$$

$$R(\pi^{\mathfrak{s}}) := \{x \in T^{\mathfrak{s}}/W^{\mathfrak{s}} \mid \#(x, \pi^{\mathfrak{s}}) > 1\}$$

It is immediate that

$$R(\rho^{\mathfrak{s}}) = \rho^{\mathfrak{s}}(T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}})$$

$R(\pi^{\mathfrak{s}})$ will be referred to as the *sub-variety of reducibility*.

In many examples $R(\rho^{\mathfrak{s}}) \neq R(\pi^{\mathfrak{s}})$. Hence in these examples it is impossible to have a bijection

$$T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

with commutativity in the diagram

A more precise statement of the conjecture is that after a simple algebraic correction ("correcting cocharacters") $\rho^{\mathfrak{s}}$ becomes isomorphic to $\pi^{\mathfrak{s}}$. Thus $\rho^{\mathfrak{s}}$ is an easily calculable map which can be algebraically corrected to give $\pi^{\mathfrak{s}}$. An implication of this is that within the algebraic variety $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ there is a flat family of sub-varieties connecting $R(\rho^{\mathfrak{s}})$ and $R(\pi^{\mathfrak{s}})$.

5. Extended quotient of the second kind

With Γ, X, Γ_x as in Section 2 above, let $\mathbf{Irr}(\Gamma_x)$ be the set of (equivalence classes of) irreducible representations of Γ_x. The *extended quotient of the second kind*, denoted $(X//\Gamma)_2$, is constructed by replacing the orbit of x (for the given action of Γ on X) by $\mathbf{Irr}(\Gamma_x)$. This is done as follows :

Set $\widetilde{X}_2 = \{(x, \tau) \,\big|\, x \in X \text{ and } \tau \in \mathbf{Irr}(\Gamma_x)\}$. Then Γ acts on $\widetilde{X}_2$.

$$\Gamma \times \widetilde{X}_2 \to \widetilde{X}_2,$$

$$\gamma(x, \tau) = (\gamma x, \gamma_* \tau),$$

where $\gamma_* \colon \mathbf{Irr}(\Gamma_x) \to \mathbf{Irr}(\Gamma_{\gamma x})$. Now we define

$$(X//\Gamma)_2 := \widetilde{X}_2/\Gamma,$$

i.e. $(X/\!/\Gamma)_2$ is the usual quotient for the action of Γ on $\widetilde{X}_2$. The projection
$\widetilde{X}_2 \to X \quad (x, \tau) \mapsto x$ is Γ-equivariant and so passes to quotient spaces to
give the projection of $(X/\!/\Gamma)_2$ onto X/Γ.

$$\rho_2 \colon (X/\!/\Gamma)_2 \longrightarrow X/\Gamma$$

Denote by triv_x the trivial one-dimensional representation of Γ_x. The inclusion

$$X \hookrightarrow \widetilde{X}_2$$
$$x \mapsto (x, \mathrm{triv}_x)$$

is Γ-equivariant and so passes to quotient spaces to give an inclusion

$$X/\Gamma \hookrightarrow (X/\!/\Gamma)_2$$

This will be referred to as the inclusion of the ordinary quotient in the
extended quotient of the second kind.

6. Comparison of the two extended quotients

With X, Γ as above, there is a non-canonical bijection $\epsilon \colon X/\!/\Gamma \to (X/\!/\Gamma)_2$
with commutativity in the diagrams

(5)

To construct the bijection ϵ, some choices must be made. We use a family
ψ of bijections

$$\psi_x : c(\Gamma_x) \to \mathbf{Irr}(\Gamma_x)$$

such that for all $x \in X$:

   (1) $\psi_x([1]) = \mathrm{triv}_x$;
   (2) $\psi_{\gamma x}([\gamma g \gamma^{-1}]) = \phi_x([g]) \circ \mathrm{Ad}_\gamma^{-1}$ for all $g \in \Gamma_x, \gamma \in \Gamma$;
   (3) $\psi_x = \psi_y$ if $\Gamma_x = \Gamma_y$ and x, y belong to the same connected component
       of the variety X^{Γ_x}.

We shall refer to such a family of bijections as a c-**Irr** system. Clearly ψ
induces a map $\widetilde{X} \to \widetilde{X}_2$ which preserves the X-coordinates. By property
(2) this map is Γ-equivariant, so it descends to a map

$$\epsilon = \epsilon_\psi : X/\!/\Gamma \to (X/\!/\Gamma)_2.$$

Observe that ϵ_ψ makes the diagrams from (5) commute, the first by con-
struction and the second by property (1). The restriction of ϵ_ψ to the fiber
over $\Gamma x \in X/\Gamma$ is ψ_x, and in particular is bijective. Therefore ϵ_ψ is bijective.
Property (3) is not really needed, it serves to exclude ugly examples.

One way to topologize $(X/\!/\Gamma)_2$ is via the X-coordinate, in other words, by
simply pulling back the topology from X/Γ via the natural projection. With
respect to this naive topology ϵ_ψ is continuous. This continuous bijection,
however, is not usually a homeomorphism. In most cases $X/\!/\Gamma$ is more
separated than $(X/\!/\Gamma)_2$.

But there are other useful topologies. Let $\mathcal{O}(X)$ be the coordinate algebra of the affine variety X and let $\mathcal{O}(X) \rtimes \Gamma$ be the crossed-product algebra for the action of Γ on $\mathcal{O}(X)$. There are canonical bijections

$$\mathbf{Irr}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow \mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma) \longleftrightarrow (X/\!/\Gamma)_2,$$

where $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is the set of primitive ideals in this algebra. The irreducible module associated to $(x, \tau) \in (X/\!/\Gamma)_2$ is

$$(6) \qquad \mathrm{Ind}_{\mathcal{O}(X) \rtimes \Gamma_x}^{\mathcal{O}(X) \rtimes \Gamma} (\mathbb{C}_x \otimes \tau).$$

The space $\mathrm{Prim}(\mathcal{O}(X) \rtimes \Gamma)$ is endowed with the Jacobson topology, which makes it a nonseparated algebraic variety. This can be transferred to a topology on $(X/\!/\Gamma)_2$, which we also call the Jacobson topology. It is slightly coarser than the naive topology described above.

The bijection ϵ_ψ is not always continuous with respect to the Jacobson topology. In fact, it follows readily from (6) that ϵ_ψ is continuous in this sense if and only the following additional condition is satisfied:

(4) Suppose that x, y lie in the same connected component of the variety X^{Γ_x}, that $\Gamma_y \supset \Gamma_x$ and that $\gamma \in \Gamma_x$. Then the Γ_y-representation $\psi_y([\gamma])$ appears in $\mathrm{Ind}_{\Gamma_x}^{\Gamma_y} \psi_x([\gamma])$.

While the first three conditions are easy to fulfill, the fourth can be rather difficult.

The two finite-type $\mathcal{O}(X/\Gamma)$-algebras $\mathcal{O}(X/\!/\Gamma)$ and $\mathcal{O}(X) \rtimes \Gamma$ are usually (i.e. if the action of Γ on X is neither trivial nor free) not Morita equivalent. In examples relevant to the representation theory of reductive p-adic groups these two finite-type $\mathcal{O}(X/\Gamma)$-algebras are equivalent via a weakening of Morita equivalence referred to as "geometric equivalence", see the appendix.

7. STATEMENT OF THE CONJECTURE

As above, $\mathcal{G}$ is a connected reductive p-adic group and $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$. {Langlands parameters}$^{\mathfrak{s}}/{}^L\mathcal{G}$ is the set of Langlands parameters for $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$, quotiented by the action of the Langlands dual group ${}^L\mathcal{G}$.

The conjecture consists of the following five statements.

(1) The infinitesimal character

$$\pi^{\mathfrak{s}} : \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

is one-to-one if and only if the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ is free.

(2) The extended quotient of the second kind $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ surjects by a canonical finite-to-one map onto {Langlands parameters}$^{\mathfrak{s}}/{}^L\mathcal{G}$.

(3) The extended quotient of the second kind $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is canonically in bijection with the Bernstein component $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$.

(4) The canonical bijection

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \longleftrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

comes from a canonical geometric equivalence of the two unital finite-type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$-algebras $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$ and $\mathcal{H}^{\mathfrak{s}}$. See the appendix for details on "geometric equivalence".

(5) A c-**Irr** system can be chosen for the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ such that the resulting bijection

$$\epsilon : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$$

when composed with the canonical bijection $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ gives a bijection

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

which has the following six properties:

Notation for Property 1:
Within the smooth dual $\mathrm{Irr}(\mathcal{G})$, we have the tempered dual
$\mathbf{Irr}(\mathcal{G})_{\mathrm{temp}} = \{$smooth tempered irreducible representations of $\mathcal{G}\}/ \sim$
$T^{\mathfrak{s}}_{\mathrm{cpt}} = $ maximal compact subgroup of $T^{\mathfrak{s}}$.
$T^{\mathfrak{s}}_{\mathrm{cpt}}$ is a compact real torus. The action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$ preserves $T^{\mathfrak{s}}_{\mathrm{cpt}}$, so we can form the compact orbifold $T^{\mathfrak{s}}_{\mathrm{cpt}}//W^{\mathfrak{s}}$.

Property 1 of the bijection $\mu^{\mathfrak{s}}$:
The bijection $\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ maps $T^{\mathfrak{s}}_{\mathrm{cpt}}//W^{\mathfrak{s}}$ onto $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$, and hence restricts to a bijection

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}_{\mathrm{cpt}}//W^{\mathfrak{s}} \longleftrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cap \mathbf{Irr}(\mathcal{G})_{\mathrm{temp}}$$

Property 2 of the bijection $\mu^{\mathfrak{s}}$:
For many $\mathfrak{s}$ the diagram

$$
\begin{array}{ccc}
T^{\mathfrak{s}}//W^{\mathfrak{s}} & \xrightarrow{\ \mu^{\mathfrak{s}}\ } & \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \\
& \searrow{\scriptstyle \rho^{\mathfrak{s}}} \quad {\scriptstyle \pi^{\mathfrak{s}}}\swarrow & \\
& T^{\mathfrak{s}}/W^{\mathfrak{s}} &
\end{array}
$$

does not commute.

Property 3 of the bijection $\mu^{\mathfrak{s}}$:
In the possibly non-commutative diagram

$$
\begin{array}{ccc}
T^{\mathfrak{s}}//W^{\mathfrak{s}} & \xrightarrow{\ \mu^{\mathfrak{s}}\ } & \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \\
& \searrow{\scriptstyle \rho^{\mathfrak{s}}} \quad {\scriptstyle \pi^{\mathfrak{s}}}\swarrow & \\
& T^{\mathfrak{s}}/W^{\mathfrak{s}} &
\end{array}
$$

the bijection $\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is continuous where $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ has the Zariski topology and $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ has the Jacobson topology — and the composition

$$\pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

is a morphism of algebraic varieties.

Property 4 of the bijection $\mu^{\mathfrak{s}}$:
There is an algebraic family

$$\theta_z : T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho^{\mathfrak{s}}, \quad \theta_{\sqrt{q}} = \pi^{\mathfrak{s}} \circ \mu^{\mathfrak{s}}, \quad \text{and} \quad \theta_{\sqrt{q}}(T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}}) = R(\pi^{\mathfrak{s}}).$$

Here q is the order of the residue field of the p-adic field F over which $\mathcal{G}$ is defined and $R(\pi^{\mathfrak{s}}) \subset T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is the sub-variety of reducibility. Setting

$$Y_z = \theta_z(T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}})$$

a flat family of sub-varieties of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ is obtained with

$$Y_1 = R(\rho^{\mathfrak{s}}), \qquad Y_{\sqrt{q}} = R(\pi^{\mathfrak{s}}).$$

Property 5 of the bijection $\mu^{\mathfrak{s}}$ (Correcting cocharacters):

For each irreducible component $\mathbf{c}$ of the affine variety $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ there is a cocharacter (i.e. a homomorphism of algebraic groups)

$$h_c : \mathbb{C}^\times \longrightarrow T^{\mathfrak{s}}$$

such that

$$\theta_z[w, t] = b(h_c(z) \cdot t)$$

for all $[w, t] \in \mathbf{c}$.

Let $b : T^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$ be the quotient map. Here, as above, points of $\widetilde{T}_{\mathfrak{s}}$ are pairs (w, t) with $w \in W^{\mathfrak{s}}$, $t \in T^{\mathfrak{s}}$ and $wt = t$. $[w, t]$ is the point in $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ obtained by applying the quotient map $\widetilde{T}^{\mathfrak{s}} \to T^{\mathfrak{s}}//W^{\mathfrak{s}}$ to (w, t).

Remark. The equality

$$\theta_z[w, t] = b(h_c(z) \cdot t)$$

is to be interpreted thus:

Let $Z_1, Z_2, \ldots, Z_r$ be the irreducible components of the affine variety $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ and let $h_1, h_2, \ldots, h_r$ be the cocharacters as in the statement of Property 5. Let

$$\nu^{\mathfrak{s}} : \widetilde{T^{\mathfrak{s}}} \longrightarrow T^{\mathfrak{s}}//W^{\mathfrak{s}}$$

be the quotient map.

Then irreducible components $X_1, X_2, \ldots, X_r$ of the affine variety $\widetilde{T^{\mathfrak{s}}}$ can be chosen with

- $\nu^{\mathfrak{s}}(X_j) = Z_j$ for $j = 1, 2, \ldots, r$
- For each $z \in \mathbb{C}^\times$ the map $m_z : X_j \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$, which is the composition

$$X_j \longrightarrow \quad T^{\mathfrak{s}} \quad \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$
$$(w, t) \longmapsto h_j(z)t \longmapsto b(h_j(z)t),$$

makes the diagram

$$
\begin{array}{ccc}
X_j & \xrightarrow{\quad \nu^{\mathfrak{s}} \quad} & Z_j \\
& \searrow{\scriptstyle m_z} \quad \swarrow{\scriptstyle \theta_z} & \\
& T^{\mathfrak{s}}/W^{\mathfrak{s}} &
\end{array}
$$

commutative. Note that $h_j(z)t$ is the product of $h_j(z)$ and t in the algebraic group $T^{\mathfrak{s}}$.

Remark. The conjecture asserts that to calculate

$$\pi^{\mathfrak{s}} \colon \mathbf{Irr}^{\mathfrak{s}}(\mathcal{G}) \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

two steps suffice:

Step 1: Calculate $\rho^{\mathfrak{s}} \colon T^{\mathfrak{s}}//W^{\mathfrak{s}} \longrightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$.

Step 2: Determine the correcting cocharacters.

The cocharacter assigned to $T^{\mathfrak{s}}/W^{\mathfrak{s}} \hookrightarrow T^{\mathfrak{s}}//W^{\mathfrak{s}}$ is always the trivial cocharacter mapping $\mathbb{C}^{\times}$ to the unit element of $T^{\mathfrak{s}}$. So all the non-trivial correcting is taking place on $T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}}$.

Notation for Property 6.

If S and V are sets, a *labelling* of S by V is a map of sets $\lambda \colon S \to V$.

Property 6 of the bijection $\mu_{\mathfrak{s}}$ (L-packets):

As in Property 5, let $\{Z_1, \ldots, Z_r\}$ be irreducible components of the affine variety $T^{\mathfrak{s}}//W^{\mathfrak{s}}$. Then a labelling $\lambda \colon \{Z_1, Z_2, \ldots Z_r\} \to V$ exists such that: For every two points $[w, t]$ and $[w', t]$ of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$:

$$\mu^{\mathfrak{s}}[w, t] \text{ and } \mu^{\mathfrak{s}}[w', t'] \text{ are in the same L-packet}$$

if and only

(i) $\theta_z[w, t] = \theta_z[w', t']$ for all $z \in \mathbb{C}^{\times}$;
(ii) $\lambda[w, t] = \lambda[w', t']$, where we lifted λ to a labelling of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ in the obvious way.

Remark. An L-packet can have non-empty intersection with more than one Bernstein component. The conjecture does not address this issue. The conjecture only describes the intersections of L-packets with any one given Bernstein component.

In brief, the conjecture asserts that — once a Bernstein component has been fixed — intersections of L-packets with that Bernstein component consisting of more than one point are "caused" by repetitions among the correcting cocharacters. If, for any one given Bernstein component, the correcting cocharacters $h_1, h_2, \ldots, h_r$ are all distinct, then (according to the conjecture) the intersections of L-packets with that Bernstein component are singletons.

A Langlands parameter can be taken to be a homomorphism of topological groups

$$\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathcal{G},$$

where F is the p-adic field over which $\mathcal{G}$ is defined, $\mathbf{W}_F$ is the Weil group of F, and ${}^L\mathcal{G}$ is the Langlands dual group of $\mathcal{G}$. Let $T({}^L\mathcal{G})$ be the maximal torus of ${}^L\mathcal{G}$. By restricting a Langlands parameter to the maximal torus of $\mathrm{SL}_2(\mathbb{C})$ a cocharacter

$$\mathbb{C}^{\times} \longrightarrow T({}^L\mathcal{G})$$

is obtained. In examples, these give the correcting cocharacters.

For any $\mathcal{G}$ and any $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ the finite group $W^{\mathfrak{s}}$ is an extended finite Coxeter group i.e. is a semi-direct product for the action of a finite abelian

group Γ on a finite Weyl group W:

$$W^{\mathfrak{s}} = W \rtimes \Gamma.$$

Due to this restriction on which finite groups can actually occur as a $W^{\mathfrak{s}}$, in examples there is often a clear preferred choice of c-**Irr** system for the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$.

Property 7 of the bijection $\mu^{\mathfrak{s}}$:
There is a canonically defined commutative triangle

(7)
$$(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$$

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow \{\text{Langlands parameters}\}^{\mathfrak{s}}/^{L}\mathcal{G}$$

in which the left slanted arrow is bijective, the right slanted arrow is surjective and finite-to-one, and the horizontal arrow is the map of the local Langlands correspondence.

Part 2. Examples

8. REMARKS ON THE SUPERCUSPIDAL CASE

Recall the group $X_{\mathrm{unr}}(\mathcal{M})$ of unramified characters of $\mathcal{M}$, its finite subgroup $\mathrm{Stab}(\sigma)$ defined in (1, and the complex algebraic torus $T^{\mathfrak{s}}$, defined in (4) as the quotient of $X_{\mathrm{unr}}(\mathcal{M})$ by $\mathrm{Stab}(\sigma)$. We first consider the special case in which $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ is a *supercuspidal inertial pair* in $\mathcal{G}$, that is, $\mathfrak{s} = [\mathcal{G}, \pi]_G$, where π is a supercuspidal irreducible representation of $\mathcal{G}$. We have

$$T^{\mathfrak{s}} \simeq \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

On the other hand, the group $W^{\mathfrak{s}}$ is the trivial group $\{1\}$. Hence we have $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 = T^{\mathfrak{s}}/W^{\mathfrak{s}} = T^{\mathfrak{s}}$.

It follows that the left slanted map in the triangle (7) is here the identity map.

In other words, *when $\mathfrak{s}$ is supercuspidal, the triangle* (7) *collapses into the horizontal map, i.e.,* the existence of the commutative triangle of maps is equivalent to the existence of the local Langlands map.

The other parts of the conjecture are trivially true when $\mathfrak{s}$ is supercuspidal, except Property 6. Each of the maps $\mu^{\mathfrak{s}}$, $\pi^{\mathfrak{s}}$, and $\rho^{\mathfrak{s}}$ being equal to the identity map, we have

$$\theta_{\sqrt{q}} = \theta_1 \, (= \mathrm{Id}).$$

It implies that only the trivial cocharacter can occur. On the other hand, the labelling should consist in a *unique label*. Hence the conditions (i) and (ii) in Property 6 are empty. It follows that Property 6 is equivalent to the following property:

Property S: When $\mathfrak{s}$ is supercuspidal, the set of representations of $\mathcal{G}$ which belong to the intersection of $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ with an L-packet is always a singleton.

Property S is obviously true if $Z(\mathcal{G})$ is compact, because in that case there is no non-trivial unramified character of $\mathcal{G}$, thus $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ itself is reduced to

a singleton. It is also valid if $\mathcal{G} = \mathrm{GL}_n(F)$, because each L-packet is a singleton.

Property S is expected to hold in general. For instance, it is true for the supercuspidal L-packets constructed in [28].

9. THE GENERAL LINEAR GROUP AND ITS INNER FORMS

Let F be a local non-archimedean field. Let $\mathcal{G}$ be an inner form of the general linear group $\mathcal{G}^* := \mathrm{GL}_n(F)$, $n \geq 1$, that is, $\mathcal{G}$ is a group of the form $\mathrm{GL}_m(D)$, where D is an F-division algebra, of dimension d^2 over its centre F, and where $n = md$ (see for instance [2, § 25]). By F-division algebra we mean a finite dimensional F-algebra with centre F, in which every nonzero element is invertible.

We have $\mathcal{G} = A^\times$, where A is a simple central F-algebra. Let V be a simple left A-module. The algebra $\mathrm{End}_A(V)$ is an F-division algebra, the opposite of which we denote by D. Considering V as a right D-vector space, we have a canonical isomorphism of F-algebras between A and $\mathrm{End}_D(V)$.

Heiermann [35] proved that every Bernstein component of the category of smooth $\mathcal{G}$-modules is equivalent to the module category of an affine Hecke algebra. Together with [60, Theorem 5.4.2] this proves a large part of the ABPS conjecture for $\mathcal{G}$: properties 1–5 from Section 7. In particular, this provides a bijection between $T^\mathfrak{s}//W^\mathfrak{s}$ and $\mathbf{Irr}(\mathcal{G})^\mathfrak{s}$ for any point $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$. Moreover, $T^\mathfrak{s}//W^\mathfrak{s}$ and $(T^\mathfrak{s}//W^\mathfrak{s})_2$ are canonically isomorphic. In subsection 9.1, we shall construct a canonical bijection from $(T^\mathfrak{s}//W^\mathfrak{s})_2$ to $\mathbf{Irr}(\mathcal{G})^\mathfrak{s}$ by following a different approach based on the fact due to Sécherre and Stevens [58] that the affine Hecke algebra which occurs in the picture here is a tensor product of affine Hecke algebras of equal parameter type.

Let $\mathbf{W}_F$ denote the Weil group of F. The complex dual group of the group $\mathcal{G}$ is $G = \mathrm{GL}_m(\mathbb{C})$. An L-group for a given p-adic group can take one of several forms. The L-group of an inner form of a (quasi-)split group may be identified with the L-group of the latter [2, § 26]. Moreover, the L-group of a split group can be taken to be equal to its complex dual group. Hence we may and we do define the L-group of $\mathcal{G}$ as

$$^L\mathcal{G} = \mathrm{GL}_n(\mathbb{C}).$$

Then a Langlands parameter for $\mathcal{G}$ is a *relevant* continuous homomorphism of topological groups

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

for which the image in G of any element is semisimple, and which commutes with the projections of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$ and G onto $\mathbf{W}_F$. Two Langlands parameters are equivalent if they are conjugate under $^L\mathcal{G}$. The set of equivalence classes of Langlands parameters is denoted $\Phi(\mathcal{G})$.

We recall that "relevant" means that if the image of Φ is contained in a Levi subgroup M of G, then M must be the L-group of a Levi subgroup $\mathcal{M}$ of $\mathcal{G}$. Hence here it means that d must divide all the n_i if $M \simeq \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_h, \mathbb{C})$. Then we set $m_i := n_i/d$ and define $\mathcal{M} = \mathrm{GL}(m_1, D) \times \cdots \times \mathrm{GL}(m_h, D)$. In the particular case of $d = 1$, *i.e.,* for the group $\mathcal{G}^*$, every

Langlands parameter is relevant. Hence we get

$$\Phi(\mathcal{G}) \subset \Phi(\mathcal{G}^*).$$

In the next three subsections we shall construct, for every Bernstein component of $\mathcal{G}$, a commutative triangle of natural bijections as Property 7 of Section 7.

9.1. Types and Hecke algebras for $\mathrm{GL}_m(D)$.

Let $\mathfrak{s}$ be any point in $\mathfrak{B}(\mathcal{G})$. Recall from [22] that an $\mathfrak{s}$-*type* is a pair $(\mathcal{K}, \lambda)$, with $\mathcal{K}$ an open compact subgroup of $\mathcal{G}$ and $(\lambda, \mathcal{V})$ is an irreducible smooth representation of $\mathcal{K}$, such that $\mathbf{Irr}^{\mathfrak{s}}(\mathcal{G})$ is precisely the set of irreducible smooth representations of $\mathcal{G}$ which contain λ. We shall denote by $(\check{\lambda}, \mathcal{V}^\vee)$ the contragredient representation of λ. The endomorphism-valued Hecke algebra $\mathcal{H}(\mathcal{G}, \lambda)$ attached to $(\mathcal{K}, \lambda)$ is defined to be the space of compactly supported functions $f \colon \mathcal{G} \to \mathrm{End}_{\mathbb{C}}(\mathcal{V}^\vee)$, such that

$$f(k_1 g k_2) = \check{\lambda}(k_1) f(g) \check{\lambda}(k_2), \quad \text{where } k_1, k_2 \in \mathcal{K} \text{ and } g \in \mathcal{G}.$$

The standard convolution gives $\mathcal{H}(\mathcal{G}, \lambda)$ the structure of an associative $\mathbb{C}$-algebra with unity. There is a canonical bijection

$$(8) \qquad\qquad \mathbf{Irr}^{\mathfrak{s}}(\mathcal{G}) \to \mathbf{Irr}(\mathcal{H}(\mathcal{G}, \lambda)).$$

Generalizing the work of Bushnell and Kutzko [23], Sécherre and Stevens have constructed in [58] an $\mathfrak{s}$-type $(\mathcal{K}^{\mathfrak{s}}, \lambda^{\mathfrak{s}})$ for each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ and explicitly described the structure of the algebra $\mathcal{H}(\mathcal{G}, \lambda^{\mathfrak{s}})$.

We shall recall the results from [58] that we need. Let $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathcal{G}}$. The Levi subgroup $\mathcal{M}$ is the stabilizer of some decomposition $V = \bigoplus_{j=1}^h V_j$ into subspaces, which gives an identification

$$(9) \qquad\qquad \mathcal{M} \simeq \prod_{j=1}^h \mathrm{GL}_{m_j}(D), \quad \text{where } m_j = \dim_D V_j.$$

We can then write

$$\sigma = \bigotimes_{j=1}^h \sigma_j,$$

where, for each j, the representation σ_j is an irreducible unitary supercuspidal representation of $\mathcal{G}_j := \mathrm{GL}_{m_j}(D)$.

We define an equivalence relation on $\{1, 2, \ldots, h\}$ by

$$(10) \qquad j \sim k \quad \text{if and only if} \quad m_j = m_k \quad \text{and} \quad [\mathcal{G}_j, \sigma_j]_{\mathcal{G}_j} = [\mathcal{G}_k, \sigma_k]_{\mathcal{G}_k},$$

where we have identified $\mathcal{G}_j$ with $\mathcal{G}_k$ whenever $m_j = m_k$. We may, and do, assume that $\sigma_j = \sigma_k$ whenever $j \sim k$, since this does not change the inertial class $\mathfrak{s}$. Denote by $S_1, S_2, \ldots, S_l$ the equivalence classes. For $i = 1, 2, \ldots, l$, we denote by e_i the cardinality of S_i. We call the integers $e_1, e_2, \ldots, e_l$ the *exponents* of $\mathfrak{s}$. Hence we get

$$(11) \qquad\qquad \mathcal{M} \simeq \prod_{i=1}^l \mathrm{GL}_{m_i}(D)^{e_i} \quad \text{and} \quad \sigma \simeq \sigma_1^{e_1} \otimes \cdots \otimes \sigma_l^{e_l},$$

where $\sigma_1, \ldots, \sigma_l$ are pairwise distinct (after unramified twist). We abbreviate $\mathfrak{s}_i := [\mathrm{GL}_{m_i}(D)^{e_i}, \sigma_i^{\otimes e_i}]_{\mathrm{GL}_{m_i e_i}(D)}$ and we say that $\mathfrak{s}$ has *exponents*

$e_1, \ldots, e_l$. In the setting of (11)

$$(12) \qquad W^{\mathfrak{s}} = N_{\mathcal{G}}(\mathcal{M}, \sigma)/\mathcal{M} = \prod_{i=1}^{l} W^{\mathfrak{s}_i} \cong \prod_{i=1}^{l} \mathfrak{S}_{e_i},$$

$$(13) \qquad \mathrm{Stab}(\sigma) = \{\chi \in X_{\mathrm{unr}}(\mathcal{M}) : \sigma \otimes \chi\} = \prod_{i=1}^{l} \mathrm{Stab}(\sigma_i)^{e_i}.$$

Recall that every unramified character of $\mathcal{G}_i = \mathrm{GL}_{m_i}(D)$ is of the form $g \mapsto |\mathrm{Nrd}(g)|_F^z$ for some $z \in \mathbb{C}$, where $\mathrm{Nrd} : \mathrm{M}_{m_i}(D) \to F$ is the reduced norm. This sets up natural isomorphisms

$$X_{\mathrm{unr}}(\mathcal{G}_i) \cong \mathbb{C} \Big/ \frac{2\pi\sqrt{-1}}{\log q_F} \mathbb{Z} \cong \mathbb{C}^{\times},$$

$$X_{\mathrm{unr}}(\mathcal{M}) \cong \prod_{i=1}^{l} (\mathbb{C}^{\times})^{e_i}.$$

Let $n(\sigma_i)$ be the torsion number of σ_i, that is, the order of $\mathrm{Stab}(\sigma_i)$. Let $T^{\mathfrak{s}}$ and T_i be the Bernstein tori associated to $\mathfrak{s}$ and $[\mathcal{G}_i, \sigma_i]_{\mathcal{G}_i}$, as in (4). There are isomorphisms

$$T_i \cong X_{\mathrm{unr}}(\mathcal{G}_i)/\mathrm{Stab}(\sigma_i) \cong \mathbb{C} \Big/ \frac{2\pi\sqrt{-1}}{n(\sigma_i)\log q_F} \mathbb{Z} \cong \mathbb{C}^{\times},$$

$$(14) \qquad T^{\mathfrak{s}} = \prod_{i=1}^{l} T^{\mathfrak{s}_i} = \prod_{i=1}^{l} (T_i)^{e_i} \cong X_{\mathrm{unr}}(\mathcal{M})/\mathrm{Stab}(\sigma).$$

As $W^{\mathfrak{s}}$ stabilizes $\sigma \in \mathbf{Irr}(\mathcal{M})$, the bijection (14) is $W^{\mathfrak{s}}$-equivariant. Although the above isomorphisms are not canonical, a consequence of the next theorem is that they can be made so by an appropriate choice of the σ_i.

Theorem 9.1. *The extended quotient of the second kind $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is canonically in bijection with the Bernstein component $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$.*

Proof. For every $i \in \{1, 2, \ldots, l\}$, as proved in [58], there exists a pair $(\mathcal{K}_i, \lambda_i)$, formed by an open compact subgroup $\mathcal{K}_i$ of $\mathcal{G}_i = \mathrm{GL}_{m_i}(D)$ and a smooth irreducible representation λ_i such that the representation $\widetilde{\lambda}_i$ extends λ_i to the normalizer $\widetilde{\mathcal{K}}_i$ of $\mathcal{K}_i$ in $\mathcal{G}_i$ and the supercuspidal representation σ_i is compactly induced from $\widetilde{\lambda}_i$:

$$(15) \qquad\qquad\qquad \sigma_i = \mathrm{c\text{–}Ind}_{\widetilde{\mathcal{K}}_i}^{\mathcal{G}_i} \widetilde{\lambda}_i.$$

The group $\mathcal{K}_i$ is a *maximal simple type* in the terminology of [57]. Its construction (given in [57]) generalizes the construction made by Bushnell and Kutzko in [21] in the case of the general linear group.

