

*Analysis of Structured Polynomial Eigenvalue
Problems*

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ANALYSIS OF STRUCTURED POLYNOMIAL EIGENVALUE PROBLEMS

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2011

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This thesis considers Hermitian/symmetric, alternating and palindromic matrix polynomials which all arise frequently in a variety of applications, such as vibration analysis of dynamical systems and optimal control problems. A classification of Hermitian matrix polynomials whose eigenvalues belong to the extended real line, with each eigenvalue being of definite type, is provided first. We call such polynomials quasidefinite. Definite pencils, definitizable pencils, overdamped quadratics, gyroscopically stabilized quadratics, (quasi)hyperbolic and definite matrix polynomials are all quasidefinite. We show, using homogeneous rotations, special Hermitian linearizations and a new characterization of hyperbolic matrix polynomials, that the main common thread between these many subclasses is the distribution of their eigenvalue types. We also identify, amongst all quasihyperbolic matrix polynomials, those that can be diagonalized by a congruence transformation applied to a Hermitian linearization of the matrix polynomial while maintaining the structure of the linearization.

Secondly, we generalize the notion of self-adjoint standard triples associated with Hermitian matrix polynomials in Gohberg, Lancaster and Rodman's theory of matrix polynomials to present spectral decompositions of structured matrix polynomials in terms of standard pairs (X, T) , which are either real or complex, plus a parameter matrix S that acquires particular properties depending on the structure under investigation. These decompositions are mainly an extension of the Jordan canonical form for a matrix over the real or complex field so we investigate the important special case of structured Jordan triples.

Finally, we use the concept of structured Jordan triples to solve a structured inverse polynomial eigenvalue problem. As a consequence, we can enlarge the collection of nonlinear eigenvalue problems [NLEVP, 2010] by generating quadratic and cubic quasidefinite matrix polynomials in different subclasses from some given spectral data by solving an appropriate inverse eigenvalue problem. For the quadratic case, we employ available algorithms to provide tridiagonal definite matrix polynomials.

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Publications

- The material in Chapter 2 is based on the paper “Hermitian matrix polynomials with real eigenvalues of definite type- part I: Classification” [3] (with Françoise Tisseur), *Linear Algebra Appl.* (2010), doi:10.1016/j.laa.2010.08.035.
- The material in Chapter 3 is based on the technical report “Standard triples for structured matrix polynomials” [4] (with Françoise Tisseur), MIMS EPrint 2011.37, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, May 2011. 20 pp., submitted to *Linear Algebra Appl.*

Dedication

To all the exceptional people involved in my academic life.

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Chapter 1

Introduction

Matrix polynomials [31], *λ -matrices* [50] and *polynomial matrices* [73] are different names for polynomials in λ whose coefficients are matrices, or equivalently, matrices whose entries are polynomials in λ with scalar coefficients. The polynomial eigenvalue problem (PEP), which is to find *eigen triples* $(\lambda, x, y) \in \mathbb{C} \cup \{\infty\} \times \mathbb{C}^n \setminus \{0\} \times \mathbb{C}^n \setminus \{0\}$ that satisfy $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$, where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + \lambda A_1 + A_0, \quad (1.1)$$

$A_j \in \mathbb{C}^{n \times n}$ for $j = 0:m$, extends the well-known standard eigenvalue problem (SEP) and the generalized eigenvalue problem (GEP), for which $P(\lambda)$ in (1.1) has the forms $-\lambda I + A$ and $\lambda A - B$ respectively, where $A, B \in \mathbb{C}^{n \times n}$. In general, the finite eigenvalues of P are the solutions of the characteristic equation

$$\det(P(\lambda)) = 0. \quad (1.2)$$

Infinity is an eigenvalue of P when $\det(A_m) = 0$. The vectors x and y are right and left eigenvectors corresponding to the eigenvalue λ , respectively. If $A_m \neq 0$ then m is the degree of P , denoted by $\deg(P)$. $P(\lambda)$ is *regular* whenever its determinant does not vanish identically. Throughout this thesis, $P(\lambda)$ is assumed to be regular with $n \times n$ matrix coefficients and degree m .

