Purity relative to classes of finitely presented modules

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Abstract

We investigate purities determined by classes of finitely presented modules including the correspondence between purities for left and right modules. We show some cases where purities determined by matrices of given sizes are different. Then we consider purities over finite-dimensional algebras, giving a general description of the relative pure-injectives which we make completely explicit in the case of tame hereditary algebras.

Introduction:

Purity for modules over general rings was defined in [5] and many relative versions of purity have been considered since then. We consider those purities which, like the original one, are determined by classes S of finitely presented modules. We present a number of characterisations of S-pure-exact sequences and of the associated classes of relatively projective and relatively injective modules. We also show the relation between the purity for left modules which is determined by S and the purity for right modules determined by S; this is said most directly in terms of the matrices presenting the modules in S.

Al-Kawarit and Cauchot [1] gave conditions under which purities determined by matrices of certain sizes are different. We obtain related results over semiperfect rings and we also consider this question in detail over finite-dimensional algebras.

Over finite-dimensional algebras we give a description of the S-pure-injective modules in terms of the type-definable category generated by τS and, in the case of tame hereditary algebras, and using results from [12] and [14], we give a complete description of these modules.

Finally we give a number of characterisations of rings whose indecomposable modules are S-pure-injective.

All rings in this paper are associative with unity, and all modules are unital. We write $_R M$ (M_R) to indicate a left (right) R-module, and we use R-Mod denote the category of all left R-modules. The endomorphism ring of a module M is denoted by $\mathrm{End}_R(M)$. We use $\mathrm{Add}(T)$ (resp., $\mathrm{add}(T)$) to denote the class of all modules that are direct summands of direct sums (resp. finite direct sums) of modules from T. Also, we use $\mathrm{Prod}(T)$ to denote the class of all modules that are direct summands of direct products of modules from T. We use the notation $\underset{n \times m}{M}(R)$ for the set of all $n \times m$ matrices over R. All matrices in this paper are matrices with finitely many rows and finitely many columns and all classes of modules are closed under isomorphisms.

A module is said to be finitely presented if it is the factor module of a free module of rank n modulo a m-generated submodule, for some $n, m \in \mathbb{Z}^+$.

Let S be a class of left R-modules. Following Warfield [18], an exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of left R-modules is said to be S-pure if the sequence $0 \rightarrow \mathrm{Hom}_R(M, A) \rightarrow \mathrm{Hom}_R(M, B) \rightarrow \mathrm{Hom}_R(M, C) \rightarrow 0$ is exact, for all $M \in S$; in this case f is said to be an S-pure monomorphism and g is said to be an S-pure epimorphism. Note that S-pure$=S \cup \{_R R\}$-pure. If $S=R$-Mod then a short exact sequence of modules is S-pure if and only if it is pure. A module M is said to be S-pure-injective (resp. S-pure-projective), if M is injective (resp. projective) relative to every S-pure exact sequence of modules. Clearly the class of S-pure-injective (resp. S-pure-projective) modules is closed under direct summands and direct products (resp. direct sums).

This paper contains five sections. In section 1, many characterizations and properties of S-purity, S-pure-injectivity and S-pure-projectivity are given. For example, we prove that, if S is a class of finitely presented modules then a module M is S-pure-projective if and only if it is projective relative to every S-pure exact sequence $0 \rightarrow K \rightarrow E \rightarrow F \rightarrow 0$ where E is S-pure-injective. Dually, M is S-pure-injective if and only if M is injective relative to every S-pure exact sequence $0 \rightarrow K \rightarrow P \rightarrow L \rightarrow 0$ where P is S-pure-projective.

In [13] purity and S-purity are compared. In particular, it is proved that S-purity and purity are equivalent if and only if S-pure-injectivity and pure-injectivity are equivalent if and only if R-mod$\subseteq$add($S \cup \{_R R\}$) [13, Theorem 2.5, p.2136]. In section 2 of this paper we compare S-purity and T-purity for arbitrary classes S and T of finitely presented left R-modules. For example, in Theorem 2.1 we prove that if S and T are classes of finitely presented left R-modules, then the following statements are equivalent: (1) every T-pure short exact sequence of left R-modules is S-pure; (2) $S \subseteq \mathrm{add}(T \cup \{_R R\})$; (3) $D_{\mathcal{C}}^* \subseteq \mathrm{Prod}((D_{\mathcal{H}} \cup R_R)^*)$ where $*$ denotes the dual of a module ;(4) the corresponding assertions for right modules. Also, in Proposition 2.4 we prove that if each indecomposable direct summand of a module in T has local endomorphism ring and each module in T is a direct sum of indecomposable modules then every S-pure short exact sequence modules is T-pure if and only if each indecomposable direct summand of a module in T is a direct summand of a module in $S \cup \{_R R\}$.

In section 3, we study (n, m)-purity over semiperfect rings. In Theorem 3.6 we give a generalization of [1, Theorem 3.5(1), p.3888] in which we prove that if (n, m) and (r, s) are any two pairs of positive integers such that $n \neq r$ and if one of the following two conditions is satisfied: (a) R is semiperfect and there exists an ideal I of R with gen(I_R)$=$max$\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j; (b) R is Krull-Schmidt and there exists a right ideal I of R with gen(I)$=$max$\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j, then: (1) (m, n)-purity and (s, r)-purity of short exact sequences of left R-modules are not equivalent; (2) (n, m)-purity and (r, s)-purity of short exact sequences of right R-modules are not equivalent.

In section 4, we study purity over finite-dimensional algebras. Firstly, we compare purities over the Kronecker algebra over an algebraically closed field k. In Proposition 4.4 we prove that if R is a finite-dimensional algebra over a field k and it is not of finite representation type, then for every $r \in \mathbb{Z}^+$, there is $n > r$ such that $(\aleph_0, n)$-purity$\neq (\aleph_0, r)$-purity for left R-modules. Let $\mathcal{H}$ be a set of matrices over a tame hereditary finite-dimensional algebra R over a field k. Conditions under which the generic module is $L_{\mathcal{H}}$-pure-injective are given in Proposition 4.10. Finally, we give a complete description of the full support topology closure of any class of indecomposable finite-dimensional modules over a tame hereditary finite-dimensional algebra R over a field k.

In the last section we give a condition on a left R-module M such that every S-pure submodule of M is a direct summand and prove that such module is a direct sum of indecomposable submodules. As a corollary of this result we give a characterizations of rings over which every indecomposable left R-module is S-pure-projective.

1 Purities

Let $n, m \in \mathbb{Z}^+$. An R-module M is said to be (n, m)-presented if it is the factor module of the module R^n modulo an m-generated submodule. Let H be an $n \times m$ matrix over R. Then right (resp. left) multiplication by H determines a homomorphism $\rho_H : {}_R R^n \to {}_R R^m$ (resp. $\lambda_H : R_R^m \to R_R^n$). Then H determines the (m, n)-presented left R-module $R^m/\mathrm{im}(\rho_H)$; we will denote it by L_H. Also, H determines the (n, m)-presented right R-module $R^n/\mathrm{im}(\lambda_H)$; we will denote it by D_H. Let $\mathcal{H}$ be a set of matrices over a ring R; we will denote by $L_{\mathcal{H}}$ the class of left R-modules $\{L_H \mid H \in \mathcal{H}\}$ and by $D_{\mathcal{H}}$ the class of right R-modules $\{D_H \mid H \in \mathcal{H}\}$. In view of proposition 1.2 below we may, where convenient, interpret $L_\emptyset$ as $\{{}_R R\}$ and $D_\emptyset$ as $\{R_R\}$,

The following theorem collects together and extends results from the literature (in particular see [8] and [20]). A proof can be found in the author's thesis [10].

Theorem 1.1. *Let R be an algebra over a commutative ring K and let E be an injective cogenerator for K-modules. Let S be a class of finitely presented left R-modules, let $\mathcal{H}$ be a set of matrices over R such that $L_{\mathcal{H}}$-purity=S-purity and let $\Sigma : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of left R-modules. Then the following statements are equivalent.*

(1) Σ is an S-pure exact sequence of left R-modules.

(2) For any two positive integers n,m, for any $n \times m$ matrix H in $\mathcal{H}$ and for all $\bar{a} \in A^n$, if the matrix equation $H\bar{x}$=$f\bar{a}$ has a solution in B^m then the equation $H\bar{x}$=$\bar{a}$ has a solution in A^m.

(3) The sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact, for all $M \in D_{\mathcal{H}}$.

(4) For any two positive integers n,m, for any $n \times m$ matrix H in $\mathcal{H}$ and for all $\bar{c} \in C^m$, if $H\bar{c}$=0, then there is $\bar{b} \in B^m$ with $g(\bar{b})$=$\bar{c}$ and $H\bar{b}$=0.

(5) For any two positive integers n,m and for any $n \times m$ matrix H in $\mathcal{H}$, for every commutative diagram of left R-modules

$$
\begin{array}{ccc}
R^n & \xrightarrow{\ \rho_H\ } & R^m \\
\alpha \downarrow & & \downarrow \\
& & \\
0 \longrightarrow A & \xrightarrow{\ f\ } & B
\end{array}
$$

there exists a homomorphism $\beta : R^m \to A$ such that α=$\beta \rho_H$.

(6) The dual exact sequence of right R-modules $0 \to C^* \to B^* \to A^* \to 0$ is $D_{\mathcal{H}}$-pure, where M^*=$\mathrm{Hom}_K(M, E)$.

We retain the notation M^* for the dual of a module with respect to ${}_K E$ as above. Let T be a class of left R-modules. Note that if $S \subseteq T \subseteq R$-Mod then every T-pure exact sequence of left R-modules is S-pure, so S-pure-injective implies T-pure-injective and S-pure-projective implies T-pure-projective.