We shall denote by $\mathfrak{o}_D$ the ring of integers of D, and by q_D the order of the residue field of D.

The group $\mathcal{K}_i = \mathcal{K}(\mathfrak{A}_i, \beta_i)$ is built from a *simple stratum* $[\mathfrak{A}_i, a_i, 0, \beta_i]$ of the F-algebra $A_i := \mathrm{End}_D(V_i)$. The definition of a simple stratum is given in [57, Def 1.18]. Here we just recall that β_i is an element of A_i such that the F-algebra $E_i := F[\beta_i]$ is a field, $\mathfrak{A}_i$ is an $\mathfrak{o}_F$-hereditary order of A_i that is normalized by $E_i^{\times}$, and a_i is a positive integer. Then the centralizer of E_i in A_i, denoted B_i, is a simple central E_i-algebra. We fix a simple left

B_i-module V_{E_i} and write D_{E_i} for the algebra opposite to $\operatorname{End}_{B_i}(V_{E_i})$. We define

$$(16) \qquad \begin{aligned} m_{E_i} &:= \dim_{D_{E_i}} V_{E_i}, \qquad q_i := q_{D_{E_i}}^{m_{E_i}}, \\ \widetilde{W}_{e_i} &:= X^*(T_i)^{e_i} \rtimes \mathfrak{S}_{e_i} = X^*(T^{\mathfrak{s}_i}) \rtimes W^{\mathfrak{s}_i}. \end{aligned}$$

Notice that $\widetilde{W}_{e_i}$ is an extended affine Weyl group of type GL_{e_i}. Now we quote [58, Main Theorem and Prop. 5.7]: there is an isomorphism of unital $\mathbb{C}$-algebras

$$(17) \qquad \mathcal{H}(\mathcal{G}, \lambda^{\mathfrak{s}}) \cong \bigotimes_{i=1}^{l} \mathcal{H}_{q_i}(\widetilde{W}_{e_i}).$$

For later use we also mention that this isomorphism preserves supports [58, Proposition 7.1 and Theorem 7.2]. The factors $\mathcal{H}_{q_i}(\widetilde{W}_{e_i})$ are (extended) affine Hecke algebras whose structure is given explicitly for instance in [21, 5.4.6, 5.6.6].

By combining Corollary A.3 from Example 3 in the Appendix with the multiplicativity property of the extended quotient of the second kind, we obtain a canonical bijection

$$(18) \qquad \bigotimes_{i=1}^{l} \mathcal{H}_{q_i}(\widetilde{W}_{e_i}) \to (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2.$$

The composition of (8), (17) and (18) gives a canonical bijection

$$\mathbf{Irr}^{\mathfrak{s}}(\mathcal{G}) \to (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2. \qquad \square$$

Let $\mathcal{H}(\mathcal{G})$ be the Hecke algebra associated to $\mathcal{G}$, *i.e.*,

$$\mathcal{H}(\mathcal{G}) := \bigcup_{\mathcal{K}} \mathcal{H}(\mathcal{G}//\mathcal{K}),$$

where $\mathcal{K}$ are open compact subgroups of $\mathcal{G}$, and $\mathcal{H}(\mathcal{G}//\mathcal{K})$ is the convolution algebra of all complex-valued, compactly-supported functions on $\mathcal{G}$ which are $\mathcal{K}$-biinvariant. The Hecke algebra $\mathcal{H}(\mathcal{G})$ is a non-commutative, non-unital, non-finitely-generated $\mathbb{C}$-algebra. It admits a canonical decomposition into ideals, the Bernstein decomposition:

$$\mathcal{H}(\mathcal{G}) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G})} \mathcal{H}(\mathcal{G})^{\mathfrak{s}}.$$

The following result extends Theorem 3.1 in [19] from $\mathrm{GL}_n(F)$ to its inner forms. This can be viewed as a "topological shadow" of conjecture 1.1.

Theorem 9.2. *Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$. Then the periodic cyclic homology of $\mathcal{H}(\mathcal{G})^{\mathfrak{s}}$ is isomorphic to the periodised de Rham cohomology of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$:*

$$\mathrm{HP}_*(\mathcal{H}(\mathcal{G})^{\mathfrak{s}}) \simeq H^*(T^{\mathfrak{s}}//W^{\mathfrak{s}}; \mathbb{C}).$$

Proof. The proof follows those of [19, Theorem 3.1]. Let $e_{\lambda}^{\mathfrak{s}}$ be the idempotent attached to the semisimple $\mathfrak{s}$-type $(\mathcal{K}^{\mathfrak{s}}, \lambda^{\mathfrak{s}})$ as in [22, Definition 2.9]:

$$e_{\lambda}^{\mathfrak{s}}(g) = \begin{cases} (\operatorname{vol} \mathcal{K}^{\mathfrak{s}})^{-1} \dim \lambda \operatorname{tr}(\tau(g^{-1})) & \text{if } g \in \mathcal{K}^{\mathfrak{s}}, \\ 0 & \text{if } g \in \mathcal{G} \setminus \mathcal{K}^{\mathfrak{s}}. \end{cases}$$

The idempotent $e_{\lambda}^{\mathfrak{s}}$ is then a *special* idempotent in the Hecke algebra $\mathcal{H}(\mathcal{G})$ according to [22, Definition 3.11]. It follows from [22, §3] that

$$\mathcal{H}^{\mathfrak{s}}(\mathcal{G}) = \mathcal{H}(\mathcal{G}) * e_{\lambda}^{\mathfrak{s}} * \mathcal{H}(\mathcal{G}).$$

We then have a Morita equivalence

$$\mathcal{H}(\mathcal{G}) * e_\lambda^{\mathfrak{s}} * \mathcal{H}(\mathcal{G}) \sim_{\text{Morita}} e_\lambda^{\mathfrak{s}} * \mathcal{H}(\mathcal{G}) * e_\lambda^{\mathfrak{s}}.$$

By [22, 2.12] we have a canonical isomorphism of unital $\mathbb{C}$-algebras:

$$(19) \qquad \mathcal{H}(\mathcal{G}, \lambda^{\mathfrak{s}}) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}} \mathcal{V} \cong e_\lambda^{\mathfrak{s}} * \mathcal{H}(\mathcal{G}) * e_\lambda^{\mathfrak{s}},$$

so that the algebra $e_\lambda^{\mathfrak{s}} * \mathcal{H}(\mathcal{G}) * e_\lambda^{\mathfrak{s}}$ is Morita equivalent to the algebra $\mathcal{H}(\mathcal{G}, \lambda^{\mathfrak{s}})$. Then, thanks to the description of latter given in (17), we may finish the proof identically as in [19, Theorem 3.1]. $\qquad\square$

Example 9.3. For the group $\text{GL}_n(F)$ we can describe the flat family of sub-varieties of $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ from Section 4 explicitly. We assume for simplicity that $\mathfrak{s} = [\text{GL}_r(F)^e, \sigma^{\otimes e}]_{\mathcal{G}}$, where $er = n$. Then the sub-variety $R(\rho^{\mathfrak{s}}) = \rho^{\mathfrak{s}}(T^{\mathfrak{s}}//W^{\mathfrak{s}} - T^{\mathfrak{s}}/W^{\mathfrak{s}})$ is the hypersurface Y_1 given by the single equation $\prod_{i \neq j}(z_i - z_j) = 0$. The sub-variety of reducibility $R(\pi^{\mathfrak{s}})$ is the variety $Y_{\sqrt{q}}$ given by the single equation $\prod_{i \neq j}(z_i - qz_j) = 0$, according to a classical theorem [16, Theorem 4.2], [67].

The polynomial equation $\prod_{i \neq j}(z_i - v^2 z_j) = 0$ determines a flat family Y_v of hypersurfaces. The hypersurface Y_1 is the *flat limit* of the family Y_v as $v \to 1$, as in [30, p. 77].

9.2. The local Langlands correspondence for $\text{GL}_m(D)$.

We recall the construction of the local Langlands correspondence for $\mathcal{G} = \text{GL}_m(D)$ from [37]. It generalizes and relies on the local Langlands correspondence for $\mathcal{G}^* = \text{GL}_n(F)$,

$$\text{rec}_{F,n} \colon \mathbf{Irr}(\text{GL}_n(F)) \to \Phi(\text{GL}_n(F)).$$

The latter was proven for supercuspidal representations in [45, 33, 36], and extended from there to $\mathbf{Irr}(GL_n(F))$ in [67].

If π is an irreducible smooth representation of $\mathcal{G}$, we shall denote its character by Θ_π. Let $\mathbf{Irr}_{\text{essL}^2}(\mathcal{G})$ denote the set of equivalence classes of irreducible essentially square-integrable representations of $\mathcal{G}$. Recall the *Jacquet–Langlands correspondence* [29, 10]:
There exists a bijection

$$\text{JL} \colon \mathbf{Irr}_{\text{essL}^2}(\mathcal{G}) \to \mathbf{Irr}_{\text{essL}^2}(\mathcal{G}^*)$$

such that for each $\pi \in \mathbf{Irr}_{\text{essL}^2}(\mathcal{G})$:

$$\Theta_\pi(g) = (-1)^{n-m}\,\Theta_{\text{JL}(\pi)}(g^*),$$

for any pair $(g, g^*) \in \mathcal{G} \times \mathcal{G}^*$ of regular semisimple elements such that g and g^* have the same characteristic polynomial. (Recall that an element in $\mathcal{G}$ is called *regular semisimple* if its characteristic polynomial admits only simple roots in an algebraic closure of F.)

Let Φ be a Langlands parameter for $\mathcal{G}$. Replacing it by an equivalent one, we may assume that there exists a standard Levi subgroup $M \subset \text{GL}_n(\mathbb{C})$ such that the image of Φ is contained in M but not in any smaller Levi subgroup. Again replacing Φ by an equivalent parameter, we can achieve that

$$M = \prod_{i=1}^{l}(\text{GL}_{n_i}(\mathbb{C}))^{e_i} \quad \text{and} \quad \Phi = \prod_{i=1}^{l} \Phi_i^{\otimes e_i},$$

where $\Phi_i \in \Phi(\mathrm{GL}_{n_i}(F))$ is not equivalent to Φ_j for $i \neq j$. Since Φ is relevant for $\mathcal{G}$, $m_i := n_i/d$ is an integer and M corresponds to the standard Levi subgroup

$$\mathcal{M} = \prod_{i=1}^{l}(\mathrm{GL}_{m_i}(D))^{e_i} \subset \mathrm{GL}_m(D).$$

By construction the image of Φ_i is not contained in any proper Levi subgroup of $\mathrm{GL}_{n_i}(F)$, so $\mathrm{rec}_{F,n_i}^{-1}(\Phi_i) \in \mathbf{Irr}_{\mathrm{essL}^2}(\mathrm{GL}_{n_i}(F))$. The Jacquet–Langlands correspondence produces

$$\sigma_i := \mathrm{JL}^{-1}\big(\mathrm{rec}_{F,n_i}^{-1}(\Phi_i)\big) \in \mathbf{Irr}_{\mathrm{essL}^2}(\mathrm{GL}_{m_i}(D)),$$

$$\sigma := \prod_{i=1}^{l} \sigma_i^{\otimes e_i} \in \mathbf{Irr}_{\mathrm{essL}^2}(\mathcal{M}).$$

The assignment $\Phi \mapsto (\mathcal{M}, \sigma)$ sets up a bijection

(20) $\Phi(\mathcal{G}) \longleftrightarrow \{(\mathcal{M}, \sigma) : \mathcal{M} \text{ a Levi subgroup of } \mathcal{G}, \sigma \in \mathbf{Irr}_{\mathrm{essL}^2}(\mathcal{M})\}/\mathcal{G}.$

It is known from [29, Theorem B.2.d] and [11] that for inner forms of $\mathrm{GL}_n(F)$ normalized parabolic induction sends irreducible square-integrable (modulo centre) representations to irreducible tempered representations. Together with the Langlands classification [43] this implies that there exists a natural bijection between $\mathbf{Irr}(\mathcal{G})$ and the right hand side of (20). It sends $(\mathcal{M}, \sigma)$ to the Langlands quotient $L\big(I_{\mathcal{M}}^{\mathcal{G}}(\sigma)\big)$, where the parabolic induction goes via a parabolic subgroup with Levi factor $\mathcal{M}$, with respect to which the central character of σ is "positive". The combination of these results yields:

Theorem 9.4. *The natural bijection*

$$\Phi(\mathcal{G}) \to \mathbf{Irr}(\mathcal{G}) : \Phi \mapsto (\mathcal{M}, \sigma) \mapsto L\big(I_{\mathcal{M}}^{\mathcal{G}}(\sigma)\big)$$

is the local Langlands correspondence for $\mathcal{G} = \mathrm{GL}_m(D)$.

We denote the inverse map by

(21) $\mathrm{rec}_{D,m} \colon \mathbf{Irr}(\mathcal{G}) \to \Phi(\mathcal{G}).$

Let $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathcal{G}}$ be an inertial equivalence class for $\mathcal{G}$. We want to understand the space of Langlands parameters

$$\Phi(\mathcal{G})^{\mathfrak{s}} := \mathrm{rec}_{D,m}(\mathbf{Irr}^{\mathfrak{s}}(\mathcal{G}))$$

that corresponds to the Bernstein component $\mathbf{Irr}^{\mathfrak{s}}(\mathcal{G})$. We may assume that $\mathcal{M}$ and σ are as in (11). Via Theorem 9.4 σ_i corresponds to

$$\mathrm{rec}_{D,m_i}(\sigma_i) = \mathrm{rec}_{F,n_i}(\mathrm{JL}(\sigma_i)) \in \Phi(\mathrm{GL}_{m_i}(D)).$$

We choose a representative $\eta_i \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{n_i}(\mathbb{C})$ and we put

$$\eta = \prod_{i=1}^{l}(\eta_i)^{\otimes e_i},$$

a representative for $\mathrm{rec}_{D,m}(I_{\mathcal{M}}^{\mathcal{G}}(\sigma))$. Since $\mathrm{JL}(\sigma_i)$ is essentially square integrable, η_i is an irreducible representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$. It follows that the centralizer of η_i in $\mathrm{M}_{n_i}(\mathbb{C})$ is $\mathrm{Z}(\mathrm{M}_{n_i}(\mathbb{C})) = \mathbb{C}I$. From this we see that the centralizer of $(\eta_i)^{\otimes e_i}$ in $\mathrm{M}_{n_i e_i}(\mathbb{C})$ is

$$\mathrm{Z}_{\mathrm{M}_{n_i e_i}(\mathbb{C})}(\eta_i^{\otimes e_i}) = \mathrm{M}_{e_i}(\mathbb{C}) \otimes \mathrm{Z}(\mathrm{M}_{n_i}(\mathbb{C})) \subset \mathrm{M}_{e_i}(\mathbb{C}) \otimes \mathrm{M}_{n_i}(\mathbb{C}) \cong \mathrm{M}_{n_i e_i}(\mathbb{C}).$$

In this way the centralizer of $\eta_i^{\otimes e_i}$ in $\mathrm{GL}_{n_i e_i}(\mathbb{C})$ becomes isomorphic to $\mathrm{GL}_{e_i}(\mathbb{C})$ and

$$(22) \qquad \mathrm{Z}_{\mathrm{GL}_n(\mathbb{C})}(\eta) \cong \prod_{i=1}^{l} \mathrm{GL}_{e_i}(\mathbb{C}).$$

This means that, for any Langlands parameter for $\mathrm{GL}_{e_i}(D)$

$$\Phi_i \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{e_i}(\mathbb{C}) \subset \mathrm{GL}_{n_i e_i}(\mathbb{C}),$$

$\Phi_i \eta_i^{\otimes e_i}$ is a Langlands parameter for $\mathrm{GL}_{m_i e_i}(D)$. More generally, for any product $\Phi = \prod_{i=1}^{l} \Phi_i$ of such maps,

$$(23) \qquad \Phi\eta \text{ is Langlands parameter for } \mathcal{G}.$$

Let $\mathbf{I}_F \subset \mathbf{W}_F$ be the inertia group and let $\mathrm{Frob} \in \mathbf{W}_F$ be a Frobenius element. Since all Langlands parameters in $\Phi(\mathcal{G})^{\mathfrak{s}}$ have the same restriction to the inertia group $\mathbf{I}_F$ (up to equivalence), it suffices to consider Φ_i which are trivial on $\mathbf{I}_F$. We will show in the next subsection that $\Phi(\mathcal{G})^{\mathfrak{s}}$ indeed consists of such products $\Phi\eta$ (up to equivalence).

Recall from [18, §10.3] that the local Langlands correspondence is compatible with twisting by central characters. The group $X_{\mathrm{unr}}(\mathrm{GL}_{m_i}(D))$ is naturally in bijection with $\mathrm{Z}(\mathrm{GL}_{n_i}(\mathbb{C}))$. Hence for any $\chi \in X_{\mathrm{unr}}(\mathrm{GL}_{m_i}(D))$

$$\mathrm{rec}_{D,m_i}(\sigma_i \otimes \chi) = \Phi_\chi \mathrm{rec}_{D,m_i}(\sigma) = \Phi_\chi \eta_i,$$

where Φ_χ is trivial on $\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})$ and $\Phi_\chi(\mathrm{Frob}) \in \mathrm{Z}(\mathrm{GL}_{n_i}(\mathbb{C}))$ corresponds to χ. By Theorem 9.4 $\Phi_\chi \eta_i$ is $\mathrm{GL}_{n_i}(\mathbb{C})$-conjugate to Φ_i if and only if $\chi \in \mathrm{Stab}(\sigma_i)$. More generally, for any $\chi \in X_{\mathrm{unr}}(\mathcal{M})$,

$$(24) \qquad \Phi_\chi \eta \text{ is } M\text{-conjugate to } \eta \text{ if and only if } \chi \in \mathrm{Stab}(\sigma).$$

Thus the unramified twists of $\sigma \in \mathbf{Irr}(\mathcal{M})$ are naturally parametrized by the torus $T^{\mathfrak{s}} \cong X_{\mathrm{unr}}(\mathcal{M})/\mathrm{Stab}(\sigma)$.

9.3. The commutative triangle for $\mathrm{GL}_m(D)$.

Recall the short exact sequence

$$1 \to \mathbf{I}_F \to \mathbf{W}_F \xrightarrow{d} \mathbb{Z} \cong \langle \mathrm{Frob} \rangle \to 1.$$

A character $\nu \colon \mathbf{W}_F \to \mathbb{C}^\times$ is said to be *unramified* if ν is trivial on $\mathbf{I}_F$. Then $\nu(w) = z_\nu^{d(w)}$ for some $z_\nu \in \mathbb{C}^\times$ ($w \in \mathbf{W}_F$). Let $X_{\mathrm{unr}}(\mathbf{W}_F)$ denote the group of all unramified characters of $\mathbf{W}_F$. Then the map $\nu \mapsto z_\nu$ is an isomorphism from $X_{\mathrm{unr}}(\mathbf{W}_F)$ to $\mathbb{C}^\times$. Denote by $R(a)$ the a-dimensional irreducible complex representation of $\mathrm{SL}_2(\mathbb{C})$. Recall that each L-parameter Φ is of the form $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_h$ with each Φ_i irreducible. The next result is Theorem 1.2 in our Introduction.

Theorem 9.5. *Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$. There is a commutative diagram*

$$
\begin{array}{ccc}
& (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 & \\
& \swarrow \qquad \searrow{\scriptstyle \varphi^{\mathfrak{s}}} & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & \longrightarrow & \Phi(G)^{\mathfrak{s}}
\end{array}
$$

in which all the arrows are natural bijections.

Proof. The bottom and left slanted maps where already established in Theorems 9.1 and 9.4. Since these are canonical bijections, we could simply define the right slanted map as the composition of the other two. Yet we prefer to give an explicit construction of $\varphi^{\mathfrak{s}}$, which highlights its geometric origin. This is inspired by [19, § 1] and [6, Theorem 4.1], with the difference that we are using here the extended quotient of the second kind $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$ instead of $(T^{\mathfrak{s}}//W^{\mathfrak{s}})$.

Let $\mathcal{M}$ be a Levi subgroup of $\mathcal{G}$ and σ an irreducible unitary supercuspidal representation of $\mathcal{M}$ such that $\mathfrak{s} = [\mathcal{M}, \sigma]_{\mathcal{G}}$. As in (11), we may assume that

$$\mathcal{M} \simeq \prod_{i=1}^{l} \mathrm{GL}_{m_i}(D)^{e_i} = \prod_{i=1}^{l} \mathcal{G}_i^{e_i} \quad \text{and} \quad \sigma \simeq \sigma_1^{e_1} \otimes \cdots \otimes \sigma_l^{e_l},$$

where the $[\mathcal{G}_i, \sigma_i]_{\mathcal{G}}$ are pairwise distinct. Recall from (14) that

$$T^{\mathfrak{s}} = \prod_{i=1}^{l} (T_i)^{e_i},$$

where each T_i is isomorphic to $\mathbb{C}^{\times}$. To make the below constructions well-defined, we must make a canonical choice for σ_i in its inertial equivalence class in $\mathbf{Irr}(\mathcal{G}_i)$. The Hecke algebra $\mathcal{H}(\mathcal{G}_i, \lambda_i)$ associated to this inertial class, as in (8), is canonically isomorphic to the ring of regular functions on $X_{\mathrm{unr}}(\mathcal{G}_i)/\mathrm{Stab}(\sigma_i) \cong \mathbb{C}^{\times}$. We choose σ_i as (15), so that it corresponds to the unit element of this torus.

By (12) the group $W^{\mathfrak{s}}$ is isomorphic to a product of symmetric groups $\mathfrak{S}_{e_i}$. Since the extended quotient of the second kind is multiplicative, we obtain

$$(25) \qquad (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 \simeq (T_1^{e_1}//\mathfrak{S}_{e_1})_2 \times \cdots \times (T_l^{e_l}//\mathfrak{S}_{e_l})_2.$$

We set $\mathfrak{s}_i := [\mathcal{G}_i^{e_i}, \sigma_i^{\otimes e_i}]_{\mathcal{G}_i'}$, for each $i \in \{1, \ldots, l\}$, where $\mathcal{G}_i' = \mathrm{GL}_{e_i m_i}(D)$. We observe that

$$(26) \qquad \Phi(\mathcal{G})^{\mathfrak{s}} \simeq \Phi(\mathcal{G}_1')^{\mathfrak{s}_1} \times \cdots \times \Phi(\mathcal{G}_l')^{\mathfrak{s}_l},$$

Put $\mathcal{M}' := \mathcal{G}_1' \times \cdots \times \mathcal{G}_l'$. Then Theorem 9.4 and (26) imply that

$$(27) \qquad \begin{array}{ccc} \mathbf{Irr}(\mathcal{G}_1')^{\mathfrak{s}_1} \times \cdots \times \mathbf{Irr}(\mathcal{G}_l')^{\mathfrak{s}_l} & \to & \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}, \\ (\pi_1', \ldots, \pi_l') & \mapsto & L\big(I_{\mathcal{M}'}^{\mathcal{G}}(\pi_1' \otimes \cdots \otimes \pi_l')\big) \end{array}$$

is a bijection. Hence we are reduced to define canonical bijections

$$(T_i^{e_i}//\mathfrak{S}_{e_i})_2 \to \Phi(\mathcal{G}_i')^{\mathfrak{s}_i}.$$

Thus we may, and do, assume that $m = er$ (so that $n = erd$) and

$$(28) \qquad \mathfrak{s} = [\mathrm{GL}_r(D)^e, \sigma^{\otimes e}]_{\mathcal{G}}.$$

Then we have $T^{\mathfrak{s}} \simeq (\mathbb{C}^{\times})^e$ and $W^{\mathfrak{s}} \simeq \mathfrak{S}_e$.

Let $t \in (\mathbb{C}^{\times})^e$. We write t in the form

$$(29) \qquad t = (\underbrace{z_1, \ldots, z_1}_{b_1 \text{ terms}}, \underbrace{z_2, \ldots, z_2}_{b_2 \text{ terms}}, \ldots, \underbrace{z_h, \ldots, z_h}_{b_h \text{ terms}}),$$

where $z_1, \ldots, z_h \in \mathbb{C}^{\times}$ are such that $z_i \neq z_j$ for $i \neq j$, and where $b_1 + b_2 + \cdots + b_h = e$. Let $W_t^{\mathfrak{s}}$ denote the stabilizer of t under the action of $W^{\mathfrak{s}}$. We have

$$(30) \qquad W_t^{\mathfrak{s}} \simeq \mathfrak{S}_{b_1} \times \mathfrak{S}_{b_2} \times \cdots \times \mathfrak{S}_{b_h}.$$

In particular every $\tau \in W^{\mathfrak{s}}$ can be written as

$$(31) \qquad \tau = \tau_1 \otimes \cdots \otimes \tau_h \quad \text{with } \tau_j \in \mathbf{Irr}(\mathfrak{S}_{b_j}).$$

Let $p_j = p(\tau_j)$ denote the partition of b_j which corresponds to τ_j. We write:

$$(32) \qquad p_j = (p_{j,1}, \ldots, p_{j,l_j}), \quad \text{where } p_{j,1} + \cdots + p_{j,l_j} = b_j.$$

Recall that we have chosen $\sigma \in \mathbf{Irr}(\mathrm{GL}_r(D))$ such that, via the Hecke algebra of a supercuspidal type, it corresponds to the unit element of the torus $X_{\mathrm{unr}}(\mathrm{GL}_r(D))/\mathrm{Stab}(\sigma)$. We fix a representative

$$(33) \qquad \eta \quad \text{for} \quad \mathrm{rec}_{D,r}(\sigma) \in \mathbf{Irr}(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})).$$

Corresponding to each pair (η, p_j) as above, where $j \in \{1, \ldots, h\}$, we have a Langlands parameter:

$$(34) \qquad \begin{aligned} &\Phi^F_{\eta,p_j} \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{b_j rd}(\mathbb{C}) \subset \mathrm{M}_{b_j}(\mathbb{C}) \otimes \mathrm{M}_{rd}(\mathbb{C}), \\ &\Phi^F_{\eta,p_j} = \big(R(p_{j,1}) \oplus \cdots \oplus R(p_{j,l_j}) \big) \otimes \eta. \end{aligned}$$

Recall that

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \simeq \big\{ (t,\tau) \,:\, t \in (\mathbb{C}^\times)^e, \tau \in \mathbf{Irr}(W^{\mathfrak{s}}_t) \big\} /W^{\mathfrak{s}}.$$

Define $\nu_j \in X_{\mathrm{unr}}(\mathbf{W}_F)$ by $\nu_j(\mathrm{Frob}) := z_j$ for each $j \in \{1, \ldots, h\}$. We fix a setwise section

$$\psi_\sigma \colon \mathbb{C}^\times \cong X_{\mathrm{unr}}(\mathrm{GL}_r(D))/\mathrm{Stab}(\sigma) \to \mathbb{C}^\times \cong X_{\mathrm{unr}}(\mathrm{GL}_r(D)).$$

We define a map

$$(35) \qquad \begin{aligned} &\varphi^{\mathfrak{s}} \colon (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 \to \Phi(\mathcal{G})^{\mathfrak{s}}, \\ &\varphi^{\mathfrak{s}}(t,\tau) = (\psi_\sigma \circ \nu_1) \otimes \Phi^F_{\eta,p_1} \oplus \cdots \oplus (\psi_\sigma \circ \nu_h) \otimes \Phi^F_{\eta,p_h}, \end{aligned}$$

where $W^{\mathfrak{s}}_t \simeq \mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_h}$ and $\tau = \tau_1 \otimes \cdots \otimes \tau_h$, with $\tau_j \in \mathbf{Irr}(\mathfrak{S}_{b_j})$, and $p_j = p(\tau_j)$ denotes the partition of b_j which corresponds to τ_j. By (24) $\varphi^{\mathfrak{s}}(t,\tau)$ does not depend on the choice of ψ_σ, up to equivalence of Langlands parameters.

The above process can be reversed. Let $\Phi \in \Phi(\mathcal{G})^{\mathfrak{s}}$. By (24) we may assume that its restriction to $\mathbf{W}_F$ is of the form

$$(\psi_\sigma)^e \circ (\nu_1 \oplus \cdots \oplus \nu_1 \oplus \nu_2 \oplus \cdots \oplus \nu_2 \oplus \cdots \oplus \nu_h \oplus \cdots \oplus \nu_h) \otimes \eta,$$

where $\nu_1, \ldots, \nu_h \in X_{\mathrm{unr}}(\mathbf{W}_F)$ are such that $\nu_i \neq \nu_j$ if $i \neq j$ (which is equivalent, since ν_i and ν_j are unramified, to $\nu_i(\mathrm{Frob}) \neq \nu_j(\mathrm{Frob})$ if $i \neq j$). For each $j \in \{1, \ldots, h\}$, let b_j denote the number of occurrences of ν_j. We set

$$t := (\nu_1(\mathrm{Frob}), \nu_1(\mathrm{Frob}), \ldots, \nu_h(\mathrm{Frob})) \in (\mathbb{C}^\times)^e.$$

For each $j \in \{1, \ldots, h\}$, let p_j be a partition of b_j, such that

$$(36) \qquad \Phi = \psi_\sigma \circ \nu_1 \otimes \Phi^F_{\eta,p_1} \oplus \cdots \oplus \psi_\sigma \circ \nu_h \otimes \Phi^F_{\eta,p_h}.$$

For $j = 1, \ldots, h$, let $\tau_j \in \mathbf{Irr}(\mathfrak{S}_{b_j})$ be the irreducible representation of $\mathfrak{S}_{b_j}$ which is parametrized by the partition p_j. We set $\tau := \tau_1 \otimes \cdots \otimes \tau_h$. The map $\Phi \mapsto (t,\tau)$ is the inverse of the map $\varphi^{\mathfrak{s}}$. Thus $\varphi^{\mathfrak{s}}$ is a bijection.

Now we have all the arrows of the diagram in Theorem 9.5, it remains to show that it commutes. Although all the arrows are canonical, this is not at all obvious, and our proof will use some results that will be established only in part 3.

In view of (25), (26) and (27), it suffices to consider $\mathfrak{s}$ as in (28). Let $\widetilde{W}_e := \mathbb{Z}^e \rtimes \mathfrak{S}_e$ and $\tilde{q} := q_{D_E}^{m_E}$ be as in (16). The left slanted bijection in Theorem 9.5 runs via the affine Hecke algebra $\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)$ associated to $\mathfrak{s}$ in (17), so we must actually look at the extended diagram

$$(37) \qquad \begin{array}{ccc} \mathbf{Irr}(\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)) & \longrightarrow & ((\mathbb{C}^\times)^e /\!/ \mathfrak{S}_e)_2 \\ {\scriptstyle \kappa^{\mathfrak{s}}} \uparrow & & \downarrow {\scriptstyle \varphi^{\mathfrak{s}}} \\ \mathbf{Irr}(\mathrm{GL}_{er}(D))^{\mathfrak{s}} & \xrightarrow{\mathrm{rec}_{D,er}} & \Phi(\mathrm{GL}_{er}(D))^{\mathfrak{s}} \end{array}$$

First we consider the essentially square-integrable representations. For those the Langlands parameter is an irreducible representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$, of the form

$$\psi_\sigma \circ \nu_z \otimes \Phi_{e,\eta}^F = \psi_\sigma \circ \nu_z \otimes R(e) \otimes \eta,$$

where $\nu_z \in X_{\mathrm{unr}}(\mathbf{W}_F)$ with $\nu_z(\mathrm{Frob}) = z$. The partition (e) of e corresponds to the sign representation of $\mathfrak{S}_e$, so

$$(38) \qquad (\varphi^{\mathfrak{s}})^{-1}(\psi_\sigma \circ \nu_z \otimes R(e) \otimes \eta) = (z, \ldots, z, \mathrm{sgn}_{\mathfrak{S}_e}).$$

Let St_e be the Steinberg representation of $\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)$. It is the unique irreducible, essentially square-integrable representation which is tempered and has real infinitesimal central character. The top map in (37), as constructed by Lusztig [46] and described in Example 3 of the Appendix, sends $(z, \ldots, z) \otimes \mathrm{St}_e$ to (38). This can be deduced with [42, Theorem 8.3], but we will also prove it later as a special case of Theorem 23.1.

