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Abstract

Let φ be an automorphism of prime order p of a finite group G, and let r be the (Prüfer) rank of the fixed-point subgroup $C_G(\varphi)$. It is proved that if G is nilpotent, then there exists a characteristic subgroup C of nilpotency class bounded in terms of p such that the rank of G/C is bounded in terms of p and r.

For infinite (locally) nilpotent groups a similar result holds if the group is torsion-free (due to Makarenko), or periodic, or finitely generated; but examples show that these additional conditions cannot be dropped, even for nilpotent groups.

As a corollary when G is an arbitrary finite group, the combination with the recent theorems of the author and Mazurov gives characteristic subgroups $R \leqslant N \leqslant G$ such that N/R is nilpotent of class bounded in terms of p, while the ranks of R and G/N are bounded in terms of p and r (under the additional unavoidable assumption that $p \nmid |G|$ if G is insoluble); in general it is impossible to get rid of the subgroup R. The inverse limit argument yields corresponding consequences for locally finite groups.

1 Introduction

Let φ be an automorphism of prime order p of a finite group G, and let $C_G(\varphi)$ be the fixed-point subgroup, which we also call the centralizer of φ in G. If φ is regular, that is, $C_G(\varphi) = 1$, then G is nilpotent by Thompson's theorem [34] and the nilpotency class is bounded in terms of p (for short, "p-bounded") by Higman's theorem [9] (an explicit bound for Higman's function was obtained by Kreknin and Kostrikin [20, 21]). It is natural to expect that if φ is in some sense almost regular, then G must be in a sense almost nilpotent. For example, if $|C_G(\varphi)| = n$, then G has a subgroup of (p, n)-bounded index that is nilpotent of p-bounded class. This result is a combination of the work of Fong [1], who bounded the index of the soluble radical using the classification of finite simple groups, the works of Hartley and Meixner [8] and Pettet [31], where (independently) the index of the Fitting subgroup was bounded for soluble groups, and the author's theorems [10, 11], where a subgroup of (p, n)-bounded index and p-bounded class was produced for nilpotent groups. The latter results were also extended to infinite (locally) nilpotent groups by Medvedev [29].

Regarding the automorphism φ as almost regular in the sense of rank means seeking restrictions on the structure of G depending on the rank of $C_G(\varphi)$ and $|\varphi| = p$. (Throughout the paper, a group has rank at most r if every finitely generated subgroup can be generated by r elements.) The author and Mazurov [17] proved a rank analogue of the Hartley–Meixner–Pettet theorem for soluble groups, as well as a rank analogue of Fong's theorem in the case where the orders of G and φ are coprime (examples show that the coprimeness condition cannot be dropped in a rank analogue of Fong's theorem).

In the present paper we in a sense complete the study of finite groups with an automorphism of prime order that is almost regular in the sense of rank, by proving a rank generalization of the Higman–Kreknin–Kostrikin theorem for nilpotent groups.

Theorem 1. *Suppose that a finite nilpotent group G admits an automorphism φ of prime order p with centralizer $C_G(\varphi)$ of rank r. Then G has a characteristic subgroup C of p-bounded nilpotency class such that G/C has (p,r)-bounded rank.*

Earlier the case $|\varphi| = 2$ was settled by Shumyatsky [33]. We produce examples showing that Theorem 1 cannot be extended to infinite nilpotent groups, even for $|\varphi| = 2$. Nevertheless, such extensions can be proved for some classes of (locally) nilpotent groups, in particular, giving a positive solution to Problem 13.58 in the Kourovka Notebook [28].

Corollary 1. *Suppose that a locally nilpotent group G admits an automorphism φ of prime order p with centralizer $C_G(\varphi)$ of finite rank r. Let $T = T(G)$ be the periodic part of G.*

(a) *Then G/T and T have characteristic nilpotent subgroups C_0 and C_1 of p-bounded class such that $(G/T)/C_0$ and T/C_1 have finite (p,r)-bounded ranks.*

(b) *If G is in addition finitely generated, then G has a characteristic nilpotent subgroup C of p-bounded class such that G/C has finite (p,r)-bounded rank.*

The torsion-free case of this corollary is mainly due to Makarenko [22, Theorem 2], while the periodic case readily follows from Theorem 1 by the inverse limit argument.

Theorem 1 strengthens the results of the author and Mazurov [17] as follows.

Corollary 2. *Suppose that a finite group G admits an automorphism φ of prime order p with centralizer $C_G(\varphi)$ of rank r. If G is insoluble, then suppose in addition that $p \nmid |G|$. Then G has characteristic subgroups $R \leqslant N \leqslant G$ such that N/R is nilpotent of p-bounded class, while R and G/N have (p,r)-bounded ranks.*

The improvement in comparison with [17] is the bound for the nilpotency class of N/R. Examples in [17] show that the coprimeness condition cannot be dropped for insoluble groups, and that one cannot get rid of the subgroup R (nor of the quotients G/C or G/N, of course) in results of this kind. The inverse limit argument allows us to derive consequences for locally finite groups.

Corollary 3. *Suppose that a locally finite group G admits an automorphism φ of prime order p such that p is coprime to the orders of elements of G. If the centralizer $C_G(\varphi)$ has finite rank r, then G has normal subgroups $R \leqslant N \leqslant G$ such that N/R is nilpotent of p-bounded class, while both G/N and R have finite (p,r)-bounded rank.*

Since for soluble groups no coprimeness condition is required in Corollary 2, we have the following.

Corollary 4. *Suppose that a periodic locally soluble group G has an element g of prime order p with centralizer $C_G(g)$ of finite rank r. Then G has normal subgroups $R \leqslant N \leqslant G$ such that N/R is nilpotent of p-bounded class, while both G/N and R have finite (p,r)-bounded rank. If in addition G is locally nilpotent, then one can choose N to be characteristic and $R = 1$.*

All the functions of p and r occurring in these results can be given explicit upper estimates, although we do not write them out.

The proof of Theorem 1 stems from the following sources. The first is the author's [11] theorem on Lie rings with an almost regular automorphism of prime order, which generalized the Higman–Kreknin–Kostrikin theorem. The proof in [11] is based on the so-called method of graded centralizers. This Lie ring result was applied in [11] to the associated Lie ring of a nilpotent periodic group G having an automorphism φ of prime order with centralizer of given order, to yield a similar group-theoretic result, albeit by quite a complicated argument.

However, for an almost regular automorphism in the sense of the rank of $C_G(\varphi)$, the associated Lie ring does not adequately reflect the hypothesis: the rank of the fixed-point subring of the induced automorphism may become much larger, with the fixed points being "scattered" over the factors of the lower central series. Only for torsion-free locally nilpotent groups do Lie algebra results yield immediate consequences in the rank problem (via the Mal'cev correspondence). Such a result was obtained by Makarenko [22], who improved the Lie algebra result of [11] for automorphism of prime order by producing a nilpotent ideal, rather than a subalgebra, with bounds for the class and codimension; this result is also used in the present paper.

Therefore, for dealing with restrictions on the rank r of $C_G(\varphi)$, a different technique of graded centralizers in group rings was developed by the author in [13]. This technique, however, works only for the action on an abelian normal subgroup and therefore originally only a "weak" conclusion was obtained in [13] depending on the derived length d of G. Namely, using also the aforementioned Lie ring theorem [11] we constructed in [13] a subnormal subgroup of p-bounded nilpotency class connected with the group by a subnormal series of (p, r, d)-bounded length with quotients of (p, r, d)-bounded rank.

Another ingredient in the proof of Theorem 1 is the recent theorem of the author and Makarenko [16] on nilpotent characteristic subgroups of bounded co-rank. This result was used in [16] to convert the aforementioned subnormal subgroup into a normal one with quotient of (p, r, d)-bounded rank. In the present paper the 'weak' bound for the rank of the quotient is miraculously transformed into a 'strong' one, independent of d.

The method of graded centralizers in Lie rings and algebras has now been fairly well covered in the literature; see the book [12] and the more recent applications of this method in the papers by the author and Makarenko on almost regular automorphisms [14, 22, 23, 24, 25, 26]. In contrast, graded centralizers in group rings previously appeared only in a less accessible publication [13]. We present this group-ring technique in § 3 by giving a new proof of the "weak" result depending on the derived length of the group. This proof is simplified compared with [13] due to the same Khukhro–Makarenko result on characteristic subgroups [16]. This result on characteristic subgroups is also instrumental in finishing the proof of Theorem 1 in § 4. The reader who is familiar with the technique of graded centralizers in group rings in [13], or simply not much interested in it, may jump from § 2, which contains all the relevant statements, directly to § 4.

