Finite and Infinite Elementary Divisors of Matrix Polynomials: A Global Approach

Zaballa, Ion and Tisseur, Francoise

2012

MIMS EPrint: **2012.78**

ISSN 1749-9097

FINITE AND INFINITE ELEMENTARY DIVISORS OF MATRIX POLYNOMIALS: A GLOBAL APPROACH *

ION ZABALLA† AND FRANÇOISE TISSEUR‡

1. Introduction. There is general agreement on the definition of the finite elementary divisors of a matrix polynomial $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$, where $\mathbb{F}$ an arbitrary field. One starts with the equivalence of matrix polynomials: $A(\lambda), B(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ are equivalent if and only if there are unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ (i.e., matrices whose determinants are units of $\mathbb{F}[\lambda]$) such that $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$ and proves the existence of a canonical form (Smith normal form) in $\mathbb{F}[\lambda]^{m\times n}$. In fact, any $m \times n$ matrix polynomial $A(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ with coefficients in an arbitrary field $\mathbb{F}$, is equivalent to a diagonal matrix polynomial called the *Smith form* of $A(\lambda)$, that is, there are unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that

$$U(\lambda)A(\lambda)V(\lambda) = D(\lambda) = \begin{bmatrix} \operatorname{diag}\big(\alpha_1(\lambda), \dots, \alpha_r(\lambda)\big) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \operatorname{rank} A(\lambda)$ and $\alpha_1(\lambda)|\cdots|\alpha_r(\lambda)$ are monic polynomials. Here, "|" stands for divisibility, so that $\alpha_j(\lambda)$ is divisible by $\alpha_{j-1}(\lambda)$. These polynomials are the *invariant factors* of $A(\lambda)$ and are uniquely determined by $A(\lambda)$.

The invariant factors of $Q(\lambda)$ can be decomposed into irreducible factors over $\mathbb{F}$ as follows [6, Chap. VI, §3]:

$$\begin{array}{rcl} \alpha_n(\lambda) & = & \phi_1(\lambda)^{m_{11}} \cdots \phi_s(\lambda)^{m_{s1}}, \\ \alpha_{n-1}(\lambda) & = & \phi_1(\lambda)^{m_{12}} \cdots \phi_s(\lambda)^{m_{s2}}, \\ \vdots & & \vdots \\ \alpha_1(\lambda) & = & \phi_1(\lambda)^{m_{1n}} \cdots \phi_s(\lambda)^{m_{sn}}, \end{array} \tag{1.1}$$

where $\phi_i(\lambda)$, $i = 1\colon s$ are distinct monic polynomials irreducible over $\mathbb{F}[\lambda]$, and

$$m_{i1} \geq m_{i2} \geq \cdots \geq m_{in} \geq 0, \quad i = 1\colon s. \tag{1.2}$$

The factors $\phi_i(\lambda)^{m_{ij}}$ with $m_{ij} > 0$ are the *finite elementary divisors* of $Q(\lambda)$.

Regarding the elementary divisors at infinity, or infinite elementary divisors, such an agreement has not been so unanimous. First of all some authors prefer to talk about the *pole-zero structure at infinity* of $Q(\lambda)$ (see, for example, Kailath [8] and Vardulakis [12]). The reason is that any polynomial has no zeros at infinity but it has always poles. A matrix polynomial may have both zeros and poles at infinity. Following the same pattern as for rational functions, the zeros and poles at infinity of $Q(\lambda)$ are defined in [8] as the poles and zeros at $\lambda = 0$ in the Smith-McMillan form

*Version of August 8, 2012.

†Departamento de Matemática Aplicada y EIO, Euskal Herriko Univertsitatea (UPV/EHU), Apdo. Correos 644, 48080 Bilbao, Spain (ion.zaballa@ehu.es). This author was supported by the Dirección General de Investigación, Proyecto de Investigación MTM2010-19356-C02-01 and Gobierno Vasco, GIC10/IT-361-10 and UPV/EHU UFI11/52.

‡School of Mathematics, The University of Manchester, Manchester, M13 9PL, UK (ftisseur@ma.man.ac.uk, http://www.ma.man.ac.uk/~ftisseur/). This author was supported by Engineering and Physical Sciences Research Council grant EP/I005293.

of the rational matrix $Q(\lambda^{-1})$. A different approach is used, for example, by Dion and Commault [4] and Vardulakis [12]. In these references, the ring of proper rational functions is considered. This is actually a local ring and it can be seen as the local ring at infinity of $\mathbb{F}[\lambda]$ ([1]). Its field of fractions is the field $\mathbb{F}(s)$ of rational functions and, with respect to the local ring at infinity, any matrix polynomial can be seen as a matrix whose elements are in its field of fractions. Hence, any matrix polynomial admits a Smith-McMillan form at infinity. Its exponents form the structure of poles and zeros at infinity of that matrix polynomial. Both definitions lead to the same objects.