The map $\kappa^{\mathfrak{s}}$ comes from the support-preserving algebra isomorphism (17). By [60, (3.19)] this map respects temperedness of representations. The argument in [60] relies on Casselman's criteria, see [26, §4.4] and [51, §2.7]. A small variation on it shows that $\kappa^{\mathfrak{s}}$ also preserves essential square-integrability. Both statements apply to $\mathrm{rec}_{D,er}^{-1}(R(e) \otimes \eta)$ because it is essentially square-integrable and unitary, hence tempered.

By the Zelevinsky classification for $\mathrm{GL}_n(F)$ [67], $\mathrm{rec}_{F,n}^{-1}(R(e) \otimes \eta)$ is a consituent of

$$I_{\mathrm{GL}_{rd}(F)^e}^{\mathrm{GL}_n(F)}\left(\nu_F^{(1-e)/2}\mathrm{JL}(\sigma) \otimes \nu_F^{(3-e)/2}\mathrm{JL}(\sigma) \otimes \cdots \otimes \nu_F^{(e-1)/2}\mathrm{JL}(\sigma)\right),$$

where $\nu_F(g^*) = |\det(g^*)|_F$. It follows that $\mathrm{rec}_{D,er}^{-1}(R(e) \otimes \eta)$ is a constituent of

$$I_{\mathrm{GL}_r(D)^e}^{\mathrm{GL}_{er}(D)}\left(\nu_D^{(1-e)/2}\sigma \otimes \nu_D^{(3-e)/2}\sigma \otimes \cdots \otimes \nu_D^{(e-1)/2}\sigma\right),$$

where $\nu_D(g) = |\mathrm{Nrd}(g)|_F$. By our choice of σ the infinitesimal central character of $\kappa^{\mathfrak{s}}\left(I_{\mathrm{GL}_r(D)^e}^{\mathrm{GL}_{er}(D)}(\sigma^{\otimes e})\right)$ is $1 \in (\mathbb{C}^\times)^e/\mathfrak{S}_e$, so $\kappa^{\mathfrak{s}}\left(\mathrm{rec}_{D,er}^{-1}(R(e) \otimes \eta)\right)$ has real infinitesimal central character. Thus $\kappa^{\mathfrak{s}}\left(\mathrm{rec}_{D,er}^{-1}(R(e) \otimes \eta)\right)$ possesses all the properties that characterize St_e in $\mathbf{Irr}(\mathcal{H}_{\tilde{q}}(\widetilde{W}_e))$, and these two representations are isomorphic.

By definition $\kappa^{\mathfrak{s}} \circ \mathrm{rec}_{D,er}^{-1}$ transforms a twist by $\psi_\sigma \circ \nu_z$ into a twist by $(z, \ldots, z)$, so

$$\kappa^{\mathfrak{s}}\left(\mathrm{rec}_{D,er}^{-1}(\psi_\sigma \circ \nu_z \otimes R(e) \otimes \eta)\right) = (z, \ldots, z) \otimes \mathrm{St}_e.$$

We conclude that the diagram (37) commutes for all essentially square-integrable representations.

Now we take t and τ as in (29) and (31). The construction of the Langlands correspondence for $\mathrm{GL}_{er}(D)$ in (20) implies

$$(39) \quad \mathrm{rec}_{D,er}^{-1}(\varphi^{\mathfrak{s}}(t,\tau)) =$$

$$L\Big(I_{\mathrm{GL}_{b_1 r}(D)\times\cdots\times\mathrm{GL}_{b_h r}(D)}^{\mathrm{GL}_{er}(D)}\big(\bigotimes_{j=1}^{h}\mathrm{rec}_{D,b_j r}^{-1}(\psi_\sigma\circ\nu_j\otimes\Phi_{\eta,p_j}^{F})\big)\Big) =$$

$$L\Big(I_{\prod_{j=1}^{h}\prod_{i=1}^{l_j}\mathrm{GL}_{p_{j,i}r}(D)}^{\mathrm{GL}_{er}(D)}\big(\bigotimes_{j=1}^{h}\bigotimes_{i=1}^{l_j}\mathrm{rec}_{D,p_{j,i}r}^{-1}(\psi_\sigma\circ\nu_j\otimes R(p_{j,i})\otimes\eta)\big)\Big).$$

The $\mathrm{GL}_{p_{j,i}r}(D)$-representation $\mathrm{rec}_{D,p_{j,i}r}^{-1}(\psi_\sigma\circ\nu_j\otimes R(p_{j,i})\otimes\eta)$ is essentially square-integrable, so by the above we know where it goes in the diagram (37). Since (17) preserves supports, it respects parabolic induction and Langlands quotients (see [60] for these notions in the context of affine Hecke algebras). In particular every subgroup $\mathrm{GL}_{p_{j,i}r}(D)\subset\mathrm{GL}_{er}(D)$ corresponds to a subalgebra $\mathcal{H}_{\tilde{q}}(\widetilde{W}_{p_{j,i}})\subset\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)$. Thus the representation of $\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)$ associated to (39) via (17) is

$$(40) \qquad L\Big(\mathrm{Ind}_{\otimes_{j=1}^{h}\otimes_{i=1}^{l_j}\mathcal{H}_{\tilde{q}}(\widetilde{W}_{p_{j,i}})}^{\mathcal{H}_{\tilde{q}}(\widetilde{W}_e)}\big(\bigotimes_{j=1}^{h}\bigotimes_{i=1}^{l_j}(z_j,\ldots,z_j)\otimes\mathrm{St}_{p_{j,i}}\big)\Big).$$

In view of the Zelevinsky classification for affine Hecke algebras of type GL_n [42, 67], the Langlands parameter of (40) is

$$\bigoplus_{j=1}^{h}z_j\otimes\bigoplus_{i=1}^{l_j}R(p_{j,i}).$$

Now Theorem 22.3, in combination with the notations (29)–(32), shows that the top map in (37) sends (40) to (t,τ). Hence the diagrams in (37) and in the statement of the theorem commute. $\qquad\qquad\square$

10. Symplectic groups

For symplectic and orthogonal groups, Heiermann [35] proved that every Bernstein component of the category of smooth modules is equivalent to the module category of an affine Hecke algebra. Together with [60, Theorem 5.4.2] this proves a large part of the ABPS conjecture for such groups: properties 1–5 from Section 7.

Both statement 2 and property 7 involve the local Langlands correspondence for symplectic and orthogonal groups, which is now proved by Arthur in [4], up-to the stabilization of the twisted trace formula for the group $\mathrm{GL}_n(F)\rtimes\langle\epsilon_n\rangle$, where u_n is the matrix defined in (41), see [4, § 3.2]. Property 6 should in addition make use of the description of the L-packets from [50].

In this section, in order to illustrate the fact that extended quotients can be easily calculated, we shall compute $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ in the case when $\mathcal{G}$ is a symplectic group and $\mathfrak{s}=[\mathcal{M},\sigma]_{\mathcal{G}}$ with $\mathcal{M}$ the Levi subgroup of a maximal parabolic of $\mathcal{G}$.

Let $u_n \in \mathrm{M}_n(F)$ be the $n \times n$ matrix defined by

$$u_n := \begin{pmatrix} & & & & \iddots \\ & & & \iddots & \\ & & \iddots & & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}.$$

For $g \in \mathrm{M}_n(F)$, we denote by ${}^t g$ the transpose matrix of g, and by ϵ_n the automorphism of $\mathrm{GL}_n(F)$ defined by

$$(41) \qquad\qquad \epsilon_n(g) := u_n \cdot {}^t g^{-1} \cdot u_n^{-1}.$$

We denote by $\mathcal{G} = \mathrm{Sp}_{2n}(F)$ the symplectic group defined with respect to the symplectic form u_{2n}:

$$\mathcal{G} := \left\{ g \in \mathrm{GL}_{2n}(F) \ : \ {}^t g u_{2n} g = u_{2n} \right\}.$$

Let k and m two integers such that $k \geq 0$, $m \geq 0$ and $k + m = n$. In this section we consider a Levi subgroup $\mathcal{M}$ of the following form:

$$\mathcal{M} := \left\{ \begin{pmatrix} g & & \\ & g' & \\ & & \epsilon_k(g) \end{pmatrix} \ : \ g \in \mathrm{GL}_k(F), \ \ g' \in \mathrm{Sp}_{2m}(F) \right\} \simeq \mathrm{GL}_k(F) \times \mathcal{G}_m.$$

Let $\mathrm{I}_n \in \mathrm{M}_n(F)$ denote the identity matrix. We set

$$(42) \qquad\qquad w_{\mathcal{M}} := \begin{pmatrix} & & \mathrm{I}_k \\ & \mathrm{I}_{2m} & \\ \mathrm{I}_k & & \end{pmatrix}.$$

The element $w_{\mathcal{M}}$ has order 2 and we have

$$(43) \qquad w_{\mathcal{M}} \cdot \begin{pmatrix} g & & \\ & g' & \\ & & \epsilon_k(g) \end{pmatrix} \cdot w_{\mathcal{M}} = \begin{pmatrix} \epsilon_k(g) & & \\ & g' & \\ & & g \end{pmatrix}.$$

We have

$$(44) \qquad\qquad \mathrm{N}_{\mathcal{G}}(\mathcal{M})/\mathcal{M} = \{1, w_{\mathcal{M}}\}.$$

Let $\pi_{\mathcal{M}}$ be an irreducible supercuspidal representation of $\mathcal{M}$. We have $\pi_{\mathcal{M}} = \rho \otimes \sigma$, where ρ is an irreducible supercuspidal representation of $\mathrm{GL}_k(F)$ and σ is an irreducible supercuspidal representation of $\mathrm{Sp}_{2m}(F)$. We set $\mathfrak{s} := [\mathcal{M}, \pi_{\mathcal{M}}]_{\mathcal{G}}$.

If $W^{\mathfrak{s}} = \{1\}$, then the parabolically induced representation $\rho \rtimes \sigma$ of $\mathcal{G}$ is irreducible.

We will assume from now on that $W^{\mathfrak{s}} \neq \{1\}$. Then it follows from (44) that

$$W^{\mathfrak{s}} = \mathrm{N}_{\mathcal{G}}(\mathcal{M})/\mathcal{M} = \{1, w_{\mathcal{M}}\}.$$

Let ρ^{ϵ_k} denote the representation of $\mathrm{GL}_k(F)$ defined by

$$\rho^{\epsilon_k}(g) := \rho(\epsilon_k(g)), \quad g \in \mathrm{GL}_k(F).$$

Then equation (43) gives:

$$(45) \qquad\qquad {}^{w_{\mathcal{M}}}(\rho \otimes \sigma) = \rho^{\epsilon_k} \otimes \sigma.$$

The theorem of Gel'fand and Kazhdan [32, Theorem 2] says that the representation ρ^{ϵ_k} is equivalent to $\rho^\vee$, the contragredient representation of ρ. Hence we get

$$(46) \qquad \qquad {}^{w_{\mathcal{M}}}(\rho \otimes \sigma) \cong \rho^\vee \otimes \sigma.$$

Lemma 10.1. *We have*

$$T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} = T^{\mathfrak{s}}/W^{\mathfrak{s}} \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2.$$

Proof. Since the group $\mathrm{Sp}_{2m}(F)$ does not have non-trivial unramified characters, we get

$$X_{\mathrm{unr}}(\mathcal{M}) \cong X_{\mathrm{unr}}(\mathrm{GL}_k(F)) \cong \mathbb{C} \Big/ \frac{2\pi\sqrt{-1}}{\log q} \mathbb{Z} \cong \mathbb{C}^\times.$$

We have

$$\mathrm{Stab}(\pi_{\mathcal{M}}) = \mathrm{Stab}(\rho).$$

It follows that

$$T^{\mathfrak{s}} \cong X_{\mathrm{unr}}(\mathcal{M})/\mathrm{Stab}(\pi_{\mathcal{M}}) = X_{\mathrm{unr}}(\mathrm{GL}_k(F))/\mathrm{Stab}(\rho) \cong \mathbb{C} \Big/ \frac{2\pi\sqrt{-1}}{n(\rho)\log q} \mathbb{Z} \cong \mathbb{C}^\times,$$

where $n(\rho)$ is the order of $\mathrm{Stab}(\rho)$.

On the other hand, since $w_{\mathcal{M}}$ belongs to $W^{\mathfrak{s}}$, we have

$${}^{w_{\mathcal{M}}}\pi_{\mathcal{M}} \cong \nu_{\mathcal{M}}\, \pi_{\mathcal{M}},$$

for some $\nu_{\mathcal{M}} \in X_{\mathrm{unr}}(\mathcal{M})$. Hence

$$(47) \qquad \qquad {}^{w_{\mathcal{M}}}\pi_{\mathcal{M}} \cong \nu\rho \otimes \sigma,$$

for some $\nu \in X_{\mathrm{unr}}(F^\times)$, where we have put $\nu\rho := (\nu \circ \det) \otimes \rho$. Then it follows from (46) that

$$\rho^\vee \otimes \sigma \cong \nu\rho \otimes \sigma,$$

that is,

$$\nu^{1/2}\rho \otimes \sigma \cong (\nu^{1/2}\rho)^\vee \otimes \sigma.$$

It implies that

$$\rho\nu^{1/2} \cong (\rho\nu^{1/2})^\vee.$$

Hence, by replacing ρ by $\rho\nu^{1/2}$ if necessary, we can always assume that the representation ρ is *self-contragredient*, that is, $\rho \cong \rho^\vee$.

We shall assume from now on, that ρ is self-contragredient. Let $\nu_k \in X_{\mathrm{unr}}(\mathrm{GL}_k(F))$. We shall denote by $[\nu_k]$ the image of ν_k in the quotient $X_{\mathrm{unr}}(\mathrm{GL}_k(F))/\mathrm{Stab}(\rho)$.

We have $\nu_k = \nu \circ \det$ for some $\nu \in X_{\mathrm{unr}}(F^\times)$. Let $g \in \mathrm{GL}_k(F)$. We obtain

$${}^{w_{\mathcal{M}}}\nu_k(g) = \nu(\det(\epsilon_k(g))) = \nu(\det({}^t g^{-1})) = \nu(\det(g^{-1})) = \nu_k(g)^{-1}.$$

Hence $\nu_k \mathrm{Stab}(\rho)$ is fixed by $w_{\mathcal{M}}$ if and only if $\nu_k^2 \in \mathrm{Stab}(\rho)$. On the other hand, we have $\mathrm{Z}_{W^{\mathfrak{s}}}(w_{\mathcal{M}}) = W^{\mathfrak{s}}$. It follows that

$$(T^{\mathfrak{s}})^{w_{\mathcal{M}}}/\mathrm{Z}_{W^{\mathfrak{s}}}(w_{\mathcal{M}}) = \big\{ [1], [\zeta_{n(\rho)}] \big\},$$

where we have put

$$\zeta_{n(\rho)} := |\det(\cdot)|_F^{\frac{\pi\sqrt{-1}}{n(\rho)\log q}},$$

that is,

$$(T^{\mathfrak{s}})^{w_{\mathcal{M}}}/\mathrm{Z}_{W^{\mathfrak{s}}}(w_{\mathcal{M}}) \simeq \{-1, 1\} \subset \mathbb{C}^\times. \qquad \qquad \square$$

Then by using [60, Theorem 5.4.2] (which is closely related to Property 4 of the bijection $\mu^{\mathfrak{s}}$) we recover from Lemma 10.1 the well-known fact that $\nu_k \rho \rtimes \sigma$ reduces for exactly two unramified characters ν_k.

11. THE IWAHORI SPHERICAL REPRESENTATIONS OF G_2

Let $\mathcal{G}$ be the exceptional group G_2. Let $\mathfrak{s}_0 = [\mathcal{T}, 1]_{\mathcal{G}}$ where $\mathcal{T} \simeq F^\times \times F^\times$ is a maximal F-split torus of $\mathcal{G}$. The following result is a special case of Theorem 22.3 in Part 3.

Theorem 11.1. *The conjecture (as stated in Section 7) is true for the point* $\mathfrak{s}_0 = [\mathcal{T}, 1]_{G_2}$.

This is such an illustrative example that we include some of the calculations in [7].

We note that $X_{\mathrm{unr}}(\mathcal{T}) \cong T$ with T a maximal torus in the Langlands dual group $G = G_2(\mathbb{C})$. The Weyl group W of G_2 is the dihedral group of order 12. The extended quotient is

$$T//W = T/W \sqcup \mathfrak{C}_1 \sqcup \mathfrak{C}_2 \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \sqcup \mathrm{pt}_5.$$

The flat family is $Y_\alpha := (1 - \alpha^2 y)(x - \alpha^2 y) = 0$. Note that $Y_{\sqrt{q}} = \mathfrak{R}$ the curve of reducibility points in the quotient variety T/W. The restriction of π_α to $T//W - T/W$ determines a finite morphism

$$\mathfrak{C}_1 \sqcup \mathfrak{C}_2 \sqcup \mathrm{pt}_1 \sqcup \mathrm{pt}_2 \sqcup \mathrm{pt}_3 \sqcup \mathrm{pt}_4 \sqcup \mathrm{pt}_5 \longrightarrow Y_\alpha.$$

EXAMPLE. The fibre of the point $(q^{-1}, 1) \in \mathfrak{R}$ via the map $\pi_{\sqrt{q}}$ is a set with 5 points, corresponding to the fact that there are 5 smooth irreducible representations of G_2 with infinitesimal character $(q^{-1}, 1)$.

The map π_α restricted to the one affine line $\mathfrak{C}_1$ is induced by the map $(z, 1) \mapsto (\alpha z, \alpha^{-2})$, and restricted to the other affine line $\mathfrak{C}_2$ is induced by the map $(z, z) \mapsto (\alpha z, \alpha^{-1} z)$. With regard to the second map: the two points $(\omega/\sqrt{q}, \omega/\sqrt{q})$, $(\omega^2/\sqrt{q}, \omega^2/\sqrt{q})$ are distinct points in $\mathfrak{C}_2$ but become identified via $\pi_{\sqrt{q}}$ in the quotient variety T/W. This implies that the image $\pi_{\sqrt{q}}(\mathfrak{C}_2)$ of one affine line has a *self-intersection point* in the quotient variety T/W. Also, the curves $\pi_{\sqrt{q}}(\mathfrak{C}_1), \pi_{\sqrt{q}}(\mathfrak{C}_2)$ intersect in 3 points. These intersection points account for the number of distinct constituents in the corresponding induced representations.

Part 3. The principal series of split reductive p-adic groups

12. INTRODUCTION TO PART 3

Let $\mathcal{G}$ be a connected reductive p-adic group, split over F, and let $\mathcal{T}$ be a split maximal torus in $\mathcal{G}$. The principal series consists of all $\mathcal{G}$-representations that are obtained with parabolic induction from characters of $\mathcal{T}$.

We denote the collection of all Bernstein components of $\mathcal{G}$ of the form $[\mathcal{T}, \chi]_{\mathcal{G}}$ by $\mathfrak{B}(\mathcal{G}, \mathcal{T})$ and call these the Bernstein components in the principal series. The union

$$\mathbf{Irr}(\mathcal{G}, \mathcal{T}) := \bigcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G}, \mathcal{T})} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

is by definition the set of all irreducible subquotients of principal series representations of $\mathcal{G}$.

Let T be the Langlands dual group of $\mathcal{T}$ and choose a uniformizer $\varpi_F \in F$. There is a bijection $t \mapsto \nu$ between points in T and unramified quasicharacters of $\mathcal{T}$, determined by the relation

$$\nu(\lambda(\varpi_F)) = \lambda(t)$$

where $\lambda \in X_*(\mathcal{T}) = X^*(T)$. The space $\mathbf{Irr}(\mathcal{T})^{[\mathcal{T},\chi]\tau}$ is in bijection with T via $t \mapsto \nu \mapsto \sigma \otimes \nu$. Hence Bernstein's torus $T^{\mathfrak{s}}$ is isomorphic to T. However, because the isomorphism is not canonical and the action of the group $W^{\mathfrak{s}}$ depends on it, we prefer to denote it $T^{\mathfrak{s}}$.

For each $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ we will construct a commutative triangle of bijections

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$$

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow \{\text{KLR parameters}\}^{\mathfrak{s}}/H$$

Here $\{\text{KLR parameters}\}^{\mathfrak{s}}$ is the set of Kazhdan–Lusztig–Reeder parameters associated to $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ and H is the stabilizer of this set of parameters in the Langlands dual group ${}^L\mathcal{G}$.

In examples, $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ is much simpler to directly calculate than either $\mathbf{Irr}(G)^{\mathfrak{s}}$ or $\{\text{KLR parameters }\}^{\mathfrak{s}}$.

Let us discuss the triangle in the case that H is connected. The bijectivity of the right slanted arrow (see Section 19) is essentially a reformulation of results of Kato [41]. It involves Weyl groups of possibly disconnected reductive groups. We will extend the Springer correspondence to such groups in Section 15.

The left slanted arrow is defined (and by construction bijective) in [46]. The horizontal map is defined and proved to be a bijection in [42, 54], see Section 20. The results in [54] are based on and extend those of [42]. Thus there are three logically independent definitions and bijectivity proofs.

The bijectivity of the horizontal arrow shows that the local Langlands correspondence is valid for each such Bernstein component $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ and describes the intersections of L-packets with $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$. Once a c-$\mathbf{Irr}$ system has been chosen for the action of $W^{\mathfrak{s}}$ on $T^{\mathfrak{s}}$, there is the bijection

$$T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \longrightarrow (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2,$$

so the L-packets can be described in terms of the extended quotient of the first kind. In Section 25 we check that the labelling of the irreducible components of $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ predicted by the conjecture is provided by the unipotent classes of H.

13. Twisted extended quotient of the second kind

Let J be a finite group and let $\alpha \in \mathrm{H}^2(J; \mathbb{C}^\times)$. Let V be a finite dimensional vector space over $\mathbb{C}$. Consider all maps $\tau \colon J \to \mathrm{GL}(V)$ such that there exists a $\mathbb{C}^\times$-valued 2-cocycle c on J with

$$\tau(j_1) \circ \tau(j_2) = c(j_1, j_2)\, \tau(j_1 j_2), \qquad [c] = \alpha$$

and τ is irreducible, where $[c]$ is the class of c in $\mathrm{H}^2(J; \mathbb{C}^\times)$.

For such a map τ let $f \colon J \to \mathbb{C}^\times$ be any map and consider the map $(f\tau)(j) := f(j)\tau(j)$. This is again such a map τ with cocycle $c \cdot f$ defined by

$$(c \cdot f)(j_1, j_2) := c(j_1, j_2) \frac{f(j_1)f(j_2)}{f(j_1 j_2)}.$$

Given two such maps τ_1, τ_2, an *isomorphism* is an intertwining operator $V_1 \to V_2$. Note that $\tau_1 \simeq \tau_2 \Rightarrow c_1 = c_2$. Given τ_1 and τ_2, we define τ_1 to be *equivalent* to τ_2 if and only if there exists $f \colon J \to \mathbb{C}^\times$ with τ_1 isomorphic to $f\tau_2$. The set of equivalence classes of maps τ is denoted $\mathbf{Irr}^\alpha(J)$.

Now let Γ be a finite group with a given action on a set X. Let $\natural$ be a given function which assigns to each $x \in X$ an element $\natural(x) \in \mathrm{H}^2(\Gamma_x; \mathbb{C}^\times)$ where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$. The function $\natural$ is required to satisfy the condition

$$\natural(\gamma x) = \gamma_* \natural(x), \qquad \forall(\gamma, x) \in \Gamma \times X$$

where $\gamma_* \colon \Gamma_x \to \Gamma_{\gamma x}$, $\alpha \mapsto \gamma\alpha\gamma^{-1}$. Now define

$$\widetilde{X}_2^\natural = \{(x, \tau) : \tau \in \mathbf{Irr}^{\natural(x)}(\Gamma_x)\}.$$

We have a map $\Gamma \times \widetilde{X}_2^\natural \to \widetilde{X}_2^\natural$ and we form the *twisted extended quotient of the second kind*

$$(X/\!/\Gamma)_2^\natural := \widetilde{X}_2^\natural / \Gamma.$$

We will apply this construction in the following two special cases.

1. Given two finite groups Γ_1, Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(\Gamma_1)$, we can form the semidirect product $\Gamma_1 \rtimes \Gamma$. Let $X = \mathbf{Irr}\,\Gamma_1$. Now Γ acts on $\mathbf{Irr}\,\Gamma_1$ and we get $\natural$ as follows. Given $x \in \mathbf{Irr}\,\Gamma_1$ choose an irreducible representation $\phi : \Gamma_1 \to \mathrm{GL}(V)$ whose isomorphism class is x. For each $\gamma \in \Gamma_x$ consider ϕ twisted by γ *i.e.,* consider $\phi^\gamma : \gamma_1 \mapsto \phi(\gamma\gamma_1\gamma^{-1})$. Since $\gamma \in \Gamma_x$, ϕ^γ is equivalent to ϕ *i.e.,* there exists an intertwining operator $T_\gamma : \phi \simeq \phi^\gamma$. For this operator we have

$$T_\gamma \circ T_{\gamma'} = c(\gamma, \gamma')T_{\gamma\gamma'}, \qquad \gamma, \gamma' \in \Gamma_x$$

and $\natural(x)$ is then the class in $\mathrm{H}^2(\Gamma_x; \mathbb{C}^\times)$ of c.

This leads to a new formulation of a classical theorem of Clifford.

Lemma 13.1. *We have a canonical bijection*

$$\mathbf{Irr}(\Gamma_1 \rtimes \Gamma) \simeq (\mathbf{Irr}\,\Gamma_1 /\!/ \Gamma)_2^\natural.$$

Proof. The proof proceeds by comparing our construction with the classical theory of Clifford; for an exposition of Clifford theory, see [52]. $\qquad\square$

Lemma 13.2. *If Γ_1 is abelian, then we have*

$$(\mathbf{Irr}\,\Gamma_1 /\!/ \Gamma)_2^\natural = (\mathbf{Irr}\,\Gamma_1 /\!/ \Gamma)_2.$$

Proof. The irreducible representations ϕ of Γ_1 are 1-dimensional, and we have $\phi^\gamma = \phi$. In that case each cohomology class $\natural(x)$ is trivial, and the projective representations of Γ_x which occur in the construction are all true representations. $\qquad\square$

2. Given a $\mathbb{C}$-algebra R, a finite group Γ and a group homomorphism $\Gamma \to \mathrm{Aut}(R)$, we can form the crossed product algebra

$$R \rtimes \Gamma := \{\sum_{\gamma \in \Gamma} r_\gamma \gamma \, : \, r_\gamma \in R\},$$

with multiplication given by the distributive law and the relation

$$\gamma r = \gamma(r)\gamma, \quad \text{for } \gamma \in \Gamma \text{ and } r \in R.$$

Let $X = \mathbf{Irr}\, R$. Now Γ acts on $\mathbf{Irr}\, R$ and as above we get $\natural$ for free. Here we have

$$\widetilde{X}_2^\natural = \{(V, \tau) \, : \, V \in \mathbf{Irr}\, R, \; \tau \in \mathbf{Irr}^{\natural(V)}(\Gamma_V)\}.$$

Lemma 13.3. *We have a canonical bijection*

$$\mathbf{Irr}(R \rtimes \Gamma) \simeq (\mathbf{Irr}\, R // \Gamma)_2^\natural.$$

Proof. The proof proceeds by comparing our construction with the theory of Clifford as stated in [52, Theorem A.6]. $\qquad\square$

Notation 13.4. We shall denote by $\tau_1 \rtimes \tau$ (resp. $V \rtimes \tau$) the element of $\mathbf{Irr}(X \rtimes \Gamma)$ which corresponds to (τ_1, τ) (resp. (V, τ)) by the bijection of Lemma 13.1 (resp. 13.3).

14. Weyl groups of disconnected groups

Let M be a reductive complex algebraic group. Then M may have a finite number of connected components, M^0 is the identity component of M, and $\mathcal{W}^{M^0}$ is the Weyl group of M^0:

$$\mathcal{W}^{M^0} := \mathrm{N}_{M^0}(T)/T$$

where T is a maximal torus of M^0. We will need the analogue of the Weyl group for the possibly disconnected group M.

Lemma 14.1. *Let M, M^0, T be as defined above. Then we have*

$$\mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

Proof. The group $\mathcal{W}^{M^0}$ is a normal subgroup of $\mathrm{N}_M(T)/T$. Indeed, let $n \in \mathrm{N}_{M^0}(T)$ and let $n' \in \mathrm{N}_M(T)$, then $n'nn'^{-1}$ belongs to M^0 (since the latter is normal in M) and normalizes T, that is, $n'nn'^{-1} \in \mathrm{N}_{M^0}(T)$. On the other hand, $n'(nT)n'^{-1} = n'nn'^{-1}(n'Tn'^{-1}) = n'nn'^{-1}T$.

Let B be a Borel subgroup of M^0 containing T. Let $w \in \mathrm{N}_M(T)/T$. Then wBw^{-1} is a Borel subgroup of M^0 (since, by definition, the Borel subgroups of an algebraic group are the maximal closed connected solvable subgroups). Moreover, wBw^{-1} contains T. In a connected reductive algebraic group, the intersection of two Borel subgroups always contains a maximal torus and the two Borel subgroups are conjugate by a element of the normalizer of that torus. Hence B and wBw^{-1} are conjugate by an element w_1 of $\mathcal{W}^{M^0}$. It follows that $w_1^{-1}w$ normalises B. Hence

$$w_1^{-1}w \in \mathrm{N}_M(T)/T \cap \mathrm{N}_M(B) = \mathrm{N}_M(T, B)/T,$$

that is,

$$\mathrm{N}_M(T)/T = \mathcal{W}^{M^0} \cdot (\mathrm{N}_M(T, B)/T).$$

Finally, we have

$$\mathcal{W}^{M^0} \cap (\mathrm{N}_M(T,B)/T) = \mathrm{N}_{M^0}(T,B)/T = \{1\},$$

since $\mathrm{N}_{M^0}(B) = B$ and $B \cap \mathrm{N}_{M^0}(T) = T$. This proves (1).

Now consider the following map:

$$(48) \qquad \mathrm{N}_M(T,B)/T \to M/M^0 \qquad mT \mapsto mM^0.$$

It is injective. Indeed, let $m, m' \in \mathrm{N}_M(T,B)$ such that $mM^0 = m'M^0$. Then $m^{-1}m' \in M^0 \cap \mathrm{N}_M(T,B) = \mathrm{N}_{M^0}(T,B) = T$ (as we have seen above). Hence $mT = m'T$.

On the other hand, let m be an element in M. Then $m^{-1}Bm$ is a Borel subgroup of M^0, hence there exists $m_1 \in M^0$ such that $m^{-1}Bm = m_1^{-1}Bm_1$. It follows that $m_1 m^{-1} \in \mathrm{N}_M(B)$. Also $m_1 m^{-1} T m m_1^{-1}$ is a torus of M^0 which is contained in $m_1 m^{-1} B m m_1^{-1} = B$. Hence T and $m_1 m^{-1} T m m_1^{-1}$ are conjugate in B: there is $b \in B$ such that $m_1 m^{-1} T m m_1^{-1} = b^{-1}Tb$. Then $n := b m_1 m^{-1} \in \mathrm{N}_M(T,B)$. It gives $m = n^{-1} b m_1$. Since $b m_1 \in M^0$, we obtain $mM^0 = n^{-1}M^0$. Hence the map (48) is surjective. $\qquad \square$

Let G be a connected complex reductive group and let T be a maximal torus in G. The Weyl group of G is denoted $\mathcal{W}^G$.

