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Minimal Indices and Minimal Bases via Filtrations

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Abstract

In this note we develop a new way of formulating the notions of minimal basis and minimal indices, based on the concept of a filtration of a vector space. The goal is to provide useful new tools for working with these important concepts, as well as to gain deeper insight into their fundamental nature. This approach also readily reveals a strong minimality property of minimal indices, from which follows a characterization of the vector polynomial bases in rational vector spaces. The effectiveness of this new formulation is further illustrated by proving two fundamental properties: the invariance of the minimal indices of a matrix polynomial under field extension, and the direct sum property of minimal indices.

Key words. singular matrix polynomial, minimal indices, minimal basis, filtration, flag.

AMS subject classification. 65F15, 15A18, 15A21, 15A22, 15A54.

1 Introduction

Minimal indices and bases are quantities commonly associated with singular matrix polynomials, and thus play a significant role in a number of applications, especially in systems and control theory [5, 11, 14, 15], but they also are important in algebraic coding theory [4, 12, 13, 15]. Although they have been defined in the literature in several different ways [5, 6], these definitions have been shown to lead to the same quantities [2]. The purpose of this note is to introduce a new formulation of the notions of minimal basis and minimal indices, with the goals of:

- developing new tools for effectively working with these important concepts, and
- simplifying the conceptual foundation, so as to smoothly unify the classical approaches to minimal indices, and to make the well-definedness of minimal indices as transparent as possible.

This new formulation takes the algorithmic approach described in [6] by Gantmacher (and attributed to Kronecker) as a starting point, but is motivated by the following simple idea. Rather than deal with the special polynomial bases produced by Kronecker's algorithm, which are far from canonical due to the many arbitrary choices made in generating them, focus instead on the *underlying subspaces* from which these choices are made. These subspaces are uniquely defined, canonical objects which more clearly and directly reveal the intrinsic nature of minimal indices, and form the building blocks of the filtration at the heart of the new formulation.

Although the motivating idea is simple, some preliminary work is required to set up the appropriate definitions, terminology and notation needed to effectively implement this idea. Section 2 begins this background work by reviewing the notion of filtration and giving some examples, then Section 3 continues by recalling the two "classical" ways of defining minimal bases and indices. Section 3 then

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goes on to introduce the new characterization of minimal bases and indices by means of a special filtration called the "degree filtration", and from this develops two new tools for proving results about minimal indices. Section 4 shows how the filtration point of view reveals a "strong" minimality property of minimal indices, which is then used to unify and simplify the connection between the two classical approaches to minimal indices. Finally, the effectiveness of these new tools is illustrated in Sections 5 and 6 by proving the invariance of the minimal indices of a matrix polynomial under field extension, and deriving the behavior of minimal indices under direct sums.

2 Filtrations

Let us begin with a quote from Hilton & Wylie's classic text on algebraic topology [8, p. 395]:

> "We now introduce the notion of a filtration. It has a rather wide application; a construct may be said to be filtered if an increasing sequence of sub-constructs is selected which exhaust the whole construct."

Clearly all kinds of mathematical objects can be filtered: topological spaces, groups, algebras, modules, chain complexes, ... However, in this paper we only need to consider filtrations of vector spaces, which we now formally define.

Definition 2.1 (Filtration of a Vector Space)**.**
A *filtration* $\mathcal{F}$ of a vector space V is an infinite nested sequence of subspaces of V,

$$\mathcal{F}: \; W_0 \subseteq W_1 \subseteq W_2 \subseteq W_3 \subseteq \; \cdots \, , \tag{2.1}$$

such that $\bigcup_{i=0}^{\infty} W_i = V$. A vector space V equipped with a filtration $\mathcal{F}$ as in (2.1) is said to be a *filtered vector space*.

For our purposes it will be convenient to allow $W_i = W_j$ for $i \neq j$ in (2.1), although in some contexts authors require the subspaces in a filtration to be distinct. In fact the filtrations of most interest to us will usually *not* have distinct subspaces. The inclusion relations in (2.1) of course imply that

$$\dim W_0 \, \leq \, \dim W_1 \, \leq \, \dim W_2 \, \leq \; \cdots \, . \tag{2.2}$$

Definition 2.2. The infinite sequence of numbers $(\dim W_0 \, , \, \dim W_1 \, , \, \dim W_2 \, , \, \ldots)$ in (2.2) will be referred to as the *dimension sequence* of the filtration $\mathcal{F}$, and denoted $\dim \mathcal{F}$.

Example 2.3.

(a) For $V = \mathbb{F}^n$, the standard basis $\{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_n\}$ induces the "standard" filtration

$$\operatorname{span}\{\mathsf{e}_1\} \subseteq \operatorname{span}\{\mathsf{e}_1, \mathsf{e}_2\} \subseteq \cdots \subseteq \operatorname{span}\{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_{n-1}\} \subseteq V \subseteq V \subseteq \cdots \, . \tag{2.3}$$

(b) Indeed *any* ordered list of vectors $\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_m\}$ induces an associated filtration of the vector space $V = \operatorname{span}\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_m\}$ as follows:

$$\operatorname{span}\{\mathsf{v}_1\} \subseteq \operatorname{span}\{\mathsf{v}_1, \mathsf{v}_2\} \subseteq \cdots \subseteq \operatorname{span}\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_{m-1}\} \subseteq V \subseteq V \subseteq \cdots \, .$$

In particular, for any matrix $A \in \mathbb{F}^{m \times n}$, the ordered columns $\{\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n\}$ induce a filtration $\mathcal{F}(A)$ of the column space $V = \operatorname{Col}(A)$:

$$\operatorname{span}\{\mathsf{a}_1\} \subseteq \operatorname{span}\{\mathsf{a}_1, \mathsf{a}_2\} \subseteq \cdots \subseteq \operatorname{span}\{\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_{n-1}\} \subseteq V \subseteq V \subseteq \cdots \, .$$

(c) A sequence of Krylov subspaces $\operatorname{span}\{\mathsf{x}\} \subseteq \operatorname{span}\{\mathsf{x}, A\mathsf{x}\} \subseteq \operatorname{span}\{\mathsf{x}, A\mathsf{x}, A^2\mathsf{x}\} \subseteq \cdots$ is certainly nested, but does not necessarily define a filtration for $V = \mathbb{F}^n$, since the condition $\bigcup_{i=0}^{\infty} W_i = \mathbb{F}^n$ may not be satisfied.

In a filtration $\mathcal{F}$, if for some index m we have $W_m = V$ (and hence $W_n = V$ for all $n \geq m$), then $\mathcal{F}$ is said to be a *finite filtration* and is sometimes written in the truncated fashion

$$W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_m = V,$$

although we will not do so here. Note that any filtration of a finite-dimensional space V is necessarily a finite filtration, because the condition $\bigcup_{i=0}^{\infty} W_i = V$ forces $W_m = V$ to hold for some index m. A filtration such that $\dim W_j = j$ for all $0 \leq j \leq \dim V$ is called a *complete filtration*; a complete filtration of $\mathbb{F}^n$ is sometimes called a *flag*.