On the other hand, Wimmer [13] defines the infinite elementary divisors of $Q(\lambda)$ as the zeros at $\lambda = 0$ in the Smith-McMillan form of the rational matrix $\lambda Q(\lambda^{-1})$. This definition differs from the nowadays more accepted: if ℓ is the degree of $Q(\lambda)$, its elementary divisors at infinity are those of $\operatorname{rev} Q(\lambda) = \lambda^\ell Q(\lambda^{-1})$ at 0, where $\operatorname{rev} Q(\lambda)$ is the *reversal* of $Q(\lambda)$. We will show in Section 2 that this is the most natural definition if one applies the usual geometric technique of using homogeneous coordinates to deal with the point at infinity. This technique consists in passing from the affine line to the projective line; a process which is usually called homogenization [9]. This is the approach that we will use here. We call it global because the homogeneous invariant factors of $Q(\lambda)$ are defined for all points of the projective line and to distinguish it from another possible approach that, using local rings, leads to the same conclusions (see [2], for example).

Section 2 is a simple generalization of [5, Sec. 1.14]. Section 3 is dedicated to analyze the problem of how the finite and infinite elementary divisors of a matrix changes under Möbius transformations. Results about this issue are scattered in the literature (see, for example, [2, Sec. 4.2], [7, Th. 7.3], [14] and, above all, [3, Lem. 10]). Recently a thorough and complete study has been carried on in [10]. We will approach this issue using homogeneous invariant polynomials. Actually, this is just a simple generalization of what is made in [3] for linear pencils.

2. Homogenization and dehomogenization of matrix polynomials. For a polynomial $f(\lambda) = f_\ell\lambda^\ell + f_{\ell-1}\lambda^{\ell-1} + \cdots + f_1\lambda + f_0 \in \mathbb{F}[\lambda]$ with coefficients in an arbitrary field $\mathbb{F}$, the *homogenization* of $f(\lambda)$ is

$$f_h(\lambda,\mu) = \mu^\ell f\left(\frac{\lambda}{\mu}\right) = f_\ell\lambda^\ell + f_{\ell-1}\lambda^{\ell-1}\mu + \cdots + f_1\lambda\mu^{\ell-1} + f_0\mu^\ell.$$

This is an homogeneous polynomial in $\mathbb{F}[\lambda,\mu]$. Conversely, if $f(\lambda,\mu) = f_\ell\lambda^\ell + f_{\ell-1}\lambda^{\ell-1}\mu + \cdots + f_1\lambda\mu^{\ell-1} + f_0\mu^\ell$ is an homogeneous polynomial in $\mathbb{F}[\lambda,\mu]$ then

$$f_d(\lambda) = f(\lambda,1) = f_\ell\lambda^\ell + f_{\ell-1}\lambda^{\ell-1} + \cdots + f_1\lambda + f_0 \in \mathbb{F}[\lambda]$$

is the *dehomogenization* of $f(\lambda,\mu)$. Likewise, if $Q(\lambda) = A_\ell\lambda^\ell + A_{\ell-1}\lambda^{\ell-1} + \cdots + A_1\lambda + A_0 \in \mathbb{F}[\lambda]^{m\times n}$ is a matrix polynomial of degree ℓ ($A_\ell \neq 0$), the homogenization of $Q(\lambda)$ is the homogeneous matrix polynomial $Q_h(\lambda,\mu) = A_\ell\lambda^\ell + A_{\ell-1}\lambda^{\ell-1}\mu + \cdots + A_1\lambda\mu^{\ell-1} + A_0\mu^\ell$ and the dehomogenization of an homogeneous matrix polynomial $Q(\lambda,\mu)$ is $Q_d(\lambda) = Q(\lambda,1)$. It is plain that the dehomogenization of the homogenization of $Q(\lambda)$ is $Q(\lambda)$.

Although the ring $\mathbb{F}[\lambda,\mu]$ is not a Bézout domain ([5, Sec. 1.2]) (and so matrices over $\mathbb{F}[\lambda,\mu]$ do not admit, in general, a Smith normal form), it is a unique factorization domain ([5, Sec. 1.3]). Then the greatest common divisor of any two elements in $\mathbb{F}[\lambda,\mu]$ can be computed and the invariant factors of any matrix with elements in $\mathbb{F}[\lambda,\mu]$ can be defined ([5, Sec. 1.14]). The following properties are direct consequences of the definitions (see [11]).