Lemma 14.2. *Let A be a subgroup of T and write $M = \mathrm{Z}_G(A)$. Then the isotropy subgroup of A in $\mathcal{W}^G$ is*

$$\mathcal{W}_A^G = \mathrm{N}_M(T)/T \cong \mathcal{W}^{M^0} \rtimes \pi_0(M).$$

In case that the group M is connected, $\mathcal{W}_A^G$ is the Weyl group of M.

Proof. Let $R(G,T)$ denote the root system of G. According to [61, § 4.1], the group $M = \mathrm{Z}_G(A)$ is the reductive subgroup of G generated by T and those root groups U_α for which $\alpha \in R(G,T)$ has trivial restriction to A together with those Weyl group representatives $n_w \in \mathrm{N}_G(T)$ ($w \in \mathcal{W}^G$) for which $w(t) = t$ for all $t \in A$. This shows that $\mathcal{W}_A^G = \mathrm{N}_M(T)/T$, which by Lemma 14.1 is isomorphic to $\mathcal{W}^{M^0} \rtimes \pi_0(M)$.

Also by [61, § 4.1], the identity component of M is generated by T and those root groups U_α for which α has trivial restriction to A. Hence the Weyl group $\mathcal{W}^{M^\circ}$ is the normal subgroup of $\mathcal{W}_A^G$ generated by those reflections s_α and

$$\mathcal{W}_A^G / \mathcal{W}^{M^\circ} \cong M/M^\circ.$$

In particular, if M is connected then $\mathcal{W}_A^G$ is the Weyl group of M. $\qquad \square$

In summary, for $t \in T$ such that $M = \mathrm{Z}_G(t)$ we have

$$(49) \qquad (T /\!/ \mathcal{W}^G)_2 = \{(t,\sigma) : t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}_t^G)\}/\mathcal{W}^G$$

$$(50) \qquad \mathbf{Irr}\,\mathcal{W}_t^G = (\mathbf{Irr}\,\mathcal{W}^{M^0} /\!/ \pi_0(M))_2^\natural$$

15. An extended Springer correspondence

Let M° be a connected reductive complex group. We take $x \in M^\circ$ unipotent and we abbreviate

$$(51) \qquad A_x := \pi_0(\mathrm{Z}_{M^0}(x)).$$

Let $x \in M^\circ$ be unipotent, $\mathcal{B}^x = \mathcal{B}^x_{M^\circ}$ the variety of Borel subgroups of M° containing x. All the irreducible components of $\mathcal{B}^x$ have the same dimension $d(x)$ over $\mathbb{R}$, see [27, Corollary 3.3.24]. Let $H_{d(x)}(\mathcal{B}^x, \mathbb{C})$ be its top homology, let ρ be an irreducible representation of A_x and write

(52) $$\tau(x, \rho) = \operatorname{Hom}_{A_x}\big(\rho, H_{d(x)}(\mathcal{B}^x, \mathbb{C})\big).$$

We call $\rho \in \mathbf{Irr}(A_x)$ geometric if $\tau(x, \rho) \neq 0$. The Springer correspondence yields a one-to-one correspondence

(53) $$(x, \rho) \mapsto \tau(x, \rho)$$

between the set of M^0-conjugacy classes of pairs (x, ρ) formed by a unipotent element $x \in M^0$ and an irreducible geometric representation ρ of A_x, and the equivalence classes of irreducible representations of the Weyl group $\mathcal{W}^{M^0}$.

Remark 15.1. *The Springer correspondence which employ here sends the trivial unipotent class to the trivial $\mathcal{W}^{M^\circ}$-representation and the regular unipotent class to the sign representation. It coincides with the correspondence constructed by Lusztig by means of intersection cohomology. The difference with Springer's construction via a reductive group over a field of positive characteristic consists of tensoring with the sign representation of $\mathcal{W}^{M^0}$, see [38].*

Choose a set of simple reflections for $\mathcal{W}^{M^\circ}$ and let Γ be a group of automorphisms of the Coxeter diagram of W. Then Γ acts on $\mathcal{W}^{M^\circ}$ by group automorphisms, so we can form the semidirect product $\mathcal{W}^{M^\circ} \rtimes \Gamma$. Furthermore Γ acts on $\mathbf{Irr}(\mathcal{W}^{M^\circ})$, by $\gamma \cdot \tau = \tau \circ \gamma^{-1}$. The stabilizer of $\tau \in \mathbf{Irr}(\mathcal{W}^{M^\circ})$ is denoted Γ_τ. As described in Section 13, Clifford theory for $\mathcal{W}^{M^\circ} \rtimes \Gamma$ produces a 2-cocycle $\natural(\tau) : \Gamma_\tau \times \Gamma_\tau \to \mathbb{C}^\times$.

We fix a Borel subgroup B_0 of M° containing T and let $\Delta(B_0, T)$ be the set of roots of (M°, T) that are simple with respect to B_0. We may and will assume that this agrees with the previously chosen simple reflections in $\mathcal{W}^{M^\circ}$. In every root subgroup U_α with $\alpha \in \Delta(B_0, T)$ we pick a nontrivial element u_α. The data $(M^\circ, T, (u_\alpha)_{\alpha \in \Delta(B_0, T)})$ are called a pinning of M°. The action of $\gamma \in \Gamma$ on the Coxeter diagram of $\mathcal{W}^{M^\circ}$ lifts uniquely to an action of γ on M° which preserves the pinning. In this way we construct the semidirect product $M := M^\circ \rtimes \Gamma$. By Lemma 14.2 we may identify $\mathcal{W}^M$ with $\mathcal{W}^{M^\circ} \rtimes \Gamma$.

For any unipotent element $x \in M^\circ$ the centralizer $\mathrm{Z}_M(x)$ acts on the variety $\mathcal{B}^x_{M^\circ}$. We say that an irreducible representation ρ_1 of $\mathrm{Z}_M(x)$ is geometric if it appears in $H_{d(x)}(\mathcal{B}^x_{M^\circ}, \mathbb{C})$. Notice that this condition forces ρ_1 to factor through the component group $\pi_0(\mathrm{Z}_M(x))$.

Proposition 15.2. *The class of $\natural(\tau)$ in $H^2(\Gamma_\tau, \mathbb{C}^\times)$ is trivial for all $\tau \in \mathbf{Irr}(\mathcal{W}^{M^\circ})$. Hence there are natural bijections between the following sets:*

$\quad \mathbf{Irr}(\mathcal{W}^{M^\circ} \rtimes \Gamma) = \mathbf{Irr}(\mathcal{W}^M)$,

$\quad \big(\mathbf{Irr}(\mathcal{W}^{M^\circ}) / / \Gamma\big)_2$,

$\quad \big\{(x, \rho_1) \mid x \in M^\circ \text{ unipotent}, \rho_1 \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_M(x))\big) \text{ geometric}\big\} / M$.

Proof. There are various ways to construct the Springer correspondence for $\mathcal{W}^{M^\circ}$, for the current proof we use the method with Borel–Moore homology.

Let Z_{M° be the Steinberg variety of M° and $H_{\mathrm{top}}(Z_{M^\circ})$ its homology in the top degree

$$2\dim_{\mathbb{C}} Z_{M^\circ} = 4\dim_{\mathbb{C}} \mathcal{B}_{M^\circ} = 4(\dim_{\mathbb{C}} M^\circ - \dim_{\mathbb{C}} B_0),$$

with rational coefficients. We define a natural algebra isomorphism

$$(54) \qquad\qquad \mathbb{Q}[\mathcal{W}^{M^\circ}] \to H_{\mathrm{top}}(Z_{M^\circ})$$

as the composition of [27, Theorem 3.4.1] and a twist by the sign representation of $\mathbb{Q}[\mathcal{W}^{M^\circ}]$. By [27, Section 3.5] the action of $\mathcal{W}^{M^\circ}$ on $H_*(\mathcal{B}^x, \mathbb{C})$ (as defined by Lusztig) corresponds to the convolution product in Borel–Moore homology.

Since M° is normal in M, the groups Γ, M and $M/Z(M)$ act on the Steinberg variety Z_{M° via conjugation. The induced action of the connected group M° on $H_{\mathrm{top}}(Z_{M^\circ})$ is trivial, and it easily seen from [27, Section 3.4] that the action of Γ on $H(Z_{M^\circ})$ makes (54) Γ-equivariant.

The groups Γ, M and $M/Z(M)$ also act on the pairs (x, ρ) and on the varieties of Borel subgroups, by

$$\mathrm{Ad}_m(x, \rho) = (mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1}),$$
$$\mathrm{Ad}_m : \mathcal{B}^x \to \mathcal{B}^{mxm^{-1}}, \ B \mapsto mBm^{-1}.$$

Given $m \in M$, this provides a linear bijection $H_*(\mathrm{Ad}_m)$:

$$\mathrm{Hom}_{A_x}(\rho, H_*(\mathcal{B}^x, \mathbb{C})) \to \mathrm{Hom}_{A_{mxm^{-1}}}(\rho \circ \mathrm{Ad}_m^{-1}, H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C})).$$

The convolution product in Borel–Moore homology is compatible with these M-actions so, as in [27, Lemma 3.5.2], the following diagram commutes for all $h \in H_{\mathrm{top}}(Z_{M^\circ})$:

$$(55) \qquad \begin{array}{ccc} H_*(\mathcal{B}^x, \mathbb{C}) & \xrightarrow{\ h\ } & H_*(\mathcal{B}^x, \mathbb{C}) \\ \downarrow{\scriptstyle H_*(\mathrm{Ad}_m)} & & \downarrow{\scriptstyle H_*(\mathrm{Ad}_m)} \\ H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C}) & \xrightarrow{\ m\cdot h\ } & H_*(\mathcal{B}^{mxm^{-1}}, \mathbb{C}). \end{array}$$

In case $m \in M^\circ\gamma$ and $m \cdot h$ corresponds to $w \in \mathcal{W}^{M^\circ}$, the element $h \in H(Z_{M^\circ})$ corresponds to $\gamma^{-1}(w)$, so (55) becomes

$$(56) \quad H_*(\mathrm{Ad}_m) \circ \tau(x, \rho)(\gamma^{-1}(w)) = \tau(mxm^{-1}, \rho \circ \mathrm{Ad}_m^{-1})(w) \circ H_*(\mathrm{Ad}_m).$$

Denoting the M°-conjugacy class of (x, ρ) by $[x, \rho]_{M^\circ}$, we can write

$$(57) \qquad \Gamma_{\tau(x,\rho)} = \{\gamma \in \Gamma \mid \tau(x, \rho) \circ \gamma^{-1} \cong \tau(x, \rho)\}$$
$$= \{\gamma \in \Gamma \mid [\mathrm{Ad}_\gamma(x, \rho)]_{M^\circ} = [x, \rho]_{M^\circ}\} =: \Gamma_{[x,\rho]_{M^\circ}}.$$

This group fits in an exact sequence

$$(58) \quad 1 \to \pi_0\big(Z_{M^\circ}(x, \rho)/Z(M^\circ)\big) \to \pi_0\big(Z_M(x, \rho)/Z(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1.$$

Assume for the moment that (58) has a splitting

$$s : \Gamma_{[x,\rho]_{M^\circ}} \to \pi_0\big(Z_M(x, \rho)/Z(M^\circ)\big).$$

By homotopy invariance in Borel–Moore homology $H_*(\mathrm{Ad}_z) = \mathrm{id}_{H_*(\mathcal{B}^x, \mathbb{C})}$ for any $z \in Z_{M^\circ}(x, \rho)^\circ Z(M^\circ)$, so $H_*(\mathrm{Ad}_m)$ is well-defined for $m \in \pi_0\big(Z_M(x, \rho)/Z(M^\circ)\big)$. In particular we obtain for every $\gamma \in \Gamma_{\tau(x,\rho)} = \Gamma_{[x,\rho]_{M^\circ}}$ a linear bijection

$$H_*(\mathrm{Ad}_{s(\gamma)}) : \mathrm{Hom}_{A_x}(\rho, H_{d(x)}(\mathcal{B}_x, \mathbb{C})) \to \mathrm{Hom}_{A_x}(\rho, H(\mathcal{B}_x, \mathbb{C})),$$

which by (56) intertwines the $\mathcal{W}^{M^\circ}$-representations $\tau(x,\rho)$ and $\tau(x,\rho)\circ\gamma^{-1}$. By construction

$$H_*(\mathrm{Ad}_{s(\gamma)}) \circ H_*(\mathrm{Ad}_{s(\gamma')}) = H_*(\mathrm{Ad}_{s(\gamma\gamma')}).$$

This proves the triviality of the 2-cocycle $\natural(\tau) = \natural(\tau(x,\rho))$, up to the existence of a splitting of (58). Thus it remains to show the following.

Lemma 15.3. *Let $\rho \in \mathbf{Irr}(\pi_0(\mathrm{Z}_{M^\circ}(x)))$ and write $\mathrm{Z}_M(x,\rho) = \{m \in \mathrm{Z}_M(x)|$ $\rho \circ \mathrm{Ad}_m^{-1} \cong \rho\}$. The following short exact sequence splits:*

$$1 \to \pi_0\big(\mathrm{Z}_{M^\circ}(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \pi_0\big(\mathrm{Z}_M(x,\rho)/\mathrm{Z}(M^\circ)\big) \to \Gamma_{[x,\rho]_{M^\circ}} \to 1.$$

Proof. First we ignore ρ. According to the classification of unipotent orbits in complex reductive groups [24, Theorem 5.9.6] we may assume that x is distinguished unipotent in a Levi subgroup $L \subset M^\circ$ that contains T. Let $\mathcal{D}(L)$ be the derived subgroup of L and define

$$L' := \mathrm{Z}_{M^\circ}(\mathcal{D}(L))(T \cap \mathcal{D}(L)) = \mathrm{Z}_{M^\circ}(\mathcal{D}(L))T.$$

Choose Borel subgroups $B_L \subset L$ and $B'_L \subset L'$ such that $x \in B_L$ and $T \subset B_L \cap B'_L$. Let $[x]_{M^\circ}$ be the M°-conjugacy class of x and $\Gamma_{[x]_{M^\circ}}$ its stabilizer in Γ. Any $\gamma \in \Gamma_{[x]_{M^\circ}}$ must also stabilize the M°-conjugacy class of L, and $T = \gamma(T) \subset \gamma(L)$, so there exists a $w_1 \in \mathcal{W}^{M^\circ}$ with $w_1\gamma(L) = L$. Adjusting w_1 by an element of $W(L,T) \subset \mathcal{W}^{M^\circ}$, we can achieve that moreover $w_1\gamma(B_L) = B_L$. Then $w_1\gamma(L') = L'$, so we can find a unique $w_2 \in W(L',T) \subset \mathcal{W}^{M^\circ}$ with $w_2w_1\gamma(B'_L) = B'_L$. Notice that the centralizer of $\Phi(B_L,T) \cup \Phi(B'_L,T)$ in $\mathcal{W}^{M^\circ}$ is trivial, because it is generated by reflections and no root in $\Phi(M^\circ,T)$ is orthogonal to this set of roots. Therefore the above conditions completely determine $w_2w_1 \in \mathcal{W}^{M^\circ}$.

The element $w_1\gamma \in \mathcal{W}^{M^\circ} \rtimes \Gamma$ acts on $\Delta(B_L,T)$ by a diagram automorphism, so upon choosing $u_\alpha \in U_\alpha \setminus \{1\}$ for $\alpha \in \Delta(B_L,T)$, it can be represented by a unique element

$$\overline{w_1\gamma} \in \mathrm{Aut}\big(\mathcal{D}(L),T,(u_\alpha)_{\alpha\in\Delta(B_L,T)}\big).$$

The distinguished unipotent class of $x \in L$ is determined by its Bala–Carter diagram. The classification of such diagrams [24, §5.9] shows that there exists an element $\bar{x}$ in the same class as x, such that $\mathrm{Ad}_{\overline{w_1\gamma}}(\bar{x}) = \bar{x}$. We may just as well assume that we had $\bar{x}$ instead of x from the start, and that $\overline{w_1\gamma} \in \mathrm{Z}_M(x)$. Clearly we can find a representative $\overline{w_2}$ for w_2 in $\mathrm{Z}_M(x)$, so we obtain

$$\overline{w_2}\,\overline{w_1\gamma} \in \mathrm{Z}_M(x) \cap \mathrm{N}_M(T) \quad \text{and} \quad w_2w_1\gamma \in \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\,T}.$$

Since $w_2w_1 \in \mathcal{W}^{M^\circ}$ is unique,

$$s : \Gamma_{[x]_{M^\circ}} \to \frac{\mathrm{Z}_M(x) \cap \mathrm{N}_M(T)}{\mathrm{Z}(M^\circ)\,T}, \ \gamma \mapsto w_2w_1\gamma$$

is a group homomorphism.

We still have to analyse the effect of $\Gamma_{[x]_{M^\circ}}$ on $\rho \in \mathbf{Irr}(A_x)$. Obviously composing with Ad_m for $m \in \mathrm{Z}_{M^\circ}(x)$ does not change the equivalence class of any representation of $A_x = \pi_0(\mathrm{Z}_{M^\circ}(x))$. Hence $\gamma \in \Gamma_{[x]_{M^\circ}}$ stabilizes ρ if

and only if any lift of γ in $Z_M(x)$ does. This applies in particular to $\overline{w_2}\,\overline{w_1\gamma}$, and therefore

$$s(\Gamma_{[x,\rho]_{M^\circ}}) \subset \big(Z_M(x,\rho) \cap N_M(T)\big)\big/\big(Z(M^\circ)\,T\big).$$

Since the torus T is connected, s determines a group homomorphism from $\Gamma_{[x,\rho]_{M^\circ}}$ to $\pi_0\big(Z_M(x,\rho)/Z(M^\circ)\big)$, which is the required splitting. $\square$

This also finishes the proof of the first part of Proposition 15.2. Now the bijection between of $\mathbf{Irr}(\mathcal{W}^M)$ and $\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})//\Gamma\big)_2$ follows from Clifford theory, see Section 13. By (57) and Lemma 15.3 the geometric representations of $\pi_0(Z_M(x))$ are precisely the representations $\rho \rtimes \sigma$ with $\rho \in \mathbf{Irr}(A_x)$ geometric and $\sigma \in \mathbf{Irr}(\Gamma_{[x,\rho]_{M^\circ}})$. This provides the bijection between $\big(\mathbf{Irr}(\mathcal{W}^{M^\circ})//\Gamma\big)_2$ and the M-conjugacy classes of pairs (x,ρ_1). $\square$

There is natural partial order on the unipotent classes in M:

$$\mathcal{O} < \mathcal{O}' \quad \text{when} \quad \overline{\mathcal{O}} \subsetneq \overline{\mathcal{O}'}.$$

Let $\mathcal{O}_x \subset M$ be the class containing x. We transfer this to partial order on our extended Springer data by defining

$$(59) \qquad\qquad (x,\rho_1) < (x',\rho_1') \quad \text{when} \quad \overline{\mathcal{O}_x} \subsetneq \overline{\mathcal{O}_{x'}}.$$

We will use it to formulate a property of the composition series of some $\mathcal{W}^M$-representations that will appear later on.

Lemma 15.4. *Let $x \in M$ be unipotent and let $\rho \rtimes \sigma$ be a geometric irreducible representation of $\pi_0(Z_M(x))$. There exist multiplicities $m_{x,\rho\rtimes\sigma,x',\rho'\rtimes\sigma'} \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathrm{Ind}_{W\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{W\rtimes\Gamma}\big(\mathrm{Hom}_{A_x}\big((\rho, H_*(\mathcal{B}^x,\mathbb{C})\big) \otimes \sigma\big) \cong$$

$$\tau(x,\rho) \rtimes \sigma \oplus \bigoplus_{(x',\rho'\rtimes\sigma')>(x,\rho\rtimes\sigma)} m_{x,\rho\rtimes\sigma,x',\rho'\rtimes\sigma'}\, \tau(x',\rho') \rtimes \sigma'.$$

Proof. Consider the vector space $\mathrm{Hom}_{A_x}\big((\rho, H_*(\mathcal{B}^x,\mathbb{C})\big)$ with the $\mathcal{W}^{M^\circ}$-action coming from (54). The proof of Proposition 15.2 remains valid for these representations. By [12, Theorem 4.4] (attributed to Borho and MacPherson) there exist multiplicities $m_{x,\rho,x',\rho'} \in \mathbb{Z}_{\geq 0}$ such that

$$(60) \qquad \mathrm{Hom}_{A_x}\big((\rho, H_*(\mathcal{B}^x,\mathbb{C})\big) \cong \tau(x,\rho) \oplus \bigoplus_{(x',\rho')>(x,\rho)} m_{x,\rho,x',\rho'}\, \tau(x',\rho').$$

By (57) and (56) $\Gamma_{[x,\rho]_{M^\circ}}$ also stabilizes the $\tau(x',\rho')$ with $m_{x,\rho,x',\rho'} > 0$, and by Lemma 15.3 the associated 2-cocycles are trivial. It follows that

$$(61) \quad \mathrm{Ind}_{W\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{W\rtimes\Gamma}\big(\mathrm{Hom}_{A_x}\big((\rho, H_*(\mathcal{B}^x,\mathbb{C})\big) \otimes \sigma\big) \cong$$

$$\tau(x,\rho) \rtimes \sigma \oplus \bigoplus_{(x',\rho')>(x,\rho)} m_{x,\rho,x',\rho'}\mathrm{Ind}_{W\rtimes\Gamma_{[x,\rho]_{M^\circ}}}^{W\rtimes\Gamma}\big(\tau(x',\rho') \otimes \sigma\big).$$

Decomposing the right hand side into irreducible representations then gives the statement of the lemma. $\square$

16. The Langlands parameter Φ

Let $\mathbf{W}_F$ denote the Weil group of F, let $\mathbf{I}_F$ be the inertia subgroup of $\mathbf{W}_F$. Let $\mathbf{W}_F^{\mathrm{der}}$ denote the closure of the commutator subgroup of $\mathbf{W}_F$, and write $\mathbf{W}_F^{\mathrm{ab}} = \mathbf{W}_F/\mathbf{W}_F^{\mathrm{der}}$. The group of units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^\times$.

We recall the Artin reciprocity map $\mathbf{a}_F : \mathbf{W}_F \to F^\times$ which has the following properties (local class field theory):

(1) The map $\mathbf{a}_F$ induces a topological isomorphism $\mathbf{W}_F^{\mathrm{ab}} \simeq F^\times$.
(2) An element $x \in \mathbf{W}_F$ is a geometric Frobenius if and only if $\mathbf{a}_F(x)$ is a prime element ϖ_F of F.
(3) We have $\mathbf{a}_F(\mathbf{I}_F) = \mathfrak{o}_F^\times$.

We now consider the principal series of $\mathcal{G}$. We recall that $\mathcal{G}$ denotes a connected reductive split p-adic group with maximal split torus $\mathcal{T}$, and that G, T denote the Langlands dual groups of $\mathcal{G}$, $\mathcal{T}$. Next, we consider conjugacy classes in G of continuous morphisms

$$\Phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$

which are rational on $\mathrm{SL}_2(\mathbb{C})$ and such that $\Phi(\mathbf{W}_F)$ consists of semisimple elements in G.

Let B_2 be the upper triangular Borel subgroup in $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ denote the variety of Borel subgroups of G containing $\Phi(\mathbf{W}_F \times B_2)$. The variety $\mathcal{B}^{\Phi(\mathbf{W}_F \times B_2)}$ is non-empty if and only if Φ factors through $\mathbf{W}_F^{\mathrm{ab}}$, see [54, §4.2]. In that case, we view the domain of Φ to be $F^\times \times \mathrm{SL}_2(\mathbb{C})$:

$$\Phi \colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G.$$

In this section we will build such a continuous morphism Φ from $\mathfrak{s}$ and data coming from the extended quotient of second kind. In Section 17 we show how such a Langlands parameter Φ can be enhanced with a parameter ρ.

Throughout this article, a Frobenius element Frob_F has been chosen and fixed. This determines a uniformizer ϖ_F via the equation $\mathbf{a}_F(\mathrm{Frob}_F) = \varpi_F$. That in turn gives rise to a group isomorphism $\mathfrak{o}_F^\times \times \mathbb{Z} \to F^\times$, which sends $1 \in \mathbb{Z}$ to ϖ_F. Let $\mathcal{T}_0$ denote the maximal compact subgroup of $\mathcal{T}$. As the latter is F-split,

$$(62) \qquad \mathcal{T} \cong F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}) \cong (\mathfrak{o}_F^\times \times \mathbb{Z}) \otimes_{\mathbb{Z}} X_*(\mathcal{T}) = \mathcal{T}_0 \times X_*(\mathcal{T}).$$

Because $\mathcal{W}$ does not act on $F^\times$, these isomorphisms are $\mathcal{W}$-equivariant if we endow the right hand side with the diagonal $\mathcal{W}$-action. Thus (62) determines a $\mathcal{W}$-equivariant isomorphism of character groups

$$(63) \qquad \mathbf{Irr}(\mathcal{T}) \cong \mathbf{Irr}(\mathcal{T}_0) \times \mathbf{Irr}(X_*(\mathcal{T})) = \mathbf{Irr}(\mathcal{T}_0) \times X_{\mathrm{unr}}(\mathcal{T}).$$

Lemma 16.1. *Let χ be a character of $\mathcal{T}$, and let $[\mathcal{T}, \chi]_{\mathcal{G}}$ be the inertial class of the pair $(\mathcal{T}, \chi)$ as in §3. Let*

$$(64) \qquad\qquad\qquad \mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}.$$

Then $\mathfrak{s}$ determines, and is determined by, the $\mathcal{W}$-orbit of a smooth morphism

$$c^{\mathfrak{s}} \colon \mathfrak{o}_F^\times \to T.$$

Proof. There is a natural isomorphism

$$\mathbf{Irr}(\mathcal{T}) = \mathrm{Hom}(F^\times \otimes_{\mathbb{Z}} X_*(\mathcal{T}), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(\mathcal{T})) = \mathrm{Hom}(F^\times, T).$$

Together with (63) we obtain isomorphisms

$$\mathbf{Irr}(\mathcal{T}_0) \cong \mathrm{Hom}(\mathfrak{o}_F^\times, T),$$

$$X_{\mathrm{unr}}(\mathcal{T}) \cong \mathrm{Hom}(\mathbb{Z}, T) = T.$$

Let $\hat\chi \in \mathrm{Hom}(F^\times, T)$ be the image of χ under these isomorphisms. By the above the restriction of $\hat\chi$ to $\mathfrak{o}_F^\times$ is not disturbed by unramified twists, so we take that as $c^{\mathfrak{s}}$. Conversely, by (63) $c^{\mathfrak{s}}$ determines χ up to unramified twists. Two elements of $\mathbf{Irr}(\mathcal{T})$ are $\mathcal{G}$-conjugate if and only if they are $\mathcal{W}$-conjugate so, in view of (64), the $\mathcal{W}$-orbit of the $c^{\mathfrak{s}}$ contains the same amount of information as $\mathfrak{s}$. $\square$

Let $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}})$ and let $M = \mathrm{Z}_H(t)$ for some $t \in T$. Recall that a unipotent element $x \in M^0$ is said to be *distinguished* if the connected center $\mathrm{Z}_{M^0}^0$ of M^0 is a maximal torus of $\mathrm{Z}_{M^0}(x)$. Let $x \in M^0$ unipotent. If x is not distinguished, then there is a Levi subgroup L of M^0 containing x and such that $x \in L$ is distinguished.

Let $X \in \mathrm{Lie}\, M^0$ such that $\exp(X) = x$. A cocharacter $h \colon \mathbb{C}^\times \to M^0$ is said to be *associated to* x if

$$\mathrm{Ad}(h(t))X = t^2 X \quad \text{for each } t \in \mathbb{C}^\times,$$

and if the image of h lies in the derived group of some Levi subgroup L for which $x \in L$ is distinguished (see [40, Rem. 5.5] or [31, Rem.2.12]).

A cocharacter associated to a unipotent element $x \in M^0$ is not unique. However, any two cocharacters associated to a given $x \in M^0$ are conjugate under elements of $\mathrm{Z}_{M^0}(x)^0$ (see for instance [40, Lem. 5.3]).

We work with the Jacobson–Morozov theorem [27, p. 183]. Let $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ be the standard unipotent matrix in $\mathrm{SL}_2(\mathbb{C})$ and let x be a unipotent element in M^0. There exist rational homomorphisms

(65) $$\gamma \colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \quad \text{with} \quad \gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = x,$$

see [27, §3.7.4]. Any two such homomorphisms γ are conjugate by elements of $\mathrm{Z}_{M^\circ}(x)$.

For $\alpha \in \mathbb{C}^\times$ we define the following matrix in $\mathrm{SL}_2(\mathbb{C})$:

$$Y_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right).$$

Then each γ as above determines a cocharacter $h \colon \mathbb{C}^\times \to M^0$ by setting

(66) $$h(\alpha) := \gamma(Y_\alpha) \quad \text{for } \alpha \in \mathbb{C}^\times.$$

Each cocharacter h obtained in this way is associated to x, see [40, Rem. 5.5] or [31, Rem.2.12]. Hence each two such cocharacters are conjugate under $\mathrm{Z}_{M^0}(x)^0$.

Lemma 16.2. *Each cocharacter h above can be identified with a cocharacter of H associated to x, where x is viewed as a unipotent element of H.*

Any two such cocharacters of H are conjugate by elements of $\mathrm{Z}_H(x)^0$.

Proof. Recall J.C. Jantzen's result [40, Claim 5.12] (see also [31] for a related study in positive good characteristic): For any connected reductive subgroup H_2 of an arbitrary connected complex Lie group H_1, the cocharacters of H_2 associated to a unipotent element $x \in H_2$ are precisely the cocharacters of H_1 associated to x which take values in H_2.

Applying this with $H_1 = H$ and $H_2 = M^0$, we get that h can be identified with a cocharacter of H, and is associated to x viewed as a unipotent element of H.

The last assertion follows from [40, Lem. 5.3]. $\qquad\square$

From now on we view the h above as cocharacters of H associated to x. Any two $\gamma\colon \mathrm{SL}_2(\mathbb{C}) \to M^0 \subset H$ as above are conjugate by elements of $Z_H(x)$.

Suppose that $\gamma'\colon \mathrm{SL}_2(\mathbb{C}) \to H$ is an optimal SL_2-homomorphism for x such that
$$\gamma'(Y_\alpha) = \gamma(Y_\alpha) \quad \text{for } \alpha \in \mathbb{C}^\times.$$
Then $\gamma' = \gamma$, see [49, Prop. 42].

Choose a geometric Frobenius ϖ_F and set $\Phi(\varpi_F) = t \in T$. Define the Langlands parameter Φ as follows:

(67) $\qquad \Phi\colon F^\times \times \mathrm{SL}_2(\mathbb{C}) \to G, \qquad (u\varpi_F^n, Y) \mapsto c^{\mathfrak{s}}(u) \cdot t^n \cdot \gamma(Y)$

for all $u \in \mathfrak{o}_F^\times$, $n \in \mathbb{Z}$, $Y \in \mathrm{SL}_2(\mathbb{C})$.

Note that the definition of Φ uses the appropriate data: the semisimple element $t \in T$, the map $c^{\mathfrak{s}}$, and the homomorphism γ (which depends on the Springer parameter x).

Since x determines γ up to M°-conjugation, $c^{\mathfrak{s}}, x$ and t determine Φ up to conjugation by their common centralizer in G. Notice also that one can recover $c^{\mathfrak{s}}, x$ and t from Φ and that

(68) $\qquad\qquad\qquad h(\alpha) = \Phi(1, Y_\alpha).$

17. Varieties of Borel subgroups

We clarify some issues with different varieties of Borel subgroups and different kinds of parameters arising from them. Let G be a connected reductive complex group and let
$$\Phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$$
be as in (67). We write
$$H = Z_G(\Phi(\mathbf{I}_F)) = Z_G(\mathrm{im}\, c^{\mathfrak{s}}),$$
$$M = Z_G(\Phi(\mathbf{W}_F)) = Z_H(t).$$

Although both H and M are in general disconnected, $\Phi(\mathbf{W}_F)$ is always contained in H° because it lies in the maximal torus T of G and H°. Hence $\Phi(\mathbf{I}_F) \subset Z(H^\circ)$.