Remark 2.4. Various types of flags, such as "Hessenberg flags", and "eigenflags" associated to a matrix, have been used as tools in a geometric approach to understanding the convergence behavior of the QR-algorithm. See, for example, [1], [9, App. L7], [10], [16].

Definition 2.5 (Compatible Basis for a Filtration)**.**
Let V be a finite-dimensional filtered vector space, with a given filtration $\mathcal{F}$ as in (2.1). Then an ordered basis $\mathcal{B} = \{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n\}$ for V is said to be *compatible* with (or adapted to) the filtration $\mathcal{F}$ if for each subspace W_ℓ in $\mathcal{F}$ there is an initial segment $\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_j\}$ of $\mathcal{B}$ (with $j \leq n$) that forms a basis for W_ℓ.

Example 2.6.

(a) Suppose $A \in \mathbb{F}^{n\times n}$ is nonsingular. Then the (ordered) columns of A form a basis compatible with the "standard" filtration (2.3) if and only if A is upper triangular.

(b) Here is a well-known result expressed in the language of filtrations:
Suppose A is nonsingular with QR-decomposition $A = QR$. Then the ordered columns of Q form a basis compatible with the filtration $\mathcal{F}(A)$ induced by the ordered columns of A. Equivalently, one could express this result simply by saying that $\mathcal{F}(A) = \mathcal{F}(Q)$.

3 Minimal Bases and Minimal Indices

3.1 Two classical approaches

Minimal bases and indices were originally introduced by Kronecker as a means to help prove the uniqueness of what we now refer to as the Kronecker canonical form (KCF) for matrix pencils $L(\lambda) = \lambda X + Y$, where $X, Y \in \mathbb{C}^{m\times n}$. We begin by recalling the main points of this theory, summarizing the development in Gantmacher [6].

First some notation. Throughout the paper $\mathbb{F}$ will denote an arbitrary field, $\mathbb{F}[\lambda]$ the ring of polynomials in the variable λ with coefficients from $\mathbb{F}$, and $\mathbb{F}(\lambda)$ the field of rational functions over $\mathbb{F}$. Then the column vectors $\mathbb{F}(\lambda)^n$ form an n-dimensional vector space over the field $\mathbb{F}(\lambda)$, and the elements $\mathsf{v}(\lambda) \in \mathbb{F}[\lambda]^n \subset \mathbb{F}(\lambda)^n$ are the *vector polynomials* in $\mathbb{F}(\lambda)^n$. The *degree* of a vector polynomial is the maximum of the degrees of its component scalar polynomials.

For a matrix pencil $L(\lambda) = \lambda X + Y \in \mathbb{F}[\lambda]^{m\times n}$, viewed as a linear transformation $\mathbb{F}(\lambda)^n \to \mathbb{F}(\lambda)^m$, consider the right nullspace of $L(\lambda)$, i.e., consider

$$\mathcal{N}_r(L) := \left\{ \mathsf{w}(\lambda) \in \mathbb{F}(\lambda)^n \,:\, L(\lambda)\mathsf{w}(\lambda) \equiv 0 \right\}.$$

Our goal is to find a basis for the subspace $\mathcal{N}_r(L)$ consisting solely of vector polynomials, but with the *minimum possible degrees*. In the Kronecker/Gantmacher development, this minimality is defined by the following "greedy" algorithm for constructing a vector polynomial basis.

Algorithm 3.1 (Kronecker-Gantmacher Construction).

- First choose any nonzero vector polynomial $\mathsf{v}_1(\lambda) \in \mathcal{N}_r(L)$ of minimal degree.
- Next choose any vector polynomial $\mathsf{v}_2(\lambda)$ in the complement $\mathcal{N}_r(L) \setminus \operatorname{span}\{\mathsf{v}_1(\lambda)\}$ of minimal degree, and extend to $\{\mathsf{v}_1(\lambda), \mathsf{v}_2(\lambda)\}$.
- Continue in this fashion until a basis for $\mathcal{N}_r(L)$ is attained, always extending $\{\mathsf{v}_1(\lambda), \ldots, \mathsf{v}_{k-1}(\lambda)\}$ to $\{\mathsf{v}_1(\lambda), \ldots, \mathsf{v}_k(\lambda)\}$ by choosing a vector polynomial $\mathsf{v}_k(\lambda)$ of minimal degree in the remaining complement $\mathcal{N}_r(L) \setminus \operatorname{span}\{\mathsf{v}_1(\lambda), \ldots, \mathsf{v}_{k-1}(\lambda)\}$.

Definition 3.2 (K-minimal Basis).
Any vector polynomial basis produced by Algorithm 3.1 is said to be a minimal basis for $\mathcal{N}_r(L)$ in the sense of Kronecker/Gantmacher, or a *K-minimal basis* for short. Such a basis is also commonly referred to as a right minimal basis for the pencil $L(\lambda)$.

It can be shown (see [6] for details) that the ordered list of degrees $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$, where $\varepsilon_j = \deg \mathsf{v}_j(\lambda)$, is the same for *every* K-minimal basis of $\mathcal{N}_r(L)$, and thus displays an intrinsic feature of $\mathcal{N}_r(L)$.

Definition 3.3 (K-minimal Indices).
The numbers $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the *K-minimal indices* of $\mathcal{N}_r(L)$, often referred to as the right minimal indices of the pencil $L(\lambda)$.

Clearly one can proceed analogously with the left nullspace of $L(\lambda)$, i.e.,

$$\mathcal{N}_\ell(L) := \left\{ \mathsf{y}(\lambda) \in \mathbb{F}(\lambda)^m \,:\, \mathsf{y}(\lambda)^T L(\lambda) \equiv 0 \right\},$$

and thus obtain left minimal bases for $L(\lambda)$, and thence the left minimal indices of $L(\lambda)$. It can be shown (again, see [6] for details) that these left and right minimal indices encode the sizes of the "singular blocks" in the KCF of $L(\lambda)$, thus proving that the "singular part" of the KCF is uniquely determined.

An examination of Algorithm 3.1 shows that $L(\lambda)$ being a matrix pencil plays no role in the discussion; it might just as well be any matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$. Thus the same development applied to the right and left nullspaces $\mathcal{N}_r(P) \subseteq \mathbb{F}(\lambda)^n$ and $\mathcal{N}_\ell(P) \subseteq \mathbb{F}(\lambda)^m$ yields right and left minimal bases for P, as well as right and left minimal indices for P. Indeed we can go one step further, and observe that even the matrix polynomial P isn't essential. One can apply the "greedy" algorithm to *any rational subspace* $V \subseteq \mathbb{F}(\lambda)^n$, and thus obtain K-minimal bases and K-minimal indices for any such V. Note that K-minimal indices are sometimes referred to in the literature as *Kronecker indices* [7, 17].

The recognition that the notions of minimal bases and indices apply to any subspace $V \subseteq \mathbb{F}(\lambda)^n$ is made explicit by Forney in [5], where he uses a somewhat different minimality principle to define minimal bases and minimal indices. Instead of building up vector polynomial bases one step at a time and invoking a "local" minimality condition at each step of the construction, Forney works more "globally" by assigning a single number to each vector polynomial basis; the *order* of a vector polynomial basis $\mathcal{B} = \{\mathsf{v}_1(\lambda), \ldots, \mathsf{v}_p(\lambda)\}$ for a subspace $V \subseteq \mathbb{F}(\lambda)^n$ is

$$\operatorname{ord}(\mathcal{B}) := \sum_{i=1}^{p} \deg \mathsf{v}_i(\lambda)\,. \tag{3.1}$$

Thus we have the following definition.