The PEP is a subclass of the more general nonlinear eigenvalue problem (NLEVP) that takes the form of finding eigen triples $(\lambda, x, y) \in \mathbb{C} \cup \{\infty\} \times \mathbb{C}^n \setminus \{0\} \times \mathbb{C}^n \setminus \{0\}$

satisfying $T(\lambda)x = 0$ and $y^*T(\lambda) = 0$, where $T : \mathbb{C} \rightarrow \mathbb{C}^{n \times n}$ is a nonlinear operator. PEPs arise in many applications in science and engineering. Examples include acoustic structural coupled systems, fluid mechanics, multiple input multiple output systems in control theory, signal processing, and constrained least square problems, for all of which details and references are listed in [90]. Frazer et al.'s book on Elementary Matrices [26] includes a collection of PEPs' applications to aerodynamical systems, for example equations of motion of an aeroplane and small motions of systems subject to aerodynamical forces. The most commonly occurring PEPs are quadratic eigenvalue problems (QEPs)

$$(\lambda^2 A_2 + \lambda A_1 + A_0)x = 0, \quad (1.3)$$

where in mechanical systems A_2 is a mass matrix, A_1 a damping matrix, A_0 a stiffness matrix, λ a resonant frequency, and x a mode shape. Consider a simple standard mass-spring damper system with a free body of mass m , a spring of stiffness coefficient k , viscous damper of damping coefficient d and assume that no external force acts on the system (see Figure 1.1). The solutions of the differential equation

$$m \frac{d^2 v}{dt^2} + d \frac{dv}{dt} + kv = 0 \quad (1.4)$$

determine the dynamics of the system, where v is the displacement of the mass relative to a fixed point. Clearly, the solutions of (1.4) have the form

$$v(t) = x e^{\lambda t},$$

where x is a right eigenvector associated with λ as an eigenvalue of the quadratic scalar polynomial $\lambda^2 m + \lambda d + k$. Such a system is *overdamped* [11] if $d^2 > 4mk$. Note that, as m, d and k are positive, the overdamped system has real negative distinct eigenvalues. Consequently, all the solutions $v(t)$ approach zero as $t \rightarrow \infty$. Similarly, systems with multiple degrees of freedom give rise to (1.3) where A_2, A_1 are Hermitian positive definite and A_0 is Hermitian positive semidefinite, which extends the idea of overdamping. In fact, overdamped quadratics form a subclass of special Hermitian matrix polynomials that have interesting stability properties under Hermitian perturbations. Such Hermitian matrix polynomials and some closely related structured

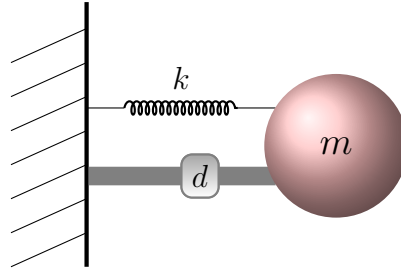


Figure 1.1: A mass-spring damped system with one degree of freedom.

matrix polynomials are the topics of concern in this work. Note that $P(\lambda)$ of degrees larger than two also arise in applications [44], [76], [84].

This chapter covers the background material, introduces the notation that is used throughout the thesis and outlines its contents.

1.1 Notations

- \mathbb{F} is the field of real or complex numbers.
- $\mathbb{F}^{n \times n}$ is the set of $n \times n$ matrices with coefficients in \mathbb{F} .
- A^T is the transpose of the matrix A .
- $A^* = (\overline{A})^T$ is the conjugate transpose of the matrix A .
- \star denotes the transpose T for real matrices and either the transpose T or the conjugate transpose $*$ for matrices with complex entries.
- $\ker A$ is the kernel of the matrix A .
- $A > 0$ denotes $A \in \mathbb{C}^{n \times n}$ that is positive definite, i.e.

$$A = A^*, \quad x^* A x > 0, \quad \forall x \in \mathbb{C}^n \setminus \{0\}.$$

- $A < 0$ denotes $A \in \mathbb{C}^{n \times n}$ that is negative definite, i.e. $-A > 0$.
- $A \in \mathbb{C}^{n \times n}$ is definite if $A > 0$ or $-A > 0$.