Lemma 2.1. *Let $x(\lambda), y(\lambda), z(\lambda) \in \mathbb{F}[\lambda]$ and let $x_h(\lambda,\mu)$, $y_h(\lambda,\mu)$ and $z_h(\lambda,\mu)$ be their homogenizations. The following properties hold true:*

(i) $z(\lambda) = x(\lambda) + y(\lambda)$ *if and only if* $z_h(\lambda,\mu) = x_h(\lambda,\mu) + y_h(\lambda,\mu)$.

(ii) $z(\lambda) = x(\lambda)y(\lambda)$ *if and only if* $z_h(\lambda,\mu) = x_h(\lambda,\mu)y_h(\lambda,\mu)$.

(iii) $x(\lambda)|y(\lambda)$ *if and only if* $x_h(\lambda,\mu)|y_h(\lambda,\mu)$.

(iv) $\gcd(x(\lambda), y(\lambda)) = z(\lambda)$ *if and only if* $\gcd(x_h(\lambda,\mu), y_h(\lambda,\mu)) = z_h(\lambda,\mu)$.

(v) $x(\lambda)$ *is prime if and only if* $x_h(\lambda,\mu)$ *is prime.*

If $x(\lambda,\mu)$ and $y(\lambda,\mu)$ are homogeneous polynomials then their greatest common divisor is also an homogeneous polynomial. Thus, if $Q(\lambda,\mu)$ is an homogeneous matrix polynomial then its invariant factors are homogeneous polynomials as well. They satisfy the property that each one is multiple of the previous one (same proof as [5, Lemma 1.14.8]). For $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ the invariant factors of its homogenization $Q_h(\lambda,\mu)$ are called the homogeneous invariant polynomials of $Q(\lambda)$.

Although the following theory can be applied for non-square or rank deficient polynomial matrices, for notational simplicity we are going to assume that $Q(\lambda)$ is a given $n\times n$ nonsingular polynomial matrix. Let its homogeneous invariant polynomials be $\gamma_1(\lambda,\mu)|\cdots|\gamma_n(\lambda,\mu)$ and write them as

$$\gamma_i(\lambda,\mu) = \mu^{e_i}\widetilde{\gamma}_i(\lambda,\mu), \quad i = 1\colon n,$$

where $e_i \geq 0$ and

$$\widetilde{\gamma}_i(\lambda,\mu) = g_{id_i}\lambda^{d_i} + g_{id_i-1}\lambda^{d_i-1}\mu + \cdots + g_{i1}\lambda\mu^{d_i-1} + g_{i0}\mu^{d_i}, \quad g_{id_i} \neq 0.$$

Since $\gcd(\mu^{e_i}, \widetilde{\gamma}_i(\lambda,\mu)) = 1$, it follows that $0 \leq e_1 \leq \cdots \leq e_n$ and $\widetilde{\gamma}_1(\lambda,\mu)|\cdots|\widetilde{\gamma}_n(\lambda,\mu)$. Given that $\widetilde{\gamma}_i(\lambda,\mu)$ is the homogenization of $g_i(\lambda) = g_{id_i}\lambda^{d_i} + g_{id_i-1}\lambda^{d_i-1} + \cdots + g_{i1}\lambda + g_{i0}$, by Lemma 2.1, $g_1(\lambda)|\cdots|g_n(\lambda)$ and these are the invariant factors of the dehomogenization of $Q_h(\lambda,\mu)$, $Q(\lambda) = Q_h(\lambda, 1)$. Furthermore,

$$Q_h(1,\mu) = \operatorname{rev} Q(\mu) = A_0\mu^\ell + A_1\mu^{\ell-1} + \cdots + A_\ell,$$

and its invariant factors are

$$\gamma_i(1,\mu) = \mu^{e_i}\left(g_{i0}\mu^{d_i} + \cdots + g_{id_i-1}\mu + g_{id_i}\right).$$

Taking into account that $g_{id_i} \neq 0$, it follows that $\mu^{e_1}, \ldots, \mu^{e_n}$ are the elementary divisors of $Q_h(1,\mu)$ at $\mu = 0$. But, in homogeneous coordinates, $(1,0)$ represents the point at infinity in the projective line $\mathbb{P}^1(\mathbb{F})$. Hence $\mu^{e_1}, \ldots, \mu^{e_n}$ with $0 \leq e_1 \leq \cdots \leq e_n$ are the elementary divisors of $Q_h(\lambda,\mu)$ at infinity. They are also called infinite elementary divisors (or elementary divisors at infinity) of $Q(\lambda)$.

The relationship between the infinite elementary divisors of $Q(\lambda)$ and its structure of poles and zeros at infinity is given in Corollary 4.41 of [12].

