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Wilkie, A J and Kirby, Jonathan and Bays, Martin

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A SCHANUEL PROPERTY FOR EXPONENTIALLY TRANSCENDENTAL POWERS

MARTIN BAYS, JONATHAN KIRBY, AND A.J. WILKIE

Abstract. We prove the analogue of Schanuel's conjecture for raising to the power of an exponentially transcendental real number. All but countably many real numbers are exponentially transcendental. We also give a more general result for several powers in a context which encompasses the complex case.

1. Introduction

We prove a Schanuel property for raising to a real power:

Theorem 1.1. *Let $\lambda \in \mathbb{R}$ be exponentially transcendental, let $\overline{y} \in (\mathbb{R}_{>0})^n$, and suppose $\overline{y}$ is multiplicatively independent. Then*

$$\mathrm{td}(\overline{y}, \overline{y}^{\lambda}/\lambda) \geqslant n.$$

Here and later, $\mathrm{td}(X/Y)$ denotes the transcendence degree of the field extension $\mathbb{Q}(X,Y)/\mathbb{Q}(Y)$ (for X, Y subsets of the ambient field, in this case $\mathbb{R}$). To say that $\overline{y}$ is multiplicatively independent means that if $m_1, \ldots, m_n \in \mathbb{Z}$ and $\prod y_i^{m_i} = 1$ then $m_i = 0$ for each i. The usual exponential function $\exp : \mathbb{R} \to \mathbb{R}$ makes the reals into an *exponential field*, formally a field of characteristic zero equipped with a homomorphism from its additive to multiplicative groups. In any exponential field $\langle F; +, \cdot, \exp\rangle$, we say that an element $x \in F$ is *exponentially algebraic* in F iff there is $n \in \mathbb{N}$, $\overline{x} = (x_1, \ldots, x_n) \in F^n$, and exponential polynomials $f_1, \ldots, f_n \in \mathbb{Z}[\overline{X}, e^{\overline{X}}]$ such that $x = x_1$, $f_i(\overline{x}, e^{\overline{x}}) = 0$ for each $i = 1, \ldots, n$, and the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix}$$

is nonzero at $\overline{x}$. If x is not exponentially algebraic in F we say it is *exponentially transcendental* in F. More generally, for a subset A of F, we can define the notion of x being *exponentially algebraic over A* with the same definition except that the f_i can have coefficients from A. Observe that the non-vanishing of the Jacobian in the reals means that $\overline{x}$ is an isolated zero of the system of equations, and hence all but countably many real numbers are exponentially transcendental. Thus a consequence of theorem 1.1 is that the numbers $\lambda, \lambda^{\lambda}, \lambda^{\lambda^2}, \lambda^{\lambda^3}, \ldots$ are algebraically independent for all but countably many λ, although, unfortunately, one does not know any explicit λ for which this is true.

This paper contains a complete proof of theorem 1.1, assuming only some knowledge of o-minimality from the reader (and using a theorem of Ax). The paper [Kir08] of the second author develops the theory of exponential algebraicity in an arbitrary exponential field, and, using that, we can prove a more general theorem.

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Theorem 1.2. *Let F be any exponential field, let $\lambda \in F$ be exponentially transcendental, and let $\overline{x} \in F^n$ be such that $\exp(\overline{x})$ is multiplicatively independent. Then*

$$\mathrm{td}(\exp(\overline{x}), \exp(\lambda\overline{x})/\lambda) \geqslant n.$$

Theorem 1.1 follows from 1.2 by taking $x_i = \log y_i$.

We define the exponential algebraic closure $\mathrm{ecl}(A)$ of a subset A of F to be the set of $x \in F$ which are exponentially algebraic over A. In [Kir08] it is shown that ecl is a pregeometry in any exponential field, and hence we have notions of dimension and independence. We also prove a general Schanuel property for raising to several independent powers, which uses a slightly subtle notion of relative linear dimension. For any subfield K of F, we can think of F as a K-vector space. For subsets X, Y of F, consider the K-linear subspaces $\langle XY\rangle_K$ and $\langle Y\rangle_K$ of F generated by $X \cup Y$ and Y respectively. We define $\mathrm{ldim}_K(X/Y)$ to be the K-linear dimension of the quotient K-vector space $\langle XY\rangle_K \,/\, \langle Y\rangle_K$.