Definition 3.4 (F-minimal Basis).
A minimal basis for a subspace $V \subseteq \mathbb{F}(\lambda)^n$ in the sense of Forney, or an *F-minimal basis*, is any vector polynomial basis for V with *minimum order* among all vector polynomial bases for V.

Forney then shows in [5] that the ordered degree sequence $0 \leq f_1 \leq f_2 \leq \cdots \leq f_p$, where $f_i = \deg \mathsf{v}_i(\lambda)$, is the same for any F-minimal basis for V, thus uniquely defining the *F-minimal indices* for V.

Remark 3.5. Note that Forney does not use the phrase F-minimal indices in [5] for the ordered degree sequence $f_1 \leq f_2 \leq \cdots \leq f_p$; instead he calls them the "invariant dynamical indices" of the subspace $V \subseteq \mathbb{F}(\lambda)^n$. Several later authors have referred to these as the *Forney indices* of V [12, 13, 15].

It is natural to wonder whether there is any simple relationship between K-minimality and F-minimality, for either bases or indices. This question was addressed in [2].

Theorem 3.6 ([2], Lemma 2.4). *Consider any subspace $V \subseteq \mathbb{F}(\lambda)^n$. Then a vector polynomial basis $\mathcal{B}$ for V is K-minimal if and only if it is F-minimal. Thus the K-minimal indices are identical to the F-minimal indices for V.*

As a consequence we see that the minimality principles used by Kronecker/Gantmacher and Forney are equivalent, and can be used interchangeably, depending on convenience.

3.2 Minimal bases from filtrations

We aim to reformulate the Kronecker/Gantmacher approach to minimal bases in such a way that it becomes *completely transparent* why all the arbitrary choices made in building a K-minimal basis always result in the same list of degrees, thereby producing a well-defined list of K-minimal indices. For an arbitrary subspace $V \subseteq \mathbb{F}(\lambda)^n$, we will see that there is an intrinsic, uniquely defined filtration induced by the notion of the *degree* of vector polynomials in V. This will be referred to as the (canonical) *degree filtration* of V, and denoted by $\mathcal{F}_{\deg}(V)$. It is from this particular filtration that we can immediately recover the K-minimal bases and K-minimal indices of V.

Here is how to define $\mathcal{F}_{\deg}(V)$. First consider various subsets of V consisting only of vector polynomials of *bounded* degree. For each integer $d \geq 0$, define

$$\mathcal{P}_d(V) := \left\{ \mathsf{v}(\lambda) \in V \,:\, \mathsf{v}(\lambda) \text{ is a vector polynomial with } \deg \mathsf{v}(\lambda) \leq d \right\}. \tag{3.2}$$

Clearly $\mathcal{P}_d(V)$ is an $\mathbb{F}$-subspace of V, but *not* an $\mathbb{F}(\lambda)$-subspace. Observe also that the inclusions

$$\mathcal{P}_0(V) \subseteq \mathcal{P}_1(V) \subseteq \mathcal{P}_2(V) \subseteq \cdots$$

clearly hold. Now to get $\mathbb{F}(\lambda)$-subspaces of V, and thence a filtration of V, we simply take the $\mathbb{F}(\lambda)$-spans of these vector polynomial subsets,

$$\mathcal{S}_d(V) := \operatorname*{span}_{\mathbb{F}(\lambda)} \mathcal{P}_d(V). \tag{3.3}$$

Note that if $\mathcal{P}_d(V)$ is non-trivial, then $\mathcal{S}_d(V)$ contains vector polynomials of *unbounded* degree, not just of degree at most d. This is because for any $\mathsf{v}(\lambda) \in \mathcal{P}_d(V)$, multiplying by the scalars $\lambda^m \in \mathbb{F}(\lambda)$ gives vector polynomials $\lambda^m \mathsf{v}(\lambda)$ that are in $\mathcal{S}_d(V)$ for every $m \in \mathbb{N}$. However, since $\mathcal{P}_d(V)$ is by definition a spanning set for $\mathcal{S}_d(V)$, we know there is always a basis of $\mathcal{S}_d(V)$ consisting solely of elements chosen from $\mathcal{P}_d(V)$; let us call any such basis a "*$\mathcal{P}_d(V)$-basis*" for $\mathcal{S}_d(V)$. Furthermore, any $\mathcal{P}_d(V)$-basis for $\mathcal{S}_d(V)$ can be *extended* to a $\mathcal{P}_{d+1}(V)$-basis for $\mathcal{S}_{d+1}(V)$. Note also that

$$\dim_{\mathbb{F}(\lambda)} \mathcal{S}_d(V) \leq \dim_{\mathbb{F}} \mathcal{P}_d(V),$$

i.e., the dimension of $\mathcal{S}_d(V)$ as an $\mathbb{F}(\lambda)$-vector space is never greater than the dimension of $\mathcal{P}_d(V)$ as an $\mathbb{F}$-vector space, and is often very much less. This is because any $\mathbb{F}$-basis for $\mathcal{P}_d(V)$ is also an $\mathbb{F}(\lambda)$-spanning set for $\mathcal{S}_d(V)$.

It is not hard to see that the condition $\bigcup_{d=0}^{\infty} \mathcal{S}_d(V) = V$ is satisfied. Suppose that $\mathsf{v}(\lambda)$ is an arbitrary element of V. Then there is some (scalar) polynomial $q(\lambda)$ such that $\mathsf{w}(\lambda) = q(\lambda)\mathsf{v}(\lambda)$ is a vector *polynomial* in V (just clear all the denominators of the entries of $\mathsf{v}(\lambda)$), with some degree $d = \deg \mathsf{w}(\lambda)$. Then $\mathsf{w}(\lambda) \in \mathcal{P}_d(V)$, and so $\mathsf{v}(\lambda) = \left(\frac{1}{q(\lambda)}\right) \cdot \mathsf{w}(\lambda)$ is in $\mathcal{S}_d(V)$. Thus every element of V is contained in some $\mathcal{S}_d(V)$, and so the equality $\bigcup_{d=0}^{\infty} \mathcal{S}_d(V) = V$ follows. Consequently the subspaces $\mathcal{S}_d(V)$ define a filtration of V.

Definition 3.7 (The Degree Filtration). The nested sequence

$$\mathcal{S}_0(V) \subseteq \mathcal{S}_1(V) \subseteq \mathcal{S}_2(V) \subseteq \cdots \subseteq \mathcal{S}_d(V) \subseteq \cdots \tag{3.4}$$

is the *degree filtration* of V, denoted $\mathcal{F}_{\deg}(V)$.

In the context of a subspace $V \subseteq \mathbb{F}(\lambda)^n$ filtered by the degree filtration, the natural bases to consider are those vector polynomial bases for V that are compatible with the filtration $\mathcal{F}_{\deg}(V)$, and at the same time provide $\mathcal{P}_d(V)$-bases for each $\mathcal{S}_d(V)$, as in the following definition.