Finally we briefly describe the situation with finite groups G admitting an almost regular automorphism φ of arbitrary finite order n. The natural conjecture is that there is a subgroup of n-bounded derived length with quotient bounded either in terms of $|C_G(\varphi)|$ or the rank of $C_G(\varphi)$. (For Lie algebras, a similar generalization of Kreknin's theorem [20] was proved by the author and Makarenko [14].) There is already a reduction to nilpotent groups, giving subgroups of n-bounded Fitting height with bounded quotient. Hartley [6] used the classification to prove that the index of the soluble radical of a finite group is bounded in terms of the order of the centralizer of an element. Numerous papers initiated by Thompson [35] culminated in virtually best-possible bounds for the Fitting height of a subgroup of bounded index in the papers of Turull [36] and Hartley–Isaacs [7]. There are also similar results for non-cyclic groups of automorphisms under unavoidable additional coprimeness conditions. For ranks, the corresponding reduction to nilpotent groups was carried out by the author and Mazurov [18]. However, so far even a group analogue of Kreknin's theorem itself, for a regular automorphism, has not been proved. The only progress for composite n is the case of $|\varphi| = 4$: regular φ was done by Kovács [19], and the almost regular case in the sense of the order $|C_G(\varphi)|$ was done by the author and Makarenko [15].

2 Preliminaries

First we state for convenience two results in [16], where general theorems on the existence of characteristic subgroups were proved.

Theorem 2 ([16, Theorem 1.2]). *Suppose that a group G has a nilpotent normal subgroup H of nilpotency class c such that the quotient group G/H has finite rank r. If H is either*

(a) *torsion-free, or*

(b) *periodic,*

then G has also a characteristic nilpotent subgroup C of nilpotency class at most c with quotient group G/C of finite (r, c)-bounded rank.[1]

This theorem enables one to replace a subnormal nilpotent subgroup S of given class c and of given "co-rank" by a normal one: if S is normal in G with G/S of rank r, while G is normal in a group F, then the characteristic subgroup C of G given by Theorem 2 is normal in F and the rank of G/C is (r, c)-bounded. An earlier result of the author in [13] produced a subnormal subgroup of p-bounded nilpotency class connected with the group by a subnormal series of (p, r, d)-bounded length with quotients of (p, r, d)-bounded rank. A repeated application of Theorem 2 gives the following.

Theorem 3 ([16, Corollary 1.4]). *If a finite nilpotent group G of derived length d admits an automorphism of prime order p with centralizer of rank r, then G has a characteristic nilpotent subgroup C of p-bounded class such that the quotient G/C has (p, r, d)-bounded rank.*

We recall some notation. If G is a p-group, then $\Omega_1(G) := \langle g \in G \mid g^p = 1\rangle$. The terms of the lower central series are denoted by $\gamma_i(G)$, starting from $\gamma_1(G) = G$. A simple commutator is denoted by $[a_1, a_2, \ldots, a_k] := [...[[a_1, a_2], a_3], \ldots, a_k]$; here the a_i may also be subgroups. If G is a group, and m a positive integer, then $G^m := \langle g^m \mid g \in G\rangle$.

We also prove here a couple of technical lemmas and list several known facts.

[1] There are examples, produced independently by the author and H. Smith, showing that Theorem 2 is false without conditions (a) or (b).

Lemma 1. *Suppose that A and B are normal subgroups of the group AB that is nilpotent of class k, while for some positive integer q both AB^q and B are nilpotent of class at most c. Then for some k-bounded number $h(k)$ the group $A^{q^{h(k)}}B$ is nilpotent of class at most c.*

Proof. Induction on k. If $k \leqslant c$, then even AB is nilpotent of class at most c.

Now let $k > c$. We claim that $A^{q^{k-1}}B$ is nilpotent of class at most $k-1$. It suffices to show that every simple commutator of weight k in generators of $A^{q^{k-1}}B$ is trivial, and for the generators we choose elements $b \in B$ and $a^{q^{k-1}}$ for $a \in A$. Since the group AB is nilpotent of class k, commutators of weight k are linear with respect to powers of their elements. A (simple) commutator of weight k involving only elements of B is trivial, since B is nilpotent of class at most $c \leqslant k-1$. Thus we can assume that there is an entry $a^{q^{k-1}}$. Using the linearity we can "spread" this exponent coefficient q^{k-1} as q-powers of all entries of type $b \in B$, if any, by repeatedly applying formulae of the type

$$[\dots, u^q, \dots, v, \dots] = [\dots, u, \dots, v, \dots]^q = [\dots, u, \dots, v^q, \dots].$$

The resulting commutator involves only elements of A and B^q and therefore is trivial, since $\gamma_k(AB^q) \leqslant \gamma_{c+1}(AB^q) = 1$.

The same conditions are now satisfied with A replaced by $A^{q^{k-1}}$ and with nilpotency class of $A^{q^{k-1}}B$ being at most $k-1$. By the induction hypothesis, $(A^{q^{k-1}})^{q^{h(k-1)}}B$ is nilpotent of class at most c. We can set $h(k) = h(k-1) + k - 1$, since then $A^{q^{h(k)}} \leqslant (A^{q^{k-1}})^{q^{h(k-1)}}$. $\square$

Lemma 2. *Let φ be an automorphisms of a finite group G of coprime order: $(|G|, |\varphi|) = 1$. If N is a normal φ-invariant subgroup, then $C_{G/N}(\varphi) = C_G(\varphi)N/N$.* $\square$

We shall also need an "infinite" analogue of Lemma 2 for nilpotent groups.

Lemma 3. *Let φ be an automorphism of prime order p of a group G. Suppose that N is a normal φ-invariant subgroup of G such that $[N, \underbrace{G, \dots, G}_{m}] = 1$.*

(a) *If $gN \in C_{G/N}(\varphi)$, then $g^{p^m} \in C_G(\varphi)N$.*

(b) *If G/N is a q-group for $q \neq p$, then $C_{G/N}(\varphi) = C_G(\varphi)N/N$.*

Proof. (a) Induction on m. Let $m = 1$, that is, $N \leqslant Z(G)$. By hypothesis, $g^{\varphi^i} = gn_i$ for $n_i \in N \leqslant Z(G)$. Then the elements g^{φ^i} commute and therefore $gg^{\varphi}g^{\varphi^2}\cdots g^{\varphi^{p-1}} \in C_G(\varphi)$. Hence $g^p \in gg^{\varphi}g^{\varphi^2}\cdots g^{\varphi^{p-1}}N \subseteq C_G(\varphi)N$. For $m > 1$ we have $g^{p^{m-1}} \in C_{G/[N,\underbrace{G,\dots,G}_{m-1}]}(\varphi)N$ by the induction hypothesis. Applying the above argument for $m = 1$ with $[N, \underbrace{G, \dots, G}_{m-1}]$ in place of N we obtain the result.

(b) This follows from (a), since $gN \in \langle g^{p^m}N\rangle$ in the q-group G/N for $q \neq p$. $\square$

Lemma 4. *If a finite abelian p-group A admits an automorphism φ of order p with centralizer of rank r, then the rank of A is at most pr.*

Proof. The rank of A is equal to the rank of $\Omega_1(A)$, which can be regarded as a vector space over the field of p elements. Since $0 = \varphi^p - 1 = (\varphi - 1)^p$, all eigenvalues of the linear transformation φ are equal to 1. The number of blocks in the Jordan normal form of φ is equal to the dimension of the centralizer $C_{\Omega_1(A)}(\varphi)$, while the size of each block does not exceed p. $\square$

The following lemma appeared independently and simultaneously in the papers of Gorchakov [3], Merzlyakov [30], and as "P. Hall's lemma" in the paper of Roseblade [32].

Lemma 5. *Let q be a prime number. The rank of a q-group of automorphisms of a finite q-group of rank r is r-bounded.*

Although in [3], [30], and [32] automorphisms of finite *abelian* q-groups were considered, the general result can be easily derived from this special case; see, for example, [33, Lemma 4.2]. Lemmas 4 and 5 imply the following.

Lemma 6. *If a finite p-group G admits an automorphism φ of order p with centralizer of rank r, then the rank of G is (p, r)-bounded.*

The following well-known fact can be easily derived from the theory of powerful q-groups.

Lemma 7. *If a finite q-group has rank r and exponent q^s, then its order is at most $q^{f(r,s)}$ for some (r, s)-bounded number $f(r, s)$.*

3 Graded centralizers in group rings

Here we give a new proof of Theorem 3 — or rather, a new proof of the previous results in [13]. This proof is simplified compared with [13] due to Theorem 2(b) on characteristic subgroups. However, the construction of graded centralizers of various levels still remains essential.

Definition. We define for brevity the *co-rank* of a normal subgroup N in a group G to be the rank of G/N.

Recall that Theorem 3 deals with a finite nilpotent group G of derived length d admitting an automorphism φ of prime order p with centralizer of rank r. Graded centralizers in group rings are used to find a subgroup of (p, r, d)-bounded co-rank which is nilpotent of (p, r, d)-bounded class. A required subgroup of p-bounded class is then obtained in the next section using an analogue of the Lie ring theorem in [11].