By construction t commutes with $\Phi(\mathrm{SL}_2(\mathbb{C})) \subset M$. For any $q^{1/2} \in \mathbb{C}^\times$ the element

(69) $\qquad\qquad\qquad t_q := t\Phi(Y_{q^{1/2}})$

satisfies the familiar relation $t_q x t_q^{-1} = x^q$. Indeed

$$
\begin{aligned}
t_q x t_q^{-1} &= t\Phi(Y_{q^{1/2}})\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\Phi(Y_{q^{1/2}}^{-1})t^{-1} \\
&= t\Phi\left(Y_{q^{1/2}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)Y_{q^{1/2}}^{-1}\right)t^{-1} \\
&= t\Phi\left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)t^{-1} = x^q.
\end{aligned}
$$

(70)

Recall that B_2 denotes the upper triangular Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. In the flag variety of M° we have the subvarieties $\mathcal{B}_{M^\circ}^x$ and $\mathcal{B}_{M^\circ}^{\Phi(B_2)}$ of Borel subgroups containing x and $\Phi(B_2)$, respectively. Similarly the flag variety of H° has subvarieties $\mathcal{B}_{H^\circ}^{t,x}$, $\mathcal{B}_{H^\circ}^{t_q,x}$ and

$$\mathcal{B}_{H^\circ}^{t,\Phi(B_2)} = \mathcal{B}_{H^\circ}^{t_q,\Phi(B_2)}.$$

Notice that $\Phi(\mathbf{I}_F)$ lies in every Borel subgroup of H°, because it is contained in $\mathrm{Z}(H^\circ)$. We abbreviate $\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})))$ and similarly for other groups.

Proposition 17.1.   (1) *The inclusion maps*

$$\begin{array}{ccccc} \mathrm{Z}_{M^\circ}(\Phi) & \to & \mathrm{Z}_{M^\circ}(\Phi(B_2)) & \to & \mathrm{Z}_{M^\circ}(x), \\ \mathrm{Z}_H(t_q,x) \quad \leftarrow \quad \mathrm{Z}_H(\Phi) & \to & \mathrm{Z}_H(t,\Phi(B_2)) & \to & \mathrm{Z}_H(t,x), \end{array}$$

are homotopy equivalences. In particular they induce isomorphisms between the respective component groups.

   (2) *The inclusions* $\mathcal{B}_{M^\circ}^{\Phi(B_2)} \to \mathcal{B}_{M^\circ}^x$ *and* $\mathcal{B}_{H^\circ}^{t_q,x} \leftarrow \mathcal{B}_{H^\circ}^{t,\Phi(B_2)} \to \mathcal{B}_{H^\circ}^{t,x}$ *are homotopy equivalences.*

Proof. It suffices to consider the statements for H and t_q, since the others can be proven in the same way.

(1) Our proof uses some elementary observations from [54, §4.3]. There is a Levi decomposition

$$\mathrm{Z}_{H^\circ}(x) = \mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))U_x$$

with $\mathrm{Z}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C}))) = \mathrm{Z}_{H^\circ}(\Phi(B_2))$ reductive and U_x unipotent. Since $t_q \in \mathrm{N}_{H^\circ}(\Phi(\mathrm{SL}_2(\mathbb{C})))$ and $\mathrm{Z}_H(x^q) = \mathrm{Z}_H(x)$, conjugation by t_q preserves this decomposition. Therefore

$$(71) \qquad \mathrm{Z}_{H^\circ}(t_q,x) = \mathrm{Z}_{H^\circ}(\Phi)\mathrm{Z}_{U_x}(t_q) = \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2))\mathrm{Z}_{U_x}(t_q).$$

We note that

$$\mathrm{Z}_{U_x}(t_q) \cap \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2)) \subset U_x \cap \mathrm{Z}_{H^\circ}(\Phi(B_2)) = 1$$

and that $\mathrm{Z}_{U_x}(t_q) \subset U_x$ is contractible, because it is a unipotent complex group. It follows that

$$(72) \qquad \mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_{H^\circ}(t_q,\Phi(B_2)) \to \mathrm{Z}_{H^\circ}(t_q,x)$$

is a homotopy equivalence. If we want to replace H° by H, we find

$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \{hH^\circ \in \pi_0(H) \mid h\Phi h^{-1} \in \mathrm{Ad}(H^\circ)\Phi\},$$

and similarly with $(t_q,\Phi(B_2))$ or (t_q,x) instead of Φ.

   Let us have a closer look at the H°-conjugacy classes of these objects. Given any Φ, we obviously know what t_q and x are. Conversely, suppose that t_q and x are given. We apply a refinement of the Jacobson–Morozov theorem due to Kazhdan and Lusztig. According to [42, §2.3] there exist homomorphisms $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ as above, which return t_q and x in the prescribed way. Moreover all such homomorphisms are conjugate under $\mathrm{Z}_{H^\circ}(t_q,x)$, see [42, §2.3.h] or Section 19. So from (t_q,x) we can

reconstruct the $\mathrm{Ad}(H^\circ)$-orbit of Φ, and this gives bijections between H°-conjugacy classes of Φ, $(t_q, \Phi(B_2))$ and (t_q, x). Since these bijections clearly are $\pi_0(H)$-equivariant, we deduce

(73)
$$\mathrm{Z}_H(\Phi)/\mathrm{Z}_{H^\circ}(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2))/\mathrm{Z}_{H^\circ}(t_q, \Phi(B_2)) = \mathrm{Z}_H(t_q, x)/\mathrm{Z}_{H^\circ}(t_q, x).$$

Equations (72) and (73) imply that

$$\mathrm{Z}_H(\Phi) = \mathrm{Z}_H(t_q, \Phi(B_2)) \to \mathrm{Z}_H(t_q, x)$$

is also a homotopy equivalence.

(2) By the aforementioned result [42, §2.3.h]

(74)
$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathcal{B}_{H^\circ}^{t_q, x}.$$

On the other hand, by (71)

(75)
$$\mathrm{Z}_{H^\circ}(t_q, x) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q)\mathrm{Z}_H(t_q, \Phi(B_2)) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} = \mathrm{Z}_{U_x}(t_q) \cdot \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}.$$

For any $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$ and $u \in \mathrm{Z}_{U_x}(t_q)$ it is clear that

$$u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \iff \Phi(B_2) \subset uBu^{-1} \iff u^{-1}\Phi(B_2)u \subset B.$$

Furthermore, since $\Phi(B_2) \subset B$ is generated by x and $\{\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mid \alpha \in \mathbb{C}^\times\}$, the right hand side is equivalent to

$$u^{-1}\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)u \in B \quad \forall \alpha \in \mathbb{C}^\times.$$

In Lie algebra terms this can be reformulated as

$$\mathrm{Ad}_{u^{-1}}(d\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \alpha \in \mathbb{C}.$$

Because u is unipotent, this happens if and only if

$$\mathrm{Ad}_{u^\lambda}(d\Phi\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)) \in \mathrm{Lie}\, B \quad \forall \lambda, \alpha \in \mathbb{C}.$$

By the reverse chain of arguments the last statement is equivalent with

$$u^\lambda \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \quad \forall \lambda \in \mathbb{C}.$$

Thus $\{u \in \mathrm{Z}_{U_x}(t_q) \mid u \cdot B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}\}$ is contractible for all $B \in \mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)}$, and we already knew that $\mathrm{Z}_{U_x}(t_q)$ is contractible. Together with (74) and (75) these imply that $\mathcal{B}_{H^\circ}^{t_q, \Phi(B_2)} \to \mathcal{B}_{H^\circ}^{t_q, x}$ is a homotopy equivalence. $\square$

For the affine Springer correspondence we will need more precise information on the relation between the varieties for G, for H and for M°.

Proposition 17.2. (1) *The variety $\mathcal{B}_{H^\circ}^{t, x}$ is isomorphic to $[\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}]$ copies of $\mathcal{B}_{M^\circ}^x$, and $\mathcal{B}_{H^\circ}^{t, \Phi(B_2)}$ is isomorphic to the same number of copies of $\mathcal{B}_{M^\circ}^{\Phi(B_2)}$.*

(2) *The group $\mathrm{Z}_{H^\circ}(t, x)/\mathrm{Z}_{M^\circ}(x)$ permutes these two sets of copies freely.*

(3) *The variety $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is isomorphic to $[\mathcal{W}^G : \mathcal{W}^{H^\circ}]$ copies of $\mathcal{B}_{H^\circ}^{t, \Phi(B_2)}$. The group $\mathrm{Z}_G(\Phi)/\mathrm{Z}_{H^\circ}(\Phi)$ permutes these copies freely.*

Proof. (1) Let A be a subgroup of T such that $M^\circ = Z_{H^\circ}(A)^\circ$ and let $\mathcal{B}_{H^\circ}^A$ denote the variety of all Borel subgroups of H° which contain A. With an adaptation of [27, p.471] we will prove that, for any $B \in \mathcal{B}_{H^\circ}^A$, $B \cap M^0$ is a Borel subgroup of M^0.

Since $B \cap M^\circ \subset B$ is solvable, it suffices to show that its Lie algebra is a Borel subalgebra of Lie M°. Write Lie $T = \mathfrak{t}$ and let

$$\text{Lie } H^\circ = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$$

be the triangular decomposition, where Lie $B = \mathfrak{n} \oplus \mathfrak{t}$. Since $A \subset B$, it preserves this decomposition and

$$\text{Lie } M^\circ = (\text{Lie } H)^A = \mathfrak{n}^A \oplus \mathfrak{t} \oplus \mathfrak{n}_-^A,$$
$$\text{Lie } B \cap M^\circ = \text{Lie } B^A = \mathfrak{n}^A \oplus \mathfrak{t}.$$

The latter is indeed a Borel subalgebra of Lie M°. Thus there is a canonical map

$$(76) \qquad \mathcal{B}_{H^\circ}^A \to \text{Flag } M^0, \quad B \mapsto B \cap M^0.$$

The group M acts by conjugation on $\mathcal{B}_{H^\circ}^A$ and (76) clearly is M-equivariant. By [27, p. 471] the M°-orbits form a partition

$$(77) \qquad \mathcal{B}_{H^\circ}^A = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_m.$$

At the same time these orbits are the connected components of $\mathcal{B}_{H^\circ}^A$ and the irreducible components of the projective variety $\mathcal{B}_{H^\circ}^A$. The argument from [27, p. 471] also shows that (76), restricted to any one of these orbits, is a bijection from the M^0-orbit onto Flag M^0.

The number of components m can be determined as in the proof of [62, Corollary 3.12.a]. The collection of Borel subgroups of M° that contain the maximal torus T is in bijection with the Weyl group $\mathcal{W}^{M^\circ}$. Retracting via (76), we find that every component $\mathcal{B}_i$ has precisely $|\mathcal{W}^{M^\circ}|$ elements that contain T. On the other hand, since $A \subset T$, $\mathcal{B}_{H^\circ}^A$ has $|\mathcal{W}^{H^\circ}|$ elements that contain T, so

$$m = [\mathcal{W}^{H^\circ} : \mathcal{W}^{M^\circ}].$$

To obtain our desired isomorphisms of varieties, we let A be the group generated by t and we restrict $\mathcal{B}_i \to \text{Flag } M^\circ$ to Borel subgroups that contain t, x (respectively $t, \Phi(B_2)$).

(2) By Proposition 17.1

$$Z_{H^\circ}(t, x)/Z_{M^\circ}(x) \cong Z_{H^\circ}(t, \Phi(B_2))/Z_{M^\circ}(\Phi(B_2)).$$

Since the former is a subgroup of M/M° and the copies under consideration are in M-equivariant bijection with the components (77), it suffices to show that M/M° permutes these components freely. Pick B, B' in the same component $\mathcal{B}_i$ and assume that $B' = hBh^{-1}$ for some $h \in M$. Since $\mathcal{B}_i$ is M°-equivariantly isomorphic to the flag variety of M° we can find $m \in M^\circ$ such that $B' = m^{-1}Bm$. Then mh normalizes B, so $mh \in B$. As B is connected, this implies $mh \in M^\circ$ and $h \in M^\circ$.

(3) Apply the proofs of parts 1 and 2 with $A = \Phi(\mathbf{I}_F)$, G in the role of H°, H° in the role of M° and $t\Phi(B_2)$ in the role of x. $\qquad\square$

18. Comparison of different parameters

In the following sections we will make use of several different but related kinds of parameters.

Kazhdan–Lusztig–Reeder parameters

For a Langlands parameter as in (67), the variety of Borel subgroups $\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}$ is nonempty, and the centralizer $Z_G(\Phi)$ of the image of Φ acts on it. Hence the group of components $\pi_0(Z_G(\Phi))$ acts on the homology $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We call an irreducible representation ρ of $\pi_0(Z_G(\Phi))$ geometric if it appears in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. We define a Kazhdan–Lusztig–Reeder parameter for G to be a such pair (Φ, ρ). The group G acts on these parameters by

$$(78) \qquad\qquad g \cdot (\Phi, \rho) = (g\Phi g^{-1}, \rho \circ \mathrm{Ad}_g^{-1})$$

and we denote the corresponding equivalence class by $[\Phi, \rho]_G$.

Affine Springer parameters

As before, suppose that $t \in G$ is semisimple and that $x \in Z_G(t)$ is unipotent. Then $Z_G(t, x)$ acts on $\mathcal{B}_G^{t,x}$ and $\pi_0(Z_G(t, x))$ acts on the homology of this variety. In this setting we say that $\rho_1 \in \mathbf{Irr}\big(\pi_0(Z_G(t, x))\big)$ is geometric if it appears in $H_{\mathrm{top}}(\mathcal{B}_G^{t,x}, \mathbb{C})$, where top refers to highest degree in which the homology is nonzero, the real dimension of $\mathcal{B}_G^{t,x}$. We call such triples (t, x, ρ_1) affine Springer parameters for G, because they appear naturally in the representation theory of the affine Weyl group associated to G. The group G acts on such parameters by conjugation, and we denote the conjugacy classes by $[t, x, \rho_1]_G$.

Kazhdan–Lusztig parameters

Next we consider a unipotent element $x \in G$ and a semisimple element $t_q \in G$ such that $t_q x t_q^{-1} = x^q$. As above, $Z_G(t_q, x)$ acts on the variety $\mathcal{B}_G^{t_q,x}$ and we call $\rho_q \in \mathbf{Irr}\big(\pi_0(Z_G(t_q, x))\big)$ geometric if it appears in $H_*\big(\mathcal{B}_G^{t_q,x}, \mathbb{C}\big)$. Triples (t_q, x, ρ_q) of this kind are known as Kazhdan–Lusztig parameters for G. Again they are endowed with an obvious G-action and we denote the equivalence classes by $[t_q, x, \rho_q]_G$.

In [42, 54] there are some indications that these three kinds of parameters are essentially equivalent. Proposition 17.1 allows us to make this precise in the necessary generality.

Lemma 18.1. *Let $\mathfrak{s}$ be a Bernstein component in the principal series, associate $c^{\mathfrak{s}} \colon \mathfrak{o}_F^{\times} \to T$ to it as in Lemma 16.1 and write $H = Z_G(c^{\mathfrak{s}}(\mathfrak{o}_F^{\times}))$. There are natural bijections between H°-equivalence classes of:*

- *Kazhdan–Lusztig–Reeder parameters for G with $\Phi\big|_{\mathfrak{o}_F^{\times}} = c^{\mathfrak{s}}$ and $\Phi(\varpi_F) \in H^{\circ}$;*
- *affine Springer parameters for H°;*
- *Kazhdan–Lusztig parameters for H°.*

Proof. Since $\mathrm{SL}_2(\mathbb{C})$ is connected and commutes with $\mathfrak{o}_F^{\times}$, its image under Φ must be contained in the connected component of H. Therefore

KLR-parameters with these properties are in canonical bijection with KLR-parameters for H° and it suffices to consider the case $H^\circ = G$.

As in (67) and (69), any KLR-parameter gives rise to the ingredients t, x and t_q for the other two kinds of parameters. As we discussed after (67), the pair (t, x) is enough to recover the conjugacy class of Φ. A refined version of the Jacobson–Morozov theorem says that the same goes for the pair (t_q, x), see [42, §2.3] or [54, Section 4.2].

To complete $\Phi, (t, x)$ or (t_q, x) to a parameter of the appropriate kind, we must add an irreducible representation ρ, ρ_1 or ρ_q. For the affine Springer parameters it does not matter whether we consider the total homology or only the homology in top degree. Indeed, it follows from Propositions 17.1 and 17.2 and [59, bottom of page 296 and Remark 6.5] that any irreducible representation ρ_1 which appears in $H_*\big(\mathcal{B}_G^{t,x}, \mathbb{C}\big)$, already appears in the top homology of this variety.

This and Proposition 17.1 show that there is a natural correspondence between the possible ingredients ρ, ρ_1 and ρ_q. $\qquad\square$

19. The affine Springer correspondence

An interesting instance of Proposition 15.2 arises when M is the centralizer of a semisimple element t in a connected reductive complex group G. As before we assume that t lies in a maximal torus T of G and we write $\mathcal{W}^G = W(G, T)$. By Lemma 14.2

$$(79) \qquad \mathcal{W}^M := \mathrm{N}_M(T)/\mathrm{Z}_M(T) \cong W \rtimes \pi_0(M)$$

is the stabilizer of t in $\mathcal{W}^G$, so the role of Γ is played by the component group $\pi_0(M)$. In contrast to the setup in Proposition 15.2, it is possible that some elements of $\pi_0(M) \setminus \{1\}$ fix W pointwise. This poses no problems however, as such elements never act trivially on T. For later use we record the following consequence of (57):

$$(80) \qquad \pi_0(M)_{\tau(x,\rho)} \cong \big(\mathrm{Z}_M(x)/\mathrm{Z}_{M^\circ}(x)\big)_\rho.$$

Recall from Section 5 that

$$\widetilde{T}_2 := \{(t, \sigma) \,:\, t \in T, \sigma \in \mathbf{Irr}(\mathcal{W}_t^G)\},$$
$$(T//\mathcal{W}^G)_2 := \widetilde{T}_2/\mathcal{W}^G.$$

We note that the rational characters of the complex torus T span the regular functions on the complex variety T:

$$\mathcal{O}(T) = \mathbb{C}[X^*(T)].$$

From (49) and (50), we infer the following rough form of the extended Springer correspondence for the affine Weyl group $X^*(T) \rtimes \mathcal{W}^G$.

Theorem 19.1. *There are canonical bijections*

$$(T//\mathcal{W}^G)_2 \simeq \mathbf{Irr}\,(X^*(T) \rtimes \mathcal{W}^G) \simeq \{(t, \tau(x, \varrho) \rtimes \psi)\}/\mathcal{W}^G$$

with $t \in T, \tau(x, \varrho) \in \mathbf{Irr}\,\mathcal{W}^{M^0}, \psi \in \mathbf{Irr}(\pi_0(M)_{\tau(x,\varrho)})$.

Now we recall the geometric realization of irreducible representations of $X^*(T) \rtimes \mathcal{W}^G$ by Kato [41]. For a unipotent element $x \in M^\circ$ let $\mathcal{B}_G^{t,x}$ be the variety of Borel subgroups of G containing t and x. Fix a Borel subgroup B of G containing T and let $\theta_{G,B} : \mathcal{B}_G^{t,x} \to T$ be the morphism defined by

$$\theta_{G,B}(B') = g^{-1}tg \text{ if } B' = gBg^{-1} \text{ and } t \in gTg^{-1}.$$

The image of $\theta_{G,B}$ is $\mathcal{W}^G t$, the map is constant on the irreducible components of $\mathcal{B}_G^{t,x}$ and it gives rise to an action of $X^*(T)$ on the homology of $\mathcal{B}_G^{t,x}$. Furthermore $\mathbb{Q}[\mathcal{W}^G] \cong H(\mathcal{Z}_G)$ acts on $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$ via the convolution product in Borel–Moore homology, as described in (54). Both actions commute with the action of $\mathrm{Z}_G(t,x)$ induced by conjugation of Borel subgroups.

Let $\rho_1 \in \mathrm{Irr}(\mathrm{Z}_G(t,x))$. By [41, Theorem 4.1] the $X^*(T) \rtimes \mathcal{W}^G$-module

$$(81) \qquad \tau(t,x,\rho_1) := \mathrm{Hom}_{\mathrm{Z}_G(t,x)}\big(\rho_1, H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})\big)$$

is either irreducible or zero. Moreover every irreducible representation of $X^*(T) \rtimes \mathcal{W}^G$ is obtained is in this way, and the data (t,x,ρ_1) are unique up to G-conjugacy. This generalizes the Springer correspondence for finite Weyl groups, which can be recovered by considering the representations on which $X^*(T)$ acts trivially.

Propositions 15.2 and 17.2 shine some new light on this:

Theorem 19.2. (1) *There are natural bijections between the following sets:*
- $\mathrm{Irr}(X^*(T) \rtimes \mathcal{W}^G) = \mathrm{Irr}(\mathcal{O}(T) \rtimes \mathcal{W}^G)$;
- $(T /\!/ \mathcal{W}^G)_2 = \big\{ (t, \tilde{\tau}) \mid t \in T, \tilde{\tau} \in \mathrm{Irr}(\mathcal{W}^M) \big\} /\mathcal{W}^G$;
- $\big\{ (t, \tau, \sigma) \mid t \in T, \tau \in \mathrm{Irr}(\mathcal{W}^{M^\circ}), \sigma \in \mathrm{Irr}(\pi_0(M)_\tau) \big\} /\mathcal{W}^G$;
- $\big\{ (t, x, \rho, \sigma) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_{M^\circ}(x))\big)$ geometric, $\sigma \in \mathrm{Irr}\big(\pi_0(M)_{\tau(x,\rho)}\big) \big\} /G$;
- $\big\{ (t, x, \rho_1) \mid t \in T, x \in M^\circ \text{ unipotent}, \rho_1 \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t,x))\big)$ geometric$\big\} /G$.

  Here a representation of $\pi_0(\mathrm{Z}_{M^\circ}(x))$ (or $\pi_0(\mathrm{Z}_G(t,x))$) is called geometric if it appears in $H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C})$ (respectively $H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})$).

(2) *The $X^*(T) \rtimes \mathcal{W}^G$-representation corresponding to (t, x, ρ_1) via these bijections is Kato's module (81).*

We remark that in the fourth and fifth sets it would be more natural to allow t to be any semisimple element of G. In fact that would give the affine Springer parameters from Lemma 18.1. Clearly G acts on the set of such more general parameters (t, x, ρ, σ) or (t, x, ρ_1), which gives equivalence relations $/G$. The two above $/G$ refer to the restrictions of these equivalence relations to parameters with $t \in T$.

Proof. (1) Recall that the isotropy group of t in $\mathcal{W}^G$ is

$$\mathcal{W}_t^G = \mathcal{W}^M = \mathcal{W}^{M^\circ} \rtimes \pi_0(M).$$

Hence the bijection between the first two sets is an instance of Clifford theory, see Section 5. The second and third sets are in natural bijection by Proposition 15.2. The Springer correspondence for $\mathcal{W}^{M^\circ}$ provides the bijection with the fourth collection. To establish a natural bijection with

the fifth collection, we first recall from (57) that $\pi_0(M)_{\tau(x,\rho)} = \pi_0(M)_{[x,\rho]_{M^\circ}}$.
By that and Proposition 15.2 every irreducible representation of

$$\pi_0(Z_G(t,x)) \cong \pi_0(Z_{M^\circ}(x)) \rtimes \pi_0(M)_{[x]_{M^\circ}}$$

is of the form $\rho \rtimes \sigma$ for ρ and σ as in the fourth set. By Proposition 17.2

$$(82) \qquad H_*(\mathcal{B}_G^{t,x}, \mathbb{C}) \cong H_*(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[Z_G(t,x)/Z_{M^\circ}(x)] \otimes \mathbb{C}^{[\mathcal{W}^G : \mathcal{W}_t^G]}$$

as $Z_G(t,x)$-representations. By [54, §3.1]

$$Z_G(t,x)/Z_{M^\circ}(x) \cong \pi_0(M)_{[x]_{M^\circ}}$$

is abelian. Hence $\operatorname{Ind}_{\pi_0(M)_{[x,\rho]_{M^\circ}}}^{\pi_0(M)_{[x]_{M^\circ}}}(\sigma)$ appears exactly once in the regular representation of this group and

$$(83) \quad \operatorname{Hom}_{\pi_0(Z_G(t,x))}\big(\rho \rtimes \sigma, H_{d(x)}(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \cong$$

$$\operatorname{Hom}_{\pi_0(Z_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C})\big) \rtimes \sigma \otimes \mathbb{C}^{[\mathcal{W}^G : \mathcal{W}_t^G]}.$$

In particular we see that ρ is geometric if and only if $\rho \rtimes \sigma$ is geometric, which establishes the final bijection.
(2) The $X^*(T) \rtimes \mathcal{W}^G$-representation constructed from $(t,x,\rho \rtimes \sigma)$ by means of our bijections is

$$(84) \qquad \operatorname{Ind}_{X^*(T) \rtimes \mathcal{W}_t^G}^{X^*(T) \rtimes \mathcal{W}^G}\big(\operatorname{Hom}_{\pi_0(Z_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C})\big) \rtimes \sigma\big).$$

On the other hand, by [41, Proposition 6.2]

$$(85) \quad H_*(\mathcal{B}_G^{t,x}, \mathbb{C}) \cong \operatorname{Ind}_{X^*(T) \rtimes \mathcal{W}^{M^\circ}}^{X^*(T) \rtimes \mathcal{W}^G}(H_*(\mathcal{B}_{M^\circ}^x, \mathbb{C}))$$

$$\cong \operatorname{Ind}_{X^*(T) \rtimes \mathcal{W}_t^G}^{X^*(T) \rtimes \mathcal{W}^G}\big(H_*(\mathcal{B}_{M^\circ}^x, \mathbb{C}) \otimes \mathbb{C}[Z_G(t,x)/Z_{M^\circ}(x)]\big)$$

as $Z_G(t,x) \times X^*(T) \rtimes \mathcal{W}^G$-representations. Together with the proof of part 1 this shows that $\tau(t,x,\rho \rtimes \sigma)$ is isomorphic to (84). $\qquad \square$

We can extract a little more from the above proof. Recall that $\mathcal{O}_x$ denotes the conjugacy class of x in M. Let us agree that the affine Springer parameters with a fixed $t \in T$ are partially ordered by

$$(t,x,\rho_1) < (t,x',\rho_1') \quad \text{when} \quad \overline{\mathcal{O}_x} \subsetneq \overline{\mathcal{O}_{x'}}.$$

Lemma 19.3. *There exist multiplicities* $m_{t,x,\rho_1,x',\rho_1'} \in \mathbb{Z}_{\geq 0}$ *such that*

$$\operatorname{Hom}_{\pi_0(Z_G(t,x))}\big(\rho_1, H_*(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \cong$$

$$\tau(t,x,\rho_1) \oplus \bigoplus_{(t,x',\rho_1') > (t,x,\rho_1)} m_{t,x,\rho_1,x',\rho_1'}\, \tau(t,x',\rho_1').$$

Proof. It follows from (85), (82) and (83) that

$$(86) \quad \operatorname{Hom}_{\pi_0(Z_G(t,x))}\big(\rho \rtimes \sigma, H_*(\mathcal{B}_G^{t,x}, \mathbb{C})\big) \cong$$

$$\operatorname{Ind}_{X^*(T) \rtimes \mathcal{W}_t^G}^{X^*(T) \rtimes \mathcal{W}^G} \operatorname{Ind}_{\mathcal{W}^{M^\circ} \rtimes \pi_0(M)_{[x,\rho]_{M^\circ}}}^{\mathcal{W}_t^G} \big(\operatorname{Hom}_{\pi_0(Z_{M^\circ}(x))}\big(\rho, H_{d(x)}(\mathcal{B}_{M^\circ}^x, \mathbb{C})\big) \otimes \sigma\big).$$

The functor $\operatorname{Ind}_{X^*(T) \rtimes \mathcal{W}_t^G}^{X^*(T) \rtimes \mathcal{W}^G}$ provides an equivalence between the categories

- $X^*(T) \rtimes \mathcal{W}_t^G$-representations with $\mathcal{O}(T)^{\mathcal{W}_t^G}$-character t;

- $X^*(T) \rtimes \mathcal{W}^G$-representations with $\mathcal{O}(T)^{\mathcal{W}^G}$-character $\mathcal{W}^G t$.

Therefore we may apply Lemma 15.4 to the right hand side of (86), which produces the required formula. $\qquad\square$

Let us have a look at the representations with an affine Springer parameter of the form $(t, x = 1, \rho_1 = \mathrm{triv})$. Equivalently, the fourth parameter in Theorem 19.2 is $(t, x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$. The $\mathcal{W}^{M^\circ}$-representation with Springer parameter $(x = 1, \rho = \mathrm{triv})$ is the trivial representation, so $(x = 1, \rho = \mathrm{triv}, \sigma = \mathrm{triv})$ corresponds to the trivial representation of $\mathcal{W}^G_t$. With (84) we conclude that the $X^*(T) \rtimes \mathcal{W}^G$-representation with affine Springer parameter $(t, 1, \mathrm{triv})$ is

$$(87) \qquad \tau(t, 1, \mathrm{triv}) = \mathrm{Ind}_{X^*(T) \rtimes \mathcal{W}^G_t}^{X^*(T) \rtimes \mathcal{W}^G}(\mathrm{triv}_{\mathcal{W}^G_t}).$$

Notice that this is the only irreducible $X^*(T) \rtimes \mathcal{W}^G$-representation with an $X^*(T)$-weight t and nonzero $\mathcal{W}^G$-fixed vectors.

20. Geometric representations of affine Hecke algebras

Let G be a connected reductive complex group, B a Borel subgroup and T a maximal torus of G contained in B. Let $\mathcal{H}(G)$ be the affine Hecke algebra with the same based root datum as (G, B, T) and with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity.

Since later on we will have to deal with disconnected reductive groups, we include some additional automorphisms in the picture. In every root subgroup U_α with $\alpha \in \Delta(B, T)$ we pick a nontrivial element u_α. Let Γ be a finite group of automorphisms of $(G, T, (u_\alpha)_{\alpha \in \Delta(B,T)})$. Since G need not be semisimple, it is possible that some elements of Γ fix the entire root system of (G, T). Notice that Γ acts on the Weyl group $\mathcal{W}^G = W(G, T)$ because it stabilizes T.

We form the crossed product $\mathcal{H}(G) \rtimes \Gamma$ with respect to the canonical Γ-action on $\mathcal{H}(G)$. We define a Kazhdan–Lusztig parameter for $\mathcal{H}(G) \rtimes \Gamma$ to be a triple (t_q, x, ρ) such that

- $t_q \in G$ is semisimple, $x \in G$ is unipotent and $t_q x t_q^{-1} = x^q$;
- ρ is an irreducible representation of the component group $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$, such that every irreducible subrepresentation of the restriction of ρ to $\pi_0(\mathrm{Z}_G(t_q, x))$ appears in $H_*(\mathcal{B}^{t_q, x}, \mathbb{C})$.