Proposition 1.2. *(as [17, Proposition 1, p.700]) Let S be a class of finitely presented left R-modules and let M be a left R-module. Then:*

(1) There exists an S-pure exact sequence of left R-modules $0 \to K \to F \to M \to 0$ with F being a direct sum of copies of modules in $S \cup \{{}_R R\}$.

(2) $\mathrm{Add}(S \cup \{{}_R R\})$ is the class of S-pure-projective left R-modules.

Corollary 1.3. *Let S be a class of finitely presented left R-modules and let $\mathcal{H}$ be a set of matrices over R such that $L_{\mathcal{H}}$-purity=S-purity. Then for any left R-module N there is an S-pure monomorphism $\alpha : N \to F^*$ such that F is a direct sum of copies of modules in $D_{\mathcal{H}} \cup \{R_R\}$. In particular, see Theorem 1.4, F^* is S-pure-injective.*

Proof. Let N be any left R-module. By the right hand version of Proposition 1.2, there is a $D_{\mathscr{H}}$-pure exact sequence of right R-modules $0 \to G \xrightarrow{f} F \xrightarrow{g} N^* \to 0$ where F is a direct sum of copies of modules in $D_{\mathscr{H}} \cup \{R_R\}$. By the right hand version of Theorem 1.1, the dual exact sequence of left R-modules $0 \to N^{**} \xrightarrow{g^*} F^* \xrightarrow{g^*} G^* \to 0$ *is* $L_{\mathscr{H}}$-pure. The canonical monomorphism $\varphi_N : N \to N^{**}$ is pure (see, e.g., [8, Corollary 1.30, p.17]) and hence it is $L_{\mathscr{H}}$-pure. Since a composition of $L_{\mathscr{H}}$-pure monomorphisms clearly is $L_{\mathscr{H}}$-pure, $g^* \varphi_N : N \to F^*$ is an $L_{\mathscr{H}}$-pure monomorphism. $\square$

Let S be a class of left (or right) R-modules. We use S^* to denote the class $\{M^* \mid M \in S\}$.

Theorem 1.4. *(as [18, Theorem 1]) Let S be a class of finitely presented left R-modules and let $\mathscr{H}$ be a set of matrices over R such that $L_{\mathscr{H}}$-purity=S-purity, then* $\mathrm{Prod}((D_{\mathscr{H}} \cup \{R_R\})^*)$ *is the class of S-pure-injective left R-modules.*

Proof. Let M be any S-pure-injective left R-module. By Corollary 1.3, there exists an S-pure, hence split, monomorphism $\alpha : M \to F^*$ where $F = \bigoplus_{i \in I} F_i$ with $F_i \in D_{\mathscr{H}} \cup \{R_R\}$. Since $F^* = (\bigoplus_{i \in I} F_i)^* \simeq \prod_{i \in I} F_i^*$ it follows that $M \in \mathrm{Prod}((D_{\mathscr{H}} \cup \{R_R\})^*)$.

Conversely, let $H \in \mathscr{H}$ and let $\Sigma : 0 \to A \to B \to C \to 0$ be any L_H-pure exact sequence of left R-modules. By Theorem 1.1, the sequence $D_H \otimes_R \Sigma : 0 \to D_H \otimes_R A \to D_H \otimes_R B \to D_H \otimes_R C \to 0$ is exact. Since E is an injective K-module, the sequence $0 \to \mathrm{Hom}_K(D_H \otimes_R C, E) \to \mathrm{Hom}_K(D_H \otimes_R B, E) \to \mathrm{Hom}_K(D_H \otimes_R A, E) \to 0$ is exact. This is isomorphic to the sequence $0 \to \mathrm{Hom}_R(C, \mathrm{Hom}_K(D_H, E)) \to \mathrm{Hom}_R(B, \mathrm{Hom}_K(D_H, E)) \to \mathrm{Hom}_R(A, \mathrm{Hom}_K(D_H, E)) \to 0$. That is, the sequence $0 \to \mathrm{Hom}_R(C, D_H^*) \to \mathrm{Hom}_R(B, D_H^*) \to \mathrm{Hom}_R(A, D_H^*) \to 0$ is exact. Therefore, D_H^* is L_H-pure-injective. By, for instance, [7, Theorem 3.2.9, p.77], R_R^* is injective and thus each module in $(D_{\mathscr{H}} \cup \{R_R\})^*$ is S-pure-injective. It follows that every module in $\mathrm{Prod}((D_{\mathscr{H}} \cup \{R_R\})^*)$ is S-pure-injective. $\square$

Proposition 1.5. *Let S be a class of finitely presented left R-modules, let $\mathscr{H}$ be a set of matrices over R such that $L_{\mathscr{H}}$-purity=S-purity and let $\Sigma : 0 \to A \to B \to C \to 0$ be any exact sequence of left R-modules. Then the following statements are equivalent:*

(1) Σ *is S-pure.*
(2) *Every S-pure-injective left R-module is injective relative to Σ.*
(3) D_H^* *is injective relative to Σ, for all $H \in \mathscr{H}$.*
(4) *Every S-pure-projective left R-module is projective relative to Σ.*

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (4)$ are obvious and $(2) \Rightarrow (3)$ is immediate from Theorem 1.4.

$(3) \Rightarrow (1)$ Let $H \in \mathscr{H}$. By hypothesis, the sequence

$$0 \to \mathrm{Hom}_R(C, \mathrm{Hom}_K(D_H, E)) \to \mathrm{Hom}_R(B, \mathrm{Hom}_K(D_H, E)) \to \mathrm{Hom}_R(A, \mathrm{Hom}_K(D_H, E)) \to 0,$$

equivalently, the sequence

$$0 \to \mathrm{Hom}_K(D_H \otimes_R C, E) \to \mathrm{Hom}_K(D_H \otimes_R B, E) \to \mathrm{Hom}_K(D_H \otimes_R A, E) \to 0$$

is exact. Since E is an injective cogenerator for K-modules, it follows (see [7, Lemma 3.2.8, p.77]) that the sequence $0 \to D_H \otimes_R A \to D_H \otimes_R B \to D_H \otimes_R C \to 0$ is exact. Thus Σ is S-pure.

$(4) \Rightarrow (1)$ This is immediate from Proposition 1.2, and the definition of S-pure exact sequence. $\square$

Theorem 1.6. *Let S be a class of finitely presented left R-modules. Then for a left R-module M:*

(1) *M is S-pure-projective if and only if it is projective relative to every S-pure exact sequence $0 \to K \to E \to F \to 0$ of left R-modules where E is S-pure-injective;*

(2) *M is S-pure-injective if and only if M is injective relative to every S-pure exact sequence $0 \to K \to P \to L \to 0$ of left R-modules where P is S-pure-projective.*

Proof. $(1)(\Rightarrow)$ is obvious.

$(\Leftarrow)$ Let $0 \to A \xrightarrow{\mu} B \xrightarrow{v} C \to 0$ be any S-pure exact sequence of left R-modules. By Corollary 1.3 and Theorem 1.4, there is an S-pure exact sequence $0 \to B \xrightarrow{\lambda} P \xrightarrow{\rho} N \to 0$ of left R-modules where P is S-pure-injective. We have the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{v} & C & \longrightarrow & 0 \\
 & & \downarrow{I_A} & & \downarrow{\lambda} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & D & \longrightarrow & 0 \\
 & & & & \downarrow{\rho} & & \downarrow{\delta} & & \\
 & & & & N & \xrightarrow{I_N} & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since μ and λ are S-pure R-monomorphisms so is $\lambda\mu$. Since $\alpha=\lambda\mu$, the exact sequence $0 \to A \xrightarrow{\alpha} P \xrightarrow{\beta} D \to 0$ is S-pure. Let $\psi \in \mathrm{Hom}_R(M,C)$. By hypothesis, there is $\gamma \in \mathrm{Hom}_R(M,P)$ such that $\beta\gamma=\varphi\psi$. We have $\rho\gamma=\delta\beta\gamma=\delta\varphi\psi = 0$ so $\mathrm{im}(\gamma) \subseteq \ker(\rho)=\mathrm{im}(\lambda)$ and hence $\gamma=\lambda\gamma'$ for some $\gamma' \in \mathrm{Hom}_R(M, B)$. Then we have $\varphi v\gamma'=\beta\lambda\gamma'=\beta\gamma=\varphi\psi$. Since φ is a monomorphism, $v\gamma'=\psi$. Hence M is S-pure-projective.

(2) The proof is dual to that of (1). $\qquad\square$

Corollary 1.7. *Let S be a class of finitely presented left R-modules. Then the following statements are equivalent:*

(1) For every S-pure exact sequence $0 \to N \to M \to K \to 0$ of left R-modules, if M is S-pure-projective, then N is S-pure-projective.

(2) For every S-pure exact sequence $0 \to N \to M \to K \to 0$ of left R-modules, if M is S-pure-injective, then K is S-pure-injective.

Proof. $(1) \Rightarrow (2)$ Let $0 \to N \xrightarrow{v} M \xrightarrow{\mu} K \to 0$ be any S-pure exact sequence of left R-modules where M is S-pure-injective. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be any S-pure exact sequence of left R-modules where B is S-pure-projective. By hypothesis, A is S-pure-projective. Let $f : A \to K$ be any R-homomorphism. Thus there is an R-homomorphism $g : A \to M$ such that $\mu g = f$. Since M is S-pure-injective, there is an R-homomorphism $h : B \to M$ such that $h\alpha = g$. Put $\lambda = \mu h$, thus $\lambda\alpha = (\mu h)\alpha = \mu(h\alpha) = \mu g = f$. Hence K is injective relative to every S-pure exact sequence $0 \to A \to B \to C \to 0$ where B is S-pure-projective. By Theorem 1.6, K is S-pure-injective.