- $A_1, A_2 \in \mathbb{C}^{n \times n}$ have opposite parity if one is positive definite and the other is negative definite. Similarly, a sequence A_0, A_1, A_2, \dots of definite matrices has alternating parity if A_j and A_{j+1} have opposite parity for all j .
- $A \geq 0$ denotes $A \in \mathbb{C}^{n \times n}$ that is positive semi-definite, i.e.

$$A = A^*, \quad x^* A x \geq 0 \quad \forall x \in \mathbb{C}^n.$$

- For $A \in \mathbb{F}^{n \times n}$ with no eigenvalues on the closed negative real axis, $A^{1/2}$ denotes the principal square root of A which is the unique square root of A all of whose eigenvalues lie in the segment $\{z : -\pi/2 < \arg(z) < \pi/2\}$ [36, Thm. 1.29].
- A^{-1} denotes the inverse of $A \in \mathbb{F}^{n \times n}$.
- I_n is the identity $n \times n$ matrix.
- e_i is the vector that is equal to the i -th column of I_n .
- If $A_m \neq 0$ then the positive integer m is the degree of $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$, denoted by $\deg(P)$, where $A_j \in \mathbb{F}^{n \times n}$ for $j = 0:m$.
- The homogenous form of the matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ is

$$P(\alpha, \beta) = \sum_{j=0}^m \alpha^j \beta^{m-j} A_j,$$

where λ is identified with any pair $(\alpha, \beta) \neq (0, 0)$ for which $\lambda = \alpha/\beta$.

- $\text{rev}P(\lambda) := \sum_{j=0}^m \lambda^{m-j} A_j$ is the reversal polynomial of $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$.
- $P^*(\lambda) := \sum_{j=0}^m \lambda^j A_j^*$.
- $P^*(\alpha, \beta) := \sum_{j=0}^m \alpha^j \beta^{m-j} A_j^*$.
- P'_λ is the first derivative of P with respect to λ . The subscript λ is dropped when there is no ambiguity.
- $\mathcal{P}(\mathbb{F}^n)$ denotes the set of all matrix polynomials with coefficient matrices in $\mathbb{F}^{n \times n}$. When the polynomials are structured with structure \mathcal{S} , the corresponding space is denoted by $\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$.

- $\sigma(P)$ is the set of the eigenvalues of P .
- $\rho(P) = \max\{|\lambda| : \lambda \in \sigma(P)\}$ is the spectral radius of P .
- $\text{sig}(S)$ denotes the signature of S , that is, the difference between the number of positive eigenvalues and the number of negative eigenvalues of S .
- If $A = (A_{ij})$ is a block $k \times \ell$ matrix with $m \times n$ blocks A_{ij} , then the block transpose of A is the block $\ell \times k$ matrix A^B with $m \times n$ blocks defined by $(A^B)_{ij} = A_{ji}$.
- The Kronecker product of $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{q \times p}$ is [43, Section 4.2],

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{F}^{mp \times nq}, \quad \text{where } A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

- The direct sum of $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ is [42, Section 0.9.2],

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{F}^{(n+m) \times (n+m)}.$$

- $\bigoplus_{j=1}^m A_j = A_1 \oplus \cdots \oplus A_m$.
- $\mathbb{L}_1(P) = \{ L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda), v \in \mathbb{C}^m \}$, where $L(\lambda) = \lambda A - B$, $A, B \in \mathbb{C}^{n \times n}$ and $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T \in \mathbb{C}^m$.
- $\mathbb{H}(P) = \{ L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) : A = A^*, B = B^* \}$.
- $\mathbb{L}_2(P) = \{ L(\lambda) : (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda), w \in \mathbb{C}^m \}$, where $L(\lambda) = \lambda A - B$, $A, B \in \mathbb{C}^{n \times n}$ and $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T \in \mathbb{C}^m$.
- $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$.

1.2 Preliminaries

This section collects the basic definitions and concepts that we need in the sequel.

1.2.1 Some structured matrix polynomials

Table 1.1 defines the structures of interest in this work. We use \star -alternating to refer to either \star -even or \star -odd polynomials, where \star denotes the transpose T for real matrices and either the transpose T or the conjugate transpose $*$ for matrices with complex entries. For real-life applications associated with these structured matrix polynomials, see [65, Section 6.4] and [13, Section 3].