The group $G \rtimes \Gamma$ acts on such parameters by conjugation, and we denote the conjugacy class of a parameter by $[t_q, x, \rho]_{G \rtimes \Gamma}$. Now we generalize [42, Theorem 7.12] and [54, Theorem 3.5.4]:

Theorem 20.1. *There exists a natural bijection between* $\mathrm{Irr}(\mathcal{H}(G) \rtimes \Gamma)$ *and* $G \rtimes \Gamma$-*conjugacy classes of Kazhdan–Lusztig parameters. The module corresponding to* (t_q, x, ρ) *is the unique irreducible quotient of the* $\mathcal{H}(G) \rtimes \Gamma$-*module*

$$\mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_q, x))}(\rho_q, H_*(\mathcal{B}^{t_q, x}, \mathbb{C})) \rtimes \rho_2,$$

where $\rho_q \in \mathrm{Irr}(\pi_0(\mathrm{Z}_G(t_q, x)))$ *and* $\rho_2 \in \mathrm{Irr}^{\natural(t_q, x, \rho_q)}(\Gamma_{[t_q, x, \rho_q]_G})$ *are such that* $\rho \cong \rho_q \rtimes \rho_2$.

Proof. First we recall the geometric constructions of $\mathcal{H}(G)$-modules by Kazhdan, Lusztig and Reeder, taking advantage of Lemma 15.3 to simplify the presentation somewhat. As in [54, §1.5], let

$$1 \to C \to \tilde{G} \to G \to 1$$

be a finite central extension such that $\tilde{G}$ is a connected reductive group with simply connected derived group. The kernel C acts naturally on $\mathcal{H}(\tilde{G})$ and

$$(88) \qquad \mathcal{H}(\tilde{G})^C \cong \mathcal{H}(G).$$

The action of Γ on the based root datum of (G, B, T) lifts uniquely to an action on the corresponding based root datum for $\tilde{G}$, so the Γ-actions on G and on $\mathcal{H}(G)$ lift naturally to actions on $\tilde{G}$ and $\mathcal{H}(\tilde{G})$. Let $\mathcal{H}_{\mathbf{q}}(\tilde{G})$ be the variation on $\mathcal{H}(\tilde{G})$ with scalars $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ instead of $\mathbb{C}$ and $q \in \mathbb{C}^\times$. In [42, Theorem 3.5] an isomorphism

$$(89) \qquad \mathcal{H}_{\mathbf{q}}(\tilde{G}) \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}})$$

is constructed, where the right hand side denotes the $\tilde{G} \times \mathbb{C}^\times$-equivariant K-theory of the Steinberg variety $\mathcal{Z}_{\tilde{G}}$ of $\tilde{G}$. Since $G \rtimes \Gamma$ acts via conjugation on $\tilde{G}$ and on $\mathcal{Z}_{\tilde{G}}$, it also acts on $K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}})$. However, the connected group G acts trivially, so the action factors via Γ. Now the definition of the generators in [42, Theorem 3.5] shows that (89) is Γ-equivariant. In particular it specializes to Γ-equivariant isomorphisms

$$(90) \qquad \mathcal{H}(\tilde{G}) \cong \mathcal{H}_{\mathbf{q}}(\tilde{G}) \otimes_{\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]} \mathbb{C}_q.$$

Let $(\tilde{t}_q, \tilde{x}) \in (\tilde{G})^2$ be a lift of $(t_q, x) \in G^2$ with $\tilde{x}$ unipotent. The $\tilde{G}$-conjugacy class of $\tilde{t}_q$ defines a central character of $\mathcal{H}(\tilde{G})$ and according to [27, Proposition 8.1.5] the associated localization is

$$\mathcal{H}(\tilde{G}) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde{G}))} \mathbb{C}_{\tilde{t}_q} \cong K^{\tilde{G} \times \mathbb{C}^\times}(\mathcal{Z}_{\tilde{G}}) \otimes_{R(\tilde{G} \times \mathbb{C}^\times)} \mathbb{C}_{\tilde{t}_q, q} \cong H_*(\mathcal{Z}^{\tilde{t}_q, q}, \mathbb{C}).$$

Any Borel subgroup of $\tilde{G}$ contains C, so $\mathcal{B}^{\tilde{t}_q, \tilde{x}} = \mathcal{B}_{\tilde{G}}^{\tilde{t}_q, \tilde{x}}$ and $\mathcal{B}^{t_q, x} = \mathcal{B}_G^{t_q, x}$ are isomorphic algebraic varieties. From [27, p. 414] we see that the convolution product in Borel–Moore homology leads to an action of $H_*(\mathcal{Z}_{\tilde{G}}^{\tilde{t}_q, q}, \mathbb{C})$ on $H_*(\mathcal{B}^{\tilde{t}_q, \tilde{x}}, \mathbb{C})$. Notice that for $\tilde{h} \in H_*(\mathcal{Z}_{\tilde{G}}^{\tilde{t}_q, q}, \mathbb{C})$ and $g \in G \rtimes \Gamma$ we have

$$g \cdot \tilde{h} \in H_*(\mathcal{Z}_{\tilde{G}}^{g\tilde{t}_q g^{-1}, q}, \mathbb{C}) \cong \mathcal{H}(\tilde{G}) \otimes_{\mathrm{Z}(\mathcal{H}(\tilde{G}))} \mathbb{C}_{g\tilde{t}_q g^{-1}}.$$

An obvious generalization of [27, Lemma 8.1.8] says that all these constructions are compatible with the above actions of $G \rtimes \Gamma$, in the sense that the following diagram commutes:

$$(91) \qquad \begin{array}{ccc} H_*(\mathcal{B}^{\tilde{t}_q, \tilde{x}}, \mathbb{C}) & \xrightarrow{\tilde{h}} & H_*(\mathcal{B}^{\tilde{t}_q, \tilde{x}}, \mathbb{C}) \\ \downarrow{H_*(\mathrm{Ad}_g)} & & \downarrow{H_*(\mathrm{Ad}_g)} \\ H_*(\mathcal{B}^{g\tilde{t}_q g^{-1}, g\tilde{x}g^{-1}}, \mathbb{C}) & \xrightarrow{g \cdot \tilde{h}} & H_*(\mathcal{B}^{g\tilde{t}_q g^{-1}, g\tilde{x}g^{-1}}, \mathbb{C}). \end{array}$$

In particular the component group $\pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q, \tilde{x}))$ acts on $H_*(\mathcal{B}^{\tilde{t}_q, \tilde{x}}, \mathbb{C})$ by $\mathcal{H}(\tilde{G})$-intertwiners. Let $\tilde{\rho}$ be an irreducible representation of this component

group, appearing in $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}}, \mathbb{C})$. In other words, $(\tilde{t}_q, \tilde{x}, \tilde{\rho})$ is a Kazhdan–Lusztig parameter for $\mathcal{H}(\tilde{G})$. According to [42, Theorem 7.12]

$$(92) \qquad \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q, \tilde{x}))}(\tilde{\rho}, H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}}, \mathbb{C}))$$

is a $\mathcal{H}(\tilde{G})$-module with a unique irreducible quotient, say $V_{\tilde{t}_q,\tilde{x},\tilde{\rho}}$.

Following [54, §3.3] we define a group $R_{\tilde{t}_q,\tilde{x}}$ by

$$(93) \qquad 1 \to \pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q, \tilde{x})) \to \pi_0(\mathrm{Z}_{\tilde{G}}(t_q, x)) \to R_{\tilde{t}_q,\tilde{x}} \to 1.$$

Since the derived group of $\tilde{G}$ is simply connected, $\mathrm{Z}_{\tilde{G}}(t_q)^\circ = \mathrm{Z}_{\tilde{G}}(\tilde{t}_q)$. Furthermore $\mathrm{Z}(\tilde{G})$ acts trivially on $H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}}, \mathbb{C})$, so we may just as well replace (93) by the short exact sequence

$$1 \to \pi_0(\mathrm{Z}_{\mathrm{Z}_G(t_q)^\circ}(x)/\mathrm{Z}(G)) \to \pi_0(\mathrm{Z}_G(t_q, x)/\mathrm{Z}(G)) \to R_{\tilde{t}_q,\tilde{x}} \to 1.$$

From Lemma 15.3 (with the trivial representation of $\pi_0(\mathrm{Z}_G(t_q)^\circ(x))$ in the role of ρ) we know that the latter exact sequence splits. Hence all the 2-cocycles of subgroups of $R_{\tilde{t}_q,\tilde{x}}$ appearing in [54, Section 3] are trivial.

Let $\tilde{\sigma}$ be any irreducible representation of $R_{\tilde{t}_q,\tilde{x},\tilde{\rho}}$, the stabilizer of the isomorphism class of $\tilde{\rho}$ in $R_{\tilde{t}_q,\tilde{x}}$. Clifford theory for (93) produces $\tilde{\rho} \rtimes \tilde{\sigma} \in \mathrm{Irr}(\pi_0(\mathrm{Z}_{\tilde{G}}(t_q, x)))$, a representation which factors through $\pi_0(\mathrm{Z}_G(t_q, x))$ because C acts trivially. Moreover by [54, Lemma 3.5.1] it appears in $H_*(\mathcal{B}^{t_q,x}, \mathbb{C})$, and conversely every irreducible representation with the latter property is of the form $\tilde{\rho} \rtimes \tilde{\sigma}$.

With the above in mind, [54, Lemma 3.5.2] says that the $\mathcal{H}(G)$-module

$$(94) \qquad \begin{aligned} M(t_q, x, \tilde{\rho} \rtimes \tilde{\sigma}) &:= \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t_q,x))}\big(\tilde{\rho} \rtimes \tilde{\sigma}, H_*(\mathcal{B}^{t_q,x}, \mathbb{C})\big) \\ &= \mathrm{Hom}_{R_{\tilde{t}_q,\tilde{x},\tilde{\rho}}}\big(\tilde{\sigma}, \mathrm{Hom}_{\pi_0(\mathrm{Z}_{\tilde{G}}(\tilde{t}_q,\tilde{x}))}(\tilde{\rho}, H_*(\mathcal{B}^{\tilde{t}_q,\tilde{x}}, \mathbb{C}))\big) \end{aligned}$$

has a unique irreducible quotient

$$(95) \qquad \pi(t_q, x, \tilde{\rho} \rtimes \tilde{\sigma}) = \mathrm{Hom}_{R_{\tilde{t}_q,\tilde{x},\tilde{\rho}}}(\tilde{\sigma}, V_{\tilde{t}_q,\tilde{x},\tilde{\rho}}).$$

According [54, Lemma 3.5.3] this sets up a bijection between $\mathrm{Irr}(\mathcal{H}(G))$ and G-conjugacy classes of Kazhdan–Lusztig parameters for G.

Remark 20.2. *The module (94) is well-defined for any $q \in \mathbb{C}^\times$, although for roots of unity it may have more than one irreducible quotient. For $q = 1$ the algebra $\mathcal{H}(G)$ reduces to $\mathbb{C}[X^*(T) \rtimes \mathcal{W}^G]$ and [27, Section 8.2] shows that Kato's module (81) is a direct summand of $M(t_1, x, \rho_1)$.*

Next we study what Γ does to all these objects. There is natural action of Γ on Kazhdan–Lusztig parameters for G, namely

$$\gamma \cdot (t_q, x, \rho_q) = \big(\gamma t_q \gamma^{-1}, \gamma x \gamma^{-1}, \rho_q \circ \mathrm{Ad}_\gamma^{-1}\big).$$

From (91) and (94) we deduce that the diagram

$$(96) \qquad \begin{array}{ccc} \pi(t_q, x, \rho_q) & \xrightarrow{\ h\ } & \pi(t_q, x, \rho_q) \\ \downarrow{H_*(\mathrm{Ad}_g)} & & \downarrow{H_*(\mathrm{Ad}_g)} \\ \pi\big(g t_q g^{-1}, g x g^{-1}, \rho_q \circ \mathrm{Ad}_g^{-1}\big) & \xrightarrow{\ \gamma(h)\ } & \pi\big(g t_q g^{-1}, g x g^{-1}, \rho_q \circ \mathrm{Ad}_g^{-1}\big) \end{array}$$

commutes for all $g \in G\gamma$ and $h \in \mathcal{H}(G)$. Hence

$$(97) \qquad \text{Reeder's parametrization of } \mathrm{Irr}(\mathcal{H}(G)) \text{ is } \Gamma\text{-equivariant.}$$

Let $\pi \in \mathrm{Irr}(\mathcal{H}(G))$ and choose a Kazhdan–Lusztig parameter such that π is equivalent with $\pi(t_q, x, \rho_q)$. Composition with γ^{-1} on π gives rise to a 2-cocycle $\natural(\pi)$ of Γ_π. Clifford theory tells us that every irreducible representation of $\mathcal{H}(G) \rtimes \Gamma$ is of the form $\pi \rtimes \rho_2$ for some $\pi \in \mathrm{Irr}(\mathcal{H}(G))$, unique up to Γ-equivalence, and a unique $\rho_2 \in \mathrm{Irr}^{\natural(\pi)}(\Gamma_\pi)$. By the above the stabilizer of π in Γ equals the stabilizer of the G-conjugacy class $[t_q, x, \rho_q]_G$. Thus we have parametrized $\mathrm{Irr}(\mathcal{H}(G) \rtimes \Gamma)$ in a natural way with $G \rtimes \Gamma$-conjugacy classes of quadruples (t_q, x, ρ_q, ρ_2), where (t_q, x, ρ_q) is a Kazhdan–Lusztig parameter for G and $\rho_2 \in \mathrm{Irr}^{\natural(\pi(t_q, x, \rho_q))}\Gamma_{[t_q, x, \rho_q]_G}$.

The short exact sequence

$$(98) \qquad 1 \to \pi_0(\mathrm{Z}_G(t_q, x)) \to \pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x)) \to \Gamma_{[t_q, x]_G} \to 1$$

yields an action of $\Gamma_{[t_q, x]_G}$ on $\mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$. Restricting this to the stabilizer of ρ_q, we obtain another 2-cocycle $\natural(t_q, x, \rho_q)$ of $\Gamma_{[t_q, x, \rho_q]_G}$, which we want to compare to $\natural(\pi(t_q, x, \rho_q))$. Let us decompose

$$H_*(\mathcal{B}^{t_q, x}, \mathbb{C}) \cong \bigoplus_{\rho_q} \rho_q \otimes M(t_q, x, \rho_q)$$

as $\pi_0((\mathrm{Z}_G(t_q, x)) \times \mathcal{H}(G)$-modules. We sum over all $\rho_q \in \mathrm{Irr}\big(\pi_0(\mathrm{Z}_G(t_q, x))\big)$ for which the contribution is nonzero, and we know that for such ρ_q the $\mathcal{H}(G)$-module $M(t_q, x, \rho_q)$ has a unique irreducible quotient $\pi(t_q, x, \rho_q)$. Since $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$ acts (via conjugation of Borel subgroups) on $H_*(\mathcal{B}^{t_q, x}, \mathbb{C})$, any splitting of (98) as sets provides a 2-cocycle $\natural$ for the action of $\Gamma_{[t_q, x, \rho_q]_G}$ on $\rho_q \otimes M(t_q, x, \rho_q)$. Unfortunately we cannot apply Lemma 15.3 to find a splitting of (98) as groups, because $\mathrm{Z}_G(t_q)$ need not be connected. Nevertheless $\natural$ can be used to describe the actions of $\Gamma_{[t_q, x, \rho_q]_G}$ on both ρ_q and $\pi(t_q, x, \rho_q)$, so as cohomology classes

$$(99) \qquad \natural(t_q, x, \rho_q) = \natural = \natural(\pi(t_q, x, \rho_q)) \in H^2(\Gamma_{[t_q, x, \rho_q]_G}, \mathbb{C}^\times).$$

It follows that every irreducible representation ρ of $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$ is of the form $\rho_q \rtimes \rho_2$ for ρ_q and ρ_2 as above. Moreover ρ determines ρ_q up to $\Gamma_{[t_q, x]_G}$-equivalence and ρ_2 is unique if ρ_q has been chosen. Finally, if ρ_q appears in $H_{\mathrm{top}}(\mathcal{B}^{t_q, x}, \mathbb{C})$ then every irreducible $\pi_0(\mathrm{Z}_G(t_q, x))$-subrepresentation of ρ does, because $\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t_q, x))$ acts naturally on $H_*(\mathcal{B}^{t_q, x}, \mathbb{C})$. Therefore we may replace the above quadruples (t_q, x, ρ_q, ρ_2) by Kazhdan–Lusztig parameters (t_q, x, ρ). $\qquad\square$

21. Spherical representations

Let G, B, T and Γ be as in the previous section. Let $\mathcal{H}(\mathcal{W}^G)$ be the Iwahori–Hecke algebra of the Weyl group $\mathcal{W}^G$, with a parameter $q \in \mathbb{C}^\times$ which is not a root of unity. This is a deformation of the group algebra $\mathbb{C}[\mathcal{W}^G]$ and a subalgebra of the affine Hecke algebra $\mathcal{H}(G)$. The multiplication is defined in terms of the basis $\{T_w \mid w \in \mathcal{W}^G\}$, as in (123).

Recall that $\mathcal{H}(G)$ also has a commutative subalgebra $\mathcal{O}(T)$, such that the multiplication maps

$$(100) \qquad \mathcal{O}(T) \otimes \mathcal{H}(\mathcal{W}^G) \longrightarrow \mathcal{H}(G) \longleftarrow \mathcal{H}(\mathcal{W}^G) \otimes \mathcal{O}(T)$$

are bijective.

The trivial representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$ is defined as

(101) $$\mathrm{triv}(T_w \gamma) = q^{\ell(w)} \quad w \in \mathcal{W}^G, \gamma \in \Gamma.$$

It is associated to the idempotent

$$p_{\mathrm{triv}} \, p_\Gamma := \sum_{w \in \mathcal{W}^G} T_w P_{\mathcal{W}^G}(q)^{-1} \sum_{\gamma \in \Gamma} \gamma |\Gamma|^{-1} \quad \in \mathcal{H}(\mathcal{W}^G) \rtimes \Gamma,$$

where $P_{\mathcal{W}^G}$ is the Poincaré polynomial

$$P_{\mathcal{W}^G}(q) = \sum\nolimits_{w \in \mathcal{W}^G} q^{\ell(w)}.$$

Notice that $P_{\mathcal{W}^G}(q) \neq 0$ because q is not a root of unity. The trivial representation appears precisely once in the regular representation of $\mathcal{H}(\mathcal{W}^G) \rtimes \Gamma$, just like for finite groups.

An $\mathcal{H}(G) \rtimes \Gamma$-module V is called spherical if it is generated by the subspace $p_{\mathrm{triv}} p_\Gamma V$ [34, (2.5)]. This admits a nice interpretation for the unramified principal series representations. Recall that $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G}, \mathcal{I})$ for an Iwahori subgroup $\mathcal{I} \subset \mathcal{G}$. Let $\mathcal{K} \subset \mathcal{G}$ be a good maximal compact subgroup containing $\mathcal{I}$. Then p_{triv} corresponds to averaging over $\mathcal{K}$ and $p_{\mathrm{triv}} \mathcal{H}(\mathcal{G}, \mathcal{I}) p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G}, \mathcal{K})$, see [34, Section 1]. Hence spherical $\mathcal{H}(\mathcal{G}, \mathcal{I})$-modules correspond to smooth $\mathcal{G}$-representations that are generated by their $\mathcal{K}$-fixed vectors, also known as $\mathcal{K}$-spherical $\mathcal{G}$-representations. By the Satake transform

(102) $$p_{\mathrm{triv}} \mathcal{H}(\mathcal{G}, \mathcal{I}) p_{\mathrm{triv}} \cong \mathcal{H}(\mathcal{G}, \mathcal{K}) \cong \mathcal{O}(T/\mathcal{W}^G),$$

so the irreducible spherical modules of $\mathcal{H}(G) \cong \mathcal{H}(\mathcal{G}, \mathcal{I})$ are parametrized by $T/\mathcal{W}^G$ via their central characters. We want to determine the Kazhdan–Lusztig parameters (as in Theorem 20.1) of these representations.

Proposition 21.1. *For every central character $(\mathcal{W}^G \rtimes \Gamma)t \in T/(\mathcal{W}^G \rtimes \Gamma)$ there is a unique irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-module, and it has Kazhdan–Lusztig parameter $(t, x = 1, \rho = \mathrm{triv})$.*

Proof. We will first prove the proposition for $\mathcal{H}(G)$, and only then consider Γ.

By the Satake isomorphism (102) there is a unique irreducible spherical $\mathcal{H}(G)$-module for every central character $\mathcal{W}^G t \in T/\mathcal{W}^G$. The equivalence classes of Kazhdan–Lusztig parameters of the form $(t, x = 1, \rho = \mathrm{triv})$ are also in canonical bijection with $T/\mathcal{W}^G$. Therefore it suffices to show that $\pi(t, 1, \mathrm{triv})$ is spherical for all $t \in T$.

The principal series of $\mathcal{H}(G)$ consists of the modules $\mathrm{Ind}_{\mathcal{O}(T)}^{\mathcal{H}(G)} \mathbb{C}_t$ for $t \in T$. This module admits a central character, namely $\mathcal{W}^G t$. By (100) every such module is isomorphic to $\mathcal{H}(\mathcal{W}^G)$ as a $\mathcal{H}(\mathcal{W}^G)$-module. In particular it contains the trivial $\mathcal{H}(\mathcal{W}^G)$-representation once and has a unique irreducible spherical subquotient.

As in Section 20, let $\tilde{G}$ be a finite central extension of G with simply connected derived group. Let $\tilde{T}, \tilde{B}$ be the corresponding extensions of T, B. We identify the roots and the Weyl groups of $\tilde{G}$ and G. Let $\tilde{t} \in \tilde{T}$ be a lift of $t \in T$. From the general theory of Weyl groups it is known that there is

a unique $t^+ \in \mathcal{W}^G \tilde{T}$ such that $|\alpha(t^+)| \geq 1$ for all $\alpha \in R(\tilde{B}, \tilde{T}) = R(B, T)$. By (91)

$$H_*(\mathcal{B}_{\tilde{G}}^{\tilde{t}}, \mathbb{C}) \cong H_*(\mathcal{B}_{\tilde{G}}^{t^+}, \mathbb{C})$$

as $\mathcal{H}(\tilde{G})$-modules. These $t^+, \tilde{B}$ fulfill [54, Lemma 2.8.1], so by [54, Proposition 2.8.2]

$$(103) \qquad M_{\tilde{t}, \tilde{x}=1, \tilde{\rho}=\mathrm{triv}} = H_*(\mathcal{B}_{\tilde{G}}^{t^+}, \mathbb{C}) \cong \mathrm{Ind}_{\mathcal{O}(\tilde{T})}^{\mathcal{H}(\tilde{G})} \mathbb{C}_{t^+}.$$

According to [53, (1.5)], which applies to t^+, the spherical vector p_{triv} generates $M_{\tilde{t}, 1, \mathrm{triv}}$. Therefore it cannot lie in any proper $\mathcal{H}(\tilde{G})$-submodule of $M_{\tilde{t}, 1, \mathrm{triv}}$ and represents a nonzero element of $\pi(\tilde{t}, 1, \mathrm{triv})$. We also note that the central character of $\pi(\tilde{t}, 1, \mathrm{triv})$ is that of $M_{\tilde{t}, 1, \mathrm{triv}}$, $\mathcal{W}^G \tilde{t} = \mathcal{W}^G t^+$.

Now we analyse this is an $\mathcal{H}(G)$-module. The group $R_{\tilde{t}, 1} = R_{\tilde{t}, \tilde{x}=1, \tilde{\rho}=\mathrm{triv}}$ from (93) is just the component group $\pi_0(\mathrm{Z}_G(t))$, so by (95)

$$\pi(\tilde{t}, 1, \mathrm{triv}) \cong \bigoplus_\rho \mathrm{Hom}_{\pi_0(\mathrm{Z}_G(t))}(\rho, \pi(\tilde{t}, 1, \mathrm{triv})) = \bigoplus_\rho \pi(t, 1, \mathrm{triv}).$$

The sum runs over $\mathbf{Irr}(\pi_0(\mathrm{Z}_G(t)))$, all these representations ρ contribute nontrivially by [54, Lemma 3.5.1]. Recall from Lemma 14.2 that $\pi_0(\mathrm{Z}_G(t))$ can be realized as a subgroup of $\mathcal{W}^G$ and from (102) that $p_{\mathrm{triv}} \in \pi(\tilde{t}, 1, \mathrm{triv})$ can be regarded as a function on $\tilde{\mathcal{G}}$ which is bi-invariant under a good maximal compact subgroup $\tilde{\mathcal{K}}$. This brings us in the setting of [25, Proposition 4.1], which says that $\pi_0(\mathrm{Z}_G(t))$ fixes $p_{\mathrm{triv}} \in \pi(\tilde{t}, 1, \mathrm{triv})$. Hence $\pi(t, 1, \mathrm{triv})$ contains p_{triv} and is a spherical $\mathcal{H}(G)$-module. Its central character is the restriction of the central character of $\pi(\tilde{t}, 1, \mathrm{triv})$, that is, $\mathcal{W}^G t \in T/\mathcal{W}^G$.

Now we include Γ. Suppose that V is a irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-module. By Clifford theory its restriction to $\mathcal{H}(G)$ is a direct sum of irreducible $\mathcal{H}(G)$-modules, each of which contains p_{triv}. Hence V is built from irreducible spherical $\mathcal{H}(G)$-modules. By (97)

$$\gamma \cdot \pi(t, 1, \mathrm{triv}) = \pi(\gamma t, 1, \mathrm{triv}),$$

so the stabilizer of $\pi(t, 1, \mathrm{triv}) \in \mathbf{Irr}(\mathcal{H}(G))$ in Γ equals the stabilizer of $\mathcal{W}^G t \in T/\mathcal{W}^G$ in Γ. Any isomorphism of $\mathcal{H}(G)$-modules

$$\psi_\gamma : \pi(t, 1, \mathrm{triv}) \to \pi(\gamma t, 1, \mathrm{triv})$$

must restrict to a bijection between the onedimensional subspaces of spherical vectors in both modules. We normalize ψ_γ by $\psi_\gamma(p_{\mathrm{triv}}) = p_{\mathrm{triv}}$. Then $\gamma \mapsto \psi_\gamma$ is multiplicative, so the 2-cocycle of $\Gamma_{\mathcal{W}^G t}$ is trivial. With Theorem 20.1 this means that the irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules whose restriction to $\mathcal{H}(G)$ is spherical are parametrized by equivalence classes of triples $(t, 1, \mathrm{triv} \rtimes \sigma)$ with $\sigma \in \mathbf{Irr}(\Gamma_{\mathcal{W}^G t})$. The corresponding module is

$$\pi(t, 1, \mathrm{triv} \rtimes \sigma) = \pi(t, 1, \mathrm{triv}) \rtimes \sigma = \mathrm{Ind}_{\mathcal{H}(G) \rtimes \Gamma_{\mathcal{W}^G t}}^{\mathcal{H}(G) \rtimes \Gamma} \big(\pi(t, 1, \mathrm{triv}) \otimes \sigma\big).$$

Clearly $\pi(t, 1, \mathrm{triv} \rtimes \sigma)$ contains the spherical vector $p_{\mathrm{triv}} p_\Gamma$ if and only if σ is the trivial representation. It follows that the irreducible spherical $\mathcal{H}(G) \rtimes \Gamma$-modules are parametrized by equivalence classes of triples $\big(t, 1, \mathrm{triv}_{\pi_0(\mathrm{Z}_{G \rtimes \Gamma}(t))}\big)$, that is, by $T/(\mathcal{W}^G \rtimes \Gamma)$. $\qquad \square$

22. Main result (the case of a connected endoscopic group)

Let χ be a smooth character of the maximal torus $\mathcal{T} \subset \mathcal{G}$. We recall that

$$\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}},$$
$$c^{\mathfrak{s}} = \hat{\chi}\big|_{\mathfrak{o}_F^{\times}},$$
$$H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}}),$$
$$W^{\mathfrak{s}} = \mathrm{Z}_{\mathcal{W}^G}(\mathrm{im}\, c^{\mathfrak{s}}).$$

Let $\{$KLR parameters$\}^{\mathfrak{s}}$ be the collection of Kazhdan–Lusztig–Reeder parameters for G such that $\Phi\big|_{\mathfrak{o}_F^{\times}} = c^{\mathfrak{s}}$. Notice that the condition forces $\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})) \subset H$. This collection is not closed under conjugation by elements of G, only $H = \mathrm{Z}_G(\mathrm{im}\, c^{\mathfrak{s}})$ acts naturally on it.

The Bernstein centre associated to $\mathfrak{s}$ is $T^{\mathfrak{s}}/W^{\mathfrak{s}}$. Since $T = T^{\mathfrak{s}}$ is a maximal torus in H, we can identify $T^{\mathfrak{s}}/W^{\mathfrak{s}}$ with the space $c(H)_{\mathrm{ss}}$ of semisimple conjugacy classes in H.

Roche [56] proved that $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is naturally in bijection with $\mathbf{Irr}(\mathcal{H}(H))$, under some restrictions on the residual characteristic p. Although Roche works with a local non-archimedean field F of characteristic 0, it follows from [1] that his arguments apply just as well over local fields of positive characteristic. We note that for unramified characters χ this result is already classical, proven (without any restrictions on p) by Borel [17].

In the current section we will prove the most important part of our conjecture in the case that H is connected. This happens for most $\mathfrak{s}$, a sufficient condition is:

Lemma 22.1. *Suppose that G has simply connected derived group and that the residual characteristic p satisfies the hypothesis in [56, p. 379]. Then H is connected.*

Proof. We consider first the case where $\mathfrak{s} = [\mathcal{T}, 1]_{\mathcal{G}}$. Then we have $c^{\mathfrak{s}} = 1, H = G$ and $W^{\mathfrak{s}} = \mathcal{W}$.

We assume now that $c^{\mathfrak{s}} \neq 1$. Then $\mathrm{im}\, c^{\mathfrak{s}}$ is a finite abelian subgroup of T which has the following structure: the direct product of a finite abelian p-group A_p with a cyclic group B_{q-1} whose order divides $q-1$. This follows from the well-known structure theorem for the group $\mathfrak{o}_F^{\times}$, see [39, §2.2]:

$$\mathrm{im}\, c^{\mathfrak{s}} = A_p \cdot B_{q-1}.$$

We have

$$H = \mathrm{Z}_{H_A}(B_{q-1}) \quad \text{where} \quad H_A := \mathrm{Z}_G(A_p).$$

Since G has simply connected derived group, it follows from Steinberg's connectedness theorem [63] that the group H_A is connected. Since A_p is a p-group, and p is not a torsion prime for the root system $R(G,T)$, the derived group of H_A is simply connected (see [64]).

Now B_{q-1} is cyclic. Applying Steinberg's connectedness theorem to the group H_A, we get that H itself is connected. $\square$

Remark 22.2. *Notice that H does not necessarily have simply connected derived group in setting of Lemma 22.1. For instance, if G is the exceptional group of type G_2 and χ is the tensor square of a ramified quadratic character of $F^{\times}$, then $H = \mathrm{SO}(4, \mathbb{C})$.*

In the remainder of this section we will assume that H is connected, Then Lemma 14.2 shows that $W^{\mathfrak{s}}$ is the Weyl group of H.

Theorem 22.3. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. In the case when $H \neq G$, assume that H is connected and that the residual characteristic p satisfies the conditions of [56, Remark 4.13]. Then there is a commutative triangle of natural bijections*

$$
\begin{array}{ccc}
 & (T//W^{\mathfrak{s}})_2 & \\
 & \swarrow \qquad \searrow & \\
\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} & \longrightarrow & \{\text{KLR parameters}\}^{\mathfrak{s}}/H
\end{array}
$$

In this triangle, the right slanted map stems from the affine Springer correspondence, the bottom horizontal map is the bijection established by Reeder [54], and the left slanted map can be constructed via the asymptotic algebra of Lusztig.