Definition 3.8 (Minimal Basis of $\mathcal{F}_{\deg}(V)$). An ordered vector polynomial basis $\mathcal{B}$ for a subspace $V \subseteq \mathbb{F}(\lambda)^n$ is said to be a *minimal basis for the degree filtration* $\mathcal{F}_{\deg}(V)$ if

(a) $\mathcal{B}$ is *compatible* with the filtration $\mathcal{F}_{\deg}(V)$, and

(b) for each $d \geq 0$, the initial segment of $\mathcal{B}$ that forms a basis for $\mathcal{S}_d(V)$ is a $\mathcal{P}_d(V)$*-basis.*

Now observe that the Kronecker/Gantmacher construction (Algorithm 3.1) for generating a K-minimal basis can be simply described as follows: first find a $\mathcal{P}_0(V)$-basis for $\mathcal{S}_0(V)$, then extend to a $\mathcal{P}_1(V)$-basis for $\mathcal{S}_1(V)$, then extend to a $\mathcal{P}_2(V)$-basis for $\mathcal{S}_2(V)$, ..., and so on inductively through the degree filtration of V, until a vector polynomial basis for all of V is attained. But this is exactly how to generate a minimal basis for the filtration $\mathcal{F}_{\deg}(V)$; indeed any minimal basis for $\mathcal{F}_{\deg}(V)$ can be viewed as being generated in this way. Thus we have the following theorem.

Theorem 3.9 (Equivalence of Minimal Basis Concepts).
An ordered vector polynomial basis $\mathcal{B}$ for a subspace $V \subseteq \mathbb{F}(\lambda)^n$ is a K-minimal basis for V if and only if $\mathcal{B}$ is a minimal basis for the degree filtration $\mathcal{F}_{\deg}(V)$.

3.3 Minimal indices from the degree filtration

In order to conveniently work with arbitrary lists of vector polynomials, we introduce one final bit of terminology.

Definition 3.10 (Degree Sequence). Suppose $\mathcal{L} = \{\mathsf{v}_1(\lambda), \mathsf{v}_2(\lambda), \ldots, \mathsf{v}_k(\lambda)\}$ with $d_i = \deg \mathsf{v}_i(\lambda)$ for $i = 1, \ldots, k$ is any finite set of vector polynomials from a subspace $V \subseteq \mathbb{F}(\lambda)^n$, and let $\mathcal{L}$ be ordered so that $d_i \leq d_{i+1}$ for $i = 1, \ldots, k-1$. Then the list of numbers $d_1 \leq d_2 \leq \cdots \leq d_k$ is the *degree sequence* of $\mathcal{L}$.

An important feature of the characterization in Theorem 3.9 is that it transparently reveals why every K-minimal basis for V has exactly the same list of degrees, and hence *why* the notion of K-minimal indices of V is *well-defined* and meaningful. Since K-minimality and minimality with respect to the degree filtration are now seen to be equivalent notions, from now on we will just use the phrase "minimal indices", as in the following definition.

Definition 3.11 (Minimal Indices). The *minimal indices* of a subspace $V \subseteq \mathbb{F}(\lambda)^n$ are the numbers $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ in the degree sequence of any K-minimal basis for V, or equivalently, in the degree sequence of any minimal basis for the degree filtration $\mathcal{F}_{\deg}(V)$.

From this definition it is clear that the number of minimal indices $\varepsilon = 0$ is just $\dim_{\mathbb{F}(\lambda)} \mathcal{S}_0(V)$, the number of minimal indices with $\varepsilon \leq 1$ is $\dim_{\mathbb{F}(\lambda)} \mathcal{S}_1(V)$, and in general for any $d \in \mathbb{N}$ the number of minimal indices with $\varepsilon \leq d$ is $\dim_{\mathbb{F}(\lambda)} \mathcal{S}_d(V)$. Thus we have a simple and intrinsic way to characterize minimal indices, which gives us the following useful tool for determining and working with them.

Theorem 3.12 (Minimal Indices from the Degree Filtration).
Let V be an arbitrary subspace of $\mathbb{F}(\lambda)^n$. Then the minimal indices of V are uniquely determined by the dimension sequence of the degree filtration $\mathcal{F}_{\deg}(V)$, and vice versa. In particular, the number of zero minimal indices $\varepsilon = 0$ is $\dim\limits_{\mathbb{F}(\lambda)} \mathcal{S}_0(V)$, while for $j \geq 1$ the number of minimal indices $\varepsilon = j$ is

$$\dim_{\mathbb{F}(\lambda)} \mathcal{S}_j(V) \; - \; \dim_{\mathbb{F}(\lambda)} \mathcal{S}_{j-1}(V) \,. \tag{3.5}$$

Conversely, the dimensions of the subspaces $\mathcal{S}_d(V)$ are uniquely determined from the minimal indices of V by

$$\dim_{\mathbb{F}(\lambda)} \mathcal{S}_d(V) \; = \; \textit{total number of minimal indices with } \varepsilon \leq d. \tag{3.6}$$

As an immediate corollary we have the following criterion for deciding if two subspaces have the *same* minimal indices.

Corollary 3.13 (Equality of Minimal Indices).
Let $\mathbb{F}$ and $\mathbb{K}$ be any two fields, and let n and q be any two positive integers. Then a pair of subspaces $V \subseteq \mathbb{F}(\lambda)^n$ and $W \subseteq \mathbb{K}(\lambda)^q$ have the same *minimal indices if and only if*

$$\dim_{\mathbb{F}(\lambda)} \mathcal{F}_{\deg}(V) \; = \; \dim_{\mathbb{K}(\lambda)} \mathcal{F}_{\deg}(W) \,,$$

i.e., if and only if the degree filtrations of V and W have identical *dimension sequences.*

In Sections 5 and 6 we will illustrate the efficacy of the tools provided by Theorem 3.12 and Corollary 3.13 by establishing two fundamental properties of minimal indices:

(i) for nullspaces of matrix polynomials, minimal indices are unchanged by passage to any field extension,

(ii) minimal indices behave nicely under direct sums; i.e., the minimal indices of $V \oplus W$, where $V \subseteq \mathbb{F}(\lambda)^n$ and $W \subseteq \mathbb{F}(\lambda)^m$, are just the concatenation of the minimal indices of V and W.

But first we will see how the degree filtration notion of minimal indices connects up with Forney's approach in [5]. This connection will be made via a property of minimal indices brought to light by the filtration view, a "strong" minimality property that leads eventually to a characterization of the vector polynomial bases in any rational subspace of $\mathbb{F}(\lambda)^n$.

4 A Strong Minimality Property of Minimal Indices

The filtration view of minimal indices provides insight that is not so readily obtained from either the Kronecker or Forney point of view, as illustrated by the following simple example.

