The Continuing Influence of Fiedler's Work on Companion Matrices

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Abstract

This is a reconstruction in article-like form of a talk given at the "Minisymposium in Honor of Miroslav Fiedler" at the 17th ILAS Conference, held at TU Braunschweig, Germany, on Thurs 25 Aug 2011.

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Introduction

Recall that the companion matrix of a scalar polynomial $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ is the $k \times k$ matrix

$$
C_p := \begin{bmatrix}
-a_{k-1} & -a_{k-2} & \ldots & -a_1 & -a_0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1 & 0
\end{bmatrix}.
$$

Well-known for a very long time are that the characteristic polynomial and minimal polynomial of C_p are both just $p(x)$. The use of this matrix C_p goes back at least to Frobenius in his "rational canonical form", so it is reasonable that we should call C_p the *Frobenius companion matrix*, both to honor Frobenius, and to clearly distinguish C_p from other matrices that can reasonably aspire to bear the title "companion matrix".

This companion matrix C_p seems such a "simple" thing to us now, so familiar and commonplace that we tend not to see it any longer as an object worthy of any further study. But Arnold Ross, a number theorist long associated with Ohio State University, and much-renowned for his influence on the development of young mathematical talent [10], would ever encourage his students to

<div align="center">"Think deeply of simple things!"</div>

I submit to you Fiedler's work on companion matrices as an exemplar of the effectiveness of this aphorism.

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The 2003 LAA paper

Let us begin by considering Fiedler's 2003 paper [7] in LAA, simply titled "A note on companion matrices". There are three key ideas in this paper. The first is the observation that C_p *has a simple factorization* $C_p = M_{k-1}M_{k-2}\cdots M_1 M_0$, where

$$
M_j := \begin{bmatrix} I_{k-j-1} & & & \\ & -a_j & 1 & \\ & 1 & 0 & \\ & & & I_{j-1} \end{bmatrix} \quad \text{for} \;\; j = 1,\ldots,k-1 \,,
$$

and $M_0 := \mathrm{diag}[\,I_{k-1}, -a_0\,]$. Note that the Fiedler factors M_j are all invertible, except possibly for M_0:

$$
M_j^{-1} := \begin{bmatrix} I_{k-j-1} & & & \\ & 0 & 1 & \\ & 1 & a_j & \\ & & & I_{j-1} \end{bmatrix} \quad \text{for} \;\; j = 1,\ldots,k-1 \,.
$$

The second key idea, and perhaps the most important, is that something interesting might result from *scrambling the order of the factors* M_ℓ. So let $\sigma = (j_0, j_1, \ldots, j_{k-1})$ be any permutation of the indices $(0, 1, 2, \ldots, k-1)$, and consider the Fiedler matrix

$$
F_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}
$$

associated with the permutation σ.

Theorem 1. *For any given (monic) scalar polynomial $p(x)$, all associated Fiedler matrices F_σ are similar to each other. (Note: But not in general permutation-similar to each other.)*

Proof. (Idea)   Work with the matrices F_σ in *product form*; equivalently, manipulate the associated permutations σ via operations corresponding to equality or similarity of these products.

In particular, using the fact that many pairs of Fiedler factors commute,

$$
M_i M_j \;=\; M_j M_i \quad \text{for} \;\; |i - j| > 1 \,,
$$

one sees immediately that many distinct permutations produce equal Fiedler matrices. The invertibility of the Fiedler factors M_j for $j = 1,\ldots,k-1$, together with the fact that AB and BA are similar whenever either A or B is invertible, implies that (most) cyclic permutations of the indices $(j_0, j_1, \ldots, j_{k-1})$ result in similar Fiedler matrices. Combining these ingredients, Fiedler shows that all the F_σ's are similar to each other.   □

Note: It can be shown that there are 2^{k-1} distinct Fiedler matrices for any degree k (monic) scalar polynomial $p(x)$, although of course there are $k!$ distinct permutations σ.