Proposition 1. *If a finite nilpotent group G of derived length d admits an automorphism φ of prime order p with centralizer $C_G(\varphi)$ of rank r, then G has a characteristic subgroup C of (p, d)-bounded nilpotency class and of (p, r, d)-bounded co-rank.*

Proof. Since the rank of a finite nilpotent group is equal to the maximum rank of its Sylow subgroups, we can assume that G is a finite q-group. If $q = p$, then the rank of G is (p, r)-bounded by Lemma 6. Thus we assume in what follows that $q \neq p$.

We use induction on d. By the induction hypothesis, $[G, G]$ has a characteristic subgroup H of nilpotency class at most $c(p, d - 1)$ such that $[G, G]/H$ has rank at most $f(p, r, d - 1)$. Then $C = C_G([G, G]/H)$ is a characteristic subgroup of G with G/C of bounded rank by Lemma 5. The quotient $[C, C]H/H$ is central in C/H and has rank at most $f(p, r, d - 1)$. If X is a characteristic subgroup of C containing H such that $X/[H, H]$ is nilpotent of class m, then by Hall's theorem [5] the group X is nilpotent of class bounded in terms of m and $c(p, d - 1)$. Therefore Proposition 1 will follow from the following proposition applied to $C/[H, H]$.

Proposition 2. *Let G be a finite q-group admitting an automorphism φ of prime order $p \neq q$ with centralizer $C_G(\varphi)$ of rank r. Suppose that G has an abelian characteristic subgroup V such that the quotient group G/V is nilpotent of class 2, and let s be the rank of $[G,G]V/V$. Then G contains a characteristic subgroup of (p,r,s)-bounded co-rank which contains V and is nilpotent of class at most $(2p-1)^2+2$.*

Definition. We say for short that S is a *subnormal subgroup of (p,r,s)-bounded co-rank* in a group G if S is connected with G by a subnormal series of (p,r,s)-bounded length with factors of (p,r,s)-bounded rank.

Proof of Proposition 2. By Theorem 2(b) it suffices to produce a subnormal subgroup of (p,r,s)-bounded co-rank that is nilpotent of class at most $(2p-1)^2+2$, so this will be our aim in what follows.

Since $[[G,G],G] \leqslant V$ by hypothesis, a subgroup X containing V is nilpotent of class at most $t+2$ if $[V, \underbrace{X,\dots,X}_{t}] = 1$. The action of G by conjugation on V induces the action of the quotient group $Q = G/V$ on V. It suffices to find a subnormal subgroup $Y \leqslant Q$ of bounded co-rank such that $[V, \underbrace{Y,\dots,Y}_{(2p-1)^2}] = 1$; then the full inverse image of Y in G will be the required subgroup. We can regard V as a $\mathbb{Z}Q\langle\varphi\rangle$-module; in these terms the last equality can be rewritten as $[V, \underbrace{Y,\dots,Y}_{(2p-1)^2}] = 0$. Henceforth for an additive subgroup $U \subseteq V$ and a subgroup $L \leqslant Q$ we denote by $[U,L]$ the additive subgroup generated by all elements of the form $u(l-1)$, $u \in U$, $l \in L$; this is precisely the mutual commutator subgroup of the subgroups U and L in the semidirect product VQ.

The construction of the required subgroup is carried out in two stages. First certain elements of levels 1, 2, …, $2p-1$ are fixed successively in certain subgroups $Q(1) = [Q,\varphi]$, $Q(2),\dots,Q(2p-1)$ of the centralizers of the elements of previous levels; these subnormal subgroups $Q(i)$ of (p,r,s)-bounded co-rank are constructed in parallel. Then the required subgroup is constructed with the help of "graded centralizers" that are defined within the enveloping algebra of a certain subgroup.

Definition (of subgroups and elements of levels at most $2p-1$)**.** We set $Q(1) = [Q,\varphi]$; since $Q = C_Q(\varphi)Q(1)$ by Lemma 2, $Q(1)$ is a φ-invariant normal subgroup of Q of co-rank at most r. The rank of $[Q(1),Q(1)]$ does not exceed s. By the Burnside Basis Theorem we can choose s generators of $[Q(1),Q(1)]$ of the form $[y,a]$, where $y,a \in Q(1)$. (Note that this is formally true even if $[Q(1),Q(1)] = 1$.)

We shall need the following technical remark. Since $Q(1) = [Q(1),\varphi]$ and $p \neq q$, the abelian q-group $Q(1)/[Q(1),Q(1)]$ consists of the images of the p-th powers of commutators of the form $[u,\varphi]$, $u \in Q(1)$. Since $[Q,Q] \leqslant Z(Q)$, it follows that the element y in a commutator $[y,a]$ can be chosen to be of the form $y = x^p$, where $x = [u,\varphi]$ for some $u \in Q(1)$ and therefore

$$xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = u^{-1}u^{\varphi}(u^{-1})^{\varphi}u^{\varphi^2}\cdots(u^{-1})^{\varphi^{p-1}}u^{\varphi^p} = 1.$$

For further references we mark this property as

$$xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1 \quad \text{for } x^p = y; \tag{1}$$

note that here $x \in \langle y \rangle$.

The elements y, a satisfying (1) fixed for our chosen generators of $[Q(1), Q(1)]$ are called *elements of level* 1 and denoted by $y(1), a(1)$, with level indicated in parenthesis. To lighten notation we do not distinguish elements $y(1)$, $a(1)$ by indices. Recall that the number of elements $y(1)$ is s.

For any $y \in Q$ the centralizer $C_Q(y)$ is a normal subgroup of bounded co-rank. Indeed, the mapping $g \to [g, y]$ is a homomorphism of the group Q into $[Q, Q]$. Its kernel is exactly $C_Q(y)$; hence the rank of $Q/C_Q(y)$ is at most s. We set

$$D(2) = \bigcap_{y(1)} \bigcap_{i=1}^{p-1} C_{Q(1)}(y(1)^{\varphi^i}),$$

where $y(1)$ runs over the fixed elements of level 1. The subgroup $D(2)$ is φ-invariant. Since each subgroup involved in the intersection has co-rank at most s in $Q(1)$ and the number of these subgroups is (p, s)-bounded, the subgroup $D(2)$ also has (p, r, s)-bounded co-rank. We define the subgroup of level 2 to be $Q(2) = [D(2), \varphi]$, which is a φ-invariant normal subgroup of $D(2)$. The co-rank of $Q(2)$ in $D(2)$ is at most r, since $D(2) = C_{D(2)}(\varphi)Q(2)$. By construction, $[y(1)^{\varphi^i}, Q(2)] = 1$ for all elements $y(1)$ of level 1 and for all i.

Since $[Q(2), Q(2)] \leqslant [Q, Q]$ and $[Q(2), \varphi] = Q(2)$, we can carry out the same construction for $Q(2)$. We fix s generators of $[Q(2), Q(2)]$ of the form $[y(2), a(2)]$, where $y(2), a(2) \in Q(2)$ and each element $y = y(2)$ satisfies (1). We set

$$D(3) = \bigcap_{y(2)} \bigcap_{i=1}^{p-1} C_{Q(2)}(y(2)^{\varphi^i}),$$

where $y(2)$ runs over the elements of level 2. Then we set $Q(3) = [D(3), \varphi]$.

Repeating this construction $2p-1$ times we obtain φ-invariant subnormal subgroups

$$Q(1) \geqslant Q(2) \geqslant \cdots \geqslant Q(2p-1)$$

of (p, r, s)-bounded co-rank. For each $j = 1, 2, \ldots, 2p-2$, generators of $[Q(j), Q(j)]$ were fixed of the form $[y(j), a(j)]$, where $y(j), a(j) \in Q(j)$, each element $y = y(j)$ satisfies (1), and their total number is bounded. By construction, subgroups of higher level centralize the fixed elements of lower level: $[y(j), Q(k)] = 1$ for $j < k$.

We now discuss "graded action" in group rings, or rather in enveloping algebras. Extending the ground ring by a primitive p-th root of unity ω, we denote by the same letter V the resulting $\mathbb{Z}[\omega]Q\langle\varphi\rangle$-module. The rank of the additive group $C_V(\varphi)$ can increase only to at most $r(p-1)$. Our aim remains the same: to find a subnormal subgroup Y of bounded co-rank in Q such that $[V, \underbrace{Y, \ldots, Y}_{(2p-1)^2}] = 0$.

Let $E = E(Q)$ be the subalgebra of $\mathrm{Hom}_{\mathbb{Z}[\omega]}(V)$ generated by Q; we call E the enveloping algebra of Q. The action of φ on Q extends naturally to E. For any φ-invariant subgroup $N \leqslant Q$ its enveloping algebra $E(N)$ is a φ-invariant subalgebra of $E(Q)$.