Table 1.1: Matrix polynomials $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$, $\det(A_m) \neq 0$ with structure \mathcal{S} .

Structure \mathcal{S}	Definition	Coefficients property
Hermitian	$P(\lambda) = P^*(\lambda)$	$A_j = A_j^*$
symmetric	$P(\lambda) = P^T(\lambda)$	$A_j = A_j^T$
\star -even	$P(\lambda) = P^\star(-\lambda)$	$A_j = (-1)^j A_j^\star$
\star -odd	$P(\lambda) = -P^\star(-\lambda)$	$A_j = (-1)^{j+1} A_j^\star$
\star -palindromic	$P(\lambda) = \lambda^m P^\star(\frac{1}{\lambda})$	$A_j = A_{m-j}^\star$
\star -antipalindromic	$P(\lambda) = -\lambda^m P^\star(\frac{1}{\lambda})$	$A_j = -A_{m-j}^\star$

In Chapter 2 we present the diverse subclasses of Hermitian λ -matrices with real eigenvalues, specifically those for which every eigenvalue λ_0 satisfies $x^* P'(\lambda_0)x \neq 0$ for all nonzero $x \in \ker P(\lambda_0)$, which we call *quasidefinite* matrix polynomials. We collect and unify existing results scattered in the literature about such polynomials, propose a new characterization of hyperbolicity and show how to avoid the requirement of having a nonsingular or a definite leading coefficient matrix of P . Moreover, we identify an important new class of diagonalizable matrix polynomials. For the sake of completeness, we provide a survey of the available algorithms for detecting some specific structures of Hermitian matrix polynomials in Figure 2.1.

The notion of a *Jordan triple* (X, J, Y) , introduced by Gohberg, Lancaster and Rodman, summarizes the spectral properties of $P(\lambda)$ (see [58, Section 14.2]). In particular if (X, J, Y) is a Jordan triple for $P(\lambda)$, then

$$\sum_{j=0}^m A_j X J^j = 0, \quad \sum_{j=0}^m J^j Y A_j = 0.$$

In Chapter 3 we investigate Jordan triples of the structured λ -matrices listed in Table 1.1, which have extra properties depending on the underlying structure. For

instance, GLR in [29] show that if (X, J, Y) is a Jordan triple for a Hermitian matrix polynomial then $Y = SX^*$ for some nonsingular $mn \times mn$ matrix S such that $S = S^*$, $JS = (JS)^*$. They call Jordan triples with such properties *self-adjoint*. We show that this type of result also holds for all the structures \mathcal{S} listed in Table 1.1. We introduce the notion of \mathcal{S} -structured standard triples and show that, apart from T -(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, if $P(\lambda)$ admits an \mathcal{S} -structured standard triple then $P(\lambda)$ has structure \mathcal{S} . Reciprocally, if $P(\lambda)$ has structure \mathcal{S} then its standard triples are \mathcal{S} -structured.

Chapter 4 highlights the use of the \mathcal{S} -structured standard triples theory in solving a structured inverse polynomial eigenvalue problem (SIPEP) of the following form: Construct a quadratic matrix polynomial $Q(\lambda)$ with structure \mathcal{S} from a given admissible list of elementary divisors. Additionally, Chapter 4 concerns generating quadratic and cubic matrix polynomials in different subclasses displayed in Figure 2.1 from some given spectral data by solving an appropriate inverse polynomial eigenvalue problem (IPEP). Furthermore, it points out that generating hyperbolic quadratic matrix polynomials that are real monic (i.e., with I_n as the leading coefficient matrix) and tridiagonal is always possible when the eigenvalues are given. As a result, we contribute to increase the collection of NLEVPs [13].

1.2.2 Elementary divisors

In order to define elementary divisors, we recall some fundamental notions from the theory of matrix polynomials [31, Chap. S1].

Theorem 1.2.1 [31, Thm. S1.1] *Every $n \times n$ matrix polynomial $P(\lambda)$ admits the representation*

$$P(\lambda) = E(\lambda)D(\lambda)F(\lambda), \quad (1.5)$$

where

$$D(\lambda) = \text{diag}(d_1(\lambda), \dots, d_r(\lambda), 0, \dots, 0), \quad (1.6)$$

is a diagonal λ -matrix with monic scalar polynomials $d_i(\lambda)$ such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$, $r = \max_{\lambda \in \mathbb{C}} \{\text{rank}P(\lambda)\}$ and the $n \times n$ matrix polynomials $E(\lambda)$, $F(\lambda)$ have constant nonzero determinants.