Example 4.1. Suppose a two-dimensional rational subspace $V \subseteq \mathbb{F}(\lambda)^n$ has minimal indices $\varepsilon_1 = 1$ and $\varepsilon_2 = 4$. Is it possible for there to exist a vector polynomial basis $\mathcal{B} = \{\mathsf{v}_1(\lambda), \mathsf{v}_2(\lambda)\}$ for V with degree sequence $(3, 3)$? From the Kronecker/Gantmacher point of view this seems to be perfectly plausible; once you choose $\mathsf{v}_1(\lambda)$ with *non*-minimal degree ($\deg \mathsf{v}_1 = 3$) to be the first basis vector in $\mathcal{B}$, then you are no longer following the greedy algorithm, so all bets are off as to what might be available for a second basis vector. The Forney view also does not seem to offer any objection to the existence of such a basis $\mathcal{B}$, since $\operatorname{ord}(\mathcal{B}) = 6$ is certainly compatible with the minimal order being $\varepsilon_1 + \varepsilon_2 = 5$.

In fact, though, it is impossible for V to have such a basis $\mathcal{B}$, and this can be seen rather easily from the filtration point of view. The minimal indices $\varepsilon_1 = 1$ and $\varepsilon_2 = 4$ immediately imply that the dimension sequence of the degree filtration of V must be $(0, 1, 1, 1, 2, 2, 2, 2, \dots)$; in particular, we would have $\dim \mathcal{S}_3(V) = 1$. But the existence of a vector polynomial basis $\mathcal{B}$ with degree sequence $(3, 3)$ would mean that $\mathcal{S}_3(V)$ would have to have dimension 2. This contradiction shows the impossibility of such a basis $\mathcal{B}$ for V. This example also hints at the presence of subtle constraints on the possible vector polynomial bases in a general rational vector space $V \subseteq \mathbb{F}(\lambda)^n$, or at least at constraints that are not so obvious from either the Forney or Kronecker/Gantmacher definitions.

The idea used to resolve Example 4.1 can be extended and refined to establish the following strong minimality property, which now fully justifies the name "minimal indices". Note that the proof given here is a modified version of an argument used in [2, Lemma 2.4].

Theorem 4.2 (Strong Minimality Property of Minimal Indices)**.**
Suppose that $V \subseteq \mathbb{F}(\lambda)^n$ is a p-dimensional subspace with minimal indices $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$, and $\mathcal{B} = \{\mathsf{v}_1(\lambda), \dots, \mathsf{v}_p(\lambda)\}$ is an arbitrary *vector polynomial basis for V. If $\delta_1 \le \delta_2 \le \cdots \le \delta_p$ is the degree sequence of $\mathcal{B}$, then*

$$\varepsilon_j \le \delta_j \quad \textit{for each } j = 1, \dots, p. \tag{4.1}$$

Proof. The proof proceeds by induction on j to show that $\varepsilon_j \le \delta_j$ holds for every $1 \le j \le p$. First observe that ε_1 is by definition the minimum degree of all vector polynomials in V, so clearly we must have $\varepsilon_1 \le \delta_1$. Now suppose that $\varepsilon_j \le \delta_j$ holds for every $1 \le j \le m$, where $1 \le m < p$. Let us see why we must also have $\varepsilon_{m+1} \le \delta_{m+1}$. Suppose not, i.e., suppose that $\varepsilon_{m+1} > \delta_{m+1}$. Then we would have $\varepsilon_m \le \delta_m \le \delta_{m+1} < \varepsilon_{m+1}$. Letting $d = \varepsilon_{m+1} - 1$, so that

$$\varepsilon_m \le \delta_m \le \delta_{m+1} \le d < \varepsilon_{m+1},$$

we see from (3.6) that $\dim_{\mathbb{F}(\lambda)} \mathcal{S}_d(V) = m$. But at the same time $\{\mathsf{v}_1(\lambda), \mathsf{v}_2(\lambda), \dots, \mathsf{v}_{m+1}(\lambda)\}$ would constitute an $\mathbb{F}(\lambda)$-linearly independent set of $m+1$ vector polynomials in $\mathcal{S}_d(V)$. This contradiction shows that $\varepsilon_{m+1} > \delta_{m+1}$ is impossible, so $\varepsilon_{m+1} \le \delta_{m+1}$ and the induction is complete. □

As an easy consequence of this strong minimality property, we can now immediately (and simultaneously) establish two of the fundamental properties of the F-minimal indices introduced by Forney in [5]: that they are well-defined, and that they are identical to the K-minimal indices. Note that this argument is completely independent of the results in [5], and substantially simplifies and unifies our understanding of the relationships between these various notions of minimal indices.

Corollary 4.3 (F-minimal and K-minimal indices are Identical)**.**
A vector polynomial basis $\mathcal{B}$ for a subspace $V \subseteq \mathbb{F}(\lambda)^n$ is F-minimal (i.e., has minimal order) if and only if its degree sequence is identical *to the minimal indices of the degree filtration $\mathcal{F}_{\deg}(V)$.*

Proof. By (4.1) in Theorem 4.2 it is clear that for every vector polynomial basis $\mathcal{B}$ for V, the order of $\mathcal{B}$ is bounded below by the sum $\mu = \sum_j \varepsilon_j$ of all the minimal indices of the degree filtration $\mathcal{F}_{\deg}(V)$. But since this lower bound μ is actually attained by any K-minimal basis, this must be the order of any F-minimal basis for V. It is also clear from (4.1) that the *only* way for a vector polynomial basis $\mathcal{B}$ to attain the minimum order μ is for its degree sequence to be *identical* to the minimal indices of the degree filtration $\mathcal{F}_{\deg}(V)$. □

The strong minimality property of minimal indices can be furthered strengthened. The converse of Theorem 4.2 can also be shown to hold, thus making it possible to *characterize* the degree sequences of all vector polynomial bases in any rational subspace $V \subseteq \mathbb{F}(\lambda)^n$ in terms of the minimal indices of V.

Theorem 4.4 (Characterization of Vector Polynomial Basis Degree Sequences).
Suppose that $V \subseteq \mathbb{F}(\lambda)^n$ is a p-dimensional subspace with minimal indices $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$. Then there exists a vector polynomial basis for V with degree sequence $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_p$ if and only if

$$\varepsilon_j \leq \delta_j \quad \textit{for} \text{ each } \ j = 1, \ldots, p\,. \tag{4.2}$$

Proof. The necessity of the conditions in (4.2) is exactly the content of Theorem 4.2. To establish the sufficiency, consider any minimal basis $\{\mathsf{v}_1(\lambda), \mathsf{v}_2(\lambda), \ldots, \mathsf{v}_p(\lambda)\}$ for V. Then

$$\mathcal{B} := \{\lambda^{\delta_1 - \varepsilon_1}\mathsf{v}_1(\lambda), \lambda^{\delta_2 - \varepsilon_2}\mathsf{v}_2(\lambda), \ldots, \lambda^{\delta_p - \varepsilon_p}\mathsf{v}_p(\lambda)\}$$

is clearly a vector polynomial basis for V, with the desired degree sequence $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_p$. □

Remark 4.5. The strong minimality property of minimal indices proved in Theorem 4.2 does not seem to be widely known. However, it does appear in the coding theory literature, as least as early as [13].