Proof. The right slanted map is the composition of Theorem 19.2.1 (applied to H) and Lemma 18.1 (with the condition $\Phi(\varpi_F) = t$). Since $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \cong \mathbf{Irr}(\mathcal{H}(H))$, we can take as the horizontal map the parametrization of irreducible $\mathcal{H}(H)$-modules by Kazhdan, Lusztig and Reeder as described in Section 20. These are both natural bijections, so there is a unique left slanted map which makes the diagram commute, and it is also natural. We want to identify it in terms of Hecke algebras.

Fix a KLR-parameter (Φ, ρ) and recall from Theorem 19.2.2 that the corresponding $X^*(T) \rtimes \mathcal{W}^H$-representation is

$$
(104) \qquad \mathrm{Hom}_{\pi_0(Z_H(t,x))}\big(\rho, H_{d(x)}(\mathcal{B}_H^{t,x}, \mathbb{C})\big).
$$

Similarly, by Theorem 20.1 the corresponding $\mathcal{H}(H)$-module is the unique irreducible quotient of the $\mathcal{H}(H)$-module

$$
(105) \qquad \mathrm{Hom}_{\pi_0(Z_H(t_q,x))}\big(\rho_q, H_*(\mathcal{B}_H^{t_q,x}, \mathbb{C})\big).
$$

In view of Proposition 17.1 both spaces are unchanged if we replace t by t_q and ρ by ρ_q, and the vector space (105) is also naturally isomorphic to

$$
(106) \qquad \mathrm{Hom}_{\pi_0(Z_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t,\Phi(B_2)}, \mathbb{C})\big).
$$

Recall Lusztig's asymptotic Hecke algebra $\mathcal{J}(H)$ from [46]. As discussed in Corollary A.3 in Example 3 of the Appendix, there are canonical bijections

$$
(107) \qquad \mathbf{Irr}(\mathcal{H}(H)) \longleftrightarrow \mathbf{Irr}(\mathcal{J}(H)) \longleftrightarrow \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H).
$$

According to [47, Theorem 4.2] $\mathbf{Irr}(\mathcal{J}(H))$ is naturally parametrized by the set of H-conjugacy classes of Kazhdan–Lusztig parameters for H. Lusztig describes the $\mathcal{J}(H)$-module associated to (t_q, x, ρ_q) in terms of equivariant K-theory, and with [27, Section 6.2] we see that its retraction to $\mathcal{H}(H)$ via

$$
\mathcal{H}(H) \xrightarrow{\phi_q} \mathcal{J}(H) \xleftarrow{\phi_1} X^*(T) \rtimes \mathcal{W}^H
$$

is none other than (105). In [46, Corollary 3.6] the a-function is used to single out a particular irreducible quotient $\mathcal{H}(H)$-module of (105). But

we saw in (94) that there is only one such quotient, which by definition is $\pi(t_q, x, \rho_q)$.

Let $\mathcal{H}_q(H)$ be the affine Hecke algebra with the same based root datum as H and with parameter $q \in \mathbb{C}^\times$. Thus

$$\mathcal{H}_q(H) = \mathcal{H}(H) \quad \text{and} \quad \mathcal{H}_1(H) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}^H].$$

The above describes the retraction of an irreducible $\mathcal{J}(H)$-module corresponding to (Φ, ρ) to $\mathcal{H}_q(H)$ for any $q \in \mathbb{C}^\times$ which is not a root of unity. But everything depends algebraically on q, so the description is valid for all $q \in \mathbb{C}^\times$, in particular for $q = 1$. Then [27, Section 8.2] implies that we obtain the $\mathcal{H}_1(H)$-module

$$(108) \qquad \qquad \operatorname{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho, H_*(\mathcal{B}_H^{t,x}, \mathbb{C})\big)$$

with the action as in (81). The right bijection in (107) sends (105) to a certain irreducible quotient of (108) (namely the unique one with minimal a-weight).

For the opposite direction, consider an irreducible $\mathcal{H}_1(H)$-module M with a-weight a_M. According to [46, Corollary 3.6] the $\mathcal{J}(H)$-module

$$\tilde{M} := \mathcal{H}_1(H)^{a_M} \otimes_{\mathcal{H}_1(H)} M,$$

is irreducible and has a-weight a_M. See [46, Lemma 1.9] for the precise definition of $\tilde{M}$.

Now we fix $t \in T$ and we will prove with induction to $\dim \mathcal{O}_x$ that $\widetilde{\tau(t, x, \rho)}$ is none other than (108). Our main tool is Lemma 19.3, which says that the constituents of (108) are $\tau(t, x, \rho)$ and irreducible representations corresponding to larger affine Springer parameters (with respect to the partial order defined via the unipotent classes $\mathcal{O}_x \subset M$). For $\dim \mathcal{O}_{x_0} = 0$ we see immediately that only the $\mathcal{J}^\mathfrak{s}$-module

$$\operatorname{Hom}_{\pi_0(\mathrm{Z}_H(t,x_0))}\big(\rho_0, H_*(\mathcal{B}_H^{t,x}, \mathbb{C})\big)$$

can contain $\tau(t, x_0, \rho_0)$, so that must be $\widetilde{\tau(t, x_0, \rho_0)}$. For $\dim \mathcal{O}_{x_n} = n$ Lemma 19.3 says that (108) can only contain $\tau(t, x_n, \rho_n)$ if $x \in \overline{\mathcal{O}_{x_n}}$. But when $\dim \mathcal{O}_x < n$

$$\widetilde{\tau(t, x_n, \rho_n)} \not\cong \operatorname{Hom}_{\pi_0(\mathrm{Z}_H(t,x))}\big(\rho, H_*(\mathcal{B}_H^{t,x}, \mathbb{C})\big),$$

because the right hand side already is $\widetilde{\tau(t, x, \rho)}$, by the induction hypothesis and the bijectivity of $M \mapsto \tilde{M}$. So the parameter of $\widetilde{\tau(t, x_n, \rho_n)}$ involves an x with $\dim \mathcal{O}_x = n$. Then another look at Lemma 19.3 shows that moreover (x, ρ) must be M-conjugate to (x_n, ρ_n). Hence $\widetilde{\tau(t, x, \rho)}$ is indeed (108).

We showed that the bijections (107) work out as

$$\begin{array}{ccccc}
\mathbf{Irr}(\mathcal{H}(H)) & \leftrightarrow & \mathbf{Irr}(\mathcal{J}(H)) & \leftrightarrow & \mathbf{Irr}(X^*(T) \rtimes \mathcal{W}^H) \\
\pi(t_q, x, \rho_q) & \leftrightarrow & \operatorname{Hom}_{\pi_0(\mathrm{Z}_H(\Phi))}\big(\rho, H_*(\mathcal{B}_H^{t, \Phi(B_2)}, \mathbb{C})\big) & \leftrightarrow & \tau(t, x, \rho),
\end{array}$$

where all the objects in the bottom line are determined by the KLR parameter (Φ, ρ). $\qquad\square$

23. MAIN RESULT (HECKE ALGEBRA VERSION)

In this section $q \in \mathbb{C}^\times$ is allowed to be any element which is not a root of unity. We study how the conjecture can be extended to the algebras and modules from Section 20. So let Γ be a group of automorphisms of G that preserves a chosen pinning, which involves T as maximal torus. With the disconnected group $G \rtimes \Gamma$ we associate three kinds of parameters:

- The extended quotient of the second kind $(T /\!/ \mathcal{W}^G \rtimes \Gamma)_2$.
- The space $\mathbf{Irr}(\mathcal{H}(G) \rtimes \Gamma)$ of equivalence classes of irreducible representations of the algebra $\mathcal{H}(G) \rtimes \Gamma$ (with parameter q).
- Equivalence classes of unramified Kazhdan–Lusztig–Reeder parameters. Let $\Phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G$ be a group homomorphism with $\Phi(\mathbf{I}_F) = 1$ and $\Phi(\mathbf{W}_F) \subset T$. As in Section 17, the component group

$$\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi)) = \pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi(\mathbf{W}_F \times B_2)))$$

acts on $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C})$. We take $\rho \in \mathbf{Irr}\big(\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi))\big)$ such that every irreducible $\pi_0(\mathrm{Z}_G(\Phi))$-subrepresentation of ρ appears in $H_*(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C})$. The set $\{$KLR parameters for $G \rtimes \Gamma\}^{\mathrm{unr}}$ of pairs (Φ, ρ) carries an action of $G \rtimes \Gamma$ by conjugation. We consider the collection $\{$KLR parameters for $G \rtimes \Gamma\}^{\mathrm{unr}}/G \rtimes \Gamma$ of conjugacy classes $[\Phi, \rho]_{G\rtimes\Gamma}$.

Theorem 23.1. *There exists a commutative diagram of natural bijections*

$$(T /\!/ \mathcal{W}^G \rtimes \Gamma)_2$$

$$\mathbf{Irr}(\mathcal{H}(G) \rtimes \Gamma) \longrightarrow \{\text{KLR parameters for } G \rtimes \Gamma\}^{\mathrm{unr}}/G \rtimes \Gamma$$

It restricts to bijections between the following subsets:

- *the ordinary quotient $T/(\mathcal{W}^G \rtimes \Gamma) \subset (T /\!/ \mathcal{W}^G \rtimes \Gamma)_2$,*
- *the collection of spherical representations in $\mathbf{Irr}(\mathcal{H}(G) \rtimes \Gamma)$,*
- *equivalence classes of KLR parameters (Φ, ρ) for $G \rtimes \Gamma$ with $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ and $\rho = \mathrm{triv}_{\pi_0(\mathrm{Z}_{G\rtimes\Gamma}(\Phi))}$.*

Proof. The corresponding statement for G, proven in Theorem 22.3, is the existence of natural bijections

$$(109) \qquad\qquad (T /\!/ \mathcal{W}^G)_2$$

$$\mathbf{Irr}(\mathcal{H}(G)) \longrightarrow \{\text{KLR parameters for } G\}^{\mathrm{unr}}/G$$

Although in Section 22 q was a prime power, we notice that the upper and right objects in (109) do not depend on q. The algebra $\mathcal{H}(G)$ does, but the bottom and left slanted maps in (109) are defined equally well for our more general $q \in \mathbb{C}^\times$, as can be seen from the proofs of Theorems 20.1 and 22.3. Thus we may use (109) as our starting point.

Step 1. The bijections in (109) *are Γ-equivariant.*
The action of Γ on $(T /\!/ \mathcal{W}^G)_2$ can be written as

$$(110) \qquad\qquad \gamma \cdot [t, \tilde{\tau}]_{\mathcal{W}^G} = [\gamma(t), \tilde{\tau} \circ \mathrm{Ad}_\gamma^{-1}]_{\mathcal{W}^G}.$$

In terms of the multiplication in $G \rtimes \Gamma$, the action on KLR parameters is

$$(111) \qquad\qquad \gamma \cdot [\Phi, \rho_1]_G = [\gamma \Phi \gamma^{-1}, \rho_1 \circ \mathrm{Ad}_\gamma^{-1}]_G$$

We recall the right slanted map in (109) from Theorem 19.2. Write $M = \mathrm{Z}_G(t)$ and $\mathcal{W}_t^G = W(M^\circ, T) \rtimes \pi_0(M)$. Then the $\mathcal{W}_t^G$-representation $\tilde{\tau}$ can be written as $\tau(x, \rho_3) \rtimes \sigma$ for a unipotent element $x \in M^\circ$, a geometric $\rho_3 \in \mathrm{Irr}(\mathrm{Z}_{M^\circ}(x))$ and a $\sigma \in \mathrm{Irr}(\pi_0(M)_{\tau(x,\rho_3)})$. The associated KLR parameter is $[\Phi, \rho_3 \rtimes \sigma]_G$, where $\Phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = x$ and Φ maps a Frobenius element of $\mathbf{W}_F$ to t.

From (56) we see that $\tau(x, \rho_3) \circ \mathrm{Ad}_\gamma^{-1}$ is equivalent with $\tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1})$, so

$$\tilde{\tau} \circ \mathrm{Ad}_\gamma^{-1} \text{ is equivalent with } \tau(\gamma x \gamma^{-1}, \rho_3 \circ \mathrm{Ad}_\gamma^{-1}) \rtimes (\sigma \circ \mathrm{Ad}_\gamma^{-1}).$$

Hence (110) is sent to the KLR parameter (111), which means that the right slanted map in (109) is indeed Γ-equivariant.

In view of Proposition 17.1 and (111), we already showed in (97) that the horizontal map in (109) is Γ-equivariant. By the commutativity of the triangle, so is the left slanted map.

Step 2. Suppose that $\pi, [t, \tilde{\tau}]_{\mathcal{W}^G}$ and $[\Phi, \rho_1]_G$ are three corresponding objects in (109). Then their stabilizers in Γ coincide:

$$\Gamma_\pi = \Gamma_{[t,\tilde{\tau}]_{\mathcal{W}^G}} = \Gamma_{[\Phi,\rho_1]_G}.$$

This follows immediately from step 1.

Step 3. Clifford theory produces 2-cocycles $\natural(\pi)$, $\natural([t, \tilde{\tau}]_{\mathcal{W}^G})$ and $\natural([\Phi, \rho_1]_G)$. These are in the same class in $H^2(\Gamma_\pi, \mathbb{C}^\times)$.

For $\natural(\pi)$ and $\natural([\Phi, \rho_1]_G)$ this was already checked in (99), where we use Proposition 17.1 to translate between Φ and (t_q, x). Comparing (94) and Theorem 19.2, we see that $\natural(\pi)$ and $\natural([t, \tilde{\tau}]_{\mathcal{W}^G})$ come from two very similar representations: the difference is that $M(t, x, \rho_1)$ is built the entire homology of a variety, while the corresponding $X^*(T) \rtimes \mathcal{W}^G$-representation uses only the homology in top degree. Also the Γ_π-actions on these modules are defined in the same way, so the two cocycles can be chosen equal.

Step 4. Upon applying $X \mapsto (X//\Gamma)_2^\natural$ to the commutative diagram (109) we obtain the corresponding diagram for $G \rtimes \Gamma$.

Here $\natural$ denotes the family of 2-cocycles constructed in steps 2 and 3. For $(T//\mathcal{W}^G)_2$ and $\mathrm{Irr}(\mathcal{H}(G))$ we know from Lemmas 13.1 and 13.3 that this procedure yields the correct parameters. That it works for Kazhdan–Lusztig–Reeder parameters was checked in the last part of the proof of Theorem 20.1. Since we have the same families of 2-cocycles on all three Γ-sets, the maps from (109) can be lifted in a natural way to the diagram for $G \rtimes \Gamma$.

The ordinary quotient is embedded in $(T//\mathcal{W}^G \rtimes \Gamma)_2$ as the collection of pairs $(t, \mathrm{triv}_{(\mathcal{W}^G \rtimes \Gamma)_t})$. By an obvious generalization of (87) these correspond to the affine Springer parameters $(t, x = 1, \rho = \mathrm{triv})$. It is clear from the above construction that they are mapped to KLR parameters (Φ, triv) with $\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})) = 1$ and $\Phi(\varpi_F) = t$. By Proposition 21.1 the latter correspond to the spherical irreducible $\mathcal{H}(G) \rtimes \Gamma$-modules. $\qquad\square$

24. MAIN RESULT (GENERAL CASE)

We return to the notation from Section 22. In general the group $H = Z_G(\operatorname{im} c^{\mathfrak{s}})$ need not be connected. It is well-known, and already used several times in the proof of Proposition 15.2, that the short exact sequence

$$(112) \qquad 1 \to H^\circ/Z(H^\circ) \to H/Z(H^\circ) \to \pi_0(H) \to 1$$

is split. More precisely, any choice of a pinning of H° (a Borel subgroup, a maximal torus, and a nontrivial element in every root subgroup associated to a simple root) determines such a splitting. We fix a pinning with T as maximal torus, and with it we fix actions of $\pi_0(H)$ on H°, on the Dynkin diagram of H° and on the Weyl group of H°. Lemma 14.2 shows that

$$(113) \qquad W^{\mathfrak{s}} = \mathcal{W}^G_{\operatorname{im} c^{\mathfrak{s}}} \cong \mathcal{W}^{H^\circ} \rtimes \pi_0(H).$$

According to [56, Section 8] $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is naturally in bijection with $\mathbf{Irr}(\mathcal{H}(H))$, where

$$(114) \qquad \mathcal{H}(H) \cong \mathcal{H}(H^\circ) \rtimes \pi_0(H).$$

Now we have collected all the material that is needed to prove our main result (Theorem 1.3 in our Introduction).

Theorem 24.1. *Let $\mathcal{G}$ be a split reductive p-adic group and let $\mathfrak{s} = [\mathcal{T}, \chi]_\mathcal{G}$ be a point in the Bernstein spectrum of the principal series of $\mathcal{G}$. Assume that the residual characteristic p satisfies the conditions of [56, Remark 4.13] when $H \neq G$. Then there is a commutative triangle of natural bijections*

$$(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$$

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow \{\text{KLR parameters}\}^{\mathfrak{s}}/H$$

The slanted maps are generalizations of the slanted maps in Theorem 22.3 and the horizontal map stems from Theorem 20.1.

We denote the irreducible $\mathcal{G}$-representation associated to a KLR parameter (Φ, ρ) by $\pi(\Phi, \rho)$.

(1) *The infinitesimal central character of $\pi(\Phi, \rho)$ is the H-conjugacy class*
$$\Phi\left(\varpi_F, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\right) \in c(H)_{\mathrm{ss}} \cong T^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

(2) *$\pi(\Phi, \rho)$ is tempered if and only if $\Phi(\mathbf{W}_F)$ is bounded, which is the case if and only if $\Phi(\varpi_F)$ lies in a compact subgroup of H.*

(3) *$\pi(\Phi, \rho)$ is essentially square-integrable if and only if $\Phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of H°.*

Recall that $\mathcal{G}$ only has irreducible square-integrable representations if $Z(\mathcal{G})$ is compact. A $\mathcal{G}$-representation is called essentially square-integrable if its restriction to the derived group of $\mathcal{G}$ is square-integrable. This is more general than square-integrable modulo centre, because for that notion $Z(\mathcal{G})$ needs to act by a unitary character.

Proof. The larger part of the commutative triangle was already discussed in (113), (114) and Theorem 23.1. It remains to show that the set {KLR

parameters$\}^\mathfrak{s}/H$ (as defined on page 43) is naturally in bijection with $\{$KLR parameters for $H^\circ \rtimes \pi_0(H)\}^{\mathrm{unr}}/H^\circ \rtimes \pi_0(H)$.

By (112) we are taking conjugacy classes with respect to the group $H/Z(H^\circ)$ in both cases. It is clear from the definitions that that in both sets the ingredients Φ are determined by the semisimple element $\Phi(\varpi_F) \in H$. This provides the desired bijection between the Φ's in the two collections, so let us focus on the ingredients ρ.

For $(\Phi, \rho) \in \{$KLR parameters$\}^\mathfrak{s}$ the irreducible representation ρ of the component group $\pi_0(Z_H(\Phi)) = \pi_0(Z_G(\Phi))$ must appear in $H_*\big(\mathcal{B}_G^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. By Proposition 17.2.3 this space is isomorphic, as a $\pi_0(Z_G(\Phi))$-representation, to a number of copies of

$$\mathrm{Ind}_{\pi_0(Z_{H^\circ}(\Phi))}^{\pi_0(Z_G(\Phi))} H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big).$$

Hence the condition on ρ is equivalent to requiring that every irreducible $\pi_0(Z_{H^\circ}(\Phi))$-subrepresentation of ρ appears in $H_*\big(\mathcal{B}_{H^\circ}^{\Phi(\mathbf{W}_F \times B_2)}, \mathbb{C}\big)$. That is exactly the condition on ρ in an unramified KLR parameter for $H^\circ \rtimes \pi_0(H)$. This establishes the properties of the commutative diagram.

(1) From the construction in Section 20 we see that the $\mathcal{H}(H^\circ)$-module with Kazhdan–Lusztig parameter (t_q, x, ρ_q) has central character

$$t_q = \Phi\big(\varpi_F, \big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\big)\big) \in c(H^\circ)_{\mathrm{ss}} \cong T/\mathcal{W}^{H^\circ}.$$

It follows that the $\mathcal{H}(H)$-module with parameter (Φ, ρ) has central character $t_q \in c(H)_{\mathrm{ss}} \cong T^\mathfrak{s}/W^\mathfrak{s}$. The corresponding $\mathcal{G}$-representation is obtained via a suitable Morita equivalence, which by definition transforms the central character into the infinitesimal character.

(2) It was checked in [20] that a member of $\mathbf{Irr}(\mathcal{G})^\mathfrak{s}$ is tempered if and only if the corresponding $\mathcal{H}(H)$-module is tempered.

For $z \in \mathbb{C}^\times$ we put $V(z) = \log|z|$. According to [42, Theorem 8.2] the $\mathcal{H}(H)$-module with parameter (Φ, ρ) is V-tempered if and only if all the eigenvalues of $t = \Phi(\varpi_F)$ on $\mathrm{Lie}\, H$ (via the adjoint representation) have absolute value 1. That [42] works with simply connected complex groups is inessential to the argument, it also applies to our H. But V-tempered (for this V) means only that the restriction of the $\mathcal{H}(H)$-module to the subalgebra $\mathcal{H}(H_{\mathrm{der}}^\circ)$ is tempered, where H_{der}° denotes the derived group of H°. The $\mathcal{H}(H)$-module is tempered if and only if moreover the subalgebra $\mathcal{H}(Z(H^0))$ acts on it by a unitary character. This is the case if and only if all the eiqenvalues of t (in some realization of H^0 as complex matrices) have absolute value 1. That in turn means that t lies in the maximal compact subgroup of $T^\mathfrak{s}$.

Since $\Phi(\mathbf{W}_F)$ is generated by the finite group $\Phi(\mathbf{I}_F)$ and $t = \Phi(\varpi_F)$, the above condition on t is equivalent to boundedness of $\Phi(\mathbf{W}_F)$.

(3) This is similar to part 2, it follows from [42, Theorem 8.3] and [20]. $\square$

Notice that the bijectons in Theorem 24.1 satisfy the statements (1) – (4) from Section 7 by construction. We will check the properties 1–6, in the upcoming sections.

Recall that $\mathbf{Irr}(\mathcal{G}, \mathcal{T})$ is the space of all irreducible $\mathcal{G}$-representations in the principal series. Considering Theorem 24.1 for all Bernstein components

in the principal series simultaneously, we will establish the local Langlands correspondence for this class of representations.

Corollary 24.2. *Let $\mathcal{G}$ be a split reductive p-adic group, with restrictions on the residual characteristic as in [56, Remark 4.13]. Then the local Langlands correspondence holds for the irreducible representations in the principal series of $\mathcal{G}$, and it is the bottom row in a commutative triangles of natural bijections*

$$(\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2$$

$$\mathbf{Irr}(\mathcal{G},\mathcal{T}) \longrightarrow \{KLR\ parameters\ for\ G\}/G$$

The restriction of this diagram to a single Bernstein component recovers Theorem 24.1. In particular the bottom arrow generalizes the Kazhdan–Lusztig parametrization of the irreducible $\mathcal{G}$-representations in the unramified principal series.

Proof. Let us work out what happens if in Theorem 24.1 we take the union over all Bernstein components $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$.

On the left we obtain (by definition) the space $\mathbf{Irr}(\mathcal{G},\mathcal{T})$. Notice that in Theorem 24.1, instead of $\{$KLR parameters$\}^{\mathfrak{s}}/H$ we could just as well take G-conjugacy classes of KLR parameters (Φ,ρ) such that $\Phi\big|_{\mathbf{I}_F}$ is G-conjugate to $c^{\mathfrak{s}}$. The union of those clearly is the space of all G-conjugacy classes of KLR parameters for G. For the space at the top of the diagram, choose a smooth character $\chi_{\mathfrak{s}}$ of $\mathcal{T}$ such that $(\mathcal{T},\chi_{\mathfrak{s}}) \in \mathfrak{s}$. Recall from Section 12 that the T in $(T//W^{\mathfrak{s}})_2$ is actually

$$T^{\mathfrak{s}} := \{\chi_{\mathfrak{s}} \otimes t \mid t \in T\},$$

where t is considered as an unramified character of $\mathcal{T}$. On the other hand $\mathbf{Irr}\,\mathcal{T}$ can be obtained by picking representatives $\chi_{\mathfrak{s}}$ for $(\mathbf{Irr}\,\mathcal{T})/T$ and taking the union of the corresponding $T^{\mathfrak{s}}$. Two such spaces $T^{\mathfrak{s}}$ give rise to the same Bernstein component for $\mathcal{G}$ if and only if they are conjugate by an element of $N_{\mathcal{G}}(\mathcal{T})$, or equivalently by an element of $\mathcal{W}^G$. Therefore

$$(\mathbf{Irr}\,\mathcal{T}//\mathcal{W}^G)_2 = \Big(\bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \mathcal{W}^G \cdot T^{\mathfrak{s}}//\mathcal{W}^G\Big)_2 = \bigcup_{\mathfrak{s}\in\mathfrak{B}(\mathcal{G},\mathcal{T})} \big(T^{\mathfrak{s}}//W^{\mathfrak{s}}\big)_2.$$

Hence the union of the commutative triangles from Theorem 24.1 gives us the desired commutative triangle. $\square$

In [18, Section 10] Borel stated several "desiderata" for the local Langlands correspondence. The properties (1), (2) and (3) of Theorem 24.1 prove some of these, whereas the others involve representations outside the principal series and therefore fall outside the scope of our results.

25. The labelling by unipotent classes

Let $\mathfrak{s} \in \mathfrak{B}(\mathcal{G},\mathcal{T})$ and construct $c^{\mathfrak{s}}$ as in Section 16. By Theorem 24.1 we can parametrize $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ with H-conjugacy classes of KLR parameters (Φ,ρ)

such that $\Phi\big|_{\mathfrak{o}_F^\times} = c^{\mathfrak{s}}$. We note that {KLR parameters}$^{\mathfrak{s}}$ is naturally labelled by the unipotent classes in H:

(115) $\quad$ {KLR parameters}$^{\mathfrak{s},[x]} := \big\{ (\Phi, \rho) \mid \Phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \text{ is conjugate to } x \big\}.$

In this way we can associate to any of the parameters in Theorem 24.1 a unique unipotent class in H:

(116) $\qquad \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} = \bigcup_{[x]} \mathbf{Irr}(\mathcal{G})^{\mathfrak{s},[x]}, \quad (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2 = \bigcup_{[x]} (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2^{[x]}.$

Via the affine Springer correspondence from Section 19 the set of equivalence classes in {KLR parameters}$^{\mathfrak{s}}$ is naturally in bijection with $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$. Recall from Section 2 that

$$\widetilde{T^{\mathfrak{s}}} = \{ (w, t) \in W^{\mathfrak{s}} \times T^{\mathfrak{s}} \mid wt = t \}$$

and $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} = \widetilde{T^{\mathfrak{s}}}/W^{\mathfrak{s}}$. In view of Section 6 $(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is also in bijection with $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, albeit not naturally.

Only in special cases a canonical bijection $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_2$ is available. For example when $G = \mathrm{GL}_n(\mathbb{C})$, the finite group $\mathcal{W}_t^H$ is a product of symmetric groups: in this case there is a canonical c-$\mathbf{Irr}$ system, according to the classical theory of Young tableaux.

In general it can already be hard to define any suitable map from {KLR parameters}$^{\mathfrak{s}}$ to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$, because it is difficult to compare the parameters ρ for different Φ's. It goes better the other way round and with $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ as target. In this way will transfer the labellings (116) to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$.

From [56, Section 8] we know that $\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$ is naturally in bijection with the equivalence classes of irreducible representations of the extended affine Hecke algebra $\mathcal{H}(H)$. To relate it to $T^{\mathfrak{s}}/\!/W^{\mathfrak{s}}$ the parametrization of Kazhdan, Lusztig and Reeder is unsuitable, it is more convenient to use the methods developed in [51, 60].

To describe it, we fix some notation. Choose a Borel subgroup $B \subset H^\circ$ containing T. Let P be a set of roots of (H°, T) which are simple with respect to B and let R_P be the root system that they span. They determine a parabolic subgroup $W_P \subset W^{\mathfrak{s}}$ and a subtorus

$$T^P := \{ t \in T \mid \alpha(t) = 1 \ \forall \alpha \in R_P \}^\circ.$$

In [60, Theorem 3.3.2] $\mathbf{Irr}(\mathcal{H}(H))$ is mapped, in a natural finite-to-one way, to equivalence classes of triples (P, δ, t). Here P is as above, $t \in T^P$ and δ is a discrete series representation of a parabolic subalgebra $\mathcal{H}_P$ of $\mathcal{H}(H)$. This t is the same as in the affine Springer parameters.

The pair (P, δ) gives rise to a *residual coset* L in the sense of [51, Appendix A]. Explicitly, it is the translation of T^P by an element $cc(\delta) \in T$ that represents the central character of δ (a W_P-orbit in a subtorus $T_P \subset T$). The element $cc(\delta)t \in L$ corresponds to t_q. The collection of residual cosets is stable under the action of $W^{\mathfrak{s}}$.

Proposition 25.1. $\quad$ (1) *There is a natural bijection between*
- *H-conjugacy classes of Langlands parameters Φ with $\Phi\big|_{\mathbf{I}_F} = c^{\mathfrak{s}}$;*
- *$W^{\mathfrak{s}}$-conjugacy classes of pairs (t_q, L) with L a residual coset for $\mathcal{H}(H)$ and $t_q \in L$.*

(2) *Let Y^P be the union of the residual cosets of T^P. The stabilizer of Y^P in $W^{\mathfrak{s}}$ is the stabilizer of R_P.*

(3) *Suppose that $w \in W^{\mathfrak{s}}$ fixes $cc(\delta)$. Then w stabilizes R_P.*

Proof. (1) Opdam constructed the maps in both directions for $\mathcal{H}(H^\circ)$. To go from $\mathcal{H}(H^\circ)$ to $\mathcal{H}(H)$ is easy, one just has to divide out the action of $\pi_0(H)$ on both sides.

Let us describe the maps for H° more explicitly. To a residual coset L Opdam [51, Proposition B.3] associates a unipotent element $x \in B$ such that $lxl^{-1} = x^q$ for all $l \in L$. Then Φ is a Langlands parameter with data t_q, x.

For the opposite direction we may assume that

$$\Phi\big(\mathbf{W}_F, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big) \subset T \quad \forall z \in \mathbb{C}^\times$$

and that $x = \Phi\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big) \in B$. Then

$$T^P := \mathrm{Z}_T\big(\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}))\big)^\circ = \mathrm{Z}_T\big(\Phi(\mathrm{SL}_2(\mathbb{C}))\big)^\circ$$

is a maximal torus of $\mathrm{Z}_{H^\circ}\big(\Phi(\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}))\big)$. We take

$$t_q = \Phi\Big(\varpi_F, \Big(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\Big)\Big) \quad \text{and} \quad L = T^P t_q.$$

This is essentially [51, Proposition B.4], but our way to write it down avoids Opdam's assumption that H° is simply connected.

(2) Clear, because any element that stabilizes Y^P also stabilizes T^P.

(3) Since $cc(\delta)$ represents the central character of a discrete series representation of $\mathcal{H}_P$, at least one element (say r) in its W_P-orbit lies in the obtuse negative cone in the subtorus $T_P \subset T$ (see Lemma 2.21 and Section 4.1 of [51]). That is, $\log|r|$ can be written as $\sum_{\alpha \in P} c_\alpha \alpha^\vee$ with $c_\alpha < 0$. Some W_P-conjugate w' of $w \in W^{\mathfrak{s}}$ fixes r and hence $\log|r|$. But an element of $W^{\mathfrak{s}}$ can only fix $\log|r|$ if it stabilizes the collection of coroots $\{\alpha^\vee \mid \alpha \in P\}$. It follows that w' and w stabilize R_P. $\square$

In particular the above natural bijection associates to any $W^{\mathfrak{s}}$-conjugacy class of residual cosets L a unique unipotent class $[x]$ in H. Conversely a unipotent class $[x]$ can correspond to more than one $W^{\mathfrak{s}}$-conjugacy class of residual cosets, at most the number of connected components of $\mathrm{Z}_T(x)$ if $x \in B$.