3. Möbius transformations. Consider now the following change of variables:

$$(\lambda,\mu) \longrightarrow (a\lambda + b\mu, c\lambda + d\mu), \tag{3.1}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}$ is nonsingular. For each polynomial $f(\lambda,\mu) \in \mathbb{F}[\lambda,\mu]$ and each matrix $F(\lambda,\mu) \in \mathbb{F}[\lambda,\mu]^{n\times n}$ define, respectively, ([3]):

$$\Pi_A(f) = f(a\lambda + b\mu, c\lambda + d\mu), \quad P_A(F) = F(a\lambda + b\mu, c\lambda + d\mu).$$

Associated with $A^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$ we can define the "inverse" change of variables: for $f(\lambda,\mu) \in \mathbb{F}[\lambda,\mu]$ and $F(\lambda,\mu) \in \mathbb{F}[\lambda,\mu]^{n\times n}$ we have, respectively,

$$\Pi_{A^{-1}}(f) = f\left(\frac{d\lambda - b\mu}{ad-bc}, \frac{-c\lambda + a\mu}{ad-bc}\right), \quad P_{A^{-1}}(F) = F\left(\frac{d\lambda - b\mu}{ad-bc}, \frac{-c\lambda + a\mu}{ad-bc}\right).$$

It is easily seen ([3, Lem. 6]) that $\Pi_A(\Pi_{A^{-1}}(f)) = \Pi_{A^{-1}}(\Pi_A(f)) = f(\lambda,\mu)$ and $P_A(P_{A^{-1}}(F)) = P_{A^{-1}}(P_A(F)) = F(\lambda,\mu)$. The following properties are straightforwardly proved (see [3, Lem. 7]).

Lemma 3.1. *Let $x(\lambda,\mu), y(\lambda,\mu), z(\lambda,\mu) \in \mathbb{F}[\lambda,\mu]$ be homogeneous polynomials. The following properties hold true:*

(i) *$x(\lambda,\mu) = y(\lambda,\mu) + z(\lambda,\mu)$ if and only if $\Pi_A(z) = \Pi_A(x) + \Pi_A(y)$.*
(ii) *$x(\lambda,\mu) = y(\lambda,\mu)z(\lambda,\mu)$ if and only if $\Pi_A(z) = \Pi_A(x)\Pi_A(y)$.*
(iii) *$x(\lambda,\mu) | y(\lambda,\mu)$ if and only if $\Pi_A(x) | \Pi_A(y)$.*
(iv) *$\gcd(x(\lambda,\mu), y(\lambda,\mu)) = z(\lambda,\mu)$ if and only if $\gcd(\Pi_A(x), \Pi_A(y)) = \Pi_A(z)$.*
(iv) *$x(\lambda,\mu)$ is prime if and only if $\Pi_A(x)$ is prime.*

Let $Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a nonsingular matrix polynomial of degree ℓ. For a given matrix $A \in \mathbb{F}^{2\times 2}$ as above, we can write

$$Q(\lambda) = (c\lambda - a)^{\ell_0}\, Q_1(\lambda), \tag{3.2}$$

where $Q_1(\lambda)$ is not a multiple of $(c\lambda - a)$ (i.e., at least one element of $Q_1(\lambda)$ is not multiple of $(c\lambda - a)$) and it is understood that $\ell_0 = 0$ if $c = 0$. Bearing in mind Lemma 3.1, a simple computation shows that if $Q_{1h}(\lambda,\mu)$ is the homogenization of $Q_1(\lambda)$ then

$$\begin{aligned} S(\lambda,\mu) = P_A(Q_h) &= \Pi_A(c\lambda - a)^{\ell_0} P_A(Q_{1h}) \\ &= (bc - ad)^{\ell_0}\mu^{\ell_0} P_A(Q_{1h}) \\ &= (bc - ad)^{\ell_0}\mu^{\ell_0} Q_{1h}(a\lambda + b\mu, c\lambda + d\mu). \end{aligned} \tag{3.3}$$

Hence, if $\deg Q_1(\lambda) = \ell_1$ then

$$\begin{aligned} R(\lambda) &:= S(\lambda,1) = (bc-ad)^{\ell_0} Q_{1h}(a\lambda + b, c\lambda + d) \\ &= (bc-ad)^{\ell_0}(c\lambda + d)^{\ell_1} Q_1\left(\tfrac{a\lambda+b}{c\lambda+d}\right). \end{aligned} \tag{3.4}$$

We will write

$$P_A^*(Q) = \frac{1}{(bc-ad)^{\ell_0}} R(\lambda) = Q_1(a\lambda + b, c\lambda + d) = (c\lambda + d)^{\ell_1} Q_{1h}\left(\frac{a\lambda + b}{c\lambda + d}\right). \tag{3.5}$$

