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A real $n \times n$ matrix M is *circulant* if $m_{ij} = m(j-i)$, for all i, j, where $m(r)$ is an n-periodic function: $m(r+n) = m(r)$ defined on the integers. For example

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

where $m(r) = r+1 \bmod 4$. M is symmetric iff $m(-r) = m(r)$ as in the example on the next page.

It will be convenient to define the (discrete) Fourier coefficients of m as usual to be,

$$a(\ell) = \sum_{r=1}^{n} m(r)\cos(2\pi\ell r/n), \quad b(\ell) = -\sum_{r=1}^{n} m(r)\sin(2\pi\ell r/n), \tag{1}$$

for each $\ell = 0, \ldots, n-1$. In other words, if $\alpha(\ell) = a(\ell) + \mathrm{i}b(\ell) \in \mathbb{C}$ then

$$\alpha(\ell) = \sum_r \mathrm{e}^{-2\mathrm{i}\pi\ell r/n} m(r),$$

which is the usual definition of discrete Fourier transform of m (hence the choice of minus sign in b). Note that the inverse Fourier transform formula gives

$$m(r) = \frac{1}{n}\sum_\ell \exp(2\mathrm{i}\pi j\ell/n)(a+\mathrm{i}b)(\ell).$$

We now wish to determine the eigenvectors of M, and it turns out that they are the same for any circulant matrix. First define the vectors $\mathbf{u}^{(\ell)}, \mathbf{v}^{(\ell)} \in \mathbb{R}^n$, with components

$$u_j^{(\ell)} = \cos(2\pi j\ell/n), \quad v_j^{(\ell)} = \sin(2\pi j\ell/n).$$

Note that $\mathbf{u}^{(n-\ell)} = \mathbf{u}^{(\ell)}$ and $\mathbf{v}^{(n-\ell)} = -\mathbf{v}^{(\ell)}$. In particular, $\mathbf{v}^{(0)} = 0$ and if n is even then $\mathbf{v}^{(n/2)} = 0$. There is therefore a total of n linearly independent vectors[1],

$$\mathbf{u}^{(\ell)}\ \left(\ell = 0, \ldots, \left[\tfrac{n}{2}\right]\right) \quad \text{and} \quad \mathbf{v}^{(\ell)}\ \left(\ell = 1, \ldots, \left[\tfrac{n-1}{2}\right]\right).$$

That is, they form a basis for $\mathbb{R}^n$.

Now calculate, for $\mathbf{u} = \mathbf{u}^{(\ell)}$,

$$\begin{aligned} (M\mathbf{u})_k &= \sum_j m(j-k)\cos(2\pi j\ell/n) \\ &= \sum_r m(r)\cos(2\pi(k+r)\ell/n) \\ &= \sum_r (m(r)\cos(2\pi r\ell/n))\cos(2\pi k\ell/n) \\ &\qquad - \sum_r (m(r)\sin(2\pi r\ell/n))\sin(2\pi k\ell/n) \\ &= a(\ell)\cos(2\pi k\ell/n) + b(\ell)\sin(2\pi k\ell/n) \\ &= a(\ell)u_k + b(\ell)v_k. \end{aligned}$$

[1] $[q]$ denotes the greatest integer less than or equal to q

Similarly for $\mathbf{v} = \mathbf{v}^{(\ell)}$, $(M\mathbf{v})_k = -b(\ell)u_k + a(\ell)v_k$. That is,

$$\begin{aligned} M\mathbf{u}^{\ell} &= a(\ell)\mathbf{u}^{(\ell)} + b(\ell)\mathbf{v}^{(\ell)}, \\ M\mathbf{v}^{\ell} &= -b(\ell)\mathbf{u}^{(\ell)} + a(\ell)\mathbf{v}^{(\ell)}. \end{aligned}$$

It follows that the complex eigenvectors of M are,

$$\zeta^{(\ell)} = \mathbf{u}^{(\ell)} \pm \mathrm{i}\mathbf{v}^{(\ell)},$$

with eigenvalues

$$\lambda(\ell) = a(\ell) \pm \mathrm{i}b(\ell), \tag{2}$$

with $a(\ell), b(\ell)$ given in (1). That is, the eigenvalues of M are the discrete Fourier coefficients $\alpha(\ell)$ of the function m.

In particular, $\mathbf{u}^{(0)} = (1\,1\,\ldots\,1)^T$ and $\mathbf{u}^{(n/2)} = (1\,(-1)\,1\ldots\,(-1))^T$ (the latter if n is even) are real eigenvectors, with respective eigenvalues $a(0) = \sum m(r)$, and $a(n/2) = \sum(-1)^r m(r)$.

Example The eigenvalues of the matrix M above are $10, -2, -2 \pm 2\mathrm{i}$.