Let $\mathfrak{U}^{\mathfrak{s}} \subset B$ be a set of representatives for the unipotent classes in H. For every $x \in \mathfrak{U}^{\mathfrak{s}}$ we choose an algebraic homomorphism

$$\gamma_x \colon \mathrm{SL}_2(\mathbb{C}) \to H \quad \text{with} \quad \gamma_x\big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big) = x \quad \text{and} \quad \gamma_x\big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big) \in T.$$

By Lemma 16.2 all choices for γ_x are conjugate. For each $x \in \mathfrak{U}^{\mathfrak{s}}$ we define

$$\{\text{KLR parameters}\}^{\mathfrak{s},x} = \{(\Phi, \rho) \mid \Phi\big|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})} = c^{\mathfrak{s}} \times \gamma_x, \Phi(\varpi_F) \in T\}.$$

We endow this set with the topology that ignores ρ and is the Zariski topology with respect to $\Phi(\varpi_F)$. In this way

$$(117) \qquad\qquad \bigsqcup_{x \in \mathfrak{U}^{\mathfrak{s}}} \{\text{KLR parameters}\}^{\mathfrak{s},x}$$

becomes a nonseparated algebraic variety with maximal separated quotient $\sqcup_{x\in\mathfrak{U}^{\mathfrak{s}}}Z_T(\operatorname{im}\gamma_x)$. Notice that (117) contains representatives for all equivalence classes in $\{$KLR parameters$\}^{\mathfrak{s}}$.

Proposition 25.2. *There exists a continuous bijection*

$$\mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

such that:

(1) *The diagram*

$$
\begin{array}{ccc}
T^{\mathfrak{s}}//W^{\mathfrak{s}} & \xrightarrow{\ \mu^{\mathfrak{s}}\ } \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \longrightarrow & \{\textit{KLR parameters}\}^{\mathfrak{s}}/H \\
\ \downarrow{\scriptstyle \rho^{\mathfrak{s}}} & & \downarrow \\
T^{\mathfrak{s}}/W^{\mathfrak{s}} & \longleftarrow & c(H)_{\mathrm{ss}}
\end{array}
$$

commutes. Here the unnamed horizontal maps are those from Theorem 24.1 and the right vertical arrow sends (Φ,ρ) to the H-conjugacy class of $\Phi(\varpi_F)$.

(2) *For every unipotent element $x \in H$ the preimage*

$$(T^{\mathfrak{s}}//W^{\mathfrak{s}})^{[x]} := (\mu^{\mathfrak{s}})^{-1}\big(\mathbf{Irr}(\mathcal{G})^{\mathfrak{s},[x]}\big)$$

is a union of connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$.

(3) *Let ϵ be the map that makes the diagram*

$$
\begin{array}{ccc}
T^{\mathfrak{s}}//W^{\mathfrak{s}} & \xrightarrow{\quad\quad\epsilon\quad\quad} & (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2 \\
& {\scriptstyle \mu^{\mathfrak{s}}}\searrow\quad \swarrow & \\
& \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} &
\end{array}
$$

commute. Then ϵ comes from a c-$\mathbf{Irr}$ system.

(4) $T^{\mathfrak{s}}//W^{\mathfrak{s}} \xrightarrow{\ \mu^{\mathfrak{s}}\ } \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \to \{\textit{KLR parameters}\}^{\mathfrak{s}}/H$ *lifts to a map*

$$\widetilde{\mu^{\mathfrak{s}}} : \widetilde{T^{\mathfrak{s}}} \to \bigsqcup_{x\in\mathfrak{U}^{\mathfrak{s}}} \{\textit{KLR parameters}\}^{\mathfrak{s},x}$$

such that the restriction of $\widetilde{\mu^{\mathfrak{s}}}$ to any connected component of $\widetilde{T^{\mathfrak{s}}}$ is algebraic and an isomorphism onto its image.

Proof. Proposition 25.1.1 yields a natural finite-to-one map from $\mathbf{Irr}(\mathcal{H}(H))$ to $W^{\mathfrak{s}}$-conjugacy classes (t_q, L), namely

$$(118) \qquad\qquad \pi(\Phi,\rho) \mapsto \Phi \mapsto (t_q, L).$$

In [60, Theorem 3.3.2] this map was obtained in a different way, which shows better how the representations depend on the parameters t, t_q, L. That was used in [60, Section 5.4] to find a continuous bijection

$$(119) \qquad\qquad \mu^{\mathfrak{s}} : T^{\mathfrak{s}}//W^{\mathfrak{s}} \to \mathbf{Irr}(\mathcal{H}(H)) \cong \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}.$$

The strategy is essentially a step-by-step creation of a c-$\mathbf{Irr}$ system for $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ and $\mathcal{H}(H)$, only the condition on the unit element and the trivial representation is not considered in [60]. Fortunately there is some freedom left, which we can exploit to ensure that $\mu^{\mathfrak{s}}(1, T^{\mathfrak{s}})$ is the family of spherical $\mathcal{H}(H)$-representations, see Section 21. This is possible because every

principal series representation of $\mathcal{H}(H)$ has a unique irreducible spherical subquotient, so choosing that for $\mu^{\mathfrak{s}}(1,t)$ does not interfere with the rest of the construction. Via Theorem 24.1 we can transfer this c-**Irr** system to a c-**Irr** system for the two extended quotients of $T^{\mathfrak{s}}$ by $W^{\mathfrak{s}}$, so property (3) holds.

By construction the triple (P, δ, t) associated to the representation $\mu^{\mathfrak{s}}(w, t)$ has the same $t \in T$, modulo $W^{\mathfrak{s}}$. That is, property (1) is fulfilled.

Furthermore $\mu^{\mathfrak{s}}$ sends every connected component of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ to a family of representations with common parameters (P, δ). Hence these representation are associated to a common residual coset L and to a common unipotent class $[x]$, which verifies property (2).

Let $\mathbf{c}$ be a connected component of $(T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}})^{[x]}$, with $x \in \mathfrak{U}^{\mathfrak{s}}$. The proof of Proposition 25.1.1 shows that $\mathbf{c}$ can be realized in $\mathrm{Z}_T(\mathrm{im}\, \gamma_x)$. In other words, we can find a suitable $w = w(\mathbf{c}) \in W^{\mathfrak{s}}$ with $T^w \subset \mathrm{Z}_T(\mathrm{im}\, \gamma_x)$. Then there is a connected component $T^w_{\mathbf{c}}$ of T^w such that

$$\mathbf{c} := \big(w, T^w_{\mathbf{c}}/\mathrm{Z}(w, \mathbf{c})\big),$$
$$\text{where } \mathrm{Z}(w, \mathbf{c}) = \{g \in \mathrm{Z}_{W^{\mathfrak{s}}}(w) \mid g \cdot T^w_{\mathbf{c}} = T^w_{\mathbf{c}}\}.$$

In this notation $\widetilde{\mathbf{c}} := (w, T^w_{\mathbf{c}})$ is a connected component of $\widetilde{T^{\mathfrak{s}}}$. We want to define $\widetilde{\mu^{\mathfrak{s}}} : \widetilde{\mathbf{c}} \to \{\text{KLR parameters}\}^{\mathfrak{s},x}$. For every $[w, t] \in \mathbf{c}$, $\mu^{\mathfrak{s}}[w, t]$ determines an equivalence class in $\{\text{KLR parameters}\}^{\mathfrak{s},x}$. Any (Φ, ρ) in this equivalence class satisfies $\Phi(\varpi_F) \in \mathrm{Z}_T(\mathrm{im}\, \gamma_x) \cap W^{\mathfrak{s}}t$. Hence there are only finitely many possibilities for $\Phi(\varpi_F)$, say $t_1, \ldots, t_k$. For every such t_i there is a unique $(\Phi_i, \rho_i) \in \{\text{KLR parameters}\}^{\mathfrak{s},x}$ with $\Phi_i(\varpi_F) = t_i$ and $\pi(\Phi_i, \rho_i) \cong \mu^{\mathfrak{s}}[w, t]$. Every element of $\widetilde{\mathbf{c}}$ lying over $[w, t] \in \mathbf{c}$ is of the form (w, t_i). (Not every t_i is eligible though, for some we would have to modify w.) We put

$$\widetilde{\mu^{\mathfrak{s}}}(w, t_i) := (\Phi_i, \rho_i).$$

Since the ρ's are irrelevant for the topology on $\{\text{KLR parameters}\}^{\mathfrak{s},x}$, $\widetilde{\mu^{\mathfrak{s}}}(\widetilde{\mathbf{c}})$ is homeomorphic to $T^w_{\mathbf{c}}$ and $\widetilde{\mu^{\mathfrak{s}}} : \widetilde{\mathbf{c}} \to \widetilde{\mu^{\mathfrak{s}}}(\widetilde{\mathbf{c}})$ is an isomorphism of affine varieties. This settles the final property (4).   □

26. CORRECTING COCHARACTERS AND L-PACKETS

In this section we construct the correcting cocharacters on the extended quotient $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$. As conjectured in Section 7, these show how to determine when two elements of $T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}}$ give rise to $\mathcal{G}$-representations in the same L-packets.

Every KLR parameter (Φ, ρ) naturally determines a cocharacter h_Φ and elements $\theta(\Phi, \rho, z) \in T^{\mathfrak{s}}$ by

$$(120) \qquad \begin{aligned} h_\Phi(z) &= \Phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big), \\ \theta(\Phi, \rho, z) &= \Phi\big(\varpi_F, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big) = \Phi(\varpi_F) h_\Phi(z). \end{aligned}$$

Although these formulas obviously do not depend on ρ, it turns out to be convenient to include it in the notation anyway. However, in this way we would end up with infinitely many correcting cocharacters, most of them with range outside T. To reduce to finitely many cocharacters with values in T, we will restrict to KLR parameters associated to $x \in \mathfrak{U}^{\mathfrak{s}}$ (117).

Recall that part (2) of Proposition 25.2 determines a labelling of the connected components of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ by unipotent classes in H. This enables us to define the correcting cocharacters: for a connected component $\mathbf{c}$ of $T^{\mathfrak{s}}//W^{\mathfrak{s}}$ with label (represented by) $x \in \mathfrak{U}^{\mathfrak{s}}$ we take the cocharacter

$$(121) \qquad h_{\mathbf{c}} = h_x : \mathbb{C}^\times \to T, \quad h_x(z) = \gamma_x \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix} \right).$$

Let $\widetilde{\mathbf{c}}$ be a connected component of $\widetilde{T^{\mathfrak{s}}}$ that projects onto $\mathbf{c}$. We define

$$(122) \qquad \begin{aligned} \widetilde{\theta}_z : \widetilde{\mathbf{c}} \to T^{\mathfrak{s}}, & \qquad (w,t) \mapsto \theta\big(\widetilde{\mu^{\mathfrak{s}}}(w,t), z\big), \\ \theta_z : \mathbf{c} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}, & \quad [w,t] \mapsto W^{\mathfrak{s}}\widetilde{\theta}_z(w,t). \end{aligned}$$

For $\widetilde{\mathbf{c}}$ as in the proof of Proposition 25.2, which we can always achieve by adjusting by element of $W^{\mathfrak{s}}$, our construction results in

$$\widetilde{\theta}_z(w,t) = t\, h_{\mathbf{c}}(z).$$

Lemma 26.1. *Let $[w,t], [w',t'] \in T^{\mathfrak{s}}//W^{\mathfrak{s}}$. Then $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet if and only if*

- *$[w,t]$ and $[w',t']$ are labelled by the same unipotent class in H;*
- *$\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$.*

Proof. Suppose that the two $\mathcal{G}$-representations $\mu^{\mathfrak{s}}[w,t] = \pi(\Phi,\rho)$ and $\mu^{\mathfrak{s}}[w',t'] = \pi(\Phi',\rho')$ belong to the same L-packet. By definition this means that Φ and Φ' are G-conjugate. Hence they are labelled by the same unipotent class, say $[x]$ with $x \in \mathfrak{U}^{\mathfrak{s}}$. By choosing suitable representatives we may assume that $\Phi = \Phi'$ and that $\{(\Phi,\rho), (\Phi,\rho')\} \subset \{\text{KLR parameters}\}^{\mathfrak{s},x}$. Then $\theta(\Phi,\rho,z) = \theta(\Phi,\rho',z)$ for all $z \in \mathbb{C}^\times$. Although in general $\theta(\Phi,\rho,z) \neq \widetilde{\theta}_z(w,t)$, they differ only by an element of $W^{\mathfrak{s}}$. Hence $\theta_z[w,t] = \theta_z[w',t']$ for all $z \in \mathbb{C}^\times$.

Conversely, suppose that $[w,t], [w',t']$ fulfill the two conditions of the lemma. Let $x \in \mathfrak{U}^{\mathfrak{s}}$ be the representative for the unipotent class which labels them. By Proposition 25.1.1 we may assume that $T^w \cup T^{w'} \subset \mathrm{Z}_T(\mathrm{im}\,\gamma_x)$. Then

$$\widetilde{\theta}_z[w,t] = t\, h_x(z) \quad \text{and} \quad \widetilde{\theta}_z[w',t'] = t'\, h_x(z)$$

are $W^{\mathfrak{s}}$ conjugate for all $z \in \mathbb{C}^\times$. As these points depend continuously on z and $W^{\mathfrak{s}}$ is finite, this implies that there exists a $v \in W^{\mathfrak{s}}$ such that

$$v(t\, h_x(z)) = t'\, h_x(z) \quad \text{for all } z \in \mathbb{C}^\times.$$

For $z = 1$ we obtain $v(t) = t'$, so v fixes $h_x(z)$ for all z. Via the Proposition 25.1.1, $h_x(q^{1/2})$ becomes an element $cc(\delta)$ for a residual coset L_x. By parts (2) and (3) of Proposition 25.1 v stabilizes the collection of residual cosets determined by x, namely the connected components of $\mathrm{Z}_T(\mathrm{im}\,\gamma_x)h_x(q^{1/2})$.

Let $(t_q, L), (t'_q, L')$ be associated to $\mu^{\mathfrak{s}}[w,t], \mu^{\mathfrak{s}}[w',t']$ by (118). Then $t_q = t h_x(q^{1/2})$ and $t'_q = t' h_x(q^{1/2})$, so the above applies. Hence v sends L to another residual coset determined by x. As $v(L)$ contains t'_q, it must be L'. Thus (t_q, L) and (t'_q, L') are $W^{\mathfrak{s}}$-conjugate, which by Proposition 25.1.1 implies that they correspond to conjugate Langlands parameters Φ and Φ'. So $\mu^{\mathfrak{s}}[w,t]$ and $\mu^{\mathfrak{s}}[w',t']$ are in the same L-packet. $\qquad\square$

Corollary 26.2. *Properties 1–6 from Section 7 hold in for $\mu^{\mathfrak{s}}$ as in Proposition 25.2, with the morphism θ_z from* (122) *and the labelling by unipotent classes in H.*

Together with Theorem 24.1 this proves the conjectures from Section 7 for all Bernstein components in the principal series of a split reductive p-adic group (with mild restrictions on the residual characteristic).

Proof. Property 1 follows from Theorem 24.1.2 and Proposition 25.2.1. The definitions of (121) and (122) establish property 5. The construction of θ_z, in combination with Theorem 24.1.1 and Proposition 25.2.1, shows that properties 2,3 and 4 are fulfilled. Property 6 is none other than Lemma 26.1. □

Appendix A. Geometric equivalence

Let X be a complex affine variety and let $k = \mathcal{O}(X)$ be its coordinate algebra. Equivalently, k is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. A k-algebra is an algebra A over the complex numbers which is a k-module (with an evident compatibility between the algebra structure of A and the k-module structure of A). A is of finite type if as a k-module A is finitely generated. This appendix will introduce — for finite type k-algebras — a weakening of Morita equivalence called *geometric equivalence.*

The new equivalence relation preserves the primitive ideal space (i.e. the set of isomorphism classes of simple A-modules) and the periodic cyclic homology. However, the new equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. The ABP conjecture (i.e. the conjecture stated in Part(1) of this paper) asserts that the finite type algebra which Bernstein constructs for any given Bernstein component of a reductive p-adic group is geometrically equivalent to the coordinate algebra of the associated extended quotient — and that the geometric equivalence can be chosen so that the resulting bijection between the Bernstein component and the extended quotient has properties as in the statement of ABP.

A.1. **k-algebras.** Let X be a complex affine variety and $k = \mathcal{O}(X)$ its coordinate algebra.

A k-*algebra* is a $\mathbb{C}$-algebra A such that A is a unital (left) k-module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda\omega)a \quad \forall(\lambda,\omega,a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1 a_2) = (\omega a_1)a_2 = a_1(\omega a_2) \quad \forall(\omega, a_1, a_2) \in k \times A \times A.$$

Denote the center of A by $\mathrm{Z}(A)$

$$\mathrm{Z}(A) := \{c \in A \mid ca = ac \; \forall a \in A\}$$

k-algebras are not required to be unital.
<u>Remark</u>.Let A be a unital k-algebra. Denote the unit of A by 1_A. $\omega \mapsto \omega 1_A$

is then a unital morphism of $\mathbb{C}$-algebras $k \to \mathrm{Z}(A)$. Thus a unital k-algebra is a unital $\mathbb{C}$-algebra A with a given unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow \mathrm{Z}(A).$$

Let A, B be two k-algebras. A morphism of k-algebras is a morphism of $\mathbb{C}$-algebras

$$f \colon A \to B$$

which is also a morphism of (left) k-modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$

Let A be a k-algebra. A representation of A [or a (left) A-module] is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V) \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that

(1) $k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital

and

(2) $(\omega a)v = \omega(av) = a(\omega v) \ \forall (\omega, a, v) \in k \times A \times V$.

Two representations

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_1, V_1) \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_1, V_1)$$

and

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_2) \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are equivalent (or isomorphic) if $\exists$ an isomorphism of $\mathbb{C}$ vector spaces $T \colon V_1 \to V_2$ which intertwines the two A-actions and the two k-actions.

Comment A.1. *A representation $\varphi \colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is* non-degenerate *iff $AV = V$. i.e. for any $v \in V$, $\exists \ v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with*

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r.$$

Comment A.2. <u>*Notation.*</u> *For economy of notation, a representation will be denoted*
$\varphi \colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. The unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

is understood to be included in the given structure.

A representation is irreducible if $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is not the zero map and $\nexists$ a sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \neq V$$

and

$$\omega w \in W \quad \forall \, (\omega, w) \in k \times W$$

and

$$aw \in W \quad \forall \, (a, w) \in A \times W$$

$\mathrm{Irr}(A)$ denotes the set of equivalence classes of irreducible representations of A.

A.2. Spectrum preserving morphisms of k-algebras. <u>Definition</u>. An ideal I in a k-algebra A is a *k-ideal* if $\omega a \in I \; \forall (\omega, a) \in k \times I$.

An ideal $I \subset A$ is *primitive* if $\exists$ an irreducible representation

$$\varphi \colon A \to \mathrm{Hom}_{\mathbb{C}}(V, V) \qquad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

with

$$I = \mathrm{Kernel}(\varphi)$$

That is,

$$0 \to I \hookrightarrow A \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{C}}(V, V)$$

is exact.

<u>Remark</u>. Any primitive ideal is a k-ideal. $\mathrm{Prim}(A)$ denotes the set of primitive ideals in A. The map $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ which sends an irreducible representation to its primitive ideal is a bijection if A is a finite type k-algebra. Since k is Noetherian, any k-ideal in a finite type k-algebra A is itself a finite type k-algebra.

<u>Definition</u>. Let A, B be two finite type k-algebras, and let $f \colon A \to B$ be a morphism of k-algebras. f is *spectrum preserving* if

(1) Given any primitive ideal $I \subset B$, $\exists$ a unique primitive ideal $L \subset A$ with $L \supset f^{-1}(I)$

and

(2) The resulting map

$$\mathrm{Prim}(B) \to \mathrm{Prim}(A)$$

is a bijection.

<u>Definition</u>. Let A, B be two finite type k-algebras, and let $f \colon A \to B$ be a morphism of k-algebras. f is *spectrum preserving with respect to filtrations* if $\exists$ k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \qquad \text{in } A$$

and k ideals

$$0 = L_0 \subset L_1 \subset \cdots \subset L_{r-1} \subset L_r = B \qquad \text{in } B$$

with $f(I_j) \subset L_j$, $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to L_j/L_{j-1}$, $(j = 1, 2, \ldots, r)$ is spectrum preserving.

A.3. Algebraic variation of k-structure. <u>Notation</u>. If A is a $\mathbb{C}$-algebra, $A[t, t^{-1}]$ is the $\mathbb{C}$-algebra of Laurent polynomials in the indeterminate t with coefficients in A.

<u>Definition</u>. Let A be a unital $\mathbb{C}$-algebra, and let

$$\Psi \colon k \to \mathrm{Z}\left(A[t, t^{-1}]\right)$$

be a unital morphism of $\mathbb{C}$-algebras. Note that $\mathrm{Z}\left(A[t, t^{-1}]\right) = \mathrm{Z}(A)[t, t^{-1}]$. For $\zeta \in \mathbb{C}^{\times} = \mathbb{C} - \{0\}$, $\mathrm{ev}(\zeta)$ denotes the "evaluation at ζ" map:

$$\mathrm{ev}(\zeta) \colon A[t, t^{-1}] \to A$$
$$\sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} \mathrm{Z}\left(A[t, t^{-1}]\right) \xrightarrow{\mathrm{ev}(\zeta)} \mathrm{Z}(A).$$

Denote the unital k-algebra so obtained by A_ζ. The underlying $\mathbb{C}$-algebra of A_ζ is A. Assume that for all $\zeta \in \mathbb{C}^\times$, A_ζ is a finite type k-algebra. Then for $\zeta, \zeta' \in \mathbb{C}^\times$, $A_{\zeta'}$ is obtained from A_ζ by an *algebraic variation of k-structure*.

A.4. **Definition and examples.** With k fixed, geometric equivalence (for finite type k-algebras) is the equivalence relation generated by the two elementary moves:

- Spectrum preserving morphism with respect to filtrations
- Algebraic variation of k-structure

Thus two finite type k-algebras A, B are *geometrically equivalent* if $\exists$ a finite sequence $A = A_0, A_1, \ldots, A_r = B$ with each A_j a finite type k-algebra such that for $j = 0, 1, \ldots, r-1$ one of the following three possibilities is valid:

(1) A_{j+1} is obtained from A_j by an algebraic variation of k-structure.
(2) There is a spectrum preserving morphism with respect to filtrations $A_j \to A_{j+1}$.
(3) There is a spectrum preserving morphism with respect to filtrations $A_{j+1} \to A_j$.

To give a geometric equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology are determined.

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad HP_*(A) \cong HP_*(B)$$

<u>Example 1.</u> Two unital finite type k-algebras A, B are *Morita equivalent* if there is an equivalence of categories

$$\left(\text{unital left } A\text{-modules}\right) \sim \left(\text{unital left } B\text{-modules}\right)$$

Any such equivalence of categories is implemented by a *Morita context* i.e. by a pair of unital bimodules

$$_AV_B \qquad _BW_A$$

together with given isomorphisms of bimodules

$$\alpha\colon V \otimes_B W \to A$$
$$\beta\colon W \otimes_A V \to B$$

A Morita context is required to satisfy certain conditions. These conditions imply that the linking algebra formed from the Morita context is a unital finite type k-algebra. The linking algebra, denoted $M_{2\times2}(_AV_B, {}_BW_A)$, is :

$$M_{2\times2}(_AV_B, {}_BW_A)$$

$$\|$$

A	V
W	B

The inclusions

$$A \hookrightarrow M_{2 \times 2}(_AV_B, _VW_A) \hookleftarrow B$$

$$a \mapsto \begin{array}{|c|c|} \hline a & 0 \\ \hline 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & b \\ \hline \end{array} \hookleftarrow b$$

are spectrum preserving morphisms of finite type k-algebras. Hence A and B are geometrically equivalent.

If $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are given the Jacobson topology, then the bijection $\mathrm{Prim}(A) \leftrightarrow \mathrm{Prim}(B)$ determined by a Morita equivalence is a homeomorphism.

Example 2. Let X be a complex affine variety, and let Y be a sub-variety. $k = \mathcal{O}(X)$ is the coordinate algebra of X, and $\mathcal{O}(Y)$ is the coordinate algebra of Y. $\mathcal{I}_Y$ is the ideal in $\mathcal{O}(X)$:

$$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \ \forall p \in Y\}$$

Set :

$$A = \begin{array}{|c|c|} \hline \mathcal{O}(X) & \mathcal{I}_Y \\ \hline \mathcal{I}_Y & \mathcal{O}(X) \\ \hline \end{array} \qquad B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$$

Thus A is the sub-algebra of the 2×2 matrices with entries in $\mathcal{O}(X)$ consisting of all those 2×2 matrices whose off-diagonal entries are in $\mathcal{I}_Y$. $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$ is the direct sum — as an algebra — of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$. Both A and B are unital finite type k-algebras. $\mathbf{Irr}(A) = \mathrm{Prim}(A)$ is X with each point of Y doubled. $\mathbf{Irr}(B) = \mathrm{Prim}(B)$ is the disjoint union of X and Y. Equipped with the Jacobson topology, $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are not homeomorphic so A and B are not Morita equivalent. However, A and B are geometrically equivalent.

Example 3. Let G be a connected reductive complex Lie group with maximal torus T. $\mathcal{W}$ denotes the Weyl group

$$\mathcal{W} = N_G(T)/T$$

and $X^*(T)$ is the character group of T. The semi-direct product $X^*(T) \rtimes \mathcal{W}$ is an affine Weyl group. In particular, it is a Coxeter group. We fix a system of generators $S \subset X^*(T) \rtimes \mathcal{W}$ such that $(X^*(T) \rtimes \mathcal{W}, S)$ is a Coxeter system. Let ℓ denote the resulting length function on $X^*(T) \rtimes \mathcal{W}$.

For each non-zero complex number q, there is the affine Hecke algebra $\mathcal{H}_q(G)$. This is an affine Hecke algebra with equal parameters and $\mathcal{H}_1(G)$ is the group algebra of the affine Weyl group:

$$\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}i,]$$

i.e., $\mathcal{H}_q$ is the algebra generated by T_x, $x \in X^*(T) \rtimes \mathcal{W}$, with relations

$$T_x T_y = T_{xy}, \quad \text{if } \ell(xy) = \ell(x) + \ell(y), \text{ and}$$

(123) $(T_s - \beta)(T_s + 1) = 0, \quad \text{if } s \in S.$

Using the action of $\mathcal{W}$ on T, form the quotient variety $T/\mathcal{W}$ and let k be its coordinate algebra,

$$k = \mathcal{O}(T/\mathcal{W})$$

For all $q \in \mathbb{C}^\times$, $\mathcal{H}_q(G)$ is a unital finite type k-algebra. Let $\mathcal{J}$ be Lusztig's asymptotic algebra. As a $\mathbb{C}$-vector space, $\mathcal{J}$ has a basis $\{t_w : w \in X^*(T) \rtimes \mathcal{W}\}$, and there is a canonical structure of associative $\mathbb{C}$-algebra on $\mathcal{J}$ (see for instance [48, § 8]).

Except for q in a finite set of roots of unity (none of which is 1) Lusztig constructs a morphism of k-algebras

$$\phi_q \colon \mathcal{H}_q(G) \longrightarrow \mathcal{J}$$

which is spectrum preserving with respect to filtrations (see [14, Theorem 9], itself based on [48]).

Let C_w, $w \in X^*(T) \rtimes \mathcal{W}$ denote the Kazhdan-Lusztig basis of $\mathcal{H}_q(G)$. For w, w', w'' in $X^*(T) \rtimes \mathcal{W}$, define $h_{w,w',w''} \in A$ by

$$C_w \cdot C_{w'} = \sum_{w'' \in X^*(T) \rtimes \mathcal{W}} h_{w,w',w''} \, C_{w''}.$$

There is a unique function $a \colon X^*(T) \rtimes \mathcal{W} \to \mathbb{N}$ such that for any $w'' \in X^*(T) \rtimes \mathcal{W}$, $v^{a(w'')} h_{w,w',w''}$ is a polynomial in v for all w, w' in $X^*(T) \rtimes \mathcal{W}$ and it has non-zero constant term for some w, w'.

Let M be a simple $\mathcal{H}_q(G)$-module (resp. $\mathcal{J}$-module). Lusztig attaches to M (see [48, p. 82]) an integer $a = a_M$ by the following two requirements:

$C_w M = 0$ (resp. $t_w M = 0$)   for all $w \in X^*(T) \rtimes \mathcal{W}$ such that $a(w) > a$;
$C_w M \neq 0$ (resp. $t_w M \neq 0$)   for some $w \in X^*(T) \rtimes \mathcal{W}$ such that $a(w) = a$.

The map ϕ_q is the unique bijection

$$M \mapsto M'$$

between the primitive ideal spaces of $\mathcal{H}_q(G)$ and $\mathcal{J}$, with the following properties: $a_M = a_{M'}$ and the restriction of M' to $\mathcal{H}_q(G)$ via ϕ_q is an $\mathcal{H}_q(G)$-module with exactly one composition factor isomorphic to M and all other composition factors of the form M'' with $a_{M''} > a_M$ (see [48, Theorem 8.1]).

The algebra $\mathcal{H}_q(G)$ is viewed as a k-algebra via the canonical isomorphism

$$\mathcal{O}(T/\mathcal{W}) \cong \mathrm{Z}(\mathcal{H}_q(G))$$

Lusztig's map ϕ_q maps $\mathrm{Z}(\mathcal{H}_q(G))$ to $\mathrm{Z}(\mathcal{J})$ and thus determines a unique k-structure for $\mathcal{J}$ such that the map ϕ_q is a morphism of k-algebras. $\mathcal{J}$ with this k-structure will be denoted $\mathcal{J}_q$. $\mathcal{H}_q(G)$ is then geometrically equivalent to $\mathcal{H}_1(G)$ by the three elementary steps

$$\mathcal{H}_q(G) \to \mathcal{J}_q \to \mathcal{J}_1 \to \mathcal{H}_1(G).$$

The second elementary step (i.e. passing from $\mathcal{J}_q$ to $\mathcal{J}_1$) is an algebraic variation of k-structure. Hence (provided q is not in the exceptional set of roots of unity) $\mathcal{H}_q(G)$ is geometrically equivalent to $\mathcal{H}_1(G) = \mathbb{C}[X^*(T) \rtimes \mathcal{W}]$.

Corollary A.3. *There is a canonical bijection*

$$(T/\!/\mathcal{W})_2 \longleftrightarrow \mathbf{Irr}(\mathcal{H}_q(G))$$

This map gives the left slanted arrow in Theorems 9.5 and 22.3.

<u>Example 4</u>. Let $\mathcal{H}_u(X \rtimes W)$ be the affine Hecke algebra of $X \rtimes W$ with unequal parameters $u = \{q_1, \ldots, q_k\}$. We assume that $q_i \in \mathbb{R}_{>0}$. Let $\mathcal{S}_u(X \rtimes W)$ be the Schwartz completion of $\mathcal{S}_u(X \rtimes W)$, as in [60, §5.4]. In this setting [60, Lemma 5.3.2] gives a morphism of Fréchet algebras

$$\mathcal{S}_1(X \rtimes W) \to \mathcal{S}_u(X \rtimes W),$$

which is spectrum preserving with respect to filtrations. However, the existence of a geometric equivalence between the $\mathcal{O}(T/W)$-algebras $\mathcal{O}(T) \rtimes W$ and $\mathcal{H}_u(T \rtimes W^{\mathfrak{s}})$ is still an open question in case u contains unequal parameters q_i.

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