For $f(\lambda) \in \mathbb{F}[\lambda]$ we will use the same notation for $\Pi_A^*(f)$. That is to say, if $f(\lambda) = (c\lambda - a)^{p_0} f_1(\lambda)$ such that $(c\lambda - a)$ and $f_1(\lambda)$ are relatively prime and $\widetilde{f}(\lambda,\mu) = \Pi_A(f_{1h})$, where f_{1h} is the homogenization of $f_1(\lambda)$, then

$$\Pi_A^*(f) = \widetilde{f}(\lambda,1) = f_{1h}(a\lambda + b, c\lambda + d) = (c\lambda + d)^{p_1} f_1\left(\frac{a\lambda + b}{c\lambda + d}\right), \tag{3.6}$$

where p_1 is the degree of $f_1(\lambda)$. In words, $P_A^*(Q)$ and $\Pi_A^*(f)$ are the dehomogenization of $P_A(Q_h)$ and $\Pi_A(f_h)$ where $Q_h(\lambda,\mu)$ and $f_h(\lambda,\mu)$ are the homogenization of $Q(\lambda)$ and $f(\lambda)$, respectively.

Notice that although $Q_h(\lambda,\mu)$ and $S(\lambda,\mu) = P_A(Q_h)$ always have the same degree, this property may not be shared by $Q(\lambda)$ and $R(\lambda) = P_A^*(Q)$. Actually we can be a little more precise.

Proposition 3.2. *With the above notation $Q(\lambda) = P_{A^{-1}}^*(P_A^*(Q))$ if and only if one (and hence all) of the following equivalent conditions holds:*

(i) $\ell_0 = 0$.

(ii) $Q(\lambda)$ *is not multiple of* $(c\lambda - a)$.

(iii) $Q_h(a\lambda + b\mu, c\lambda + d\mu)$ *is not a multiple of* μ.

Proof. The equivalence of (i) and (ii) follows from the definition (cf. (3.2)), and the equivalence of (ii) and (iii) from item (iv) of Lemma 3.1. Now, let $R(\lambda) = P_A^*(Q)$ and $L(\lambda) = P_{A^{-1}}^*(R)$. According to (3.5), $R(\lambda) = Q_{1h}(a\lambda + b, c\lambda + d)$ and its homogenization (by (i) and (ii) of Lemma 3.1) $R_h(\lambda,\mu) = Q_{1h}(a\lambda + b\mu, c\lambda + d\mu) = P_A(Q_{1h})$. Hence, $P_{A^{-1}}(R_h) = P_{A^{-1}}(P_A(Q_{1h})) = Q_{1h}(\lambda,\mu)$. So, $L(\lambda) = Q_{1h}(\lambda, 1) = Q_1(\lambda)$. It is clear that $Q(\lambda) = Q_1(\lambda)$ if and only if $\ell = 0$. ∎

Given $Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, we aim to study the relationship between the finite and infinite elementary divisors of $Q(\lambda)$ and $R(\lambda) = P_A^*(Q)$. The following result, whose proof is exactly the same as that of [3, Lem. 10] for matrix pencils, answers this question for the homogeneous invariant polynomials of $Q(\lambda)$.

Lemma 3.3. *If* $\gamma_1(\lambda,\mu)|\cdots|\gamma_n(\lambda,\mu)$ *are the homogeneous invariant polynomials of* $Q(\lambda)$ *then* $\Pi_A(\gamma_1)|\cdots|\Pi_A(\gamma_n)$ *are the homogeneous invariant factors of* $S(\lambda,\mu) = P_A(Q_h)$*, where* $Q_h(\lambda,\mu)$ *is the homogenization of* $Q(\lambda)$.

With the help of Lemma 3.3 we can analyse right away how the finite and infinite elementary divisors of $Q(\lambda)$ and $R(\lambda) = P_A^*(Q)$ are related. We should bear in mind that the change of variables (3.1) can be seen as a transformation in the projective plane. Such a transformation brings the point at infinity $(1,0)$ to the point (a,c) and the point $(-d,c)$ to the point at infinity. In particular, if $c = 0$ then the point at infinity remains unchanged.

