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Hodgins, J.E. and Plymen, Roger

1996

MIMS EPrint: **2011.92**

ISSN 1749-9097

The Representation theory of p-adic $GL(n)$ and Deligne-Langlands parameters

J.E. HODGINS AND R.J. PLYMEN
Department of Mathematics, The University of Manchester, U.K.

1 Introduction

In this article we cover an episode in the representation theory of $GL(n)$ defined over a p-adic field with finite residue class field. We concentrate on the irreducible tempered representations admitting non-zero Iwahori-fixed vectors. We describe the space of these representations in terms of Deligne-Langlands parameters. In [6], Kazdhan and Lusztig prove the Deligne-Langlands conjecture for split reductive p-adic groups with connected centre. For $GL(n)$, this conjecture amounts to a parametrization of such representations by certain pairs (s, N) satisfying the equation $sNs^{-1} = qN$ where q is the cardinality of the residue field. We discuss these parameters in §3. In §4 and §5 we discuss the theory of Zelevinsky segments and prove results concerning the form of irreducible representations of $GL(n)$ admitting non-zero Iwahori-fixed vectors. In the final section we define the Brylinski quotient $Bryl(n)$ for the space $\mathbb{T}^n$ equipped with the natural action of the symmetric group S_n and prove that the space of Deligne-Langlands parameters of these representations is homeomorphic to $Bryl(n)$.

This article is a re-interpretation of [8], (Section 7), in terms of the Deligne-Langlands parameters: Section 7 in [8] is a report on joint work with P. Baum and N. Higson.

2 Notation

Throughout this article, we will use certain widely agreed notations. However, for the sake of completeness, we give the following summary.

Firstly, we denote by F a local non-Archimedean field of characteristic zero, whereby F is a finite extension of the p-adic field $\mathbb{Q}_p$. The residue field of F will be denoted $\mathbb{F}_q$, and has cardinality q. The field F is equipped with the standard norm denoted $|\cdot|_F$.

We will denote by $GL(n)$ the group of $n \times n$ matrices with entries in F and non-zero determinant. We use I to denote the Iwahori subgroup of $GL(n)$ and $St(n)$ will denote the Steinberg representation of the group $GL(n)$. We will also require the complex general linear group $GL(n, \mathbb{C})$.

When referring to a semisimple element s of $GL(n, \mathbb{C})$, we will sometimes express it in the form $s = \mathrm{diag}(x_1, \dots x_n)$, which denotes the diagonal matrix having entry x_i in the ith diagonal position, and zero elsewhere.

By a partition of a positive integer n, we will mean a collection of integers $\alpha = \{n_1, \dots n_k\}$, possibly with repetitions, such that

$$n = n_1 + \dots + n_k = r_1.n_1 + \dots r_l.n_l$$

where the integers r_i occur in the case of α containing repetitions. More relevant properties of partitions will be discussed in §7, where we will also refer to the compact torus $\mathbb{T}^n$ consisting of n-tuples of complex numbers of modulus 1.

As we later consider some quite involved representation theory, it seems prudent to give a brief account of some of the basic facts we will require on the representations of $GL(n)$. An (admissible) representation is *supercuspidal* if each of its matrix coefficients is compactly supported modulo centre. A representation is tempered if it occurs in the support of the Plancherel measure on the unitary dual of $GL(n)$. These definitions are not vital for the work that follows and more details on these representations can be found in [5]. Let K be a subgroup of $GL(n)$. Then a representation π is said to admit K-fixed vectors if the set $\{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$ is non-zero.

3 Deligne-Langlands parameters

In this section we give an account of the results of Kazdhan and Lusztig from [6]. A fuller summary of this work can be found in [10]. Given a connected split reductive group with split centre, defined over a p-adic field

F, with finite residue class field containing q elements, a result of Borel and Matsumoto states that the category of admissible complex representations of G admitting non-zero I-fixed vectors is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of the Hecke algebra $\mathbf{H}_q$ with respect to the Iwahori subgroup I. The results of [6] hold for all connected split reductive groups with connected centre, but in the case of $GL(n)$ the result is somewhat simpler, and can be stated as follows. IRREDUCIBLE

3.1 Proposition For the group $GL(n)$, the space of admissible representations admitting non-zero I-fixed vectors is in bijective correspondance with the space of $GL(n,\mathbb{C})$-conjugacy classes of pairs (s,N), where s is a semisimple element in $GL(n,\mathbb{C})$ and N is a nilpotent element of the corresponding complex Lie algebra, and (s,N) satisfy the equation

$$sNs^{-1} = qN$$

where q is the cardinality of the residue field of F. These pairs (s,N) are the Deligne-Langlands parameters.

This result is proved in [6] by explicitly constructing all $\mathbf{H}_q$-modules via these parameters.