$(2) \Rightarrow (1)$ Let $0 \to N \xrightarrow{v} M \xrightarrow{\mu} K \to 0$ be any S-pure exact sequence of left R-modules where M is S-pure-projective. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be any S-pure exact sequence of left R-modules where B is S-pure-injective. By hypothesis, C is S-pure-injective. Let $f : N \to C$ be any R-homomorphism. Thus there is an R-homomorphism $g : M \to C$ such that $gv = f$. Since M is S-pure-projective, there is an R-homomorphism $h : M \to B$ such that $\beta h = g$. Put $\lambda = hv$, thus $\beta\lambda = \beta hv = gv = f$. Hence N is projective relative to every S-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules where B is S-pure-injective. By Theorem 1.6, N is S-pure-projective. $\qquad\square$

2 Comparing purities

Theorem 2.1. *Let S and T be classes of finitely presented left R-modules and let $\mathscr{G}$ and $\mathscr{H}$ be sets of matrices over R such that $L_{\mathscr{G}}$-purity=S-purity and $L_{\mathscr{H}}$-purity=T-purity. Then the following*

statements are equivalent.

(1) *Every T-pure short exact sequence of left R-modules is S-pure.*

(2) *Every T-pure exact sequence* $0 \to A \to B \to C \to 0$ *of left R-modules where B is T-pure-injective is S-pure.*

(3) *Every S-pure-projective left R-module is T-pure-projective.*

(4) $S \subseteq \text{add}(T \cup \{{}_R R\})$.

(5) *Every T-pure exact sequence* $0 \to A \to B \to C \to 0$ *of left R-modules where B is T-pure-projective is S-pure.*

(6) *Every S-pure-injective left R-module is T-pure-injective.*

(7) $D^*_{\mathscr{G}} \subseteq Prod((D_{\mathscr{H}} \cup R_R)^*)$.

(8) *The corresponding assertions for right modules.*

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (5)$ are obvious.

$(2) \Rightarrow (3)$ Let M be any S-pure-projective left R-module and let $\Sigma : 0 \to A \to B \to C \to 0$ be any T-pure exact sequence of left R-modules where B is T-pure-injective. By hypothesis, Σ is S-pure and hence the sequence $0 \to \text{Hom}_R(M,A) \to \text{Hom}_R(M,B) \to \text{Hom}_R(M,C) \to 0$ is exact. Thus M is projective relative to every T-pure exact sequence $\Sigma : 0 \to A \to B \to C \to 0$ of left R-modules where B is T-pure-injective. By Theorem 1.6, M is T-pure-projective.

$(3) \Rightarrow (4)$ This follows by Proposition 1.2.

$(4) \Rightarrow (1)$ Let $\Sigma : 0 \to A \to B \to C \to 0$ be any T-pure exact sequence of left R-modules and let $M \in S$. By assumption and Proposition 1.2 M is T-pure-projective. Thus the sequence $0 \to \text{Hom}_R(M,A) \to \text{Hom}_R(M,B) \to \text{Hom}_R(M,C) \to 0$ is exact. Therefore Σ is S-pure.

$(5) \Rightarrow (6)$ Let M be any S-pure-injective left R-module and let $\Sigma : 0 \to A \to B \to C \to 0$ be any T-pure exact sequence of left R-modules where B is T-pure-projective. By hypothesis, Σ is S-pure and hence the sequence $0 \to \text{Hom}_R(C,M) \to \text{Hom}_R(B,M) \to \text{Hom}_R(A,M) \to 0$ is exact. It follows by Theorem 1.6 that M is T-pure-injective.

$(6) \Rightarrow (7)$ Let $M \in D^*_{\mathscr{G}}$, thus M is an S-pure-injective left R-module (by Theorem 1.4). By hypothesis, M is T-pure-injective so by Theorem 1.4 we have that $M \in \text{Prod}((D_{\mathscr{H}} \cup R_R)^*)$.

$(7) \Rightarrow (1)$ Let $\Sigma : 0 \to A \to B \to C \to 0$ be any T-pure exact sequence of left R-modules. Let $G \in \mathscr{G}$, thus by hypothesis, $D^*_G \in \text{Prod}((D_{\mathscr{H}} \cup R_R)^*)$, hence D^*_G is T-pure-injective, in particular D^*_G is injective relative to Σ. By Proposition 1.5, Σ is S-pure.

$(1) \Rightarrow (8)$ Let $\Sigma : 0 \to A \to B \to C \to 0$ be any $D_{\mathscr{H}}$-pure exact sequence of right R-modules. By the right hand version of Theorem 1.1, the exact sequence of left R-modules $\Sigma^* : 0 \to C^* \to B^* \to A^* \to 0$ is T-pure. By hypothesis, Σ^* is S-pure and hence by Theorem 1.1 again, Σ is $D_{\mathscr{G}}$-pure.

$(8) \Rightarrow (1)$ This follows by right/left symmetry.

$\square$

The following corollary is immediately obtained from Theorem 2.1.

Corollary 2.2. *Let S and T be classes of finitely presented left R-modules and let $\mathscr{G}$ and $\mathscr{H}$ be sets of matrices over R such that $L_{\mathscr{G}}$-purity=S-purity and $L_{\mathscr{H}}$-purity=T-purity. Then the following statements are equivalent:*

(1) *T-purity =S-purity for short exact sequences of left R-modules.*

(2) *S-pure-projectivity=T-pure-projectivity for left R-modules.*

(3) $\text{add}(S \cup \{{}_R R\}) = \text{add}(T \cup \{{}_R R\})$.

(4) *S-pure-injectivity=T-pure-injectivity for left R-modules.*

(5) $\text{Prod}(\{D^*_G \mid G \in \mathscr{G} \cup \{\underset{1 \times 1}{0}\}\}) = \text{Prod}(\{D^*_H \mid H \in \mathscr{H} \cup \{\underset{1 \times 1}{0}\}\})$.

(6) *The corresponding assertions on the right.*

A short exact sequence (Σ) of left (resp. right) R-modules is called (m,n)-pure if it remains exact when tensored with any (m,n)-presented right (resp. left) R-module. A left R-module M is said to be (m,n)-pure-projective (resp. (m,n)-pure-injective) if it is projective (resp. injective) relative to every (m,n)-pure exact sequence of left R-modules . A short exact

sequence (Σ) of left (or right) R-modules is called $(\aleph_0, n)$-pure exact (resp. $(m, \aleph_0)$-pure exact) if, for each positive integer m (resp. n) (Σ) is (m, n)-pure [1]. Observe that the (m, n)-pure exact sequences of left R-modules are exactly the $L_{\mathscr{H}}$-pure exact sequences, where $\mathscr{H} = M_{m \times n}(R)$, and the (n, m)-pure exact sequences of right R-modules are exactly the $D_{\mathscr{H}}$-pure exact sequences of right modules. Also, $(\aleph_0, n)$-pure exact sequences of left R-modules are exactly the $L_{\mathscr{H}}$-pure exact sequences, where $\mathscr{H} = \bigcup_{m \in \mathbb{Z}^+} M_{m \times n}(R)$ and then the $(n, \aleph_0)$-pure exact sequences of right R-modules are exactly the $D_{\mathscr{H}}$-pure exact sequences. Note that for left modules (n, m)-presented implies (m, n)-pure-projective, where as for right modules (n, m)-presented implies (n, m)-pure-projective. For all $n, m, s, t \in \mathbb{Z}^+$ with $n \geq s$ and $m \geq t$, since every (t, s)-presented right R-module is (m, n)-presented it follows that every (m, n)-pure exact sequence of left R-modules is (t, s)-pure.

Corollary 2.3. *Let $n, m, s, t \in \mathbb{Z}^+$. Then the following statements are equivalent:*

(1) Every (m, n)-pure short exact sequence of left R-modules is (s, t)-pure.

(2) Every (n, m)-pure short exact sequence of right R-modules is (t, s)-pure.

(3) Every (s, t)-pure-projective (resp. (s, t)-pure-injective) left R-module is (m, n)-pure-projective (resp. (m, n)-pure-injective).

(4) Every (t, s)-presented left R-module is in $\mathrm{add}(\{M \mid M$ is an (n, m)-presented left R-module$\})$.

(5) Every (s, t)-presented right R-module is in $\mathrm{add}(\{M \mid M$ is an (m, n)-presented right R-module$\})$.

Proof. Take $S = L_{\mathscr{G}}$ and $T = L_{\mathscr{H}}$ where $\mathscr{G} = M_{s \times t}(R)$ and $\mathscr{H} = M_{m \times n}(R)$ and apply Theorem 2.1. $\square$

Proposition 2.4. *Let S and T be classes of finitely presented left R-modules. Consider the following statements:*

(1) Every S-pure short exact sequence of left R-modules is T-pure.

(2) Each indecomposable direct summand of a module in T is in $\mathrm{add}(S \cup \{_R R\})$.

(3) Each indecomposable direct summand of a module in T is a direct summand of a module in $S \cup \{_R R\}$.

Then (1) implies (2) and

(a) If each indecomposable direct summand of a module in T has local endomorphism ring then (2) implies (3).

(b) If each module in T is a direct sum of indecomposable modules then (3) implies (1).

Proof. $(1) \Rightarrow (2)$ This follows by Theorem 2.1.