Let us write the homogeneous invariant polynomials of $Q(\lambda)$ in the following form

$$\gamma_i(\lambda,\mu) = \mu^{e_i}\widetilde{\gamma}_i(\lambda,\mu) = \mu^{e_i}\left(\mu^{d_i} g_i\left(\frac{\lambda}{\mu}\right)\right), \quad i = 1:n,$$

where $\gcd(\mu, \widetilde{\gamma}_i(\lambda,\mu)) = 1$, $d_i = \deg \widetilde{\gamma}_i(\lambda,\mu) = \deg g_i(\lambda)$ and $\widetilde{\gamma}_i(\lambda,\mu)$ is the homogenization of $g_i(\lambda)$. As mentioned above, $\mu^{e_1}|\cdots|\mu^{e_n}$ are the infinite elementary divisors of $Q(\lambda)$ and $g_1(\lambda)|\cdots|g_n(\lambda)$ are its (finite) invariant factors. Now, factor $g_i(\lambda)$ into powers of prime polynomials:

$$g_i(\lambda) = \sigma_0(\lambda)^{f_{i0}}\sigma_1(\lambda)^{f_{i1}}\cdots\sigma_t(\lambda)^{f_{it}}, \quad i = 1:n,$$

where $0 \le f_{1j} \le f_{2j} \le \cdots \le f_{nj}$, $j = 0:t$ and $\sigma_0(\lambda) = \left(\lambda - \frac{a}{c}\right)$ if $c \neq 0$ and a/c is an eigenvalue of $Q(\lambda)$ and $\sigma_0(\lambda) = 1$ otherwise. Thus $\sigma_i(\lambda)^{f_{ij}}$ $i = 1:n$, $j = 0:t$ are the finite elementary divisors of $Q(\lambda)$.

By Lemma 3.3,

$$\alpha_i(\lambda,\mu) := \Pi_A(\gamma_i) = \gamma_i(a\lambda + b\mu, c\lambda + d\mu) = (c\lambda + d\mu)^{e_i}\Pi_A(\widetilde{\gamma}_i), \quad i = 1:n$$

are the homogeneous invariant factors of $S(\lambda,\mu) = P_A(Q_h)$. By Lemma 3.1, if $\sigma_{ih}(\lambda,\mu)$ is the homogenization of $\sigma_i(\lambda)$ and $\nu_i(\lambda,\mu) = \Pi_A(\sigma_{ih})$ then

$$\Pi_A(\widetilde{\gamma}_i) = \nu_0(\lambda,\mu)^{f_{i0}}\nu_1(\lambda,\mu)^{f_{i1}}\cdots\nu_t(\lambda,\mu)^{f_{it}} \quad i = 0:t.$$

Notice that by Lemmas 2.1 and 3.1, $\sigma_{ih}(\lambda,\mu)$ and $\nu_i(\lambda,\mu)$ are prime polynomials. Therefore

$$\alpha_i(\lambda,\mu) = (c\lambda + d\mu)^{e_i}\nu_0(\lambda,\mu)^{f_{i0}}\nu_1(\lambda,\mu)^{f_{i1}}\cdots\nu_t(\lambda,\mu)^{f_{it}}, \quad i = 1:t \tag{3.7}$$

is a factorization of $\alpha_i(\lambda,\mu)$ in powers of prime polynomials.

Assume now that

$$Q(\lambda) = (c\lambda - a)^{\ell_0} Q_1(\lambda)$$

with $\ell_0 \geq 0$ and at least one element of $Q_1(\lambda)$ prime with $c\lambda - a$. If $\ell_0 > 0$ then $\sigma_0(\lambda) \neq 1$. Recall (cf (3.3) and (3.5)) that $S(\lambda, \mu) = P_A(Q_h) = (bc-ad)^{\ell_0}\mu^{\ell_0} Q_{1h}(a\lambda + b\mu, c\lambda + d\mu)$ and $R(\lambda) = P_A^*(Q) = Q_{1h}(a\lambda + b, c\lambda + d)$. Hence, by Lemma 2.1, the homogenization of $R(\lambda)$ is

$$R_h(\lambda, \mu) = Q_{1h}(a\lambda + b\mu, c\lambda + d\mu) = \frac{(bc-ad)^{\ell_0}}{\mu^{\ell_0}} S(\lambda, \mu).$$

Therefore, the homogeneous invariant polynomials of $R(\lambda)$ are $\frac{1}{\mu^{\ell_0}}\alpha_i(\lambda, \mu)$, $i = 1 : n$. In order to obtain the finite and infinite elementary divisors of $R(\lambda)$, we compute the prime polynomials in the prime factorization (3.7) of $\alpha_i(\lambda, \mu)$. We split the study into two cases according as $c = 0$ or $c \neq 0$.