Theorem 1.3. *Let F be any exponential field, let* ker *be the kernel of its exponential map, let C be an* ecl*-closed subfield of F, and let $\overline{\lambda}$ be an m-tuple which is exponentially algebraically independent over C. Then for any tuple $\overline{z}$ from F:*

$$\mathrm{td}(\exp(\overline{z})/C, \overline{\lambda}) + \mathrm{ldim}_{\mathbb{Q}(\overline{\lambda})}(\overline{z}/\ker) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}/\ker) \geqslant 0.$$

The reader who is interested only in the real case may ignore all the references to [Kir08]. On the other hand, the reader who is unfamiliar with o-minimality may prefer to ignore that part of this paper and instead refer to the algebraic proof of proposition 2.1 in [Kir08].

2. A Schanuel property for exponentiation

We need the following relative Schanuel property for exponentiation itself.

Proposition 2.1. *Let F be an exponential field and let $\overline{\lambda} \in F^m$ be exponentially algebraically independent. Let $B \subseteq F$ be such that $B \cup \overline{\lambda}$ is a basis for F with respect to the pregeometry* ecl*. Let $C = \mathrm{ecl}(B)$. Then for any $\overline{z} \in F^n$,*

$$\mathrm{td}(\overline{\lambda}, \overline{z}, \exp(\overline{\lambda}), \exp(\overline{z})/C) - \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}, \overline{z}/C) \geqslant m.$$

Proof. Theorem 1.2 of [Kir08] states that $\mathrm{td}(\overline{\lambda}, \overline{z}, \exp(\overline{\lambda}), \exp(\overline{z})/C) - \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}, \overline{z}/C)$ is at least the dimension of the $(m+n)$-tuple $(\overline{\lambda}, \overline{z})$ over C with respect to the pregeometry ecl. Since $\overline{\lambda}$ is ecl-independent over C by assumption, this dimension is at least m. □

We give a more direct proof of proposition 2.1 in the real case. Firstly, by theorem 4.2 of [JW08], a real number x is in the exponential algebraic closure $\mathrm{ecl}(A)$ of a subset A of $\mathbb{R}$ iff it lies in the definable closure of A in the structure $\mathbb{R}_{\exp} = \langle \mathbb{R}; +, \cdot, \exp\rangle$. Definable closure is always a pregeometry in an o-minimal field, so ecl is a pregeometry on $\mathbb{R}_{\exp}$.

For each $i = 1, \ldots, m$, let $K_i = \mathrm{ecl}(B \cup \overline{\lambda} \smallsetminus \lambda_i)$, so $C = \bigcap_{i=1}^m K_i$. Then for each i, $\lambda_i \notin K_i$, but for each $a \in \mathbb{R}$ there is a function $\theta : \mathbb{R} \to \mathbb{R}$, definable in $\mathbb{R}_{\exp}$ with parameters from K_i, such that $\theta(\lambda_i) = a$. By o-minimality of $\mathbb{R}_{\exp}$, θ is differentiable at all but finitely many $x \in \mathbb{R}$, and hence this exceptional set is contained in K_i. Thus θ is differentiable on an open interval containing λ_i. Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is another such function with $\psi(\lambda_i) = a$. Again by o-minimality, the boundary of the set $\{x \in \mathbb{R} \mid \psi(x) = \theta(x)\}$ is finite and contained in K_i, so θ and ψ agree on an open interval containing λ_i. It follows that there is a well-defined function $\partial_i : \mathbb{R} \to \mathbb{R}$ which sends a to $\frac{d\theta}{dx}(\lambda_i)$, where θ is any function definable in $\mathbb{R}_{\exp}$ with parameters from K_i such that $\theta(\lambda_i) = a$. It is straightforward to check that ∂_i is a derivation on the field $\mathbb{R}$, with field of constants K_i. Furthermore, we also clearly have that $\partial_i(\exp(a)) = \partial_i(a)\exp(a)$ for any $a \in \mathbb{R}$, and that $\partial_i(p_j) = \delta_{ij}$, the Kronecker delta.