(a) Assume that each indecomposable direct summand M of a module in T has local endomorphism ring, thus by hypothesis, $M \in \mathrm{add}(S \cup \{_R R\})$. Suppose that M is a direct summand of $\bigoplus_{i \in I} F_i$ where $F_i \in S \cup \{_R R\}$, for all $i \in I$ and I is a finite set and let B be a submodule of $\bigoplus_{i \in I} F_i$ such that $M \oplus B = \bigoplus_{i \in I} F_i$. Since $\mathrm{End}_R(M)$ is local we have (see, e.g., [8, Theorem 2.8, p.37]) that M has the finite exchange property. So (see, e.g., [8, Lemma 2.7, p.37]) there is an index $j \in I$ and a direct sum decomposition $F_j = B_j \oplus C_j$ of F_j with $M \simeq C_j$. Hence M is a direct summand of a module in $S \cup \{_R R\}$.

(b) This follows directly using Proposition 1.5. $\square$

A ring R is said to be Krull-Schmidt if every finitely presented left (or right) R-module is a direct sum of modules with local endomorphism rings (see [8, p.97]).

Corollary 2.5. *Let R be a left Krull-Schmidt ring and let n, m be positive integers. Then the following statements are equivalent:*

(1) (m, n)-purity = $(\aleph_0, n)$-purity for short exact sequences of left R-modules.

(2) For each $s \in \mathbb{Z}^+$, each indecomposable (n, s)-presented left R-module is a direct summand of an (n, m)-presented left R-module.

(3) (n,m)-*purity*=$(n,\aleph_0)$-*purity for short exact sequences of right R-modules.*

(4) *For each $s \in \mathbb{Z}^+$, each indecomposable (s,n)-presented right R-module is a direct summand of an (m,n)-presented right R-module.*

Proof. Put $S = L_{\mathscr{G}}$ and $T = L_{\mathscr{H}}$, where $\mathscr{G}=M_{m\times n}(R)$ and $\mathscr{H} = \bigcup_{t\in\mathbb{Z}^+} M_{t\times n}(R)$. Since R is Krull-Schmidt, each indecomposable direct summand of a module in T has local endomorphism ring and each module in T is a direct sum of indecomposable modules. Hence the result follows on applying Proposition 2.4 and Corollary 2.3.

$\square$

3 (m,n)-**Purity over semiperfect rings**

Let M be a finitely presented left (or right) R-module, we denote by $\mathrm{gen}(M)$ its minimal number of generators and by $\mathrm{rel}(M)$ the minimal number of relations on these generators. Therefore there is an exact sequence $R^{\mathrm{rel}(M)} \to R^{\mathrm{gen}(M)} \to M \to 0$ and it follows easily that $\mathrm{rel}(M)$ is the minimal number of relations on any generating set of M.

Remark 3.1. *Let M be a finitely presented left R-module and let N be a direct summand of M. Then it is easy to see that $\mathrm{gen}(N) \leq \mathrm{gen}(M)$ and $\mathrm{rel}(N) \leq \mathrm{rel}(M)+\mathrm{gen}(M)$.*

Proposition 3.2. *Let H be any matrix over a ring R such that $\mathrm{End}_R(L_H)$ is local and L_H is not projective. Set $\mathscr{H}=\bigcup\{M_{r\times q}(R)\,|\,q<\mathrm{gen}(L_H)\text{ or }r+q<\mathrm{rel}(L_H)\}$. Then L_H is an L_H-pure-projective left R-module which is not $L_{\mathscr{H}}$-pure-projective and hence not $L_{\mathscr{G}}$-pure-projective for any $\mathscr{G} \subseteq \mathscr{H}$. In particular L_H-purity and $L_{\mathscr{H}}$-purity are not equivalent.*

Proof. By Proposition 1.2, L_H is L_H-pure-projective and, if L_H is $L_{\mathscr{H}}$-pure-projective, then $L_H \in \mathrm{Add}(L_{\mathscr{H}} \cup \{_R R\})$. Since $\mathrm{End}_R(L_H)$ is local, L_H is, as in Proposition 2.4, a direct summand of a module in $L_{\mathscr{H}} \cup \{_R R\}$. Thus either L_H is a direct summand of L_G, where $\underset{r\times q}{G} \in \mathscr{H}$ or L_H is projective. If L_H is a direct summand of L_G, by Remark 3.1, $\mathrm{gen}(L_H) \leq \mathrm{gen}(L_G) \leq q$ and $\mathrm{rel}(L_H) \leq \mathrm{rel}(L_G)+ \mathrm{gen}(L_G) \leq r + q$ and this contradicts $G \in \mathscr{H}$. $\square$

Note that if M is a left R-module, I is a left ideal of R and $\alpha \in \mathrm{End}_R(M)$ then there is an induced homomorphism $\overline{\alpha} : M/IM \to M/IM$ which is an isomorphism if α is an isomorphism.

Let R be a ring and let J be its Jacobson radical. Recall that R is semiperfect if R/J is semisimple and idempotents lift modulo J. Say that an idempotent $e \in R$ is local if eRe is a local ring. We have (e.g., [20, 42.6, p.375]) that R semiperfect if and only if $R=e_1 R\oplus e_2 R\oplus\cdots\oplus e_n R$, for local orthogonal idempotents e_i.
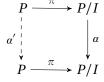
Lemma 3.3. *Let $m \in \mathbb{Z}^+$. Suppose that one of the following two conditions is satisfied.*

(1) The ring R is semiperfect and I is a nonzero ideal with $\mathrm{gen}(I_R) = m$ and $I \subseteq e_j R$ for some local idempotent e_j of R.

(2) The ring R is Krull-Schmidt and I is a nonzero right ideal with $\mathrm{gen}(I) = m$ and $I \subseteq e_j R$ for some local idempotent e_j of R.

Then $e_j R/I$ is a finitely presented right R-module with $\mathrm{gen}(e_j R/I)=1$, $\mathrm{rel}(e_j R/I)=m$ and $\mathrm{End}_R(e_j R/I)$ is a local ring.

Proof. Let $P=e_j R$. Then $\mathrm{gen}(P/I)=1$ and clearly $\mathrm{rel}(P/I)=\mathrm{gen}(I)=m$.

In case (1): Since $\mathrm{End}_R(e_j R) \simeq e_j R e_j$ it follows that $\mathrm{End}_R(P)$ is a local ring. Let $\alpha \in \mathrm{End}_R(P/I)$ and consider the following diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{\ \pi\ } & P/I \\
{\scriptstyle\alpha'}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\
P & \xrightarrow{\ \pi\ } & P/I
\end{array}
$$

where π is the natural epimorphism. By projectivity of P, there exists an R-homomorphism $\alpha' : P \rightarrow P$ such that $\pi\alpha' = \alpha\pi$ and $\alpha'(I) \subseteq I$. Since $\mathrm{End}_R(P)$ is a local ring, either α' or $1_P\text{-}\alpha'$ is an isomorphism. The inverse of that isomorphism will, as noted above, induce an isomorphism on $P/I = P/PI$ which will be an inverse of α or $1_{(P/I)} - \alpha$, as appropriate. Hence $\mathrm{End}_R(e_j R/I)$ is a local ring.

In case (2): Since $e_j R$ is a local right R-module, every homomorphic image of $e_j R$ is indecomposable [16, Proposition 4.1, p.246]. Hence $e_j R/I$ is indecomposable. Since R is Krull-Schmidt, $\mathrm{End}_R(e_j R/I)$ is a local ring.

$\square$

Let R be any ring and M be any finitely presented right R-module. An Auslander-Bridger dual of M is denoted by $D(M)$ and defined as follows. Choose an exact sequence $Q \xrightarrow{\phi} P \rightarrow M \rightarrow 0$ in which P and Q are finitely generated projective right R-modules. Define $D(M)$ to be the cokernel of the homomorphism $\phi^+ : P^+ \rightarrow Q^+$ where $X^+ = \mathrm{Hom}_R(X, R_R)$, for any right R-module X [19]. Although $D(M)$ depends on the choice of exact sequence, if $D'(M)$ is another such dual of M then $D(M) \oplus A \simeq D'(M) \oplus B$ for some finitely generated projective modules, A, B.

Lemma 3.4. *Let $m \in \mathbb{Z}^+$ and let M be any $(1,m)$-presented right R-module. Then $D(M)$ is a (n,m)-pure-projective left R-module, for all $n \in \mathbb{Z}^+$.*

Proof. Applying $\mathrm{Hom}_R(-, R_R)$ to a presentation $R_R^m \xrightarrow{\lambda_H} R_R^1 \rightarrow M \rightarrow 0$ of M gives the presentation $_R R^1 \xrightarrow{\rho_H} _R R^m \rightarrow D(M) \rightarrow 0$ of $D(M)$. Thus $D(M)$ is $(m,1)$-presented hence $(1,m)$-pure-projective, hence (n,m)-pure-projective for all $n \geq 1$.

$\square$

Proposition 3.5. *Let $m \in \mathbb{Z}^+$. Suppose that one of the following two conditions is satisfied.*
(1) The ring R is semiperfect and I is a nonzero ideal with $\mathrm{gen}(I_R) = m+1$ and $I \subseteq e_j R$ for some local idempotent e_j of R.
(2) The ring R is Krull-Schmidt and I is a nonzero right ideal with $\mathrm{gen}(I) = m+1$ and $I \subseteq e_j R$ for some local idempotent e_j of R.
Then $D(e_j R/I)$ is not an $L_{\mathscr{H}}$-pure-projective left R-module, where $\mathscr{H} = \bigcup \{ \underset{s \times t}{M}(R) \mid s, t \in \mathbb{Z}^+ \text{ with } t < m+1 \}$.