The third key point is that some permutations produce especially interesting results. We already know that $\sigma_1 = (k-1, k-2, \ldots, 1, 0)$ gives us $F_{\sigma_1} = M_{k-1}M_{k-2}\cdots M_1 M_0 = C_p$, the Frobenius companion matrix. Since the Fiedler factors M_j are all symmetric, it is not hard to see that the permutation $\sigma_2 = (0, 1, \ldots, k-2, k-1)$ at the other extreme yields

$$
F_{\sigma_2} \;:=\; M_0 M_1 \cdots M_{k-2} M_{k-1} \;=\; (M_{k-1}M_{k-2}\cdots M_1 M_0)^T \;=\; C_p^T \,,
$$

sometimes known as the second (Frobenius) companion form.

But much more interesting than either of these are the Fiedler matrices associated with the permutations $\tau = (1, 3, 5, \ldots, 0, 2, 4, \ldots)$, with all the odd indices gathered together and all the even indices gathered together. Here's an example for degree $k = 6$.

Example 2.

$$F_\tau \;=\; M_1 M_3 M_5 M_0 M_2 M_4 \;=\; \begin{bmatrix} -a_5 & -a_4 & 1 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & -a_3 & 0 & -a_2 & 1 & \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & -a_1 & 0 & -a_0 \\ & & & 1 & 0 & 0 \end{bmatrix}.$$

This Fiedler matrix is *penta-diagonal*; indeed for *any* degree k, the Fiedler matrix associated with this permutation τ is penta-diagonal. To see this, observe that both $M_{\mathrm{odd}} := M_1 M_3 M_5 \cdots$ and $M_{\mathrm{even}} := M_0 M_2 M_4 \cdots$ are tridiagonal, so their product $F_\tau = M_{\mathrm{odd}} M_{\mathrm{even}}$ is penta-diagonal. For very high degree polynomials $p(x)$, this companion form will have *much lower bandwidth* than the Frobenius companion form, and thus has potential advantages for numerical computation.

In summary, then, in [7] Fiedler has significantly expanded the available palette of companion forms beyond the classical Frobenius companion forms, providing opportunities for further development from both theoretical and numerical points of view.

Subsequent Developments

Since 2003, two main lines of development of the ideas in [7] can be clearly perceived: one within the world of matrices, the other extending these results to matrix polynomials. Although I will focus most of my attention on the latter, let me first make a few brief comments about developments within the world of matrices.

A next step was taken by Fiedler himself in [8], investigating the class of matrices expressible as products $M_\sigma = M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$, where $\sigma = (j_0, j_1, \dots, j_{k-1})$ is any permutation of $(0, 1, \dots, k-1)$, but now

$$M_j := \begin{bmatrix} I_{k-j-1} & & \\ & C_j & \\ & & I_{j-1} \end{bmatrix} \qquad \text{for } j = 1, \dots, k-1\,,$$

with C_j an *arbitrary* 2×2 matrix.

Analogous to the situation with the companion matrices F_σ, for any fixed choice of the C_j's it can be shown that the spectrum of M_σ is *unchanged by permutation* of the factors M_j. Also, every M_σ was shown in [8] to have "complementary zig-zag shape", a generalization of the notion of Hessenberg form. This class of matrices has important connections with a number of topics of current interest, including:

- unitary Hessenberg matrices, CMV matrices, Green's matrices, orthogonal polynomials

- rank structure, quasi-separability, ... (see [3, 15] and the references therein)

Matrix Polynomials and Linearizations

Before turning to the extension of Fiedler's ideas to (square) matrix polynomials, let me first recall some basic concepts. An $n \times n$ matrix polynomial of degree k over an arbitrary field $\mathbb{F}$ is denoted by

$$P(\lambda) \;=\; \lambda^k A_k + \lambda^{k-1} A_{k-1} + \cdots + \lambda A_1 + A_0\,, \quad \text{with coefficients } A_i \in \mathbb{F}^{n \times n}.$$

A key distinction is between regular and singular matrix polynomials:

- $P(\lambda)$ is *regular* if $\det P(\lambda)$ is *not* the identically zero polynomial, *singular* if it is. Equivalently, $P(\lambda)$ is regular if it is invertible as a matrix with entries in the field of rational functions $\mathbb{F}(\lambda)$, singular if it is not invertible.