3.2 Remark Proposition 3.1 as stated is valid only for the p-adic group $GL(n)$, and was proved by Zelevinsky by the theory of segments (see §4 and [3], [11]). However, the result proved by Kazdhan and Lusztig in [6] holds for all connected split reductive groups with split centre defined over F. In this general case it is neccessary to introduce a third parameter. This third parameter takes the form of a certain representation of the component group of the simultaneous centralizer in ${}^LG$ of s and u, where u is a unipotent element of ${}^LG$ such that $\exp(u) = N$. If we consider the case of ${}^LG = GL(n,\mathbb{C})$, we can see that in fact we have

$$C(s,u)/C(s,u)_0 = 1$$

for all pairs (s,N) satisfying $sNs^{-1} = qN$, and so for p-adic $GL(n)$ the third parameter is not required. More details on the form of this third parameter can be found in [6], For readers with an understanding of the original Langlands parameters as certain representations of the Weil-Deligne group, it is worth noting that the pairs (s, N) are obtained by evaluating the representation of the Weil-Deligne group at a certain element of W_F [5], [9].

3.3 Remark We note that, since we are concerned with $GL(n, \mathbb{C})$- conjugacy classes of pairs (s, N), it will suffice to consider conjugacy class representatives of nilpotent elements in the Lie algebra, in which case we need only consider the nilpotent elements N in Jordan canonical form.

3.4 Example Let us consider the implications for s and N of the equation $sNs^{-1} = qN$ for the group $GL(2)$. The form of the nilpotent element N, in Jordan canonical form, is determined by the partitions of 2, and so will take one of the two following forms,

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore we may evaluate the corresponding elements s such that the governing equation holds, and we obtain the following pairs (s, N):

$$\left(\begin{bmatrix} zq & 0 \\ 0 & z \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \text{ and } \left(\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

As we have discussed, a pair (s, N) is determined by the form of the nilpotent element N, and this in turn is determined by a partition of n, from which we can construct N in Jordan canonical form. Therefore, it seems natural at this juncture to discuss the form of a pair (s, N) given a partition $n = n_1 + \ldots + n_k$.

Given such a partition, we form the nilpotent element N, and simple calculations concerning the equation $sNs^{-1} = qN$ yield the form of s as the following diagonal matrix:

$$s = \mathrm{diag}(z_1 q^{n_1 - 1}, \ldots z_1 q, z_1, z_2 q^{n_2 - 1}, \ldots z_2, \ldots z_k q^{n_k - 1}, \ldots z_k)$$

where each z_i is a complex number for $i = 1, \ldots k$. We discuss the relationship between the parameters (s, N) and the representations of $GL(n)$ in §5, and we will also note certain properties of the Weyl group action on the space of parameters in §6.

4 The theory of segments

In the paper [11], Zelevinsky gives a classification of the irreducible representations of $GL(n)$ via a construction called segments. In this section, we give a brief overview of these results and at the end of §6 we will briefly discuss the links between the classification by Kazdhan and Lusztig and the theory of segments. More details on Zelevinsky segments can be found in the paper [5].

For a partition $n = n_1 + \ldots n_k$, let P denote the parabolic subgroup of $GL(n)$ with Levi factor $M \simeq GL(n_1) \times \ldots \times GL(n_k)$. Given a representation π of $GL(n)$, we write $\pi(s)$ to denote the representation $\pi \otimes (|\cdot|_F^s \circ \det)$ for some complex number s. We now quote a result of Bernstein and Zelevinsky on the detection of reducibility of induced representations.

4.1 Proposition [3](Theorem 4.2) Let $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_k$ be an irreducible representation of M with σ_i supercuspidal for all i. The induced representation $\mathrm{Ind}_P^{GL(n)}\sigma$ is reducible if and only if for some pair of indices i, j, with $i \neq j$ we have $n_i = n_j$ and $\sigma_i = \sigma_j(1)$.

4.2 Definition For a partition $n = m + \ldots + m = r.m$, and for an irreducible supercuspidal representation σ of $GL(m)$, a *segment* is a finite set of representations of $GL(m)$ of the form

$$\{\sigma, \sigma(1), \ldots \sigma(r-1)\} = [\sigma, \sigma(r-1)] = \Delta$$

For such a partition $n = r.m$, and for P the corresponding parabolic subgroup of $GL(n)$ as above, given a segment Δ we make the following definition

$$\mathrm{Ind}_P^{GL(n)}(\Delta) := \mathrm{Ind}_P^{GL(n)}(\sigma \otimes \sigma(1) \otimes \ldots \otimes \sigma(r-1))$$

where the induction is normalized induction from P to $GL(n)$.

4.3 Proposition [3],[5] (1.2.2) Given a segment Δ, the induced representation

$$\mathrm{Ind}_P^{GL(n)}(\Delta)$$

has a unique irreducible quotient $Q(\Delta)$ and a unique irreducible subrepresentation $Z(\Delta)$.