Proof. By Lemma 3.3, $\mathrm{End}_R(e_j R/I)$ is a local ring and hence $\mathrm{End}_R(D(e_j R/I))$ is local [19, Theorem 2.4, p.196]. Since $\mathrm{gen}(e_j R/I) = 1$ and $\mathrm{rel}(e_j R/I) = m+1$ (by Lemma 3.3), it follows easily that $\mathrm{gen}(D(e_j R/I)) = m+1$ and $\mathrm{rel}(D(e_j R/I)) = 1$. Since $e_j R/I$ is not projective, neither is $D(e_j R/I)$ so, by Proposition 3.2, $D(e_j R/I)$ is not $L_{\mathscr{H}}$-pure-projective. $\square$

The following theorem is a generalization of [1, Theorem 3.5(1), p.3888].

Theorem 3.6. *Let (n,m) and (r,s) be any two pairs of positive integers such that $n \neq r$. Suppose that one of the following two conditions is satisfied:*
(a) R is semiperfect and there exists an ideal I of R with $\mathrm{gen}(I_R) = \max\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j
(b) R is Krull-Schmidt and there exists a right ideal I of R with $\mathrm{gen}(I) = \max\{n, r\}$ and $I \subseteq e_j R$ for some local idempotent e_j.
Then:
(1) (m,n)-purity and (s,r)-purity of short exact sequences of left R-modules are not equivalent;
(2) (n,m)-purity and (r,s)-purity of short exact sequences of right R-modules are not equivalent.

Proof. (1) Without loss of generality, we can assume that $n < r$. By Lemma 3.4 and Proposition 3.5, $D(e_j R/I)$ is (s,r)-pure-projective and not (m,n)-pure-projective. Thus (m,n)-pure-projectivity and (s,r)-pure-projectivity of left R-modules are not equivalent and hence by Corollary 2.3, (m,n)-purity and (s,r)-purity for left R-modules are not equivalent.

(2) By (1) and Corollary 2.3. $\qquad\square$

Corollary 3.7. *Let R be a local ring, let I be a finitely generated ideal of R and set $\text{gen}(I_R) = r$, then for all $n < r$ and for all m,s:*

(1) (m,n)-*purity and* (s,r)-*purity for left R-modules are not equivalent.*
(2) (n,m)-*purity and* (r,s)-*purity for right R-modules are not equivalent.*

Proof. Since R is local it is a semiperfect and 1 is a local idempotent. By Theorem 3.6, the result holds. $\qquad\square$

Let M be a finitely generated left module over a semiperfect ring R. Warfield in [19] defined $\text{Gen}(M)$ to be the number of summands in a decomposition of M/JM as a direct sum of simple modules where $J = J(R)$. If M is a finitely presented left module over a semiperfect ring R, and $f : P \to M$ a projective cover, with $K = \ker(f)$, then Warfield defined $\text{Rel}(M)$ by $\text{Rel}(M)=\text{Gen}(K)$. If M is a left R-module and $x \in M$, we say x is a local element if Rx is a local module. The number of elements in any minimal generating set of local elements of M is exactly $\text{Gen}(M)$ [19, Lemma 1.11]. One may use these to obtain similar results, for example the following.

Proposition 3.8. *Let H be a matrix over a semiperfect ring R such that L_H is not projective and $\text{End}_R(L_H)$ is a local ring and let $\mathcal{H} =\{K \mid K$ is a matrix with $\text{Gen}(L_H) > \text{Gen}(L_K)$ or $\text{Rel}(L_H) >\text{Rel}(L_K)\}$. Then L_H is not $L_{\mathcal{H}}$-pure-projective.*

Proof. Assume that L_H is $L_{\mathcal{H}}$-pure-projective, thus by Proposition 1.2, $L_H \in \text{add}(L_{\mathcal{H}} \cup \{_R R\})$. Since $\text{End}_R(L_H)$ is a local ring, L_H is as in Proposition 2.4, a direct summand of a module in $L_{\mathcal{H}} \cup \{_R R\}$. Thus either L_H is a direct summand of L_D, where $D \in \mathcal{H}$ or L_H is a direct summand of $_R R$. Since L_H is not projective, L_H is a direct summand of L_D, thus by [19, Lemma 1.10, p.192], $\text{Gen}(L_H) \leq \text{Gen}(L_D)$ and $\text{Rel}(L_H) \leq \text{Rel}(L_D)$ and this contradicts $D \in \mathcal{H}$. Therefore, L_H is not $L_{\mathcal{H}}$-pure-projective. $\qquad\square$

Remark. *Since, if K is an $r \times q$ matrix, we have $\text{Gen}(R).q \geq \text{Gen}(R).\text{gen}(L_K) \geq \text{Gen}(L_K)$ and similarly for relations, if H is as in Proposition 3.8 then L_H is not $L_{\mathcal{H}_i}$-pure-projective for any of the sets of matrices:*

$\mathcal{H}_1 =\{K \mid \text{Gen}(L_H) > \text{Gen}(R)\,\text{gen}(L_K)$ or $\text{Rel}(L_H) > \text{Gen}(R)\,\text{rel}(L_K)\}$;
$\mathcal{H}_2 =\{\underset{r \times q}{K} \mid r,q \in \mathbb{Z}^+$ such that $\text{Gen}(L_H) > q\text{Gen}(R)$ or $\text{Rel}(L_H) > \text{Rel}(L_K)\}$;
$\mathcal{H}_3 = \bigcup \{\underset{r \times q}{M}(R) \mid r,q \in \mathbb{Z}^+$ such that $\text{Gen}(L_H) > q\text{Gen}(R)$ or $\text{Rel}(L_H) > r\text{Gen}(R)\}$.

4 Purity over finite-dimensional algebras

In this section we assume some knowledge of the representation theory of finite-dimensional algebras, for which see [2], [3] for example. Let R be a Krull-Schmidt ring and let M be any finitely presented left R-module. We will use $\text{ind}(M)$ to denote the class of (isomorphism types of) indecomposable direct summands of M. If S is a class of finitely presented left R-modules, we define $\text{ind}(S) = \bigcup_{M \in S} \text{ind}(M)$.

Proposition 4.1. *Let R be a Krull-Schmidt ring and let S be a class of finitely presented left R-modules. Then the following statements are equivalent for a left R-module M:*

(1) *M is S-pure-projective.*
(2) *M is $\text{ind}(S)$-pure-projective.*
(3) *M isomorphic to a direct sum of modules in $\text{ind}(S \cup \{_R R\})$.*

Proof. Since R is a Krull-Schmidt ring, each element in $S \cup \{_R R\}$ is a direct sum of modules in $\text{ind}(S \cup \{_R R\})$ so this follows by Proposition 1.2. □

The following corollary is immediate from Proposition 4.1 and Theorem 2.1.

Corollary 4.2. *Let R be a Krull-Schmidt ring and let S and T be two classes of finitely presented left R-modules. Then T-purity implies S-purity if and only if $\text{ind}(S) \subseteq \text{ind}(T \cup \{_R R\})$.*

Let $R = k\tilde{A}_1$ be the Kronecker algebra over an algebraically closed field k. Left R-modules may be viewed as representations of the quiver $_1\bullet \overset{\alpha}{\underset{\beta}{\leftleftarrows}} \bullet_2$. The preinjective and preprojective indecomposable finite-dimensional left R-modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by I_n (resp. P_n) the finite-dimensional indecomposable preinjective (resp. preprojective) left R-module with dimension vector $(n, n+1)$ (resp. $(n+1, n)$). Also, for $n \in \mathbb{Z}^+$ we will use $R_{\lambda,n}$ to denote the finite-dimensional indecomposable regular left R-module with dimension vector (n, n) and parameter $\lambda \in k \cup \{\infty\}$ where $R_{\lambda,1}$ is the module $k \overset{1}{\underset{\lambda}{\leftleftarrows}} k$ for $\lambda \in k$ and $R_{\infty,1} = k \overset{0}{\underset{1}{\leftleftarrows}} k$.

Example 4.3. *Let $R = k\tilde{A}_1$ be the Kronecker algebra over an algebraically closed field k. Let $n, r \in \mathbb{Z}^+$ and let $S_1 = \{P_i \mid i \leq n\}$, $S_2 = \{I_i \mid i \leq n-1\}$, $S_3 = \{R_{\lambda,i} \mid i \leq n \text{ and } \lambda \in k \cup \{\infty\}\}$ and $S_4 = \{R_{\lambda,1} \mid \lambda \in k \cup \{\infty\}\} \cup \{P_0, P_1\}$. Then:*

(i) $S_1 \cup S_2 \cup S_3$-purity= $(\aleph_0, n)$-purity= $(2n+1, n)$-purity, for short exact sequences of left R-modules.

(ii) S_4-purity= $(1,1)$-purity for left R-modules.

(iii) $(1,1)$-purity is not equivalent to $(\aleph_0, n)$-purity for left R-modules.

(iv) $(n,1)$-purity is not equivalent to $(r,2)$-purity for left R-modules.

Proof. (i) Let $\mathscr{H} = \bigcup_{m \in \mathbb{Z}^+} M_{m \times n}(R)$. It follows directly from the description of the finite-dimensional indecomposable modules and Remark 3.1 that $\text{ind}(L_{\mathscr{H}}) = S_1 \cup S_2 \cup S_3$. Thus, by Proposition 4.1 we have that $S_1 \cup S_2 \cup S_3$-purity=$L_{\mathscr{H}}$-purity=$(\aleph_0, n)$-purity for left R-modules.