By Ax's theorem [Ax71, theorem 3], $\mathrm{td}(\overline{\lambda}, \overline{z}, \exp(\overline{\lambda}), \exp(\overline{z})/C) - \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}, \overline{z}/C)$ is at least the rank of the matrix

$$\begin{pmatrix} \partial_1 z_1 & \cdots & \partial_1 z_n & \partial_1 \lambda_1 & \cdots & \partial_1 \lambda_m \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_m z_1 & \cdots & \partial_m z_n & \partial_m \lambda_1 & \cdots & \partial_m \lambda_m \end{pmatrix}$$

which is m since the right half is just the $m \times m$ identity matrix. That completes the proof of proposition 2.1 in the real case. The general case works the same way, but a different and much more involved argument is used in [Kir08] to produce the derivations ∂_i without using o-minimality.

3. Linear disjointness

The other key ingredient in the proofs is the concept of linear disjointness. We briefly recall the definition and some basic properties.

Definition 3.1. Let F be a field, and let K, L, and E be subfields of F with $E \subseteq K \cap L$. Then K is *linearly disjoint from L over E*, written $K \perp_E L$, iff every tuple $\overline{k}$ of elements of K that is E-linearly independent is also L-linearly independent.

Lemma 3.2.

(i) *$K \perp_E L$ iff $L \perp_E K$*
(ii) *$K \perp_E L$ iff for any tuple $\overline{l}$ from L, $\mathrm{ldim}_K(\overline{l}) = \mathrm{ldim}_E(\overline{l})$*
(iii) *If $\overline{k}$ is algebraically independent over L, then $E(\overline{k}) \perp_E L$.*

Proof. (i) and (ii) are straightforward; (iii) is proposition VIII 3.3 of [Lan93]. □

Lemma 3.3. *Suppose $K \perp_E L$. Then for any tuple $\overline{x}$ from F and any subset $A \subseteq L$,*

$$\mathrm{ldim}_K(\overline{x}/L) - \mathrm{ldim}_E(\overline{x}/L) \leqslant \mathrm{ldim}_K(\overline{x}/A) - \mathrm{ldim}_E(\overline{x}/A).$$

Proof. Let $\overline{l} \in L$ be a finite tuple such that $\mathrm{ldim}_K(\overline{x}/\overline{l}A) = \mathrm{ldim}_K(\overline{x}/L)$ and $\mathrm{ldim}_E(\overline{x}/\overline{l}A) = \mathrm{ldim}_E(\overline{x}/L)$.

Now:

$$\begin{aligned} \mathrm{ldim}_K(\overline{x}/A) - \mathrm{ldim}_K(\overline{x}/\overline{l}A) &= \mathrm{ldim}_K(\overline{l}/A) - \mathrm{ldim}_K(\overline{l}/\overline{x}A) && \text{(by the addition formula)} \\ &= \mathrm{ldim}_E(\overline{l}/A) - \mathrm{ldim}_K(\overline{l}/\overline{x}A) && \text{(by Lemma 3.2(ii))} \\ &\geqslant \mathrm{ldim}_E(\overline{l}/A) - \mathrm{ldim}_E(\overline{l}/\overline{x}A) \\ &= \mathrm{ldim}_E(\overline{x}/A) - \mathrm{ldim}_E(\overline{x}/\overline{l}A) && \text{(by the addition formula).} \end{aligned}$$

□

4. Proofs of the main theorems

Proof of theorem 1.3. By proposition 2.1, for any tuple $\overline{z}$ from F we have:

$$\mathrm{td}(\overline{z}, \exp(\overline{z}), \overline{\lambda}, \exp(\overline{\lambda})/C) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}, \overline{\lambda}/C) \geqslant m.$$