The unique irreducible quotient $Q(\Delta)$ is called the Langlands quotient. We now quote a result of Bernstein characterising the square-integrable representations of $GL(n)$.

4.4 Proposition [11] (Theorem 9.3) Every square-integrable representation of $GL(n)$ has the form $Q(\Delta)$ for some segment $\Delta = [\sigma, \sigma(r-1)]$ where $\sigma(\frac{r-1}{2})$ is unitary.

4.5 Example Let us consider an example. If we consider the trivial partition $n = 1 + \ldots + 1$ and $\sigma = |\cdot|_F^{\frac{1-n}{2}}$, then $P = B$, the standard Borel subgroup and we have the segment

$$\Delta = \{|\cdot|_F^{\frac{1-n}{2}}, |\cdot|_F^{\frac{3-n}{2}}, \ldots |\cdot|_F^{\frac{n-1}{2}}\}$$

and $Q(\Delta)$ is the Steinberg representation. See [11] (3.2, 9.2) where the notation is $Q(\Delta) = \langle\Delta\rangle^t$. Thus, in the case of $n = 2$, the Steinberg $St(2)$ corresponds to $Q(\Delta)$ for

$$\Delta = \{|\cdot|_F^{-\frac{1}{2}}, |\cdot|_F^{\frac{1}{2}}\}$$

We now introduce more structure on the set of segments.

4.6 Definition Consider two segments

$$\Delta_1 = [\sigma_1, \sigma_1(r_1 - 1)] \text{ and } \Delta_2 = [\sigma_2, \sigma_2(r_2 - 1)]$$

Then we have the following definitions:

1. Δ_1 and Δ_2 are *linked* if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment.
2. Δ_1 *precedes* Δ_2 if Δ_1 and Δ_2 are linked and $\sigma_2 = \sigma_1(t)$ for some positive integer t.

We now quote the main result of Zelevinsky [11] (Section 9)

4.7 Proposition Consider segments $\Delta_1, \ldots \Delta_k$, and assume that for $i < j$ we have that Δ_i does not precede Δ_j. Then

1. The induced representation $\mathrm{Ind}_P^{GL(n)}(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k))$ admits a unique irreducible quotient

$$Q(\Delta_1, \ldots \Delta_k)$$

2. If $\Delta'_1, \ldots \Delta'_l$ is another such collection of segments, then
$$Q(\Delta_1, \ldots \Delta_k) \simeq Q(\Delta'_1, \ldots \Delta'_l)$$
if and only if $k = l$ and $\Delta_i = \Delta_{\tau(i)}$ for each i and for some permutation τ of $\{1, \ldots k\}$.

3. Every irreducible admissible representation of $GL(n)$ is isomorphic to some $Q(\Delta_1, \ldots \Delta_k)$.

4. The induced representation $\mathrm{Ind}_P^{GL(n)}(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k))$ is irreducible if and only if no two of the segments are linked.

We now consider how tempered representations of $GL(n)$ manifest themselves in the theory of segments. We have the following result [4], [5] (Proposition 2.2.1)

4.8 Proposition The tempered representations of $GL(n)$ are precisely the representations
$$\mathrm{Ind}_P^{GL(n)}(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k))$$
where each $Q(\Delta_i)$ is square-integrable for $1 \leq i \leq k$.

We note that if $Q(\Delta)$ is square-integrable, then $\Delta = [\sigma(\frac{1-r}{2}), \sigma(\frac{r-1}{2})]$ for some unitary supercuspidal σ by Proposition 4.4. Two such segments cannot be linked, and so by Proposition 4.7 part 4, $\mathrm{Ind}_P^{GL(n)}(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k))$ is irreducible and so equal to its unique irreducible quotient $Q(\Delta_1, \ldots \Delta_k)$. Also, because no such segments are linked, we note that the tempered representations are irreducibly induced from discrete series representations.

The theory of segments has the following compatibility.

4.9 Lemma Consider a segment Δ arising from a partition $n = 1 + \ldots + 1$, and let χ be an unramified unitary character of $GL(1)$. Then we have
$$Q(\Delta \otimes \chi) = Q(\Delta) \otimes \chi \circ \det$$

4.10 Example For example, consider the group $GL(3)$ and the segment $\Delta = \{|\cdot|_F^{-1}, 1, |\cdot|_F\}$ arising from the partition $3 = 1 + 1 + 1$, then we recall from Example 4.5 that
$$Q(\{|\cdot|_F^{-1}, 1, |\cdot|_F\}) = St(3)$$

and therefore we can observe that

$$Q(\{|\cdot|_F^{-1}\otimes\chi,\chi,|\cdot|_F\otimes\chi\}) = St(3)\otimes(\chi\circ\det)$$

for an unramified unitary character χ of $GL(1)$.