- Case $c = 0$. In this case $d \neq 0$ because $ac - bd \neq 0$, $\ell_0 = 0$ and $\sigma_0(\lambda) = 1$. Then for $i = 1\colon n$

  $$\frac{1}{\mu^{\ell_0}}\alpha_i(\lambda, \mu) = d^{e_i}\mu^{e_i}\prod_{j=1}^{t}\mu_j(\lambda, \mu)^{f_{ij}} = d^{e_i}\mu^{e_i}\prod_{j=1}^{t}\sigma_{jh}(a\lambda + b\mu, d\mu)^{f_{ij}}.$$

  The elementary divisors at infinity of $R(\lambda)$ are those of $Q(\lambda)$: $\mu^{e_1}|\cdots|\mu^{e_n}$ and the finite elementary divisors of $R(\lambda)$ are the dehomogenization of $\sigma_{jh}(a\lambda + b\mu, d\mu)^{f_{ij}}$, i.e.,

  $$d^{f_{ij}s_j}\sigma_j\left(\frac{a}{d}\lambda + \frac{b}{d}\right)^{f_{ij}} = \Pi_A^*\left(\sigma_j(\lambda)^{f_{ij}}\right), \quad i = 1 : n,\ j = 1 : t,$$

  where $s_j = \deg \sigma_j(\lambda)$.
- Case $c \neq 0$. Now it may happen that $\ell_0 > 0$ and $\sigma_0(\lambda) \neq 1$. According to (3.7), μ^{e_i} is transformed into $(c\lambda + d\mu)^{e_i}$ and $\sigma_i(\lambda)$ into $\nu_i(\lambda, \mu)$, $i = 0, 1, \ldots, t$. But

  $$\nu_0(\lambda, \mu) = \Pi_A\left(\lambda - \frac{a}{c}\mu\right) = a\lambda + b\mu - \frac{a}{c}(c\lambda + d\mu) = \frac{bc - ad}{c}\mu,$$

  and for $i = 1 : t$,

  $$\nu_i(\lambda, \mu) = \Pi_A(\sigma_{ih}) = \sigma_{ih}(a\lambda + b\mu, c\lambda + d\mu) = (c\lambda + d\mu)^{s_i}\sigma_i\left(\frac{a\lambda + b\mu}{c\lambda + d\mu}\right),$$

  where $s_i = \deg \sigma_i(\lambda)$.

  (a) The dehomogenization of $(c\lambda + d\mu)^{e_i}$ is $(c\lambda + d)^{e_i}$ and so, the elementary divisors at infinity of $Q(\lambda)$, $\mu^{e_1}|\cdots|\mu^{e_n}$, are transformed into the following finite elementary divisors of $R(\lambda)$:

  $$\left(\lambda + \frac{d}{c}\right)^{e_1}|\cdots|\left(\lambda + \frac{d}{c}\right)^{e_n}.$$

  (b) The elementary divisors of $Q(\lambda)$ at $\frac{a}{c}$: $\left(\lambda - \frac{a}{c}\right)^{f_{10}}|\cdots|\left(\lambda - \frac{a}{c}\right)^{f_{n0}}$ are transformed into elementary divisors at infinity of $S(\lambda, \mu)$: $\mu^{f_{10}}|\cdots\mu^{f_{n0}}$.

But since $R_h(\lambda,\mu) = \frac{1}{\mu^{\ell_0}} S(\lambda,\mu)$, the elementary divisors at infinity of $R_h(\lambda,\mu)$ and $R(\lambda)$ are:

$$\mu^{f_{10}-\ell_0} \mid \mu^{f_{20}-\ell_0} \mid \cdots \mid \mu^{f_{n0}-\ell_0}.$$

It must be noticed that if $\ell_0 > 0$ then all elements of $Q(\lambda)$ are multiple of $(c\lambda - a)^{\ell_0}$. This means that all (finite) invariant factors of $Q(\lambda)$ are multiple of $(c\lambda - a)^{\ell_0}$ and so $f_{10} \geq \ell_0$. Actually, since all elements of $Q_1(\lambda)$ are prime with $(c\lambda - a)$, we have $\ell_0 = f_{10}$. In conclusion, the elementary divisors at infinity of $R(\lambda)$ are:

$$1 \mid \mu^{f_{20}-f_{10}} \mid \cdots \mid \mu^{f_{n0}-f_{10}}.$$

(c) The remaining elementary divisors $\sigma_j(\lambda)^{f_{ij}}$, $i = 1 : n$, $j = 1 : t$, are transformed into the dehomogenization of $\nu_i(\lambda,\mu)$:

$$(c\lambda+d)^{f_{1j}s_j}\sigma_j\left(\frac{a\lambda+b\mu}{c\lambda+d\mu}\right)^{f_{1j}} \mid \cdots \mid (c\lambda+d)^{f_{nj}s_j}\sigma_j\left(\frac{a\lambda+b\mu}{c\lambda+d\mu}\right)^{f_{nj}}, \quad j = 1:t.$$

By (3.6)

$$(c\lambda+d)^{f_{ij}s_j}\sigma_j\left(\frac{a\lambda+b\mu}{c\lambda+d\mu}\right)^{f_{ij}} = \Pi_A^*\left(\sigma_j(\lambda)^{f_{ij}}\right).$$

We have proven the following result.