Using the fact that the additive group of V is a q-group, which is p-divisible as $q \neq p$, we can decompose V into the sum of the analogues of eigenspaces:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1},$$

where $V_i = \{v \in V \mid v\varphi = \omega^i v\}$, since for each $v \in V$ we have

$$v = \sum_{i=0}^{p-1} v_i, \quad \text{where } v_i = \frac{1}{p}\sum_{k=0}^{p-1} \omega^{-ki} v\varphi^k \in V_i.$$

We call the additive subgroups V_i for brevity *φ-components* of V, and the elements v_i *φ-components* of v. Clearly, $V_0 = C_V(\varphi)$. Since the additive group of E is also a q-group, similarly

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_{p-1}$$

is the sum of its φ-components $E_i = \{e \in E \mid e^{\varphi} = \omega^i e\}$. (It would probably be better to speak of V as a $\mathbb{Q}_p[\omega]Q\langle\varphi\rangle$-module, and E as a subalgebra of $\mathrm{Hom}_{\mathbb{Q}_p[\omega]}(V)$, where $\mathbb{Q}_p = \{m/p^n \mid m, n \in \mathbb{Z}\}$.)

It is easy to see that $V_iE_j \subseteq V_{i+j\,(\mathrm{mod}\,p)}$: for $v \in V_i$ and $e \in E_j$ we have

$$(ve)\varphi = (v\varphi)(\varphi^{-1}e\varphi) = (\omega^i v)(\omega^j e) = \omega^{i+j} ve.$$

Similarly, $E_iE_j \subseteq E_{i+j\,(\mathrm{mod}\,p)}$.

Definition (of graded centralizers in group rings)**.** For a fixed $v \in V_i$ the mapping $\vartheta_v : e \to ve$ of the component E_{p-i} into the component V_0 is a homomorphism of additive groups. Since the rank of V_0 is at most $r(p-1)$, the rank of the additive quotient group $E_{p-i}/\mathrm{Ker}\,\vartheta_v$ is also at most $r(p-1)$. (Here we do not exclude the case of $i = 0$, where $E_p = E_0$.) The kernel $\mathrm{Ker}\,\vartheta_v$ can be regarded as a "graded centralizer" of the element v.

We shall be constructing certain additive subgroups $C_{p-i} \leqslant E_{p-i}$ as intersections $C_{p-i} = \bigcap_v \mathrm{Ker}\,\vartheta_v$ of such kernels over certain sets of fixed elements in V_i. If the number of these elements is bounded, then the rank of the additive quotient group E_{p-i}/C_{p-i} is also bounded, since it is at most the sum of the co-ranks of the kernels involved in the intersection.

Suppose that an additive subgroup C_i of bounded co-rank is chosen in the φ-component E_i for each $i = 0, 1, \ldots, p-1$. Let I be the two-sided ideal of the algebra E generated by all the C_i. Then the additive group E/I has bounded rank. The group Q acts by right multiplication on E, and therefore also on E/I. The kernel of this action K is a normal subgroup of Q. By Lemma 5 the rank of Q/K is bounded in terms of the rank of E/I, since this is a q-group of automorphisms of the additive q-group E/I.

Lemma 8. *The following equality holds: $K = \{h \in Q \mid 1 - h \in I\}$.*

Proof. If $h \in K$, then $1h \equiv 1 \,(\mathrm{mod}\, I)$, that is, $1 - h \in I$. If $1 - h \in I$, then $u1 - uh \in uI = I$, that is, $uh \equiv u \,(\mathrm{mod}\, I)$ for any $u \in E$, which means precisely that $h \in K$. □

The same construction of the kernel of the action on the quotient algebra described above can be carried out for any φ-invariant subgroup $N \leqslant Q$ with respect to the φ-components of the enveloping algebra $E(N)$.

We begin constructing the required subgroup. Let

$$F = \{f \in E \mid fe = ef \text{ for all } e \in Q(2p-1)\}$$

be the centralizer in E of the subgroup $Q(2p-1)$, and let F_j be its φ-components, $j = 0, 1, \ldots, p-1$. For each $j = 0, 1, \ldots, p-1$ we consider the additive subgroup of V_0 generated by the set $V_{p-j}F_j = \{vf \mid v \in V_{p-j},\ f \in F_j\}$. Since $V_{p-j}F_j \subseteq V_0$, the rank of $V_{p-j}F_j$ is at most $r(p-1)$. Since this is a q-group, we can choose $r(p-1)$ elements generating this subgroup from the original generating set $V_{p-j}F_j$. We fix such a generating set of $r(p-1)$ elements $v_{p-j,i}f_{j,i}$, where $v_{p-j,i} \in V_{p-j}$, $f_{j,i} \in F_j$, and the numbering indices i satisfy $1 \leqslant i \leqslant r(p-1)$.

For each of our fixed elements $v_{p-j,i}$, let $\vartheta_{v_{p-j,i}}$ be the homomorphism of the φ-component $E_j(2p-1)$ of the enveloping algebra $E(2p-1) = E(Q(2p-1))$ of the subgroup $Q(2p-1)$ into V_0 defined by the rule $e \to v_{p-j,i}e$ for $e \in E_j(2p-1)$. For each $j = 0, 1, \ldots, p-1$ we set

$$C_j = \bigcap_{i=1}^{r(p-1)} \operatorname{Ker} \vartheta_{v_{p-j,i}}.$$

All the additive quotient groups $E_j(2p-1)/C_j$ have (p,r)-bounded rank.

Now let I be the ideal of the algebra $E(2p-1)$ generated by the additive subgroups $C_0, C_1, \ldots, C_{p-1}$. The additive quotient group $E(2p-1)/I$ has (p,r)-bounded rank. By Lemma 8 the set $K = \{x \in Q(2p-1) \mid 1 - x \in I\}$ is the kernel of the action of $Q(2p-1)$ by right multiplication on $E(2p-1)/I$. The rank of $Q(2p-1)/K$ is (p,r)-bounded.

It is technically convenient to consider separately the action of $[K,K]$ on V and the case where $[Q,Q] = 1$.

Lemma 9. *We have*

$$[V, \underbrace{[K,K], \ldots, [K,K]}_{2p-1}] = 0. \tag{2}$$

Proof. Equality (2) is equivalent to the fact that

$$v(1-z_1)(1-z_2)\cdots(1-z_{2p-1}) = 0 \tag{3}$$

for any $v \in V$ and $z_j \in [K,K]$ (here j is a numbering index). Since $K \leqslant Q(2p-1) \leqslant Q(2p-2) \leqslant \cdots$, we can express each element $z_j \in [K,K]$ in (3) as a products of fixed generators of the form $[y(j), a(j)]$ of level j. We do this successively for $j = 1, \ldots, 2p-2$ — saving the last factor $1 - z_{2p-1}$ — performing certain transformations at each step.

First we replace the element z_1 in the first bracket in (3) by a product of fixed generators of the form $[y(1), a(1)]$ of level 1. Then we apply repeatedly the formula

$$1 - gh = (1-g)(1+h) - (1-g) + (1-h), \tag{4}$$

to express $1 - z_1$ as a linear combination of products each of which has a factor of the form $1 - [y(1), a(1)]$. Substituting this expression into (3) we can transfer all the "superfluous" factors (like $1+h$) to the right end, since all the factors arising by formulae (4) have the form $1 \pm w$ with $w \in [K,K]$ and therefore commute with any elements. We obtain a linear combination of products each of which has an initial segment of the form

$$v(1 - [y(1), a(1)])(1-z_2)\cdots(1-z_{2p-1}). \tag{5}$$

We set temporarily $y = y(1)$, $a = a(1)$ and use the formula

$$1 - [y, a] = 1 - [y^{-1}, a^{-1}] = 1 - yay^{-1}a^{-1} = (ay - ya)y^{-1}a^{-1} =$$

$$((1 - a)(1 - y) - (1 - y)(1 - a))y^{-1}a^{-1}.$$

Here $[y, a] = [y^{-1}, a^{-1}]$ since the nilpotency class is 2. We substitute the expression for $1-[y, a]$ thus obtained instead of the first bracket in (5). The factor $y^{-1}a^{-1}$ can be transferred to the right end of the product, since all elements $z_i \in [K, K]$ belong to the centre. We obtain a linear combination of products each of which has an initial segment either of the form

$$v(1 - a)(1 - y)(1 - z_2)(1 - z_3)\cdots(1 - z_{2p-1})$$

or

$$v(1 - y)(1 - a)(1 - z_2)(1 - z_3)\cdots(1 - z_{2p-1}) = v(1 - y)(1 - z_2)(1 - z_3)\cdots(1 - z_{2p-1})(1 - a)$$

(in the second case we used again the fact that all the z_i are central). Re-denoting $v(1 - a)$ by v in the first case, we see that it suffices to prove that every product of the form

$$v(1 - y(1))(1 - z_2)(1 - z_3)\cdots(1 - z_{2p-1}) \qquad (6)$$

is equal to 0.