Representation (1.5) as well as the diagonal matrix $D(\lambda)$ from (1.6) is called the *Smith form* of $P(\lambda)$. The diagonal matrix $D(\lambda)$ is unique while the matrix polynomials $E(\lambda)$ and $F(\lambda)$ are not defined uniquely. The diagonal elements $d_1(\lambda), \dots, d_r(\lambda)$ in the Smith form are called the *invariant polynomials* of $P(\lambda)$. Each invariant polynomial can be written as

$$d_i(\lambda) = (\lambda - \lambda_{i1})^{\alpha_{i1}} \cdots (\lambda - \lambda_{ik_i})^{\alpha_{ik_i}}, \quad i = 1:r,$$

where the complex numbers $\lambda_{i1}, \dots, \lambda_{ik_i}$ are distinct and $\alpha_{i1}, \dots, \alpha_{ik_i}$ are positive integers. The factors $(\lambda - \lambda_{ij})^{\alpha_{ij}}$, $j = 1:k_i$, $i = 1:r$, are the *elementary divisors* of $P(\lambda)$.

1.2.3 Linearization

A standard way of treating the PEP to find *eigenpairs* $(\lambda, x) \in \mathbb{C} \cup \{\infty\} \times \mathbb{C}^n \setminus \{0\}$ that satisfy $P(\lambda)x = \left(\sum_{j=0}^m \lambda^j A_j\right)x = 0$, both theoretically and numerically, is to convert $P(\lambda)$ into an equivalent linear matrix pencil $L(\lambda) = \lambda A - B \in \mathbb{C}^{mn \times mn}$ using the process known as *linearization*. To be more specific, L is a linearization of P if it satisfies

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix} \quad (1.7)$$

for some matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants. It is clear that the eigenvalues of L and P coincide. As an example, if A_m is nonsingular, the pencil $\lambda I - C$ with

$$C = \begin{bmatrix} -A_m^{-1} & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & I_n & 0 \end{bmatrix} \quad (1.8)$$

is the *first companion form* linearization of $A_m^{-1}P(\lambda)$. In recent work [70] vector spaces of pencils have been studied, namely,

$$\mathbb{L}_1(P) = \{ L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda), v \in \mathbb{C}^m \}, \quad (1.9)$$

and $\mathbb{DL}(P) := \{L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) : A^{\mathcal{B}} = A, B^{\mathcal{B}} = B\}$, where $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T \in \mathbb{C}^m$ and $A^{\mathcal{B}}$ is the block transpose of A [37, Def. 2.1]. $\mathbb{L}_1(P)$ generalizes the first companion form (indeed for \mathcal{C} in (1.8), $\lambda I - \mathcal{C} \in \mathbb{L}_1(A_m^{-1}P)$ with $v = e_1$) and supplies a rich source of interesting linearizations.

The following result provides a condition that $L \in \mathbb{DL}(P)$ be a linearization of P .

Theorem 1.2.2 (eigenvalue exclusion theorem) [70, Thm. 6.7]

Suppose that $P(\lambda)$ is a regular matrix polynomial of degree m and $L(\lambda) \in \mathbb{DL}(P)$ with vector $v \in \mathbb{C}^m$. Then $L(\lambda)$ is a linearization for $P(\lambda)$ if and only if no root of the v -polynomial

$$\mathbf{p}(x; v) := v_1 x^{m-1} + v_2 x^{m-2} + \dots + v_{m-1} x + v_m \quad (1.10)$$

is an eigenvalue of $P(\lambda)$, where, by convention, $\mathbf{p}(x; v)$ has a root at ∞ whenever $v_1 = 0$.

It is worth mentioning that in (1.7), the identity block can be of a size less than $(m-1)n$, which introduces the *trimmed* linearizations that allow partial deflation of ∞ as an eigenvalue (i.e., obtaining as short as possible Jordan chains associated with ∞) [14]. For lower bounds of the size of the identity block appearing in the definition of linearization, see [23].