5 Tempered Representations of $GL(n)$

We now delve into the representation theory of the p-adic group $GL(n)$. We introduce the notation

$$Q(\Delta_1)\times\ldots\times Q(\Delta_k) = \mathrm{Ind}_P^{GL(n)}(Q(\Delta_1)\otimes\ldots\otimes Q(\Delta_k))$$

All representations considered in this section satisfy the conditions of Proposition 4.8, and so we have

$$Q(\Delta_1)\times\ldots\times Q(\Delta_k)\simeq Q(\Delta_1,\ldots\Delta_k)$$

For example, in the case of $GL(4)$ and $\Delta = \{|\cdot|_F^{-\frac{1}{2}},|\cdot|_F^{\frac{1}{2}}\}$, we have

$$Q(\Delta,\Delta)\simeq \mathrm{Ind}_P^{GL(n)}(Q(\Delta)\otimes Q(\Delta)) = St(2)\times St(2)$$

as $Q(\Delta) = St(2)$ by Example 4.5. As stated in the introduction, we are concerned with tempered representations of $GL(n)$ which admit non-zero I-fixed vectors. We now state and prove the following result concerning such representations using the theory of segments introduced in the previous section.

5.1 Proposition [8] Let $n = n_1+\ldots+n_k$ be a partition of n, and let $w_1,\ldots w_k\in i\mathbb{R}$. Then the representation

$$(St(n_1)\otimes(|\cdot|_F^{w_1}\circ\det))\times\ldots\times(St(n_k)\otimes(|\cdot|_F^{w_k}\circ\det))$$

of $GL(n)$ is unitary, irreducible, tempered and admits non-zero I-fixed vectors. Conversely, all such representations are of this form.

Proof (Modelled on [8]) By Proposition 4.8, each tempered representation of $GL(n)$ is of the form $Q(\Delta_1)\times\ldots\times Q(\Delta_k)$ where $Q(\Delta_i)$ is square-integrable for each $i = 1, \ldots k$. We now use transitivity of parabolic induction [3] and Borel's theorem [2] to calculate that (modulo unramified unitary twist) we must have

$$\Delta_i = \{ |\cdot|_F^{\frac{1-n_i}{2}}, \ldots |\cdot|_F^{\frac{n_i-1}{2}} \}$$

with $i = 1, \ldots k$. But then $Q(\Delta_i)$ is the Steinberg representation of $GL(n_i)$ by Example 4.5. Note that $Q(\chi \otimes \Delta_i) = (\chi \circ \det) \otimes Q(\Delta_i)$ as in Lemma 4.9, and that $Q(\Delta_1) \times \ldots \times Q(\Delta_k)$ is irreducible by Proposition 4.1. □

5.2 Remark We must now unravel the combinatorics governing the representations above. Again, considering the partition $n = r_1.n_1 + \ldots + r_l.n_l$, and forming the subgroup M, we note that the Weyl group of M is

$$W(M) = S_{r_1} \times \ldots \times S_{r_l}$$

The action of this Weyl group permutes blocks of equal size. We will enlarge upon this in §6.

We now discuss the relationship between a parameter (s, N) and the corresponding representation π of $GL(n)$. From the above, we have the form of all irreducible tempered representations of $GL(n)$ admitting a non-zero Iwahori-fixed vector. We recall from Example 4.5 that the Steinberg representation $St(n)$ of $GL(n)$ occurs as the unique irreducible quotient $Q(\Delta)$ of the segment $\Delta = \{ |\cdot|_F^{\frac{1-n}{2}}, |\cdot|_F^{\frac{3-n}{2}}, \ldots |\cdot|_F^{\frac{n-1}{2}} \}$. The pair (s, N) corresponding to the Steinberg representation $St(n)$ is given by

$$\left(s = \begin{bmatrix} q^{\frac{n-1}{2}} & & & & \\ & q^{\frac{n-3}{2}} & & & \\ & & \ddots & & \\ & & & q^{\frac{3-n}{2}} & \\ & & & & q^{\frac{1-n}{2}} \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & & \ldots & 0 \\ & 0 & 1 & \ldots & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \right)$$