Expanding using the addition formula gives

$$\begin{aligned} &\mathrm{td}(\overline{\lambda}/C) + \mathrm{td}(\overline{z}/C, \overline{\lambda}) + \mathrm{td}(\exp(\overline{z})/C, \overline{\lambda}, \overline{z}) \\ &\qquad + \mathrm{td}(\exp(\overline{\lambda})/C, \overline{\lambda}, \overline{z}, \exp(\overline{z})) - \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}/C, \overline{z}) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}/C) \geqslant m. \end{aligned}$$

Since $\overline{\lambda}$ is algebraically independent over C, we have $\mathrm{td}(\overline{\lambda}/C) = m$, and we deduce

$$\text{(1)}\quad \mathrm{td}(\overline{z}/C, \overline{\lambda}) + \mathrm{td}(\exp(\overline{z})/C, \overline{\lambda}) + \mathrm{td}(\exp(\overline{\lambda})/C, \exp(\overline{z})) - \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}/C, \overline{z}) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}/C) \geqslant 0.$$

We also have:

$$\text{(2)}\qquad \mathrm{td}(\exp(\overline{\lambda})/C, \exp(\overline{z})) \leqslant \mathrm{ldim}_{\mathbb{Q}}(\overline{\lambda}/C, \overline{z})$$

because if $\lambda_1, \ldots, \lambda_t$ form a $\mathbb{Q}$-linear basis for $\overline{\lambda}$ over $(C, \overline{z})$, then for $i > t$, $\exp(\lambda_i)$ is in the algebraic closure of $(C, \exp(\overline{z}), \exp(\lambda_1), \ldots, \exp(\lambda_t))$. A similar argument shows

$$\text{(3)}\qquad \mathrm{td}(\overline{z}/C, \overline{\lambda}) \leqslant \mathrm{ldim}_{\mathbb{Q}(\overline{\lambda})}(\overline{z}/C)$$

since if z_i is in the $\mathbb{Q}(\overline{\lambda})$-linear span of $(z_1, \ldots, z_t, C)$ then z_i is in the algebraic closure of $(C, \overline{\lambda}, z_1, \ldots, z_t)$.

Combining (1) with (2) and (3) gives

$$\mathrm{td}(\exp(\overline{z})/C, \overline{\lambda}) + \mathrm{ldim}_{\mathbb{Q}(\overline{\lambda})}(\overline{z}/C) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}/C) \geqslant 0.$$

By lemma 3.2(iii), $\mathbb{Q}(\overline{\lambda})$ is linearly disjoint from C over $\mathbb{Q}$. Also $\ker \subseteq \mathrm{ecl}(\emptyset) \subseteq C$, so, by lemma 3.3,

$$\mathrm{td}(\exp(\overline{z})/C, \overline{\lambda}) + \mathrm{ldim}_{\mathbb{Q}(\overline{\lambda})}(\overline{z}/\ker) - \mathrm{ldim}_{\mathbb{Q}}(\overline{z}/\ker) \geqslant 0$$

as required. □

Proof of theorem 1.2. By theorem 1.3, taking $\overline{z} = (\overline{x}, \lambda\overline{x})$,

$$\begin{aligned}\mathrm{td}(\exp(\overline{x}), \exp(\lambda\overline{x})/\lambda) &\geqslant \mathrm{ldim}_{\mathbb{Q}}(\overline{x}, \lambda\overline{x}/\ker) - \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}, \lambda\overline{x}/\ker)\\ &= \mathrm{ldim}_{\mathbb{Q}}(\overline{x}/\ker) + \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \ker) - \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\ker)\\ &= n + \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \ker) - \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\ker).\end{aligned}$$

Thus it suffices to prove that $\mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \ker) \geqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\ker)$. Let $\overline{k}$ be a finite tuple from ker such that $\mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \ker) = \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \overline{k})$ and $\mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\ker) = \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\overline{k})$.