Let $M \in S_1 \cup S_2 \cup S_3$. It can be checked that, if $M \in S_1$ then $\text{rel}(M) \leq 2n-1$, if $M \in S_2$ then $\text{rel}(M) \leq 2n+1$ and if $M \in S_3$ then $\text{rel}(M) \leq 2n$ and hence $\text{rel}(M) \leq 2n+1$ in all cases. Since $\text{gen}(M) \leq n$, each module in $S_1 \cup S_2 \cup S_3$ is $(n, 2n+1)$-presented. Thus $(2n+1, n)$-purity=$S_1 \cup S_2 \cup S_3$-purity= $(\aleph_0, n)$-purity.

(ii) Let $\lambda \in k \cup \{\infty\}$ and let $M \in R_{\lambda,1} \oplus P_0$. Since the sequence $_R R \overset{(\alpha+\lambda\beta)\times-}{\longrightarrow} {}_R R \to M \to 0$ is exact, M is $(1,1)$-presented and hence $R_{\lambda,1}$ is a direct summand of a $(1,1)$-presented module. Thus every module in S_4 is a direct summand of a $(1,1)$-presented module. Conversely, let N be any indecomposable direct summand of a $(1,1)$-presented left R-module, thus $\text{gen}(N) = 1$ and $\text{rel}(N) \leq 2$ (by Remark 3.1) and hence either $N = P_0$ or $N = P_1$ or $N = R_{\lambda,1}$ for some $\lambda \in k \cup \{\infty\}$. Thus N is a direct summand of a module in $S_4 \cup \{_R R\}$. By Proposition 2.4, S_4-purity= $(1,1)$-purity.

(iii) Assume that $(1,1)$-purity=$(\aleph_0, n)$-purity for some $n \in \mathbb{Z}^+$. Thus, by (i) and (ii) above we have that S_4-purity=$S_1 \cup S_2 \cup S_3$-purity. This contradicts Corollary 4.2, because $I_0 \in S_1 \cup S_2 \cup S_3$ and $I_0 \notin S_4$.

(iv) Note that $R_R = e_1 R \oplus e_2 R$, where $e_1 R$ (resp. $e_2 R$) is the preprojective right R-module of dimension vector $(0,1)$ (resp. $(1,2)$). Let $I_R = J(e_2 R)$, since $I_R = \alpha R \oplus \beta R$ it follows that $\text{gen}(I_R) = 2$. By Theorem 3.6 we have that $(n,1)$-purity and $(r,2)$-purity for left R-modules are not equivalent.

□

Proposition 4.4. *Let R be a finite-dimensional algebra over a field k. If R is not of finite representation type, then for every $r \in \mathbb{Z}^+$, there is $n > r$ such that $(\aleph_0, n)$-purity$\neq (\aleph_0, r)$-purity for left R-modules.*

Proof. Suppose that R is not of finite representation type. Assume that there is $r \in \mathbb{Z}^+$ such that for all $n > r$ then $(\aleph_0, n)$-purity= $(\aleph_0, r)$-purity for left R-modules. Since R is a finite-dimensional algebra and it is not finite representation type it follows from [3, Corollary 1.5, p.194] that there is a finitely generated indecomposable left R-module M such that $\mathrm{gen}(M) \geq r+1$. By assumption, $(\aleph_0, \mathrm{gen}(M))$-purity= $(\aleph_0, r)$-purity for left R-modules and hence by Corollary 4.2, $M \in \mathrm{ind}(\{(r, s)\text{- presented left } R\text{-modules} \mid s \in \mathbb{Z}^+\})$, which is a contradiction. $\qquad\square$

Let R be an algebra over a field k. From now, we use M^* to denote $\mathrm{Hom}_k(M, k)$ for any R-module M.

Proposition 4.5. *Let R be a finite-dimensional algebra over a field k and let $\mathscr{H}$ be a set of matrices over R. Then a left R-module M is $L_{\mathscr{H}}$-pure-injective if and only if M is a direct summand of a direct product of modules in* $\mathrm{ind}(\{D_H^* \mid H \in \mathscr{H} \cup \{\underset{1\times 1}{0}\}\})$.

Proof. This follows by Theorem 1.4 since each module D_H^* is a finite direct sum of indecomposable modules. $\qquad\square$

We now describe these modules in terms of $\mathrm{ind}(\{L_H \mid H \in \mathscr{H} \cup \{\underset{1\times 1}{0}\}\})$.

Theorem 4.6. *Let R be a finite-dimensional algebra over a field k and let S be a set of indecomposable finite-dimensional modules. Then the S-pure-injective left R-modules are the direct summands of direct products of modules in $\tau S \cup R$-inj, where τ is the Auslander-Reiten translate and R-inj denotes the set of indecomposable injective left R-modules.*

Proof. The Auslander-Reiten translate of a module M is given by the formula $\tau M = (DM)^*$ where DM is the Auslander-Bridger dual (=transpose) of M obtained from a minimal projective resolution of M. In particular $\tau L_H = (D_H)^*$ so this follows from Proposition 4.5. $\qquad\square$

Corollary 4.7. *Let R be a finite-dimensional algebra over a field k, let $\mathscr{H}$ be a set of matrices over R. If $\mathrm{ind}\{\mathrm{Hom}_k(D_H, k) \mid H \in \mathscr{H} \cup \{\underset{1\times 1}{0}\}\}$ is finite then it is the set of indecomposable $L_{\mathscr{H}}$-pure-injective left R-modules and every $L_{\mathscr{H}}$-pure-injective module is a direct sum of copies of these modules.*

Proof. This follows since if M is indecomposable of finite length over its endomorphism ring then every product of copies of M is a direct sum of copies of M (see, for example, [12, Theorem 4.4.28, p.180]).

$\qquad\square$

Recall (see[12, 3.4.7]) that a subclass T of R-Mod is said to be definable if it is closed under direct products, direct limits and pure submodules. A class T of pure-injective modules closed under direct products, direct summands and isomorphisms is definable if and only if each direct sum of modules in T is pure-injective, that is if and only if each element in T is Σ-pure-injective (see, for example, [12, 4.4.12]). In this case every module in T is a direct sum of indecomposable modules. Let S be a class of finitely presented left R-modules. We denote by S-Pinj the class of S-pure-injective left R-modules.

Corollary 4.8. *Let R be a finite-dimensional algebra over a field k and let $\mathscr{H}$ be a set of matrices over R. Then $L_{\mathscr{H}}$-Pinj is a definable subclass of R-Mod if and only if each direct sum of modules in $\mathrm{ind}(\{D_H^* \mid H \in \mathscr{H}\})$ is pure-injective.*

Proof. Let $T = \mathrm{ind}(\{D_H^* \mid H \in \mathscr{H}\})$ and $T' = T \cup R$-inj.
One direction follows from the remarks above and Proposition 4.5.
($\Leftarrow$). By hypothesis, each direct sum of modules in T is pure-injective. Since R is a left Noetherian ring, each direct sum of modules in R-inj is injective. Thus each direct sum of

modules in T' is pure-injective and hence is Σ-pure-injective. Let $M \in L_{\mathcal{H}}$-Pinj. By Proposition 4.5, there exists a subfamily $\{M_i\}_{i \in I}$ of T' such that M is a direct summand of $\prod\limits_{i \in I} M_i$. By the proof above, $\bigoplus\limits_{i \in I} M_i$ is Σ-pure-injective. Since $\prod\limits_{i \in I} M_i$ is in the definable subcategory generated by $\bigoplus\limits_{i \in I} M_i$ it follows from [12, Proposition(4.4.12), p.176] that $\prod\limits_{i \in I} M_i$ is Σ-pure-injective. It follows that M is Σ-pure-injective and hence each element in $L_{\mathcal{H}}$-Pinj is Σ-pure-injective. Therefore $L_{\mathcal{H}}$-Pinj is a definable subclass of R-Mod.

$\square$

Every finite-dimensional module is Σ-pure-injective and by [9, Theorem 4.6, p.750] every direct sum of preinjective modules is Σ-pure-injective. The equivalence of (1) and (2) in the next result therefore follows from the description of the Σ-pure-injective modules in [11, Theorem 2.1, p.847] and the equivalence with (3) follows since the duality $\mathrm{Hom}_k(-, k)$ interchanges preprojective and preinjective modules and sends regular modules to regular modules.

Proposition 4.9. *Let R be a tame hereditary finite-dimensional algebra over a field k and let $\mathcal{H}$ be a set of matrices over R. Then the following statements are equivalent.*
(1) $L_{\mathcal{H}}$-Pinj is a definable subclass of R-Mod.
(2) The set of preprojective or regular modules in $\mathrm{ind}(\{D_H^* \mid H \in \mathcal{H}\})$ is finite.
(3) The set of preinjective or regular modules in $\mathrm{ind}(\{D_H \mid H \in \mathcal{H}\})$ is finite.

Let $_R$pinj be the isomorphism classes of indecomposable pure-injective left R-modules and let $T \subseteq R$-ind, the class of all finitely presented indecomposable left R-modules. We use $\mathrm{fsc}(T)$ (resp. $\overline{T}$) to denote the closure of T in the full support topology (resp. the Ziegler topology). Recall that $\mathrm{fsc}(T)$ is the class $\mathrm{Prod}(T) \cap _R$pinj. See for instance [12, Sections 5.1.1 and 5.3.7] for details about the Ziegler topology and the full support topology.