An important subclass of regular polynomials are ones that are unimodular:

- $E(\lambda)$ is *unimodular* if $\det E(\lambda)$ is a *nonzero constant.*

The notion of linearization of a matrix polynomial plays a central role for both theory and computation:

- A matrix pencil $L(\lambda) = \lambda X + Y$ is a *linearization* for $P(\lambda)$ if

$$E(\lambda)L(\lambda)G(\lambda) \;=\; \mathrm{diag}[\, P(\lambda), I_r\,]$$

  for some $r \in \mathbb{N}$ and unimodular $E(\lambda)$ and $G(\lambda)$. A linearization $L(\lambda)$ is a *strong linearization* if, in addition, $\mathrm{rev}_1 L(\lambda) := \lambda Y + X$ is a linearization for

$$\mathrm{rev}_k P(\lambda) := \lambda^k A_0 + \lambda^{k-1}A_1 + \cdots + \lambda A_{k-1} + A_k\,.$$

The key property of any linearization of P is that it has the same finite elementary divisors as P, while a strong linearization has the *same finite and infinite elementary divisors* as P.

Extension to Regular Matrix Polynomials

The first step in extending the ideas of Fiedler to matrix polynomials was taken in [1] by Antoniou and Vologiannidis. They considered *regular* $n \times n$ matrix polynomials

$$P(\lambda) = \lambda^k A_k + \lambda^{k-1}A_{k-1} + \cdots + \lambda A_1 + A_0 \quad \text{over the field } \mathbb{C}.$$

The Fiedler factors are now *block matrices*:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \quad \text{for } j = 1, \ldots, k-1,$$

$M_0 := \mathrm{diag}[\, I_{n(k-1)}, -A_0\,]$, and one extra block matrix, $M_k := \mathrm{diag}[\, A_k, I_{n(k-1)}\,]$, which is needed because matrix polynomials cannot, without loss of generality, be assumed to be monic.

For any permutation $\sigma = (j_0, j_1, \ldots, j_{k-1})$ of the indices $(0, 1, 2, \ldots, k-1)$, one can now define the associated *Fiedler pencil*

$$F_\sigma(\lambda) \;:=\; \lambda M_k \;-\; M_{j_0} M_{j_1} \cdots M_{j_{k-1}}\,.$$

Theorem 3. *For any* regular *matrix polynomial $P(\lambda)$ over $\mathbb{C}$, every one of its associated Fiedler pencils $F_\sigma(\lambda)$ is a* strong linearization *for P.*

Proof. (Strategy) Show that each F_σ is *strictly equivalent* to the Frobenius companion pencil. (Note that the regularity assumption is essential for the success of this strategy, which cannot be extended to the singular case.) $\qquad\Box$

Among these Fiedler pencils are of course the *block-penta-diagonal* ones; for $k = 6$ we have

Example 4.

$$F_\tau(\lambda) = \lambda M_6 - M_1 M_3 M_5 M_0 M_2 M_4 = \begin{bmatrix} \lambda A_6 + A_5 & A_4 & -I_n & & & \\ -I_n & \lambda I_n & 0 & 0 & & \\ 0 & A_3 & \lambda I_n & A_2 & -I_n & \\ & -I_n & 0 & \lambda I_n & 0 & 0 \\ & & & 0 & A_1 & \lambda I_n & A_0 \\ & & & & -I_n & 0 & \lambda I_n \end{bmatrix}.$$

Antoniou and Vologiannidis also introduced in [1] a kind of "generalized" Fiedler pencil; exploiting the fact that every M_j for $j = 1, \ldots, k-1$ is invertible, we can "shift" some of the M_j factors to the λ-term. For example, $F_\sigma(\lambda) := \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is strictly equivalent to

$$\widetilde{F}_\sigma(\lambda) = \lambda M_{j_1}^{-1} M_{j_0}^{-1} M_k M_{j_{k-1}}^{-1} - M_{j_2} \cdots M_{j_{k-2}},$$

so $\widetilde{F}_\sigma(\lambda)$ is also a strong linearization. Antoniou and Vologiannidis show that these generalized Fiedler pencils can have additional nice properties; consider the following example for a general square polynomial $P(\lambda)$ of degree $k = 5$.