5 Minimal Indices and Field Extensions

Historically, the original reason to introduce the notions of minimal bases and indices was as a tool to help clarify the properties of the Kronecker canonical form, a classical result for matrix pencils over algebraically closed fields (see [6]). In a recent investigation of a new equivalence relation on matrix polynomials [3], matrix pencils $L(\lambda)$ over an *arbitrary* field $\mathbb{F}$ were under consideration, and we wanted to make use of the Kronecker canonical form of $L(\lambda)$, viewed as a pencil over the algebraic closure $\overline{\mathbb{F}}$. In this context, a key question is whether the minimal indices of L can be affected by the change of field from $\mathbb{F}$ to $\overline{\mathbb{F}}$. The goal of this section is to resolve this issue, by proving the following invariance result for the minimal indices of a matrix polynomial over an arbitrary field.

Theorem 5.1 (Invariance of Minimal Indices under Field Extension).
Suppose $P(\lambda)$ is an $m \times n$ matrix polynomial over a field $\mathbb{F}$, and $\widetilde{\mathbb{F}} \supseteq \mathbb{F}$ is an extension field. Let

$$V := \{\mathsf{v}(\lambda) \in \mathbb{F}(\lambda)^n : P(\lambda)\mathsf{v}(\lambda) \equiv 0\} \subseteq \mathbb{F}(\lambda)^n$$

be the right nullspace of $P(\lambda)$ viewed as a matrix polynomial over $\mathbb{F}$, and

$$W := \{\mathsf{w}(\lambda) \in \widetilde{\mathbb{F}}(\lambda)^n : P(\lambda)\mathsf{w}(\lambda) \equiv 0\} \subseteq \widetilde{\mathbb{F}}(\lambda)^n$$

be the right nullspace of $P(\lambda)$ viewed as a matrix polynomial over $\widetilde{\mathbb{F}}$. Then the minimal indices of V and W are identical.

The strategy of the proof is to show that the dimension sequences $\dim \mathcal{F}_{\deg}(V)$ over the field $\mathbb{F}(\lambda)$, and $\dim \mathcal{F}_{\deg}(W)$ over the field $\widetilde{\mathbb{F}}(\lambda)$, are identical; Corollary 3.13 then implies the desired conclusion. Before we get to the proof of the theorem we need some preliminary results.

Lemma 5.2. *Suppose $\mathbb{K}$ is a field, $\widetilde{\mathbb{K}} \supseteq \mathbb{K}$ is a field extension, and $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell \in \mathbb{K}^n \subseteq \widetilde{\mathbb{K}}^n$ are column vectors in $\mathbb{K}^n$ (hence also in $\widetilde{\mathbb{K}}^n$). Then*

$$\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell \textit{ are linearly independent in } \mathbb{K}^n \iff \mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell \textit{ are lin. indep. in } \widetilde{\mathbb{K}}^n\,.$$

Proof. Line up the ℓ column vectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell$ side-by-side to form an $n \times \ell$ matrix

$$A = [\mathsf{v}_1 \ \mathsf{v}_2 \ \ldots \ \mathsf{v}_\ell] \in \mathbb{K}^{n \times \ell} \subset \widetilde{\mathbb{K}}^{n \times \ell}.$$

Then $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell$ are linearly independent in $\mathbb{K}^n$ $\Leftrightarrow$ A has an $\ell \times \ell$ submatrix $\widehat{A}$ such that $\det \widehat{A} \neq 0$ $\Leftrightarrow$ $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_\ell$ are linearly independent in $\widetilde{\mathbb{K}}^n$. □

Lemma 5.3. *Let $P(\lambda)$, V, and W be as in* Theorem 5.1. *Then for each $d \in \mathbb{N}$, we have*

$$\dim_{\mathbb{F}} \mathcal{P}_d(V) = \dim_{\widetilde{\mathbb{F}}} \mathcal{P}_d(W)\,.$$

Proof. Let $P(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k$ be an $m \times n$ matrix polynomial over a field $\mathbb{F}$. Extending a technique used by Gantmacher [6] to analyze singular pencils, consider the following (possibly rectangular) block-Toeplitz matrices built from the coefficient matrices of $P(\lambda)$:

$$M_0 := \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{bmatrix}, \quad M_1 := \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \\ \vdots & A_1 \\ A_k & \vdots \\ 0 & A_k \end{bmatrix}, \ \ldots\ , M_d := \begin{bmatrix} A_0 & & & \\ A_1 & A_0 & & \\ \vdots & A_1 & \ddots & \\ A_k & \vdots & \ddots & A_0 \\ & A_k & & A_1 \\ & & \ddots & \vdots \\ & & & A_k \end{bmatrix}, \ \ldots \tag{5.1}$$

where M_d has $d+1$ block columns. Now suppose $\mathsf{v}(\lambda) = \mathsf{v}_0 + \lambda \mathsf{v}_1 + \lambda^2 \mathsf{v}_2 + \cdots + \lambda^d \mathsf{v}_d$ is a vector polynomial of degree at most d with $\mathsf{v}_i \in \mathbb{F}^n$ for $i = 0, \ldots, d$, and $\widetilde{\mathsf{v}} := [\,\mathsf{v}_0^T\ \mathsf{v}_1^T\ \ldots\ \mathsf{v}_d^T\,]^T$ is the vector in $\mathbb{F}^{(d+1)n}$ formed by vertically stacking up all the coefficient vectors v_i. Then it is not hard to see that $\mathsf{v}(\lambda) \in \mathcal{P}_d(V)$ if and only if

$$A_0\mathsf{v}_0 = 0\,, \quad A_1\mathsf{v}_0 + A_0\mathsf{v}_1 = 0\,, \quad A_2\mathsf{v}_0 + A_1\mathsf{v}_1 + A_0\mathsf{v}_2 = 0\,, \ \ldots\ , \quad A_k\mathsf{v}_d = 0$$

if and only if $M_d\,\widetilde{\mathsf{v}} = 0$. From this it follows that the map

$$\begin{array}{rcl} \mathcal{P}_d(V) & \longrightarrow & \ker M_d \\ \mathsf{v}(\lambda) & \longmapsto & \widetilde{\mathsf{v}} \end{array}$$

is a linear isomorphism of $\mathbb{F}$-vector spaces, and so $\dim_{\mathbb{F}} \mathcal{P}_d(V) = \dim_{\mathbb{F}} \ker M_d$.

The same argument, viewing M_d and the coefficients of $P(\lambda)$ as matrices with entries in any extension field $\widetilde{\mathbb{F}}$, shows that $\dim_{\widetilde{\mathbb{F}}} \mathcal{P}_d(W) = \dim_{\widetilde{\mathbb{F}}} \ker M_d$. Now the rank/nullity theorem implies that $\dim \ker M_d = (d+1)n - \operatorname{rank} M_d$. But the rank of a matrix is the size of the largest square submatrix with nonzero determinant, so rank is insensitive to field extensions. Thus we have

$$\dim_{\mathbb{F}} \mathcal{P}_d(V) \ = \ \dim_{\mathbb{F}}\ \ker M_d \ = \ (d+1)n - \operatorname{rank} M_d \ = \ \dim_{\widetilde{\mathbb{F}}}\ \ker M_d \ = \ \dim_{\widetilde{\mathbb{F}}} \mathcal{P}_d(W)\,,$$

and the proof is complete. □

With these two lemmas in hand, we now return to the proof of Theorem 5.1.