Proposition 4.10. *Let R be a tame hereditary finite-dimensional algebra over a field k. Let $\mathcal{H}$ be a set of matrices over R such that $L_{\mathcal{H}}$-Pinj is definable. Then the following statements are equivalent.*
(1) The generic module is $L_{\mathcal{H}}$-pure-injective.
(2) The set of preinjective left R-modules in $\mathrm{ind}(\{D_H^* \mid H \in \mathcal{H}\})$ is infinite.
(3) All but at most $n(R) - 2$ Prüfer modules are $L_{\mathcal{H}}$-pure-injective, where $n(R)$ is the number of isomorphism classes of simple R-modules.
(4) At least one Prüfer R-module is $L_{\mathcal{H}}$-pure-injective.

Proof. $(1) \Rightarrow (2)$. Let $T = \mathrm{ind}(\{D_H^* \mid H \in \mathcal{H}\})$. Assume that the set of preinjective left R-modules in T is finite. Since $L_{\mathcal{H}}$-Pinj is a definable it follows from Proposition 4.9 that T is finite. By Corollary 4.7, the generic module cannot be $L_{\mathcal{H}}$-pure-injective.

$(2) \Rightarrow (3)$. Let X be the class of all indecomposable $L_{\mathcal{H}}$-pure-injective modules. Since $L_{\mathcal{H}}$-Pinj is definable it follows from [12, Theorem 5.1.1, p.211] that X is a closed set of the Ziegler topology. Since X contains infinitely many non-isomorphic preinjective modules, by [14, Corollary, p.113], all but at most $n(R) - 2$ Prüfer modules belong to X, where $n(R)$ is the number of isomorphism classes of simple R-modules.

$(3) \Rightarrow (4)$. This is obvious.

$(4) \Rightarrow (1)$. Assume that there is a Prüfer module which is $L_{\mathcal{H}}$-pure-injective. As noted in (3), X is a closed set of the Ziegler topology and by hypothesis, it contains at least one module which is not of finite length. By [14, Theorem, p.106], the generic module belongs to X. $\square$

Remark 4.11. *If R is the Kronecker algebra over a field k then condition (3) above becomes: (3) Every Prüfer module is $L_{\mathcal{H}}$-pure-injective.*

Lemma 4.12. *Let $T \subseteq R$-ind. If $\mathrm{Prod}(T)$ is definable then $\overline{T} = \mathrm{fsc}(T)$.*

Proof. Suppose that Prod(T) is definable. It is clear that fsc(T) $\subseteq \overline{T}$. Since $T \subseteq$ Prod(T) it follows that $D(T) \subseteq$ Prod(T), where $D(T)$ is the definable subcategory generated by T. Thus $\overline{T} \subseteq$ fsc(T) and hence $\overline{T} =$ fsc(T). $\qquad\square$

Remark 4.13. *Let T be a class of pure-injective left R-modules and let $S \subseteq T$. If* Prod(T) *is a definable subclass of R-Mod then so is* Prod(S).

Corollary 4.14. *Let R be a tame hereditary finite-dimensional algebra over a field k and let $\boldsymbol{I}_1$ be a class of indecomposable preinjective left R-modules. Then* fsc($\boldsymbol{I}_1$) $= \overline{\boldsymbol{I}_1}$.

Proof. By [21, Theorem 3.2, p.351], Prod($\boldsymbol{I}$) is definable, where $\boldsymbol{I}$ is the class of all indecomposable preinjective left R-modules. Since $\boldsymbol{I}_1 \subseteq \boldsymbol{I}$ it follows from Remark 4.13 that Prod($\boldsymbol{I}_1$) is definable. By Lemma 4.12, fsc($\boldsymbol{I}_1$) $= \overline{\boldsymbol{I}_1}$. $\qquad\square$

Remark 4.15. *Let R be a finite-dimensional algebra over a field k and let $T \subseteq R$-ind. Then T is the class of all indecomposable finite-dimensional left R-modules in* Prod(T). *This follows from [12, Corollary 5.3.33, p.250]*

The following fact is known; it can be found stated in [15, p.47]. We include a proof here.

Proposition 4.16. *Let R be a tame hereditary finite-dimensional algebra over a field k and let $\boldsymbol{P}_1$ be a class of indecomposable preprojective left R-modules. Then* fsc($\boldsymbol{P}_1$) $= \boldsymbol{P}_1$.

Proof. Let $M \in$ fsc($\boldsymbol{P}_1$). Thus M is a direct summand of $\prod_{i \in I} P_i$ where $P_i \in \boldsymbol{P}_1$. Choose a non-zero element $a \in M$, so $a_j \neq 0$ for some $j \in I$, where a_j is the jth component in a. Define $\alpha : M \to P_j$ by $\alpha = \pi_j i$ where $i : M \to \prod_{i \in I} P_i$ is the inclusion and $\pi_j : \prod_{i \in I} P_i \to P_j$ is the projection. Since $\alpha(a) = a_j \neq 0$ it follows that $\mathrm{Hom}_R(M, \boldsymbol{P}) \neq 0$, where $\boldsymbol{P}$ is the class of all indecomposable preprojective left R-modules. By [6, Lemma 1, p.46], M has a preprojective direct summand, and hence M is finite-dimensional and therefore we have from Remark 4.15 that $M \in \boldsymbol{P}_1$.

$\qquad\square$

Lemma 4.17. *Let R be a tame hereditary finite-dimensional algebra over a field k and let $\boldsymbol{R}_1$ be a class of indecomposable regular left R-modules. Then:*
(1) *The generic module does not belong to* fsc($\boldsymbol{R}_1$).
(2) *There is no Prüfer R-module in* fsc($\boldsymbol{R}_1$).

Proof. (1) Assume that the generic module $G \in$ fsc($\boldsymbol{R}_1$), thus $G \in$ Prod($\boldsymbol{R}_1$). As in the proof of Proposition 4.16 it follows that $\mathrm{Hom}_R(G, \boldsymbol{R}_1) \neq 0$, contradicting [15, p.46]. Therefore $G \notin$ fsc($\boldsymbol{R}_1$).

(2) Assume that there is a Prüfer module M such that $M \in$ fsc($\boldsymbol{R}_1$). By [14, Proposition 3, p.110], the generic module G is a direct summand of M^I for some I so $G \in$ Prod($\boldsymbol{R}_1$) and this contradicts (1) above. Thus there is no Prüfer module in fsc($\boldsymbol{R}_1$). $\qquad\square$

Let R be a tame hereditary finite-dimensional algebra over a field k and let S be a simple regular left R-module (that is, a module which is simple in the category of regular modules). We use $S[\infty]$ (resp. $\hat{S}$) to denote the Prüfer (resp. adic) left R-module corresponding to S, see [14, p.106] for the definitions of these modules. Also, we use T_S to denote the class $T_S = \{M \mid M$ is an indecomposable regular left R-module with $\mathrm{Hom}_R(M, S) \neq 0\}$.

Theorem 4.18. *Let R be a tame hereditary finite-dimensional algebra over a field k. Let $\boldsymbol{R}_1$ be a class of indecomposable regular left R-modules and let S be a simple regular left R-module. Then the following statements are equivalent:*
(1) $\hat{S} \in$ fsc($\boldsymbol{R}_1$).
(2) $\boldsymbol{R}_1 \cap T_S$ *is infinite.*
(3) $\hat{S} \in$ fsc($\boldsymbol{R}_1 \cap T_S$).

Proof. $(1) \Rightarrow (2)$. Suppose that $\hat{S} \in \mathrm{fsc}(\boldsymbol{R}_1)$. Assume that $\boldsymbol{R}_1 \cap T_S$ is finite. Let $D = \{M \mid \mathrm{Hom}_R(M, S) = 0\}$. By [6, Examples, p.42], D is a definable subclass of R-Mod and hence $C = D \cap_R \mathrm{pinj}$ is a closed set in the Ziegler topology. Since $\boldsymbol{R}_1 \cap T_S$ is a finite class of finite-dimensional indecomposable modules it follows from [6, 2.5] that $\boldsymbol{R}_1 \cap T_S$ is a closed set in the Ziegler topology and hence $C \cup (\boldsymbol{R}_1 \cap T_S)$ is. Thus $C \cup (\boldsymbol{R}_1 \cap T_S)$ is a closed set in the full support topology. Since $\boldsymbol{R}_1 \subseteq C \cup (\boldsymbol{R}_1 \cap T_S)$ it follows that $\mathrm{fsc}(\boldsymbol{R}_1) \subseteq \mathrm{fsc}(C \cup (\boldsymbol{R}_1 \cap T_S)) = C \cup (\boldsymbol{R}_1 \cap T_S)$. Since $\mathrm{Hom}_R(\hat{S}, S) \neq 0$ it follows that $\hat{S} \notin C \cup (\boldsymbol{R}_1 \cap T_S)$ and hence $\hat{S} \notin \mathrm{fsc}(\boldsymbol{R}_1)$ and this contradicts the hypothesis. Thus $\boldsymbol{R}_1 \cap T_S$ is infinite.

$(2) \Rightarrow (3)$ Suppose that $\boldsymbol{R}_1 \cap T_S$ is infinite, thus $(\boldsymbol{R}_1 \cap T_S)^*$ is an infinite class of regular right R-modules. Let $X \in (\boldsymbol{R}_1 \cap T_S)^*$, thus $X = M^*$ for some $M \in \boldsymbol{R}_1 \cap T_S$. Hence $\mathrm{Hom}_R(M, S) \neq 0$. Thus $\mathrm{Hom}_R(S^*, X) \neq 0$ for all $X \in (\boldsymbol{R}_1 \cap T_S)^*$. By [14, Proposition 1, p.107], $S^*[\infty]$ is the direct limit of a chain of monomorphisms $X_1 \to X_2 \to X_3 \to \cdots$ with $X_i \in (\boldsymbol{R}_1 \cap T_S)^*$. Therefore, by [9, Proposition 2.1, p.736], there is a pure exact sequence $0 \to N \to \bigoplus_{j \in J} Y_j \to S^*[\infty] \to 0$ with $Y_j \in (\boldsymbol{R}_1 \cap T_S)^*$. Therefore the exact sequence $0 \to (S^*[\infty])^* \to (\bigoplus_{j \in J} Y_j)^* \to N^* \to 0$ is split. By [6, Examples, p.44], $\hat{S} = (S^*[\infty])^*$ and hence $\hat{S}$ is a direct summand of $\prod_{j \in J} N_j^*$. Thus $\hat{S} \in \mathrm{Prod}(\boldsymbol{R}_1 \cap T_S)$ and this implies that $\hat{S} \in \mathrm{fsc}(\boldsymbol{R}_1 \cap T_S)$.