Example 5.

$$S(\lambda) = \lambda M_5 M_3^{-1} M_1^{-1} - M_4 M_2 M_0 = \begin{bmatrix} \lambda A_5 + A_4 & -I_n & & & \\ -I_n & 0 & \lambda I_n & & \\ & \lambda I_n & \lambda A_3 + A_2 & -I_n & \\ & & -I_n & 0 & \lambda I_n \\ & & & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix}$$

This pencil $S(\lambda)$ is not only a strong linearization for $P(\lambda)$, it is also block-*tri-diagonal*. Furthermore, whenever $P(\lambda)$ is symmetric (Hermitian), then so is $S(\lambda)$. We might reasonably refer to $S(\lambda)$ as a *symmetric (Hermitian) companion form* for degree 5 polynomials.

Extension to All Square Matrix Polynomials

In [4], De Terán, Dopico, and Mackey revisited Fiedler pencils, with a focus on their properties when $P(\lambda)$ is a *singular* (square) matrix polynomial. By an analysis completely different than the one used in [1], it was shown that Fiedler pencils are strong linearizations for *any* square matrix polynomial.

Theorem 6. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial, regular or singular, over an arbitrary field $\mathbb{F}$. Then every one of the associated Fiedler pencils $F_\sigma(\lambda)$ is a strong linearization for $P(\lambda)$.*

Proof. (Idea) Explicitly build up unimodular transformations $E(\lambda)$ and $G(\lambda)$, "eliminating" one Fiedler factor M_i at a time from the product representation

$$F_\sigma(\lambda) := \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}},$$

to show that $E(\lambda) F_\sigma(\lambda) G(\lambda) = \mathrm{diag}[P(\lambda), I_r]$. $\qquad\qquad\square$

Further Properties of Fiedler Pencils

Many additional properties of Fiedler pencils relevant for their use in numerical computation were also established in [4].

- *Eigenvector recovery* for regular polynomials $P(\lambda)$:

  Can eigenvectors of $P(\lambda)$ be recovered from eigenvectors of an associated Fiedler pencil? An affirmative answer is given in [4], where it is shown how to obtain an eigenvector of P simply by *extracting the "correct block"* out of a corresponding eigenvector of any $F_\sigma(\lambda)$.

- *Minimal indices* for singular polynomials $P(\lambda)$:

  Minimal indices of a singular $P(\lambda)$ are certain scalar invariants associated with the right and left nullspaces of $P(\lambda)$, viewed as a matrix over the field of rational functions $\mathbb{F}(\lambda)$. The central question in this context: how, if at all, are the minimal indices of $P(\lambda)$ *related* to those

of any of its Fiedler linearizations $F_\sigma(\lambda)$? In [4] it is shown that there is a simple relationship, which depends only on the permutation $\sigma = (j_0, j_1, \ldots, j_{k-1})$ defining the Fiedler pencil $F_\sigma(\lambda)$ under consideration. The relevant features of σ concern the relative order within the product $M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ of all pairs of factors M_i and M_{i+1} with consecutive subscripts. We say that

$$M_{j_0} M_{j_1} \cdots M_{j_{k-1}} = \begin{cases} \cdots M_i \cdots M_{i+1} \cdots, & \text{then } \sigma \text{ has a } consecution \text{ at } i\,, \\ \cdots M_{i+1} \cdots M_i \cdots, & \text{then } \sigma \text{ has an } inversion \text{ at } i\,. \end{cases}$$

Note that these notions were already highlighted as significant features of σ by Fiedler in [7]. Further define $\mathfrak{c}(\sigma)$ to be the total number of consecutions in σ, and $\mathfrak{i}(\sigma)$ to be the total number of inversions in σ. (Note: it is easy to see that $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = k - 1$, for any σ.) Then the simple relationship between the minimal indices of $P(\lambda)$ and those of any of its Fiedler companion linearizations $F_\sigma(\lambda)$ is now easily stated:

– Going from P to F_σ: $\begin{cases} \text{Left minimal indices are all uniformly increased by } \mathfrak{c}(\sigma) \\ \text{Right minimal indices are all uniformly increased by } \mathfrak{i}(\sigma) \end{cases}$

- *Strict equivalence* of Fiedler pencils:
  - For a regular polynomial P, all strong linearizations are strictly equivalent, so of course all Fiedler pencils must be, too.