Proof. (of Theorem 5.1)
The strategy of the proof is to show that for each $d \in \mathbb{N}$, the $\mathbb{F}(\lambda)$-dimension of $\mathcal{S}_d(V)$ is the same as the $\widetilde{\mathbb{F}}(\lambda)$-dimension of $\mathcal{S}_d(W)$, and hence that

$$\dim_{\mathbb{F}(\lambda)} \mathcal{F}_{\deg}(V) \ = \ \dim_{\widetilde{\mathbb{F}}(\lambda)} \mathcal{F}_{\deg}(W)\,,$$

i.e., the degree filtrations $\mathcal{F}_{\deg}(V)$ and $\mathcal{F}_{\deg}(W)$ have identical dimension sequences. It then follows from Corollary 3.13 that the minimal indices of V and W are identical, as desired.

To get a handle on the spaces $\mathcal{S}_d(V)$ and $\mathcal{S}_d(W)$, we begin by considering the vector polynomial spaces $\mathcal{P}_d(V)$ and $\mathcal{P}_d(W)$. Let $\mathcal{B}$ be any $\mathbb{F}$-basis for the $\mathbb{F}$-vector space $\mathcal{P}_d(V)$. Then clearly $\mathcal{B} \subseteq \mathcal{P}_d(W)$, and by Lemma 5.2 (with $\mathbb{K} = \mathbb{F}$ and $\widetilde{\mathbb{K}} = \widetilde{\mathbb{F}}$) it follows that $\mathcal{B}$ is an $\widetilde{\mathbb{F}}$-linearly independent subset of $\mathcal{P}_d(W)$. The equality of the dimensions $\dim_{\mathbb{F}} \mathcal{P}_d(V)$ and $\dim_{\widetilde{\mathbb{F}}} \mathcal{P}_d(W)$ from Lemma 5.3 now implies that $\mathcal{B}$ is also an $\widetilde{\mathbb{F}}$-basis for the $\widetilde{\mathbb{F}}$-vector space $\mathcal{P}_d(W)$.

Since $\mathcal{P}_d(V)$ is $\mathbb{F}$-generated by $\mathcal{B}$, and $\mathcal{S}_d(V)$ is $\mathbb{F}(\lambda)$-generated by $\mathcal{P}_d(V)$, the fact that $\mathbb{F} \subseteq \mathbb{F}(\lambda)$ implies that $\mathcal{B}$ is an $\mathbb{F}(\lambda)$-spanning set for $\mathcal{S}_d(V)$. The same kind of argument shows that $\mathcal{B}$ is also an $\widetilde{\mathbb{F}}(\lambda)$-spanning set for $\mathcal{S}_d(W)$.

Inside of the spanning set $\mathcal{B}$ we can now find a subset $\widehat{\mathcal{B}} \subseteq \mathcal{B}$ that forms an $\mathbb{F}(\lambda)$-basis for $\mathcal{S}_d(V)$, and hence is a *maximal* $\mathbb{F}(\lambda)$-linearly independent subset of $\mathcal{S}_d(V)$. Using Lemma 5.2 again, this time with $\mathbb{K} = \mathbb{F}(\lambda)$ and $\widetilde{\mathbb{K}} = \widetilde{\mathbb{F}}(\lambda)$, we see that $\widehat{\mathcal{B}}$ is not only an $\widetilde{\mathbb{F}}(\lambda)$-linearly independent subset of $\mathcal{S}_d(W)$, it is actually a *maximal* $\widetilde{\mathbb{F}}(\lambda)$-linearly independent subset of $\mathcal{S}_d(W)$. (If not, then there would be a strictly larger $\widetilde{\mathbb{F}}(\lambda)$-linearly independent subset $\mathcal{B}'$ of $\mathcal{B}$, i.e., $\widehat{\mathcal{B}} \subset \mathcal{B}' \subseteq \mathcal{B}$, which by Lemma 5.2 would contradict the maximality of $\widehat{\mathcal{B}}$ as an $\mathbb{F}(\lambda)$-linearly independent subset of $\mathcal{S}_d(V)$.) Thus we see that $\widehat{\mathcal{B}}$ is simultaneously an $\mathbb{F}(\lambda)$-basis for $\mathcal{S}_d(V)$ as well as an $\widetilde{\mathbb{F}}(\lambda)$-basis for $\mathcal{S}_d(W)$, showing that $\dim_{\mathbb{F}(\lambda)} \mathcal{S}_d(V) = \dim_{\widetilde{\mathbb{F}}(\lambda)} \mathcal{S}_d(W)$, and the proof is complete. □

Remark 5.4. Note that Theorem 5.1, despite the simple nature of its statement, has not to our knowledge appeared before in the literature. Indeed, the search for a clear proof of this result was the primary motivation for developing the filtration formulation of minimal indices in the first place.

Remark 5.5. Clearly the invariance of minimal indices under field extension also holds for the *left* nullspace of any matrix polynomial $P(\lambda)$, since the left nullspace of $P(\lambda)$ is the same as the right nullspace of $P^T(\lambda)$.

6 Minimal Indices and Direct Sums

In the course of developing the Kronecker canonical form in [6], Gantmacher remarks *without proof* that:

> The complete system of indices for the columns (rows) of a quasi-diagonal matrix is obtained as the union of the corresponding systems of minimal indices of the individual diagonal blocks. [6, Vol. II, p.39]

In my view this assertion is not obvious, and requires some proof, especially since it is an essential component of the overall argument for the KCF. In this section we close this "gap" by showing that minimal indices behave nicely under direct sums more generally; in particular, we show that the minimal indices of $V \oplus W$ are just the concatenation of the minimal indices of V and W. The filtration view of minimal indices allows for a completely straightforward proof of this result, although some preliminary discussion of how filtrations interact with direct sums is needed. Thus we begin with the simple notion of the direct sum of filtrations, applicable to any pair of filtered vector spaces.

Definition 6.1 (Direct Sum of Filtrations)**.**
Suppose V and W are filtered vector spaces over the same field $\mathbb{K}$, with given filtrations

$$\mathcal{F}(V):\ V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \quad \text{and} \quad \mathcal{F}(W):\ W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots,$$

respectively. Then the *direct sum filtration* $\mathcal{F}(V) \oplus \mathcal{F}(W)$ of the $\mathbb{K}$-vector space $V \oplus W$ is defined to be

$$V_0 \oplus W_0 \ \subseteq\ V_1 \oplus W_1 \ \subseteq\ V_2 \oplus W_2 \ \subseteq\ \cdots \quad .$$

In Section 3 we introduced some special vector spaces V having a canonical filtration, namely subspaces $V \subseteq \mathbb{F}(\lambda)^n$ equipped with the degree filtration $\mathcal{F}_{\deg}(V)$. An obvious question is whether the degree filtration of the direct sum $V \oplus W$ has any nice relationship to the degree filtrations of the individual spaces V and W. The next result gives the simple answer.