Now let us consider a partition $n = n_1 + \ldots + n_k$, and a corresponding representation $\pi = \pi_1 \times \ldots \times \pi_k$ where each π_i takes the form

$$\pi_i = St(n_i) \otimes (|\cdot|_F^{w_i} \circ \det)$$

for some $w_i \in i\mathbb{R}$. We note that $|\cdot|_F^{w_i} \circ \det$ gives rise to a factor of q^{w_i} and so we have that the parameter (s, N) corresponding to the representation $\pi_i = St(n_i) \otimes (|\,.\,|_F^{w_i} \circ \det)$ is given by

$$s = \begin{bmatrix} q^{\frac{n_i-1}{2}+w_i} & & & & \\ & q^{\frac{n_i-3}{2}+w_i} & & & \\ & & \ddots & & \\ & & & q^{\frac{3-n_i}{2}+w_i} & \\ & & & & q^{\frac{1-n_i}{2}+w_i} \end{bmatrix}$$

with N as above, and it can be seen that the condition $sNs^{-1} = qN$ holds. Therefore the form of the parameter (s, N) for the representation $\pi = \pi_1 \times \ldots \times \pi_k$ is of the form

$$s = \operatorname{diag}(q^{\frac{n_1-1}{2}+w_1}, \ldots, q^{\frac{1-n_1}{2}+w_1}, \ldots, q^{\frac{n_k-1}{2}+w_k}, \ldots q^{\frac{1-n_k}{2}+w_k})$$

and N is the $n \times n$ matrix in Jordan canonical form corresponding to the partition $n = n_1 + \ldots n_k$.

5.3 Example Let us consider the discussion above in terms of the irreducible tempered representations of $GL(2)$ and $GL(3)$ admitting non-zero Iwahori-fixed vectors. For the group $GL(2)$, we note from Proposition 5.1 that these representations must be of the form

$$\pi_1 = St(2) \otimes (|\cdot|_F^{w} \circ \det)$$

corresponding to the null-partition, or of the form

$$\pi_2 = (|\cdot|_F^{w_1} \circ \det) \times (|\cdot|_F^{w_2} \circ \det)$$

for $w, w_1, w_2 \in i\mathbb{R}$ all determined modulo $\frac{2\pi i}{\log q}$. We can now write down explicitly the parameters (s, N) corresponding to these repersentations. They are

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w} & \\ & q^{-\frac{1}{2}+w} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), \left(\begin{bmatrix} q^{w_1} & \\ & q^{w_2} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

We now note that the action of the Weyl group on the parameter corresponding to the representation π_2 permutes the diagonal entries, and so in fact the ordering of w_1 and w_2 is unimportant. We will enlarge on this comment, and indeed formalise it in §6.

We now turn our attention to representations of $GL(3)$. Again, by Proposition 5.1, the form of the representations we consider are

$$\pi_1 = St(3) \otimes (|\cdot|_F^{w} \circ \det)$$

$$\pi_2 = (St(2) \otimes (|\cdot|_F^{w_1} \circ \det)) \times (|\cdot|_F^{w_2} \circ \det)$$

$$\pi_3 = (|\cdot|_F^{w_3} \circ \det) \times (|\cdot|_F^{w_4} \circ \det) \times (|\cdot|_F^{w_5} \circ \det)$$

corresponding to partitions $3 = 2+1 = 1+1+1$ for $w, w_1 \ldots w_5 \in i\mathbb{R}$. The Deligne-Langlands parameters for these representations are then given by

$$s_1 = \mathrm{diag}(q^{1+w}, q^{w}, q^{-1+w})$$

$$s_2 = \mathrm{diag}(q^{\frac{1}{2}+w_1}, q^{-\frac{1}{2}+w_1}, q^{w_2})$$

$$s_2 = \mathrm{diag}(q^{w_3}, q^{w_4}, q^{w_5})$$

with the nilpotent elements N_1, N_2, N_3 in Jordan canonical form corresponding to the partitions as above.

6 $Bryl(n)$ and the space $\{(s, N)\}$

In this section, we define the Brylinski quotient and prove the main Theorem of this article.

The Brylinski quotient: Firstly we note some elementary facts about partitions of integers and then give the definition of the Brylinski Quotient. Given a positive integer n, a partition of n is a set of positive integers $\alpha = \{n_1, \ldots n_k\}$, possibly with repetitions, such that $n = n_1 + \ldots + n_k$. The order of the elements in α is irrelevant. If we now have $n \in \mathbb{N}^+$ and a partition α, then we define $d(\alpha)$ to be the number of distinct elements in α. Therefore, as we may write the partition α as $n = r_1.n_1 + \ldots r_l.n_l$, we have $d(\alpha) = l$. Thus, for example, if $n = 7$ and $\alpha = \{4, 1, 1, 1\}$, then $7 = 4 + 1 + 1 + 1 = 1.4 + 3.1$ whereby $d(\alpha) = 2$.

The Brylinski quotient can be defined in great generality [1], but for this article it will be sufficient to give the definition in a specific case.

6.1 Definition For the space $\mathbb{T}^n$ equipped with the natural action of S_n, the Brylinski quotient is defined by $Bryl(n) = Bryl(\mathbb{T}^n; S_n)$. We therefore have

$$Bryl(n) = \bigsqcup_{\alpha} (\mathbb{T}^n)^\gamma / Z(\gamma)$$

where the disjoint union is taken over all partitions α of n, where $\gamma \in S_n$ has cycle type α. Thus the disjoint union is taken over all conjugacy classes in S_n. The set $(\mathbb{T}^n)^\gamma = \{t \in \mathbb{T}^n | \gamma t = t\}$ is the γ-fixed set, and $Z(\gamma)$ is the centralizer of γ in S_n.