Let $A_0 := \langle \lambda\overline{x}, \overline{k} \rangle_{\mathbb{Q}}$. Then $\mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}, \overline{k}/\overline{x}, \lambda^{-1}\overline{k}) = \mathrm{ldim}_{\mathbb{Q}}(A_0/A_0 \cap \lambda^{-1}A_0)$. Inductively define $A_{i+1} := A_i \cap \lambda^{-1}A_i$ for $i \in \mathbb{N}$. Suppose for some i that $A_{i+1} = A_i$. Then multiplication by λ induces a $\mathbb{Q}$-linear automorphism of A_i. It follows that for any $f(\lambda) \in \mathbb{Q}[\lambda]$, multiplication by $f(\lambda)$ is a $\mathbb{Q}$-linear endomorphism of A_i. This endomorphism has trivial kernel because $f(\lambda)$ is not a zero divisor of the field (unless $f(\lambda) = 0$), and A_i is finite-dimensional, so it is invertible. Its inverse must be multiplication by $f(\lambda)^{-1}$, and hence A_i is a $\mathbb{Q}(\lambda)$-vector space. Since λ is transcendental, $\mathrm{ldim}_{\mathbb{Q}} \mathbb{Q}(\lambda)$ is infinite, so $A_i = \{0\}$. So $\mathrm{ldim}_{\mathbb{Q}} A_{i+1} < \mathrm{ldim}_{\mathbb{Q}} A_i$ unless $A_i = \{0\}$. Thus for some $N \in \mathbb{N}$ we have $A_N = \{0\}$.

For each i we have a chain of subspaces $A_{i+1} \subseteq A_{i+1} + \lambda A_{i+1} \subseteq A_i$, so

$$\begin{aligned}\mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1}) &= \mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \mathrm{ldim}_{\mathbb{Q}}(A_{i+1} + \lambda A_{i+1}/A_{i+1})\\ &= \mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \mathrm{ldim}_{\mathbb{Q}}(\lambda A_{i+1}/A_{i+1} \cap \lambda A_{i+1})\\ &= \mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \mathrm{ldim}_{\mathbb{Q}}(\lambda A_{i+1}/\lambda A_{i+2})\\ &= \mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) + \mathrm{ldim}_{\mathbb{Q}}(A_{i+1}/A_{i+2}).\end{aligned}$$

Thus inductively we obtain

$$\mathrm{ldim}_{\mathbb{Q}}(A_0/A_1) = \sum_{i=0}^{N} \mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}).$$

Now for each i,

$$\mathrm{ldim}_{\mathbb{Q}}(A_i/A_{i+1} + \lambda A_{i+1}) \geqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1} + \lambda A_{i+1}) = \mathrm{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1})$$

hence

$$\mathrm{ldim}_{\mathbb{Q}}(A_0/A_1) \geqslant \sum_{i=0}^{N} \mathrm{ldim}_{\mathbb{Q}(\lambda)}(A_i/A_{i+1}) = \mathrm{ldim}_{\mathbb{Q}(\lambda)}(A_0)$$

that is,

$$\mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}, \overline{k}/\overline{x}, \lambda^{-1}\overline{k}) \geqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}, \lambda^{-1}\overline{k}). \tag{4}$$

But

$$\mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}, \lambda^{-1}\overline{k}) = \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}, \overline{k}) = \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\overline{k}) + \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{k}) \tag{5}$$

and

$$\begin{aligned} \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}, \overline{k}/\overline{x}, \lambda^{-1}\overline{k}) &\leqslant \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}, \overline{k}/\overline{x}) \\ &= \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{k}, \overline{x}) + \mathrm{ldim}_{\mathbb{Q}}(\overline{k}/\overline{x}) \\ &\leqslant \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{k}, \overline{x}) + \mathrm{ldim}_{\mathbb{Q}}(\overline{k}) \\ &= \mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{k}, \overline{x}) + \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{k}) \qquad (6) \end{aligned}$$

the last line holding by lemma 3.2(ii), since $\mathbb{Q}(\lambda) \perp_{\mathbb{Q}} C$ and $\overline{k} \subseteq \ker \subseteq C$.

Putting together (4), (5), and (6) gives $\mathrm{ldim}_{\mathbb{Q}}(\lambda\overline{x}/\overline{x}, \ker) \geqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\overline{x}/\ker)$ as required. □

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