The first companion form linearization $\lambda I - \mathcal{C}$ of $A_m^{-1}P(\lambda)$ with \mathcal{C} in (1.8) is easily constructed from the coefficients of $P(\lambda)$ but it does not reflect its structure. For example, \mathcal{C} is not Hermitian when $P(\lambda)$ is Hermitian. This might lead to problems such as the loss of spectral symmetry when eigenvalues are computed in finite precision arithmetic. Linearizations that respect the structure are the subject of Section 2.2.4 and Section 3.2.1. A huge number of linearizations is available in the literature; see for example [5], [7], [8], [24], [37], [69], and [70].

Concerning solving GEPs and related computational issues such as the best algorithms regarding speed and accuracy for different structures of interest, we refer to [10], [32], [41], [47], [67], [74], [75] and [87]. Schreiber's thesis [85] provides a review of the available methods for solving NLEVPs, among which PEPs have experienced a surge of research activity in the last few years, motivated by its applications and the development of new theoretical ideas and algorithms.

Chapter 2

Classification of Quasidefinite Matrix Polynomials

2.1 Introduction

The spectral properties of Hermitian matrix polynomials with real eigenvalues have been extensively studied, through classes such as the definite or definitizable pencils, definite, hyperbolic, or quasihyperbolic matrix polynomials, and overdamped or gyroscopically stabilized quadratics [25], [38], [51], [59], [72], [90]. Indeed, we have found inconsistency in the definitions of some of these classes, in particular whether the leading coefficient have to be nonsingular or positive definite. This chapter gives a unified treatment of these and related classes that uses the *eigenvalue type* (or *sign characteristic*) as a common thread. Equivalent conditions are given for each class in a consistent format. We show that these classes form a hierarchy, all of which are contained in the new class of quasidefinite matrix polynomials. As well as collecting and unifying existing results, we make several new contributions. We propose a new characterization of hyperbolicity in terms of the distribution of the eigenvalue types on the real line. We show that definite matrix polynomials can be characterized in terms of the eigenvalue types—something that does not seem well known for definite pencils. We prove interlacing inequalities for the eigenvalues of the submatrices of hyperbolic matrix polynomials of any degree and point out that each hyperbolic quadratic matrix polynomial is strictly isospectral to a real monic tridiagonal quadratic matrix polynomial. By analyzing their effect on eigenvalue type, we show that homogeneous

rotations allow results for matrix polynomials with nonsingular or definite leading coefficient to be translated into results with no such requirement on the leading coefficient, which is important for treating definite and quasidefinite polynomials. We use definite linearizations of definite matrix polynomials [38] as another tool in obtaining the relations between the many subclasses of quasidefinite matrix polynomials which allow producing the diagram in Figure 2.1 that provides a diagrammatic summary of most of the results in this chapter. We also give a sufficient and necessary condition for a quasihyperbolic matrix polynomial to be strictly isospectral to a real diagonal quasihyperbolic matrix polynomial of the same degree, and show that this condition is always satisfied in the quadratic case and for any hyperbolic matrix polynomial, thereby identifying an important new class of diagonalizable matrix polynomials.

The chapter is organized as follows. We recall in Section 2.2 the notion of eigenvalue type and show how homogeneous rotations of matrix polynomials as well as linearizations in $\mathbb{H}(P)$ may affect the eigenvalue type. Section 2.3 investigates definite matrix polynomials and their subclasses, while Section 2.4 deals with quasidefinite polynomials and their subclasses. Finally, quasidefinite matrix polynomials that can be diagonalized by structure preserving congruences are identified in Section 2.5 followed by a summary and further discussion in Section 2.6.