  - But for a singular polynomial P, the Fiedler pencils $F_\sigma(\lambda)$ and $F_\tau(\lambda)$ are strictly equivalent if and only if $\mathfrak{c}(\sigma) = \mathfrak{c}(\tau)$, equivalently, if $\mathfrak{i}(\sigma) = \mathfrak{i}(\tau)$. Consequently, for a singular P, the 1st and 2nd Frobenius companion pencils are *never* strictly equivalent.

What is a "Companion Form", Anyway?

In this talk, and in the literature more generally, the phrases "companion matrix", "companion linearization", and "companion form" are used to refer to a number of objects that are more or less closely related or analogous in some way to the Frobenius companion matrix (or pencil). So many examples, each yearning for the title "companion", suggests that it is high time to agree on a definition for what this term means. I propose the following, based on extracting the key features from the prototype example – the Frobenius companion pencil.

Definition 7 (Companion form).
A *companion form* for degree k matrix polynomials is a *uniform template* for building a pencil $\mathcal{C}_P$ from the data of any (square) matrix poly P of degree k, using *no matrix operations* on the coefficients of P. $\mathcal{C}_P$ should be a *strong linearization* for *every* (square) P of degree k, regular or singular, over an arbitrary field $\mathbb{F}$.

Note: one might also require the template for building $\mathcal{C}_P$ to be a block matrix in which every nonzero block is either $\pm I$ or $\pm A_i$ (i.e., $\pm$ one of the coefficient matrices of P). In my view this seems unnecessarily restrictive, although certainly more precise.

Using this definition one can now concisely summarize the central result:

Every Fiedler pencil ("basic" or "generalized") **is a companion form**.

Useful extensions of the notion of companion form can also be formulated:

- A *structured companion form* for structure class $\mathcal{S}$ is a companion form $\mathcal{C}_P$ with the additional property that $\mathcal{C}_P \in \mathcal{S}$ whenever $P \in \mathcal{S}$.

- A uniform template for building a *quadratic* polynomial $\mathcal{Q}_P$ (rather than a pencil $\mathcal{C}_P$) from the data in P, such that $\mathcal{Q}_P$ and P always have identical finite and infinite elementary divisors, might reasonably be termed a *quadratic companion form*.

Recent Developments

There has been a flurry of activity in this area in the last few years .... witnessed by no less than *four* more talks on Fiedler pencils and matrices at this conference alone.

- Two talks in the minisymposium MS4.2, by F. De Terán and by S. Vologiannidis.

- Two talks in the contributed session CS8, by M.I. Bueno and by J. Pérez-Álvaro.

Here is a sampler of topics from this recent activity, some to be discussed in these talks.

- Further development of the properties of "generalized" Fiedler pencils for square matrix polynomials (recall that these pencils go back to [1]):

    - Explicit recovery formulas for eigenvectors and minimal bases [2].

    - Development of *structure-preserving companion forms* for several classes of structured matrix polynomials (e.g., alternating, palindromic, Hermitian, skew-symmetric). For a variety of different reasons, though, these structured companion forms can only exist for *odd* degree polynomials! (see [1, 5, 12, 13, 14])

    - Development of *quadratic companion forms* [11].

- Modifying the "basic" Fiedler pencils so as to be strong linearizations for *rectangular* matrix polynomials [6].

- Investigation of "extended" Fiedler pencils, in which the "Fiedler factors" M_i (or their inverses) are now allowed to be used more than once in defining a pencil. (see [16])

    - characterization of *operation-free products* of Fiedler factors (this time influenced by Fiedler's work in [9]), in which each nonzero block of the product is either $\pm A_i$ or $\pm I$.

    - yield companion forms that are *less sparse* than "basic" or "generalized" Fiedler pencils.

    - but "extended" Fiedler pencils may present more possibilities for structure preservation?

I hope this has given you some idea of how Fiedler's work on companion matrices continues to influence our understanding of matrix polynomials and their linearizations.

Conclusion

So let me offer my thanks and respects to
Miroslav Fiedler ....

May he continue to provide us with wonderful ideas,
brimming with possibilities for further development, by ....

thinking deeply about simple things.

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