Then we replace the element z_2 in (6) by a product of fixed generators of the form $[y(2), a(2)]$ of level 2. Applying formula (4) in the same way we express the element $1 - z_2$ as a linear combination in which each summand has a factor of the form $1 - [y(2), a(2)]$. On substituting this expression instead of $(1 - z_2)$ in (6) and transferring to the right end all the "superfluous" factors (which have the form $1 \pm w$ for $w \in [K, K]$ and therefore commute with any elements) we obtain a linear combination of products each of which has an initial segment either of the form

$$v(1 - y(1))(1 - y(2))(1 - a(2))(1 - z_3)\cdots(1 - z_{2p-1}) =$$

$$= v(1 - y(1))(1 - y(2))(1 - z_3)\cdots(1 - z_{2p-1})(1 - a(2))$$

or

$$v(1 - y(1))(1 - a(2))(1 - y(2))(1 - z_3)\cdots(1 - z_{2p-1}) =$$

$$= v(1 - a(2))(1 - y(1))(1 - y(2))(1 - z_3)\cdots(1 - z_{2p-1}).$$

In the first case we have used the fact that all the z_i are central. In the second case we used the fact that the element $a(2) \in Q(2)$ centralizes the fixed element $y(1)$ of smaller level by the definition of the subgroup $Q(2)$. Re-denoting in the second case $v(1 - a(2))$ by v we obtain that it suffices to prove that any product of the form

$$v(1 - y(1))(1 - y(2))(1 - z_3)\cdots(1 - z_{2p-1})$$

is equal to 0. After $2p - 2$ steps we obtain that it suffices to prove that any product of the form

$$v(1 - y(1))(1 - y(2))\cdots(1 - y(2p - 2))(1 - z_{2p-1}) \qquad (7)$$

is equal to 0.

We now use the fact that the element z_{2p-1} in (7) belongs to K. Since $K = \{x \in Q(2p-1) \mid 1-x \in I\}$, where I is the ideal generated by $C_0, \dots, C_{p-1}$, the element $1-z_{2p-1}$ is equal to a linear combination of elements of the form gc_jh, where $c_j \in C_j$, $j = 0, 1, \dots, p-1$, and g, h are arbitrary elements of $E(2p-1)$. We substitute this expression instead of the last bracket in (7). Since by definition the subgroup $Q(2p-1)$, and therefore also its enveloping algebra, centralizes all elements $y(i)$ of smaller levels $i \leqslant 2p-2$, the elements g in the factors gc_jh can be transferred to the left over all the elements $y(i)$. Expressing vg as a sum of φ-components in each summand of the resulting linear combination we obtain that it suffices to prove that every element of the form

$$v_{i_0}(1-y(1))(1-y(2))\cdots(1-y(2p-2))c_j \tag{8}$$

is equal to zero, where $v_{i_0} \in V_{i_0}$ for some $i_0 = 0, 1, \dots, p-1$.

By the property (1) each element $y = y(i)$ has the form $y = x(i)^p = x^p$, where $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$. Then

$$1-y = xx^{\varphi}\cdots x^{\varphi^{p-1}} - x^p. \tag{9}$$

(We have temporarily dropped the level indicator to lighten the notation.) We decompose the element x in (9) into the sum of φ-components $x = x_0 + x_1 + \cdots + x_{p-1}$, substitute the expressions $x^{\varphi^k} = x_0 + \omega^k x_1 + \cdots + \omega^{(p-1)k}x_{p-1}$, and expand all the brackets. This results in a linear combination of monomes of degree p in the φ-components x_i. Since the monomes x_0^p in the decompositions of $xx^{\varphi}\cdots x^{\varphi^{p-1}}$ and x^p cancel out, each summand of the resulting linear combination involves at least one φ-component x_i with non-zero index $i \neq 0$. We substitute the corresponding expression instead of each bracket in (8) and expand all the brackets. Using again the level indicators in parenthesis (and under the braces) and using indices only to indicate the φ-components which the elements belong to, we obtain that it suffices to prove that any product of the form

$$v_{i_0}\ \underbrace{x_0(1)\cdots x_0(1)x_{i_1}(1)\cdots}_{(1)}\ \cdot\ \cdot\ \cdot\ \underbrace{x_0(2p-2)\cdots x_0(2p-2)\cdots x_{i_{2p-2}}(2p-2)\cdots}_{(2p-2)}\ c_j \tag{10}$$

is equal to zero, where $\underbrace{x_0(k)\cdots x_0(k)x_{i_k}(k)\cdots}_{(k)}$ is a monome from the decomposition of the bracket $(1-y(k))$ described above, in which we distinguish the first from the left φ-component $x_{i_k}(k)$ with non-zero index $i_k \neq 0$ (there may not be preceding elements with zero index). Let $\overline{x}_{i_k}(k)$ denote the initial segment $x_0(k)\cdots x_0(k)x_{i_k}(k)$ of this monome. Then the product (10) takes the form

$$v_{i_0}\,\underbrace{\overline{x}_{i_1}(1)\cdots}_{(1)}\ \cdot\ \cdot\ \cdot\ \underbrace{\overline{x}_{i_{2p-2}}(2p-2)\cdots}_{(2p-2)}c_j, \tag{11}$$

where dots over each brace of level k indicate factors of the form $x_t(k)$ for various $t = 0, 1, \dots, p-1$.

We now recall the centralizer properties of the subgroups $Q(i)$. If $i < k$, then by definition $y(k) \in Q(k) \leqslant C_Q(y(i)^{\varphi^t})$ for any t. If elements $x(i)$, $x(k)$ are such that $x(i)^p = y(i)$,

$x(k)^p = y(k)$, then also $[x(i)^{\varphi^t}, x(k)^{\varphi^u}] = 1$ for any t, u, since $x \in \langle y \rangle$ if $x^p = y$. Since the φ-components of x are linear combinations of the elements x^{φ^t}, we obtain that φ-components $x_l(i)$, $x_m(k)$ of different levels $i \neq k$ commute (for any l, m).

Similarly, the subgroup $Q(2p-1)$ centralizes all elements of the form $y(k)^{\varphi^t}$ for levels $k \leqslant 2p-2$ and therefore it centralizes also all the elements $x(k)^{\varphi^t}$. Hence all the φ-components of the elements $x(k)$, which are linear combinations of the elements x^{φ^i}, are also centralized by $Q(2p-1)$. In particular, the element c_j, which belongs to the enveloping algebra of the subgroup $Q(2p-1)$, commutes with all the φ-components $x_s(k)$ for $k \leqslant 2p-2$.

Every element $\overline{x}_{i_k}(k)$ in (11), being a product of φ-components of level k, commutes with φ-components of other levels and with the element c_j; in this sense, $\overline{x}_{i_k}(k)$ can also be regarded as an element of level k.

We now collect all the elements $\overline{x}_{i_k}(k)$ in (11) at the beginning after v_{i_0} in the same order, followed by c_j. For that we transfer the elements $\overline{x}_{i_2}(2)$, $\overline{x}_{i_3}(3) \dots, \overline{x}_{i_{2p-2}}(2p-2)$, c_j, one at a time, successively to the left. Each of these elements $\overline{x}_{i_k}(k)$ or c_j is always transferred over some φ-components $x_t(i)$ of *smaller levels* $i < k$, with which $\overline{x}_{i_k}(k)$ commutes. As a result, it is sufficient to prove that

$$v_{i_0}\overline{x}_{i_1}(1)\overline{x}_{i_2}(2)\cdots\overline{x}_{i_{2p-2}}(2p-2)c_j \;=\; 0. \tag{12}$$

We shall need the following elementary lemma.

Lemma 10 ([21]). *If $j_1, j_2, \dots, j_{p-1}$ are any non-zero (not necessarily distinct) residues modulo a prime number p, then every residue modulo p can be obtained as the sum over some subset of the set $\{j_1, j_2, \dots, j_{p-1}\}$.*

All indices of the elements $\overline{x}_{i_k}(k)$ in (12) are non-zero and these elements, being of different levels, commute with each other, and they all commute with $c_j \in E(2p-1)$, since their levels are at most $2p-2$. First, applying Lemma 10 we permute the first $p-1$ elements $\overline{x}_{i_k}(k)$ in (12) to obtain an initial segment of the form

$$u_{p-j} = v_{i_0}\overline{x}_{i_{k_1}}(k_1)\cdots\overline{x}_{i_{k_t}}(k_t), \qquad \text{where} \quad i_{k_1}+\cdots+i_{k_t} \equiv -j-i_0 \,(\mathrm{mod}\, p),$$

so that this initial segment belongs to V_{p-j}. Secondly, since there are at least $p-1$ remaining elements $\overline{x}_{i_k}$ (outside that initial segment u_{p-j}), we again apply Lemma 10 permuting these elements and the element c_j to obtain an initial segment

$$u_{p-j}\overline{x}_{i_{l_1}}(l_1)\cdots\overline{x}_{i_{l_s}}(l_s)c_j, \qquad \text{where} \quad i_{l_1}+\cdots+i_{l_s} \equiv j \,(\mathrm{mod}\, p), \tag{13}$$

so that $\overline{x}_{i_{l_1}}(l_1)\cdots\overline{x}_{i_{l_s}}(l_s) \in E_j$.