Lemma 6.2 (Degree Filtration of Direct Sum).
Let $V \subseteq \mathbb{F}(\lambda)^n$ and $W \subseteq \mathbb{F}(\lambda)^k$ be arbitrary subspaces, so that $V \oplus W$ is a subspace of $\mathbb{F}(\lambda)^{n+k}$. Then

$$\mathcal{F}_{\deg}(V \oplus W) \;=\; \mathcal{F}_{\deg}(V) \oplus \mathcal{F}_{\deg}(W)\,. \tag{6.1}$$

Proof. To determine the degree filtration of $V \oplus W$, we must first understand the vector polynomial subsets $\mathcal{P}_d(V \oplus W)$. Any element of $V \oplus W$ has the form $\mathsf{z} = \begin{bmatrix} \mathsf{v} \\ \mathsf{w} \end{bmatrix}$, where $\mathsf{v} \in V$ and $\mathsf{w} \in W$. Thus any vector polynomial $\mathsf{z} \in \mathcal{P}_d(V \oplus W)$ must be built from some $\mathsf{v} \in \mathcal{P}_d(V)$ together with some $\mathsf{w} \in \mathcal{P}_d(W)$. Consequently we see that $\mathcal{P}_d(V \oplus W) \,=\, \mathcal{P}_d(V) \oplus \mathcal{P}_d(W)$.

Next we consider the $\mathbb{F}(\lambda)$-span of $\mathcal{P}_d(V \oplus W)$, or equivalently of $\mathcal{P}_d(V) \oplus \mathcal{P}_d(W)$, in order to obtain the $\mathbb{F}(\lambda)$-subspace $\mathcal{S}_d(V \oplus W)$ that is part of the degree filtration $\mathcal{F}_{\deg}(V \oplus W)$. It is easy to see that any linear combination of elements z_i from the direct sum $\mathcal{P}_d(V) \oplus \mathcal{P}_d(W)$ can be expressed as the direct sum of a linear combination from $\mathcal{P}_d(V)$ together with a linear combination from $\mathcal{P}_d(W)$:

$$\sum_i c_i \mathsf{z}_i \;=\; \sum_i c_i \begin{bmatrix} \mathsf{v}_i \\ \mathsf{w}_i \end{bmatrix} \;=\; \begin{bmatrix} \sum c_i \mathsf{v}_i \\ \sum c_i \mathsf{w}_i \end{bmatrix} \;=\; \sum_i c_i \mathsf{v}_i \oplus \sum_i c_i \mathsf{w}_i\,.$$

Conversely, any direct sum of a linear combination from $\mathcal{P}_d(V)$ and a linear combination from $\mathcal{P}_d(W)$ can be written as a linear combination of elements from $\mathcal{P}_d(V) \oplus \mathcal{P}_d(W)$:

$$\sum_i b_i \mathsf{v}_i \oplus \sum_j c_j \mathsf{w}_j \;=\; \begin{bmatrix} \sum b_i \mathsf{v}_i \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \sum c_j \mathsf{w}_j \end{bmatrix} \;=\; \sum_i b_i \begin{bmatrix} \mathsf{v}_i \\ 0 \end{bmatrix} + \sum_j c_j \begin{bmatrix} 0 \\ \mathsf{w}_j \end{bmatrix}.$$

Together these observations imply that

$$\operatorname*{span}_{\mathbb{F}(\lambda)} \big[\, \mathcal{P}_d(V) \oplus \mathcal{P}_d(W) \,\big] \;=\; \operatorname*{span}_{\mathbb{F}(\lambda)} \big[\, \mathcal{P}_d(V) \,\big] \;\oplus\; \operatorname*{span}_{\mathbb{F}(\lambda)} \big[\, \mathcal{P}_d(W) \,\big] \;=:\; \mathcal{S}_d(V) \oplus \mathcal{S}_d(W)\,.$$

Thus we have

$$\mathcal{S}_d(V \oplus W) \;:=\; \operatorname*{span}_{\mathbb{F}(\lambda)} \big[\, \mathcal{P}_d(V \oplus W) \,\big] \;=\; \operatorname*{span}_{\mathbb{F}(\lambda)} \big[\, \mathcal{P}_d(V) \oplus \mathcal{P}_d(W) \,\big] \;=\; \mathcal{S}_d(V) \oplus \mathcal{S}_d(W)\,.$$

Assembling these together for all $d \in \mathbb{N}$ shows that (6.1) holds. □

Now that we know how degree filtrations behave with respect to direct sums, it is just one easy further step to see how the minimal indices of a direct sum are related to the minimal indices of the summands.

Theorem 6.3 (Minimal Indices of Direct Sum).
Suppose $V \subseteq \mathbb{F}(\lambda)^n$ and $W \subseteq \mathbb{F}(\lambda)^k$ are arbitrary subspaces, so $V \oplus W$ is a subspace of $\mathbb{F}(\lambda)^{n+k}$. Then the minimal indices of $V \oplus W$ are just the concatenation of the minimal indices of V with those of W.

Proof. From (6.1) it follows that the dimension sequence of $\mathcal{F}_{\deg}(V \oplus W)$ is just the (entry-wise) sum of the dimension sequences of $\mathcal{F}_{\deg}(V)$ and $\mathcal{F}_{\deg}(W)$, i.e., that

$$\dim \mathcal{F}_{\deg}(V \oplus W) \;=\; \dim \mathcal{F}_{\deg}(V) \,+\, \dim \mathcal{F}_{\deg}(W)\,. \tag{6.2}$$

The desired result now follows from (6.2) and the minimal index formula (3.5) in Theorem 3.12:

$$\begin{aligned}
\#\,(\text{min. indices } \varepsilon = d \text{ for } V \oplus W) \;&=\; \dim \mathcal{S}_d(V \oplus W) \,-\, \dim \mathcal{S}_{d-1}(V \oplus W) \\
&=\; \left\{ \begin{array}{l} \big[\dim \mathcal{S}_d(V) + \dim \mathcal{S}_d(W)\big] \\ \qquad - \,\big[\dim \mathcal{S}_{d-1}(V) + \dim \mathcal{S}_{d-1}(W)\big] \end{array} \right\} \\
&=\; \left\{ \begin{array}{l} \big[\dim \mathcal{S}_d(V) - \dim \mathcal{S}_{d-1}(V)\big] \\ \qquad + \,\big[\dim \mathcal{S}_d(W) - \dim \mathcal{S}_{d-1}(W)\big] \end{array} \right\} \\
&=\; \left\{ \begin{array}{l} \#\,(\text{min. indices } \varepsilon = d \text{ for } V) \\ \quad + \;\; \#\,(\text{min. indices } \varepsilon = d \text{ for } W) \end{array} \right\},
\end{aligned}$$

and the theorem is proved. □

7 Conclusions

We have shown how the Kronecker/Gantmacher approach to the minimal bases and indices of any subspace $V \subseteq \mathbb{F}(\lambda)^n$ can be reformulated in a more intrinsic fashion using the degree filtration of V. This reformulation unifies and simplifies our understanding of the classical approaches to minimal bases and indices, clarifies the relationship between these approaches, and provides new tools with which to prove basic properties of minimal indices. These new tools have been utilised to show that the minimal indices of any singular matrix polynomial are unchanged by field extension, and to prove the direct sum property of minimal indices. The filtration point of view has also provided deeper insight into minimal indices, bringing an under-recognized strong minimality property clearly into the light, thereby leading to a characterization of the vector polynomial bases in rational vector spaces.

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