Symmetric circulant matrices A circulant matrix M is symmetric if and only if $m(-r) = m(r)$. This in turn is equivalent to $b(\ell) = 0$ (for all ℓ)—as should be familiar from ordinary Fourier series. Hence the eigenvalues in (2) are simply

$$\lambda(\ell) = a(\ell) \tag{3}$$

(the eigenvalues of a symmetric matrix are of course real). For example, for

$$M = \begin{pmatrix} 1 & 2 & 5 & 2 \\ 2 & 1 & 2 & 5 \\ 5 & 2 & 1 & 2 \\ 2 & 5 & 2 & 1 \end{pmatrix}$$

where $m(0) = 1$, $m(1) = m(-1) = 2$, $m(2) = 5$, the eigenvalues are $10, 2, -4, -4$.

Using the analysis above with $b(\ell) = 0$, we have

$$\begin{aligned} M\mathbf{u}^{(\ell)} &= a(\ell)\mathbf{u}^{(\ell)} \\ M\mathbf{v}^{(\ell)} &= a(\ell)\mathbf{v}^{(\ell)}. \end{aligned}$$

It follows that $\mathbf{u}^{(0)}$ and if n is even $\mathbf{u}^{n/2}$ are in general simple eigenvectors, while $\mathbf{u}^{(\ell)}, \mathbf{v}^{(\ell)}$ share the same eigenvalue $a(\ell)$ for $0 < \ell < n/2$. Consequently, for $n > 2$ every $n \times n$ symmetric circulant matrix has eigenvalues of multiplicity at least 2.

Exercise: What are the corresponding properties of skew-symmetric circulant matrices?

Representation theory These results can be related to the representation theory of the cyclic group (or dihedral group in the symmetric case), acting on $\mathbb{R}^n$ by

$$\rho(x_1,\dots,x_n) = (x_n,x_1,\dots,x_{n-1}), \quad \text{and} \quad \kappa\cdot(x_1,\cdots,x_n) = (x_n,\dots,x_1), \tag{4}$$

where ρ generates the cyclic group $\mathbf{C}_n$ and ρ,κ together generate the dihedral group $\mathbf{D}_n$. The fact that M is circulant is equivalent to $M\rho = \rho M$, and it being in addition symmetric means it commutes with κ.

To describe the representation more explicitly, we describe how it decomposes as a sum of distinct irreducible representations. Let A_0 denote the 1-dimensional trivial representation. For each $\ell = 1,\dots,[(n-1)/2]$ let A_ℓ be the following irreducible 2-dimensional real representation of $\mathbf{D}_n$: let the rotation ρ act by rotation through $2\pi\ell/n$ and let κ act by some reflexion (this is independent of the choice of reflexion as any two reflexions are conjugate, and the resulting irreps will be equivalent). Finally, A_0 is the trivial representation and if n is even, let $A_{n/2}$ denote the 1-dimensional representation where ρ and κ act by multiplication by -1. These irreducible representations are also irreducible for the cyclic group $\mathbf{C}_n$ (ignoring κ).

It is important here to understand the commuting linear maps. Each of these A_ℓ are irreducible representations for both the dihedral and cyclic groups. In the case of the dihedral group, any linear map on A_ℓ commuting with ρ,κ must be a scalar multiple of the identity (irreducible representations with this property are said to be *absolutely irreducible*). On the other hand, if a matrix just commutes with ρ, it can be any scalar multiple of the identity multiplied by a rotation.

Proposition 1 *The representation described in (4) above decomposes as a sum of irreducible representations as follows:*

$$\mathbb{R}^n = A_0 \oplus A_1 \oplus \cdots \oplus A_{[n/2]}.$$

This is called the *isotypic decomposition* of $\mathbb{R}^n$ for this action (or representation). To prove the proposition, it suffices to show that the characters agree, and this is left as an exercise. The following result is also an exercise.

Proposition 2 *The component A_ℓ is spanned by the vectors u_ℓ, v_ℓ.*

We now return to the real symmetric circulant matrix M. The following result is easy to check (for example it can be shown explicitly for 2 generators of the group, a rotation of order n and a reflexion).

Proposition 3 *A matrix M is circulant iff it commutes with the action of* $\mathbf{C}_n$*, and it is symmetric and circulant iff it commutes with* $\mathbf{D}_n$*.*

It follows from Schur's lemma that M preserves each component of the isotypic decomposition of $\mathbb{R}^n$, and moreover if M is symmetric on each the action of M is just multiplication by a scalar multiple of the identity. Of course, these scalar multiples are just the eigenvalues given in (3). In general λ_0 and (if n is even) $\lambda_{n/2}$ are simple, while the other λ_ℓ are double eigenvalues.

Remark If M is not symmetric, then it only commutes with the cyclic subgroup $\mathbf{C}_n$ of $\mathbf{D}_n$. In that case the isotypic decomposition of $\mathbb{R}^n$ given in Proposition 1 remains the same, but the irreducibles are no longer absolutely irreducible, and consequently in general M will have 1 real eigenvalue (2 if n is even) corresponding to the 1-dimensional irreducible A_0 (and $A_{n/2}$), whereas the other eigenvalues may be complex.