As noted above, the subgroup $Q(2p-1)$ centralizes all the φ-components $x_s(k)$ for $k \leqslant 2p-2$. In other words, all these φ-components centralize $Q(2p-1)$, that is, belong to F, and therefore $\overline{x}_{l_1}\cdots\overline{x}_{l_s} \in F_j$. Hence the element $u_{p-j}\overline{x}_{l_1}\cdots\overline{x}_{l_s}$ belongs to $V_{p-j}F_j$ and therefore it is equal to a linear combination of the elements $v_{p-j,i}f_{j,i}$, where $v_{p-j,i} \in V_{p-j}$ and $f_{j,i} \in F_j$ are our fixed elements. Thus the product (13) is equal to a linear combination of the elements $v_{p-j,i}f_{j,i}c_j$. The elements $f_{j,i}$ and c_j commute because c_j belongs to $C_j \subseteq E(2p-1)$ and $E(2p-1)$ is centralized by $f_{j,i} \in F$, since F centralizes $Q(2p-1)$. Therefore

$$v_{p-j,i}f_{j,i}c_j = v_{p-j,i}c_jf_{j,i} = 0,$$

since $v_{p-j,i}c_j = 0$ for $c_j \in C_j$ by the construction of $C_j \subseteq \mathrm{Ker}\,\vartheta_{v_{p-j,i}}$. $\square$

Lemma 11. *If* $[Q,Q]=1$*, then* $[V,\underbrace{[K,\varphi],\dots,[K,\varphi]}_{2p-1}]=0$.

Proof. The proof is similar to that of Lemma 9 but simpler due to the commutativity of Q. We need to show that

$$v(1-z_1)\cdots(1-z_{2p-1})=0 \tag{14}$$

for any $z_i\in[K,\varphi]$. Since $1-z_{2p-1}\in I$, the element $1-z_{2p-1}$ is a linear combination of elements of the form gc_jh, where $c_j\in C_j$ for $j=0,\,1,\,\dots,p-1$. We substitute this into (14), transfer the elements g to the left over all the elements $1-z_i$, re-denote vg by v, and then decompose v into the sum of φ-components. As a result, it suffices to prove that any product of the form

$$v_{i_0}(1-z_1)\cdots(1-z_{2p-2})c_j \tag{15}$$

is equal to zero, where $v_{i_0}\in V_{i_0}$ for some $i_0=0,\,1,\,\dots,p-1$.

Since $[K,\varphi]$ is an abelian q-group, for each $z=z_i$ in (15) there is $x\in[K,\varphi]$ such that $x^p=z$ and $x=[u,\varphi]$. Then

$$xx^{\varphi}\cdots x^{\varphi^{p-1}}=u^{-1}u^{\varphi}(u^{-1})^{\varphi}u^{\varphi^2}\cdots(u^{-1})^{\varphi^{p-1}}u^{\varphi^p}=1$$

and therefore

$$1-z=xx^{\varphi}\cdots x^{\varphi^{p-1}}-x^p. \tag{16}$$

We decompose x into the sum of φ-components $x=x_0+\cdots+x_{p-1}$, substitute the expressions $x^{\varphi^i}=x_0+\omega^ix_1+\omega^{2i}x_2\cdots+\omega^{(p-1)i}x_{p-1}$ into (16), expand all brackets, and cancel out $x_0^p-x_0^p$. The result is a linear combination of monomes of degree p in the elements x_i such that each of these monomes has a factor x_i with non-zero index $i\neq 0$. We substitute the corresponding linear combination instead of each of the brackets in (15). Since all the elements involved commute, we can collect some of those factors x_i with non-zero indices $i\neq 0$ at the beginning, followed by c_j. Since each of the $2p-2$ brackets in (15) contributes at least one such factor, it is now sufficient to prove that any product of the form

$$v_{i_0}x_{i_1}\cdots x_{i_{2p-2}}c_j,\qquad\text{where}\quad i_k\not\equiv 0\,(\mathrm{mod}\,p)\ \text{ for all }\ k. \tag{17}$$

is equal to zero. Here the indices i_k only indicate the φ-components which the elements belong to.

We can apply to (17) the same kind of "collection process" as applied above to (12). We arrive at the same conclusion using the fact that here $F_j=E_j$, since Q is abelian. □

Completion of the proof of Proposition 2. Note that $[K,\varphi]$ is a subnormal subgroup of co-rank at most r in K by Lemma 2 and therefore of (p,r,s)-bounded co-rank in Q, since K has (p,r,s)-bounded co-rank. Thus, Lemma 11 proves Proposition 2 in the case of abelian Q.

The quotient group $K/[K,K]$ acts on each of the quotient modules

$$U_k=[V,\underbrace{[K,K],\dots,[K,K]}_{k}]\Big/[V,\underbrace{[K,K],\dots,[K,K]}_{k+1}]$$

(starting with $U_0 = V/[V,[K,K]]$). By Lemma 9 there are at most $2p-1$ non-trivial of them. By Lemma 11 applied to U_k and K in place of V and Q there exist subgroups L_k of (p,r,s)-bounded co-rank in K (and therefore in Q) such that

$$[U_k, \underbrace{L_k, \dots, L_k}_{2p-1}] = 0 \qquad \text{for each } k. \tag{18}$$

Then the intersection $Y = \bigcap_{k=0}^{2p-2} L_k$ is the required subgroup, since its co-rank in Q is (p,r,s)-bounded and $[V, \underbrace{Y, \dots, Y}_{(2p-1)^2}] = 0$ by (18) and Lemma 9.

Thus Propositions 1 and 2 are proved. □

4 Proof of the main results

Finite groups. First we finish the proof of Theorem 3 for completeness and for the benefit of the reader.

Proof of Theorem 3. Recall that G is a finite nilpotent group of derived length d admitting an automorphism φ of prime order p such that $C_G(\varphi)$ has rank r. By Proposition 1 we may assume from the outset that the group G is nilpotent of (p,d)-bounded class $c = c(p,d)$. This allows us to use an analogue of the theorem in [11] on a Lie ring L with an automorphism φ of prime order. Actually in [11] we were dealing either with the case of finite fixed-point subring (centralizer) $C_L(\varphi)$ of order m, or the case of a Lie algebra L with centralizer of finite dimension m. Then the Lie ring (algebra) L contains a subring (subalgebra) of (p,m)-bounded index in the additive group (of (p,m)-bounded codimension) which is nilpotent of p-bounded class at most $g(p)$. However, an analysis of the proof in [11] shows that with minimal changes it yields also the following theorem.

Theorem 4. *If a Lie ring L the additive group of which is a q-group admits an automorphism φ of prime order p such that the additive group of the fixed-point subring $C_L(\varphi)$ has finite rank m, then L contains a φ-invariant subring M which is nilpotent of p-bounded class at most $g(p)$ such that the additive quotient group L/M has finite (p,m)-bounded rank.*

We return to the proof of Theorem 3. Clearly, we can assume that G is a finite q-group and, due to Lemma 6, that $q \neq p$. We can obviously assume that the nilpotency class c of the group G is greater than the p-bounded number $g(p)$ in Theorem 4.

Let $L = L(G) = \bigoplus \gamma_i(G)/\gamma_{i+1}(G)$ be the associated Lie ring of the group G. The automorphism φ induces an automorphism of L, which we denote by the same letter. The rank of the additive group $C_L(\varphi)$ is (p,r,d)-bounded, because $C_L(\varphi) = \bigoplus C_{\gamma_i(G)/\gamma_{i+1}(G)}(\varphi)$, where the number of summands is equal to the nilpotency class, which is (p,d)-bounded, and $C_{\gamma_i(G)/\gamma_{i+1}(G)}(\varphi) = C_{\gamma_i(G)}(\varphi)\gamma_{i+1}(G)/\gamma_{i+1}(G)$ by Lemma 2. Applying Theorem 4 we obtain a subring M of bounded co-rank in the additive group of L such that the nilpotency class of M is at most $g(p)$.

The image $\overline{M}$ of M in the additive quotient group $L/\gamma_2(L)$ can also be regarded as a subgroup of the quotient group $G/\gamma_2(G)$. Let N be the full inverse image of $\overline{M}$ in G. It follows from the definition of the operations in $L = L(G)$ that the nilpotency of class at most $g(p)$ of the Lie subring M implies that $\gamma_{g(p)+1}(N) \leqslant \gamma_{g(p)+2}(G)$, whence $\gamma_c(N) \leqslant \gamma_{c+1}(G) = 1$, since $g(p) < c$ by our assumption.

Thus, N is a normal subgroup of (p, r, d)-bounded co-rank in G such that the nilpotency class of N is at most $c-1$. Applying Theorem 2(b) we replace N by a characteristic subgroup of nilpotency class at most $c-1$ and of bounded co-rank (recall that c is (p, d)-bounded). If $c - 1 > g(p)$, the same argument can be applied to N in place of G, and so on. We repeat this procedure until we arrive at a characteristic subgroup of nilpotency class at most $g(p)$; this subgroup will have (p, r, d)-bounded co-rank, since the number of steps in this process is (p, d)-bounded. □

We now prove the main result of the paper, which is derived from the "weak" Theorem 3, a little short of a miracle, like lifting oneself by pulling one's own hair.