Suppose now we have a partition α of n, and that α consists of r_1 elements equal to n_1, up to r_l elements equal to n_l. Then we can observe that $n = r_1.n_1 + \ldots r_l.n_l$. Let γ be an element of S_n of cycle type α. The centralizer $Z(\gamma)$ is a product of wreath products

$$Z(\gamma) = (\mathbb{Z}/n_1 \wr S_{r_1}) \times \ldots \times (\mathbb{Z}/n_l \wr S_{r_l})$$

Let $Sym^m\mathbb{T}$ be the space of unordered m-tuples $\{t_1, \ldots t_m\}$, where each $t_i \in \mathbb{T}$ for $i = 1, \ldots m$. We now observe, as in [8], that

$$\begin{aligned} Bryl(n) &= \bigsqcup_{\alpha} (\mathbb{T}^n)^\gamma / Z(\gamma) \\ &= \bigsqcup_{\alpha} \frac{\{(t_1, \ldots t_1, \ldots, t_l, \ldots t_l)\}}{(\mathbb{Z}/n_1 \wr S_{r_1}) \times \ldots \times (\mathbb{Z}/n_l \wr S_{r_l})} \\ &= \bigsqcup_{\alpha} \frac{\{(t_1, \ldots t_1, \ldots, t_l, \ldots t_l)\}}{S_{r_1} \times \ldots \times S_{r_l}} \\ &= \bigsqcup_{\alpha} Sym^{r_1}\mathbb{T} \times \ldots \times Sym^{r_l}\mathbb{T} \end{aligned}$$

where each t_i occurs n_i times for a partition $\alpha = \{n_1, \ldots n_1, \ldots n_k, \ldots n_k\}$.

6.2 Proposition For the space $\mathbb{T}^n$ equipped with the natural action of S_n, the Brylinski quotient is given by

$$Bryl(n) = \bigsqcup_{\alpha} Sym^{r_1}\mathbb{T} \times \ldots \times Sym^{r_l}\mathbb{T}$$

where the disjoint union is over all partitions $\alpha = r_1.n_1 + \ldots\ldots r_l.n_l$.

The proof of the main Theorem: Now let us return our discussion to the subject of representations. From §5, Proposition 5.1, we have the form of all tempered representations of $GL(n)$ admitting non-zero I-fixed vectors and also the form of their respective Deligne-Langlands parameters (s, N).

Let us begin by considering a partition $\alpha = \{n_1, \ldots n_k\}$. Then, by Proposition 5.1, all tempered representations π of $GL(n)$ admitting non-zero I-fixed vectors are constructed from unramified unitary twists of the Steinberg representations of the group $GL(n_i)$ and are of the form

$$\pi = (St(n_1) \otimes (|\cdot|_F^{w_1} \circ \det)) \times \ldots \times (St(n_k) \otimes (|\cdot|_F^{w_k} \circ \det))$$

where $w_1, \ldots w_k \in i\mathbb{R}$.

As we have seen, these representations correspond to the pairs (s, N) where N is the $n \times n$ matrix in Jordan canonical form given by the partition α, and s is the diagonal matrix of the form

$$s = \mathrm{diag}(q^{\frac{n_1-1}{2}+w_1}, \ldots q^{\frac{1-n_1}{2}+w_1}, \ldots q^{\frac{n_k-1}{2}+w_k}, \ldots q^{\frac{1-n_k}{2}+w_k})$$

We can now rearrange the semisimple element s by denoting $q^{\frac{n_i-1}{2}+w_i}$ by z_i each $i = 1, \ldots k$. Therefore we may now write s in the form

$$s = \mathrm{diag}(z_1 q^{n_1-1}, \ldots z_1 q, z_1, \ldots z_k q^{n_k-1}, \ldots z_k q, z_k)$$

We also note that since each w_i is a pure imaginary number, since π is tempered, we have that q^{w_i} is a complex number of modulus 1. Thus we have now recovered the form of the parameters (s, N) from §3. We now turn our attention once again to the action of the Weyl group on the semisimple elements $s \in GL(n, \mathbb{C})$. Consider a partition α in the form $n = r_1.n_1 + \ldots + r_l.n_l$, giving rise to a pair (s, N) of the form we have just described. This semisimple element is acted on by Weyl group elements, and the action permutes the diagonal blocks that are of equal size. The two pairs (s, N) and $(\gamma s \gamma^{-1}, \gamma N \gamma^{-1} = N)$ lie in the same conjugacy class, and therefore give the same parameter. Thus we can see that the space of $GL(n, \mathbb{C})$-conjugacy classes of pairs (s, N) satisfying $sNs^{-1} = qN$ takes the form of an unordered

r_i-tuple of complex numbers of modulus 1 for each non-zero r_i in the α. We can map the space $\{(s,N)\}$ naturally into the Brylinski quotient by mapping the pair (s,N), with N in Jordan canonical form given by the partition α into the component of $Bryl(n)$ arising from the partition α,

$$(s,N) \mapsto Sym^{r_1}\mathbb{T} \times \ldots \times Sym^{r_l}\mathbb{T}$$

Therefore, we have proved our main result.