$(3) \Rightarrow (1)$. This is obvious. $\qquad \square$

Corollary 4.19. *Let R be a tame hereditary finite-dimensional algebra over a field k and let $\boldsymbol{R}_1$ be a class of indecomposable regular left R-modules. Then $\mathrm{fsc}(\boldsymbol{R}_1) = \boldsymbol{R}_1 \cup \{\hat{S} \mid \boldsymbol{R}_1 \cap T_S \text{ is infinite}\}$.*

Proof. This follows by Theorem 4.18, Remark 4.15 and Lemma 4.17. $\qquad \square$

In the following corollary we give a complete description of the closure of any subclass of R-ind in the full support topology and hence, by Theorem 4.6, a description of the indecomposable S-pure-injective modules for any purity defined by a class S of finitely presented modules.

Corollary 4.20. *Let R be a tame hereditary finite-dimensional algebra over a field k. Let $\boldsymbol{I}_1$ (resp. $\boldsymbol{P}_1$, resp. $\boldsymbol{R}_1$) be a class of indecomposable preinjective (resp. preprojective, resp. regular) left R-modules. Then $\mathrm{fsc}(\boldsymbol{I}_1 \cup \boldsymbol{P}_1 \cup \boldsymbol{R}_1) = \overline{\boldsymbol{I}_1} \cup \boldsymbol{P}_1 \cup \boldsymbol{R}_1 \cup \{\hat{S} \mid \boldsymbol{R}_1 \cap T_S \text{ is infinite}\}$.*

Proof. This follows from Corollary 4.14, Proposition 4.16 and Corollary 4.19. $\qquad \square$

5 Rings whose indecomposable modules are S-pure-projective.

Let T be a set. A family F of subsets of T is said to be directed if for any $U, V \in F$, there exists $W \in F$ such that $U \subseteq W$ and $V \subseteq W$.

By using Theorem 1.1, we can prove the following lemma.

Lemma 5.1. *Let S be a class of finitely presented left R-modules and let $\{N_i\}_{i \in I}$ be any directed family of S-pure submodules of a left R-module M. Then $N = \bigcup_{i \in I} N_i$ is an S-pure submodule of M.*

Let N be a submodule of a left R-module M and let T be a set of submodules of M. We will use $N(T)$ to denote the submodule $N(T) = N + \sum_{A \in T} A$. The next lemma follows using Lemma 5.1.

Lemma 5.2. *Let S be a class of finitely presented left R-modules, let N be a submodule of a left R-module M and let T be a set of submodules of M. If $N(F)$ is an S-pure submodule of M for all finite subsets F of T, then $N(T)$ is an S-pure submodule of M.*

Definition 5.3. *Let S be a class of finitely presented left R-modules, let N be a submodule of a left R-module M and let T_0 be the set of all indecomposable submodules of M. A subclass $T \subseteq T_0$ said to be S-N-independent (in M) if $N(T) = N \oplus (\sum_{B \in T} \oplus B)$ and $N(T)$ is S-pure submodule of M. This will be the case if and only if every finite subset of T is S-N-independent M.*

Theorem 5.4. *Let S be any set of finitely presented left R-modules and let M be a left R-module. Suppose that every S-pure submodule M_0 of M for which M/M_0 is indecomposable is a direct summand of M. Then every S-pure submodule of M is a direct summand of M and M is a direct sum of indecomposable submodules.*

Proof. (The following proof is based on an argument in [4, Proposition 1.13, p.53]). Let N be any S-pure submodule of M. If $N = M$ then N is a direct summand of M. Assume that $N \neq M$, thus there is $x \in M \backslash N$. Let $F = \{K \mid N \subseteq K, x \notin K$ and K is an S-pure submodule of $M\}$. Since $N \in F$ it follows that F is a non-empty family. Let $\{M_i\}_{i \in I}$ be any directed subfamily of F and let $A = \bigcup_{i \in I} M_i$. It is clear that $N \subseteq A$ and $x \notin A$. By Lemma 5.1, A is an S-pure submodule of M and hence $A \in F$. By Zorn's lemma, F has a maximal element, say M_0, thus M_0 is an S-pure submodule of M with $N \subseteq M_0$ and $x \notin M_0$. We will prove that M/M_0 is indecomposable.

Assume that M/M_0 is not indecomposable, thus there are two non-zero submodules M_1/M_0, M_2/M_0 of M/M_0 such that $M/M_0 = (M_1/M_0) \oplus (M_2/M_0)$. Therefore $M_0 \subsetneq M_1$, $M_0 \subsetneq M_2$ and $M_1 \cap M_2 = M_0$. Since M_1/M_0 and M_2/M_0 are direct summands of M/M_0 they are S-pure submodules of M/M_0. Since M_0 is an S-pure submodule of M it follows from [20, 33.3(4), p.276] that M_1 and M_2 are S-pure submodules of M. Thus, by maximality of M_0, we have that $x \in M_1 \cap M_2$ and this is a contradiction.

Hence M/M_0 is a non-zero indecomposable left R-module. By assumption, M_0 is a direct summand of M, say $M = N_0 \oplus M_0$. Thus $N_0 \simeq M/M_0$ is a non-zero indecomposable submodule of M with $N + N_0 = N \oplus N_0$. Since N is an S-pure submodule of M and $N \subseteq M_0 \subseteq M$ it follows that N is S-pure submodule of M_0 and hence $N \oplus N_0$ is an S-pure submodule of $N_0 \oplus M_0 = M$. Thus, for any proper S-pure submodule N of M, there exists a non-zero indecomposable submodule N_0 of M such that $N \cap N_0 = 0$ and $N \oplus N_0$ is an S-pure submodule of M.

Let T be the family of all S-N-independent subsets in M. Since $\{0\} \in T$ it follows that T is non-empty. Let D be any directed subfamily of T and let U be the union of all members of D. Then $U \in T$ since every finite subset of U is S-N-independent. By Zorn's lemma, T has a maximal element, say W. Now we will prove that $N(W) = M$. Assume that $N(W) \neq M$, thus $N(W)$ is a proper S-pure submodule of M. Hence there exists a non-zero indecomposable submodule B of M such that $N(W) \cap B = 0$ and $N(W) + B = N(W) \oplus B = N \oplus (\sum_{A \in W} \oplus A) \oplus B$ is an S-pure submodule of M, as seen above. Hence $W \cup \{B\}$ properly contains W and is S-N-independent in M. This contradicts the maximality of W in T. Therefore, $N(W) = M$. Since $N(W) = N \oplus (\sum_{A \in W} \oplus A)$ it follows that N is a direct summand of M and $M/N \simeq \sum_{A \in W} \oplus A$ is a direct sum of indecomposable submodules. If we take $N = 0$ then we see that M is a direct sum of indecomposable submodules. $\square$

Corollary 5.5. *Let S be any set of finitely presented left R-modules. Then the following statements are equivalent:*
- (1) *Every indecomposable left R-module is S-pure-projective.*
- (2) *For any left R-module M, every S-pure submodule of M is a direct summand of M.*
- (3) *Every left R-module is S-pure-projective.*
- (4) *Every left R-module is S-pure-injective.*
- (5) *Every left R-module is a direct sum of modules in* $\mathrm{ind}(S \cup \{_R R\})$.

Proof. $(1) \Rightarrow (2)$. Let M be any left R-module and let N be any S-pure submodule of M such that M/N is indecomposable. By hypothesis, M/N is S-pure-projective hence the S-pure exact se-

quence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ splits and hence N is a direct summand of M. By Theorem 5.4, every S-pure submodule of M is a direct summand of M.

$(2) \Rightarrow (3)$. Let M be any left R-module and let $\Sigma : 0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$ be any S-pure exact sequence of left R-modules. By hypothesis, im(f) is a direct summand of N and hence Σ is split so M is S-pure-projective.

$(3) \Rightarrow (5)$. Assume that every left R-module is S-pure-projective, thus every left R-module is pure-projective. By [4, Proposition 4.4, p.73], R is a left Artinian ring and hence R is Krull-Schmidt, by [12, p.164]. Let M be any left R-module. By hypothesis and Proposition 4.1, M isomorphic to a direct sum of modules in ind($S \cup \{_R R\}$). Thus every left R-module is a direct sum of modules in ind($S \cup \{_R R\}$).

$(5) \Rightarrow (1)$. Assume that every left R-module is a direct sum of modules in ind($S \cup \{_R R\}$). Let M be an indecomposable left R-module, thus M is a direct sum of modules in ind($S \cup \{_R R\}$). Since each module in ind($S \cup \{_R R\}$) is S-pure-projective and the class of S-pure-projective left R-modules is closed under direct sums (by [20, p.278]) it follows that M is S-pure-projective. Hence every indecomposable left R-module is S-pure-projective.

$(2) \Leftrightarrow (4)$. By using [20, 33.7, p.279]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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