Proof of Theorem 1. Recall that G is a finite nilpotent group admitting an automorphism φ of prime order p such that $C_G(\varphi)$ has rank r; we need to find a characteristic subgroup C of p-bounded nilpotency class and of (p, r)-bounded co-rank. Since the rank of a finite nilpotent group is equal to the maximum of the ranks of its Sylow subgroups, we can assume from the outset that G is a finite q-group. Here we do not exclude the case $q = p$.

Let M be a subgroup of maximum order among normal φ-invariant subgroups of nilpotency class at most $g(p)$, where $g(p)$ is the function in Theorem 3. We claim that the rank of G/M is (p, r)-bounded. Let K be a Thompson critical subgroup of G/M, which is a characteristic subgroup of class at most 2 containing its centralizer (see, for example, [4, Theorem 5.3.11]). Since $(G/M)/Z(K)$ acts faithfully on K, the rank of $(G/M)/Z(K)$ is bounded in terms of the rank of K by Lemma 5. Hence it suffices to prove that the rank of K is (p, r)-bounded.

Let L be the full inverse image of $\Omega_1(K)$. It is sufficient to prove that the rank of L/M is (p, r)-bounded, because then the rank of a maximal abelian normal subgroup A/M of K will also be (p, r)-bounded and the rank of K is bounded in terms of the rank of A/M by Lemma 5.

Note that the exponent of L/M is at most q or 4, since this group is nilpotent of class at most 2 and is generated by elements of order q.

The derived length of L is at most $3 + \log_2 g(p)$. By Theorem 3 the group L contains a characteristic subgroup N of nilpotency class at most $g(p)$ such that the rank of L/N is (p, r)-bounded, say, by $f_1(p, r)$.

By Lemma 7, $|L/MN| \leqslant q^{f_2(p,r)}$ for some (p, r)-bounded number $f_2(p, r)$, since the exponent of L/MN is q (or 4), while the rank is at most $f_1(p, r)$.

The group MN is nilpotent of class at most $2g(p)$ being the product of two normal subgroups of class at most $g(p)$. Since N^q (or N^4) is contained in M, the product MN^q (or MN^4) is nilpotent of class at most $g(p)$. Hence, by Lemma 1, for some p-bounded number $g_2(p)$ the subgroup $M^{q^{g_2(p)}}N$ is nilpotent of class at most $g(p)$.

By the maximal choice of M we now have $|M^{q^{g_2(p)}}N| \leqslant |M|$. To simplify notation, let the bar denote images in $\overline{G} = G/M^{q^{g_2(p)}}$. Then the last inequality becomes $|\overline{N}| \leqslant |\overline{M}|$, which implies

$$|\overline{N}/(\overline{N}\cap\overline{M})|\leqslant|\overline{M}/(\overline{N}\cap\overline{M})|. \tag{19}$$

The right-hand side is equal to $|\overline{MN}/\overline{N}|$, which is at most $q^{f_3(p,r)}$ for some (p,r)-bounded number $f_3(p,r)$ by Lemma 7, since the rank of $\overline{MN}/\overline{N}$ is (p,r)-bounded, while the exponent is at most $q^{g_2(p)}$. The left-hand side of (19) is equal to $|\overline{MN}/\overline{M}|$ and $\overline{MN}/\overline{M}\cong MN/M$. Thus, $|MN/M|\leqslant q^{f_3(p,r)}$.

We finally have

$$|L/M|\leqslant|L/MN|\cdot|MN/M|\leqslant q^{f_2(p,r)}q^{f_3(p,r)}.$$

Hence the rank of L/M is at most $f_2(p,r)+f_3(p,r)$.

Thus, the rank of G/M is (p,r)-bounded, while the nilpotency class of M is at most $g(p)$. By Theorem 2(b) the group G has also a characteristic subgroup C of nilpotency class at most $g(p)$ such that the rank of G/C is (p,r)-bounded. □

Infinite groups. First we show that Theorem 1 cannot be extended to infinite nilpotent groups, even for an automorphism of order 2.

Example. Let q_{ij} be pairwise different odd primes, $i,j=1,2,\dots$ We fix a positive integer n. Let $\langle b_{ij1}\rangle\times\langle b_{ij2}\rangle$ be an abelian homocyclic group with $|b_{ij1}|=|b_{ij2}|=q_{ij}^n$. Let $\langle a_i\rangle$ be an infinite cyclic group. We define the action of a_i on $\langle b_{ij1}\rangle$ as that of an automorphism of order q_{ij}^{n-1} so that $b_{ij1}^{a_i}=b_{ij1}^{q+1}$, and on $\langle b_{ij2}\rangle$ as the inverse automorphism: $b_{ij2}^{a_i}=b_{ij2}^{1/(q+1)}$, where $1/(q+1)$ is the inverse of $q+1$ modulo q^n. The semidirect product $(\langle b_{ij1}\rangle\times\langle b_{ij2}\rangle)\langle a_i\rangle$ admits an automorphism φ of order 2 that inverts a_i and transposes b_{ij1} and b_{ij2}; the centralizer of φ is cyclic of order q_{ij}^n. We define the diagonal action of the semidirect product $\langle a_i\rangle\langle\varphi\rangle$ on the direct product $\prod_j\langle b_{ij1}\rangle\times\langle b_{ij2}\rangle$. Then we define the diagonal action of $\langle\varphi\rangle$ on the direct product $G=\prod_i\prod_j(\langle b_{ij1}\rangle\times\langle b_{ij2}\rangle)\langle a_i\rangle$. The group G is nilpotent of class n. The centralizer $C_G(\varphi)$ is locally cyclic. However, with n varying, it is clear that there can be no normal subgroup of bounded nilpotency class with quotient of finite rank.

Proof of Corollary 1. Now G is a locally nilpotent group admitting an automorphism φ of prime order p such that $C_G(\varphi)$ has rank r. Recall that $T=T(G)$ is the torsion part of G. First we consider the torsion-free locally nilpotent group G/T. Let $\widehat{G}$ be the Mal'cev completion of G/T, which is obtained by adjoining all roots of non-trivial elements of G/T (see [27] or [2]); the automorphism φ has a unique extension to an automorphism of $\widehat{G}$, denoted by the same letter. Since the roots are unique, the centralizer $C_{\widehat{G}}(\varphi)$ is the completion of the centralizer $C_{G/T}(\varphi)$. If $g^\varphi=gt$ for $t\in T$, then $H=\langle g,t,t^\varphi,\dots,t^{\varphi^{p-1}}\rangle$ is a φ-invariant nilpotent subgroup and $g\in C_{H/(H\cap T)}(\varphi)$. By Lemma 3(a) we have $g^{p^m}\in C_H(\varphi)(T\cap H)\subseteq C_G(\varphi)T$. Hence $C_{G/T}(\varphi)$ is contained in the completion of $C_G(\varphi)T/T$. Thus, $C_{\widehat{G}}(\varphi)$ is the completion of $C_G(\varphi)T/T$.

The rank of a radicable locally nilpotent torsion-free group is equal to the length of a normal series with factors isomorphic to $\mathbb{Q}$; see [2]. Hence the rank of $C_{\widehat{G}}(\varphi)$ is at most the rank of $C_G(\varphi)$, and therefore the rank of $C_{G/T}(\varphi)$ is at most r. By Makarenko's result [22, Theorem 2], the group G/T has a normal subgroup of p-bounded nilpotency class $c(p)$ and of (p,r)-bounded co-rank. By Theorem 2(a) we can replace this normal subgroup by a characteristic one with the same properties, which gives the required subgroup C_0 in part (a) of the corollary.

Since the group T is locally finite, by the inverse limit argument Theorem 1 implies the existence of a normal subgroup $D \leqslant T$ such that T/D has (p, r)-bounded rank and D is nilpotent of p-bounded class (see, for example, the proof of Corollary 1 in [17]). By Theorem 2(b) we can replace this normal subgroup by a characteristic one with the same properties, which gives the required subgroup C_1 in part (a).

In part (b) of the corollary, G is a finitely generated nilpotent group. Then the periodic part $T = T(G)$ is a finite group. Let T_q be the Sylow q-subgroups of T for primes q.

We can replace G by the inverse image of the characteristic subgroup C_0 given by part (a) and thus assume from the outset that G/T is nilpotent of p-bounded class at most $g(p)$.

By Lemma 6 the rank of T_p is (p, r)-bounded and the rank of $G/C_G(T_p)$ is then also (p, r)-bounded by Lemma 5. Since $C_G(T_p)$ is a characteristic subgroup of G, we can replace G by $C_G(T_p)$ and assume from the outset that $T_p \leqslant Z(G)$.

The periodic part $\bar{T}$ of the quotient G/T_p is clearly isomorphic to the Hall p'-subgroup of T, while $(G/T_p)/\bar{T}$ is isomorphic to G/T. As shown above, the rank of $C_{G/T}(\varphi)$ is at most r. Therefore the rank of $C_{G/T_p}(\varphi)$ is at most $2r$. Indeed, in a finitely generated nilpotent group all subgroups are finitely generated. If H is a subgroup of $C_{G/T_p}(\varphi)$, the quotient $H/(H \cap \bar{T})$ is generated by r elements as a section of $C_{G/T}(\varphi)$, while $H \cap \bar{T}$ is also generated by r elements as a subgroup of $C_{T/T_p}(\varphi)$.