6.3 Theorem The space of Deligne-Langlands parameters of those irreducible tempered representations of $GL(n)$ admitting non-zero I-fixed vectors is homeomorphic to $Bryl(n)$

$$\{(s,N)\} \cong Bryl(n)$$

6.4 Remark This is a re-interpretation of [8] (7.7) in terms of Deligne-Langlands parameters.

Examples: We now give a review of the theory installed in the previous sections, by examining the consequences in terms of the groups $GL(2)$, $GL(3)$ and $GL(4)$. As we have seen how the parameters (s,N) relate to the representations of $GL(n)$, we will construct these parameters, and explicitly demonstrate the homeomorphism of Theorem 6.3.

6.5 Example Let us consider the case of $GL(2)$. We consider nilpotent elements N in Jordan canonical form to construct $GL(2,\mathbb{C})$-conjugacy class representatives of the pairs (s,N). As we have seen, we arrive at the two following pairs:

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w} & 0 \\ 0 & q^{-\frac{1}{2}+w} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \text{ and } \left(\begin{bmatrix} q^{w_1} & 0 \\ 0 & q^{w_2} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

with $w, w_1, w_2 \in i\mathbb{R}$. Conjugating by a Weyl group element, we observe the following,

$$\gamma s \gamma^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q^{w_1} & 0 \\ 0 & q^{w_2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} q^{w_2} & 0 \\ 0 & q^{w_1} \end{bmatrix}$$

and therefore, we cannot distinguish between the elements s and $\gamma s \gamma^{-1}$ in the set of $GL(2,\mathbb{C})$-conjugacy classes of pairs (s,N). Let us now turn our attention to the Brylinski quotient $Bryl(2)$. Since the only partitions of 2 are the trivial partition and $2 = 1 + 1$, we have that

$$Bryl(2) = \mathbb{T} \bigsqcup Sym^2\mathbb{T}$$

Therefore we observe the map $\{(s, N)\} \longrightarrow Bryl(2)$ as follows

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w} & 0 \\ 0 & q^{-\frac{1}{2}+w} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \mapsto q^w \in \mathbb{T}$$

$$\left(\begin{bmatrix} q^{w_1} & 0 \\ 0 & q^{w_2} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \mapsto [q^{w_1}, q^{w_2}] \in Sym^2\mathbb{T}$$

for $w, w_1, w_2 \in i\mathbb{R}$, and so we see the relation between the space of parameters and the Brylinski quotient.

6.6 Example Now consider the case of $GL(3)$. We construct the Brylinski quotient, considering the three partitions $3 = 3 = 2 + 1 = 1 + 1 + 1$, and we arrive at the following

$$Bryl(3) = \mathbb{T} \bigsqcup (\mathbb{T} \times \mathbb{T}) \bigsqcup Sym^3\mathbb{T}$$

Now let us construct our pairs (s, N). Again we need only consider the case of the nilpotent element being of Jordan canonical form, and again the relationship of the pairs (s, N) with actual representations of $GL(3)$ is taken as understood. We also exhibit which pairs map to which component of $Bryl(3)$ by way of the following natural maps

$$\left(\begin{bmatrix} q^{1+w} & & \\ & q^w & \\ & & q^{-1+w} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}\right) \mapsto q^w \in \mathbb{T}$$

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w_1} & & \\ & q^{-\frac{1}{2}+w_1} & \\ & & q^{w_2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}\right) \mapsto [q^{w_1}, q^{w_2}] \in \mathbb{T} \times \mathbb{T}$$

$$\left(\begin{bmatrix} q^{w_1} & & \\ & q^{w_2} & \\ & & q^{w_3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}\right) \mapsto [q^{w_1}, q^{w_2}, q^{w_3}] \in Sym^3\mathbb{T}$$

We now begin to see the role of the Weyl group in determining the pairs (s, N) more clearly, since we are now unable to distinguish between the semisimple element $\mathrm{diag}(q^{w_1}, q^{w_2}, q^{w_3})$ and, for example, $\mathrm{diag}(q^{w_3}, q^{w_1}, q^{w_2})$ due to the Weyl group action, and hence we must map elements of this form into $Sym^3\mathbb{T}$.