Theorem 3.4. *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a nonsingular matrix polynomial, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$ be nonsingular and $R(\lambda) = P_A^*(Q)$. Then the finite and infinite elementary divisors of $R(\lambda)$ and $Q(\lambda)$ are related as follows:*

1. *Let $\sigma(\lambda)$ is an irreducible polynomial such that if $c \neq 0$ then $\sigma(\lambda) \neq \left(\lambda - \frac{a}{c}\right)$. If $\sigma(\lambda)^{f_1} \mid \cdots \mid \sigma_n(\lambda)^{f_n}$ ($f_i \geq 0$) are the elementary divisors of $Q(\lambda)$ with respect to $\sigma(\lambda)$ then $\Pi_A\left((\sigma(\lambda)^{f_1}\right) \mid \cdots \mid \Pi_A\left(\sigma(\lambda)^{f_n}\right)$ are the elementary divisors of $R(\lambda) = P_A^*(Q)$ with respect to $\Pi_A(\sigma(\lambda)) = (c\lambda+d)^s\sigma\left(\frac{a\lambda+b}{c\lambda+d}\right)$, $s = \deg\sigma(\lambda)$.*
2. *If $c \neq 0$ and $\left(\lambda - \frac{a}{c}\right)^{f_1} \mid \cdots \mid \left(\lambda - \frac{a}{c}\right)^{f_n}$ ($f_i \geq 0$) are the elementary divisors of $Q(\lambda)$ with respect to $\left(\lambda - \frac{a}{c}\right)$ then $1 \mid \mu^{f_2-f_1} \mid \cdots \mid \mu^{f_n-f_1}$ are the infinite elementary divisors of $R(\lambda) = P_A^*(Q)$.*
3. *If $c \neq 0$ and $\mu^{f_1} \mid \mu^{f_2} \mid \cdots \mid \mu^{f_n}$ ($f_i \geq 0$) are the infinite elementary divisors of $Q(\lambda)$ then $\left(\lambda + \frac{d}{c}\right)^{f_1} \mid \cdots \mid \left(\lambda + \frac{d}{c}\right)^{f_n}$ are the elementary divisors of $R(\lambda) = P_A^*(Q)$ with respect to $\left(\lambda + \frac{d}{c}\right)$.*
4. *If $c = 0$ the infinite elementary divisors of $Q(\lambda)$ and $R(\lambda) = P_A^*(Q)$ are the same.*

REFERENCES

[1] A. Amparan, S. Marcaida and I. Zaballa *An Interpretation of Rosenbrock's Theorem via Local Rings*, Linear Algebra - Theorems and Applications, Hassan Abid Yasser (Ed.), ISBN: 978-953-51-0669-2, InTech, 2012. Available from: http://www.intechopen.com/books/linear-algebra-theorems-and-applications/an-interpretation-of-rosenbrock-s-theorem-via-local-rings

[2] ———, *On the structure invariants of proper rational matrices with prescribed finite poles*, 2012. Available from http://www.mae.ehu.es/s0100/gigamcon/es/contenidos/informacion/gamainf/es_gamainfc/adjuntos/prpfp_sometido.pdf.

[3] I. Cabral and F. C. Silva, *Unified theorems on completions of matrix pencils*, Linear Algebra Appl., 159 (1991), pp. 43–54.

[4] J. Dion and C. Commault, *Smith-Mcmillan factorizations at infinity of rational matrix functions and their control interpretation*, Systems and Control Letters, 1 (1982), pp. 312–320.

[5] S. Friedland, *Spectral Theory of Matrices I. General Matrices*, Mathematics Research Center. University of Wisconsin-Madison, 1980.

[6] F. R. Gantmacher, *The Theory of Matrices*, vol. one, Chelsea, New York, 1959.

[7] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2009. Unabridged republication of book first published by Academic Press in 1982.

[8] T. Kailath, *Linear Systems*, Prentice Hall, New York, 1980.

[9] E. Kunz *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Boston, 1985.

[10] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Möbius transformations of matrix polynomials*. Technical report, 2012. In preparation.

[11] S. Tan and J. Vandewalle, *Novel theory for polynomial and rational matrices at infinity. Part 1. Polynomial matrices* Int. J. Control, 48 (2) (1988) pp. 545–559.

[12] A. I. G. Vardulakis, *Linear Multivariable Control*, Wiley, New York, 1991.

[13] H. K. Wimmer, *The structure of nonsingular polynomial matrices*, Math. Systems Theory, 14 (1981), pp. 367–379.

[14] J. C. Zúñiga Anaya, *Diagonalization of quadratic matrix polynomials*, Systems and Control Letters, 59 (2010), pp. 105–113.