If we prove the assertion for G/T_p, with the rank of $C_{G/T_p}(\varphi)$ being at most $2r$, finding a characteristic subgroup C such that G/C has $(p, 2r)$-bounded rank and C/T_p is nilpotent of p-bounded class $d(p)$, then C is nilpotent of class at most $d(p) + 1$, since $T_p \leqslant Z(G)$ by our assumption. Therefore we can assume from the outset that $T_p = 1$; we re-denote by the same letter r the rank of $C_G(\varphi)$.

Since G is now a finitely generated nilpotent group with periodic part being a finite p'-group, there exists a p'-number m such that $G^m \cap T = 1$. Since G/T is nilpotent of class $g(p)$, we have

$$[G^m, \underbrace{G, \ldots, G}_{g(p)}] \leqslant G^m \cap T = 1. \tag{20}$$

By Lemma 3(b) we have $C_{G/G^m}(\varphi) = C_G(\varphi)G^m/G^m$ and therefore $C_{G/G^m}(\varphi)$ has rank at most r. By Theorem 1 applied to the finite group G/G^m we obtain a characteristic subgroup C such that G/C has (p, r)-bounded rank and C/G^m is nilpotent of p-bounded class $c(p)$. In view of (20), then C is nilpotent of class $c(p) + g(p)$ and is thus the required subgroup. □

Proof of Corollary 2. As noted in [17], in the soluble case the rank of $G/O_{p'}(G)$ is (p, r)-bounded. Therefore we can assume from the outset that $G = O_{p'}(G)$. By the main results of [17], there exist characteristic subgroups $R \leqslant N_1 \leqslant G$ such that the ranks of R and G/N_1 are (p, r)-bounded, while N_1/R is nilpotent. By Lemma 2 we have $C_{N_1/R}(\varphi) = C_{N_1}(\varphi)R/R$ and therefore the rank of $C_{N_1/R}(\varphi)$ is at most r. It remains to apply Theorem 1 to the group N_1/R. □

Proof of Corollaries 3 *and* 4. These results follow from Corollary 2 by the inverse limit argument as in [17]. The improvement for locally nilpotent groups is already established in Corollary 1(a). □

References

[1] P. Fong, On orders of finite groups and centralizers of p-elements, *Osaka J. Math.* 13 (1976), 483–489.

[2] V. M. Glushkov, On some questions of the theory of nilpotent and locally nilpotent torsion-free groups, *Mat. Sb.* 30 (1952), 79–104 (Russian).

[3] Yu. M. Gorchakov, On existence of abelian subgroups of infinite ranks in locally soluble groups, *Dokl. Akad. Nauk SSSR* 146 (1964), 17–22; English transl., *Math. USSR Doklady* 5 (1964), 591–594.

[4] D. Gorenstein, *Finite Groups* (Chelsea, New York, 1968).

[5] P. Hall, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* 2 (1958) 787–801.

[6] B. Hartley, A general Brauer–Fowler theorem and centralizers in locally finite groups, *Pacific J. Math.* 152 (1992), 101–118.

[7] B. Hartley and I. M. Isaacs, On characters and fixed points of coprime operator groups, *J. Algebra* 131 (1990), 342–358.

[8] B. Hartley and T. Meixner, Finite soluble groups containing an element of prime order whose centralizer is small, *Arch. Math. (Basel)* 36 (1981), 211–213.

[9] G. Higman, Groups and rings which have automorphisms without non-trivial fixed elements, *J. London Math. Soc. (2)* 32 (1957), 321–334.

[10] E. I. Khukhro, Finite p-groups admitting an automorphism of order p with a small number of fixed points, *Mat. Zametki* 38 (1985), 652–657; English transl., *Math. Notes.* 38 (1986), 867–870.

[11] E. I. Khukhro, Groups and Lie rings admitting an almost regular automorphism of prime order, *Mat. Sbornik* 181 (1990), 1197–1219; English transl., *Math. USSR Sbornik* 71 (1992), 51–63.

[12] E. I. Khukhro, *Nilpotent Groups and their Automorphisms* (de Gruyter, Berlin, 1993).

[13] E. I. Khukhro, Finite solvable and nilpotent groups with a restriction on the rank of the centralizer of an automorphism of prime order, *Sibirsk. Mat. Zh.* 41 (2000), 451–469; English transl. in *Siberian Math. J.* 41 (2000), 373–388.

[14] E. I. Khukhro and N. Yu. Makarenko, Almost solubility of Lie algebras with almost regular automorphisms, *J. Algebra* 277 (2004), 370–407.

[15] E. I. Khukhro and N. Yu. Makarenko, Large characteristic subgroups satisfying multilinear commutator identities, *J. London Math. Soc.* 75 (2007), 635–646.

[16] E. I. Khukhro and N. Yu. Makarenko, Characteristic nilpotent subgroups of bounded co-rank and automorphically-invariant nilpotent ideals of bounded codimension in Lie algebras, *Quarterly J. Math. Oxford* 58 (2007), 229–247

[17] E. I. Khukhro and V. D. Mazurov, Finite groups with an automorphism of prime order whose centralizer has small rank, *J. Algebra* 301 (2006), 474–492.

[18] E. I. Khukhro and V. D. Mazurov, Automorphisms with centralizers of small rank, *Proc. Int. Conf. Groups St. Andrwes 2005*, vol. 2, Cambridge Univ. Press, Cambridge, 2007, 564–585.

[19] L. G. Kovács, Groups with regular automorphisms of order four, *Math. Z.* 75 (1961), 277–294.

[20] V. A. Kreknin, The solubility of Lie algebras with regular automorphisms of finite period, *Dokl. Akad. Nauk SSSR* 150 (1963), 467–469; English transl., *Math. USSR Doklady* 4 (1963), 683–685.

[21] V. A. Kreknin and A. I. Kostrikin, Lie algebras with regular automorphisms, *Dokl. Akad. Nauk SSSR* 149 (1963), 249–251; English transl., *Math. USSR Doklady* 4 (1963), 355–358.

[22] N. Yu. Makarenko, A nilpotent ideal in Lie rings with an automorphism of prime order, *Sibirsk. Mat. Zh.* 46 (2005), 1360–1373; English transl., *Siberian Math. J.* 46 (2005), 1097–1107.

[23] N. Yu. Makarenko, Graded Lie algebras with few nontrivial components, *Sibirsk. Mat. Zh.* 48 (2007), 116–137; English transl., *Siberian Math. J.* 48 (2007), 95–111.

[24] N. Yu. Makarenko and E. I. Khukhro, Lie rings that admit an automorphism of order 4 with a small number of fixed points, *Algebra i Logika* 35 (1996), no. 1, 41–78; English transl., *Algebra and Logic* 35 (1996), no. 1, 21–43.

[25] N. Yu. Makarenko and E. I. Khukhro, Nilpotent groups that admit an almost regular automorphism of order four, *Algebra i Logika* 35 (1996), no. 3, 314–333; English transl., *Algebra and Logic* 35 (1996), no. 3, 176–187.

[26] N. Yu. Makarenko and E. I. Khukhro, Lie rings that admit an automorphism of order 4 with a small number of fixed points. II, *Algebra i Logika* 37 (1998), no. 2, 144–166; English transl. in *Algebra and Logic* 37 (1998), no. 2, 78–91

[27] A. I. Mal'cev, Nilpotent torsion-free groups, *Izv. Akad. Nauk. SSSR, Ser. Mat.* 13 (1949), 201–212 (Russian).

[28] V. D. Mazurov and E. I. Khukhro, *Unsolved problems in Group Theory. Kourovka Notebook*, 13th ed., Institute of Mathematics, Novosibirsk, 1995.

[29] Yu. A. Medvedev, Groups and Lie algebras with almost regular automorphisms, *J. Algebra* 164 (1994), 877–885.

[30] Yu. I. Merzlyakov, On locally soluble groups of finite rank, *Algebra i Logika* 3 (1964), No 2, 5–16 (Russian).

[31] M. R. Pettet, Automorphisms and Fitting factors of finite groups, *J. Algebra* 72 (1981), 404–412.

[32] J. E. Roseblade, On groups in which every subgroup is subnormal, *J. Algebra* 2 (1965), 402–412.

[33] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, *Arch. Math. (Basel)* 71 (1998), 425–432.

[34] J. Thompson, Finite groups with fixed-point-free automorphisms of prime order, *Proc. Nat. Acad. Sci. U.S.A.* 45 (1959), 578–581.

[35] J. Thompson, Automorphisms of solvable groups, *J. Algebra* 1 (1964), 259–267.

[36] A. Turull, Fitting height of groups and of fixed points, *J. Algebra* 86 (1984), 555–566.

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