6.7 Example Finally, let us consider the example of $GL(4)$. Again, we calculate the Brylinski quotient, and then calculate the pairs (s, N), observing the natural maps into the relevant components of $Bryl(4)$. The unordered partitions of 4 are $4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. We note that this is the first occasion on which we observe a repetition in the partition which is not the repetition of a 1. For the Brylinski quotient we have

$$Bryl(4) = \mathbb{T} \bigsqcup (\mathbb{T} \times \mathbb{T}) \bigsqcup Sym^2\mathbb{T} \bigsqcup (\mathbb{T} \times Sym^2\mathbb{T}) \bigsqcup Sym^4\mathbb{T}$$

Calculation of the pairs (s, N), and then mapping into $Bryl(4)$ gives us

$$\left(\begin{bmatrix} q^{\frac{3}{2}+w} & & & \\ & q^{\frac{1}{2}+w} & & \\ & & q^{-\frac{1}{2}+w} & \\ & & & q^{-\frac{3}{2}+w} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \right) \mapsto q^w \in \mathbb{T}$$

$$\left(\begin{bmatrix} q^{1+w_1} & & & \\ & q^{w_1} & & \\ & & q^{-1+w_1} & \\ & & & q^{w_2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \right) \mapsto [q^{w_1}, q^{w_2}] \in \mathbb{T} \times \mathbb{T}$$

For the next pair (s, N), formed from the nilpotent element N in the conjugacy class corresponding to the partition $4 = 2 + 2$, the Weyl group action will permute the 2×2 blocks, and so we map the pair (s, N) into the component $Sym^2\mathbb{T}$ as follows.

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w_1} & & & \\ & q^{-\frac{1}{2}+w_1} & & \\ & & q^{\frac{1}{2}+w_2} & \\ & & & q^{-\frac{1}{2}+w_2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \right) \mapsto [q^{w_1}, q^{w_2}] \in Sym^2\mathbb{T}$$

In the case of the pairs arising form the partition $4 = 2 + 1 + 1$, the Weyl group action will permute the diagonal entries q^{w_2} and q^{w_3}, but the 2×2 diagonal block $\mathrm{diag}(q^{\frac{1}{2}+w_1}, q^{-\frac{1}{2}+w_1})$ will remain fixed, and therefore we map into $\mathbb{T} \times Sym^2\mathbb{T}$.

$$\left(\begin{bmatrix} q^{\frac{1}{2}+w_1} & & & \\ & q^{-\frac{1}{2}+w_1} & & \\ & & q^{w_2} & \\ & & & q^{w_3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \right) \mapsto [q^{w_1}, q^{w_2}, q^{w_3}] \in \mathbb{T} \times Sym^2\mathbb{T}$$

Finally, in the case of the zero nilpotent element, the Weyl group action permutes all diagonal entries, and so we have the map below.

$$\left(\begin{bmatrix} q^{w_1} & & & \\ & q^{w_2} & & \\ & & q^{w_3} & \\ & & & q^{w_4} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \right) \mapsto [q^{w_1}, q^{w_2}, q^{w_3}, q^{w_4}] \in Sym^4\mathbb{T}$$

and thus we observe the homeomorphism of Theorem 6.3.

6.8 Remark As a final point, we note that the role q plays in the form of s mimics the development of a Zelevinsky segment. For example, in the case of $GL(3)$, the semisimple element $s = \mathrm{diag}(q^{1+w}, q^{w}, q^{-1+w})$ corresponds to a segment of length 3, whereas a semisimple element of the form $s = \mathrm{diag}(q^{w_1}, q^{w_2}, q^{w_3})$ corresponds to 3 segments, all of length 1. In fact, considering Example 4.5 we can state the segments explicitly as follows. In the case of the segment of length 3, we have

$$\Delta = \{|\cdot|_F^{w-1}, |\cdot|_F^{w}, |\cdot|_F^{w+1}\}$$

which corresponds to the representation $St(3) \otimes |\cdot|_F^{w} \circ \det$, and for the segments of length 1 we have

$$\Delta = \{\{|\cdot|_F^{w_1}\}, \{|\cdot|_F^{w_2}\}, \{|\cdot|_F^{w_3}\}\}$$

corresponding to the representation induced from

$$(|\cdot|_F^{w_1} \circ \det) \times (|\cdot|_F^{w_2} \circ \det) \times (|\cdot|_F^{w_3} \circ \det)$$

and thus we can observe how the Deligne-Langlands parameters correspond with Zelevinsky's theory of segments.

References

There are many texts available on the Langlands programme. The books listed below are only those used directly in obtaining the results in this article. Other texts may be found which deal with the background and basic definitions of this area in much greater detail. These include the two parts of Proceedings of the Symposia on Pure Mathematics, Volume 33.

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