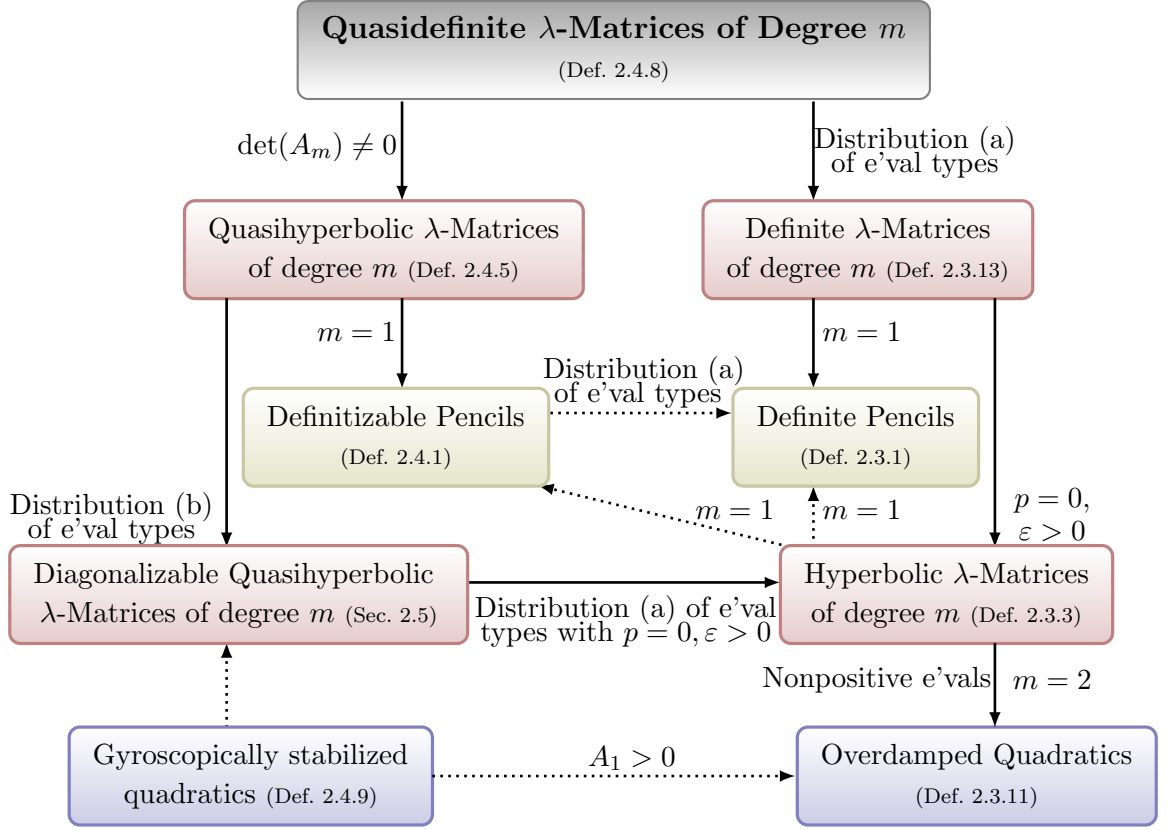
2.2 Preliminaries

An $n \times n$ Hermitian matrix polynomial P of degree m as in (1.1) has mn eigenvalues, which are all finite when A_m is nonsingular. Infinite eigenvalues occur when A_m is singular and zero eigenvalues are present when A_0 is singular. Because P is Hermitian, $\sigma(P)$ is symmetric with respect to the real axis.

Since we concentrate here on Hermitian matrix polynomials whose eigenvalues are all real and of definite type, we begin this section with a brief review of the eigenvalue types and sign characteristic (detailed discussions can be found in [30], [31]).

2.2.1 Eigenvalue type and sign characteristic

We start by defining the concept of eigenvalue type. Here $P'(\lambda)$ denotes the first derivative of P with respect to λ .



Distribution (a) : $\underbrace{\lambda_{mn} \leq \dots \leq \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\varepsilon \text{ type}}} < \dots < \underbrace{\lambda_{jn+p} \leq \dots \leq \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\varepsilon \text{ type, } 1 \leq j \leq m-1}} < \dots < \underbrace{\lambda_p \leq \dots \leq \lambda_1}_{\substack{p \text{ eigenvalues} \\ \text{of } -\varepsilon \text{ type}}}$,

with $0 \leq p < n$ and where “ $\alpha\varepsilon$ type” denotes positive type when $\alpha\varepsilon > 0$ and negative type otherwise, λ_{p+1} being of ε type.

Distribution (b) : there is a grouping of the eigenvalues into n subsets $\{\lambda_{i_j} : j = 1:m\}_{i=1}^n$ that can be ordered so that $\underbrace{\lambda_{i_m} < \dots < \lambda_{i_2} < \lambda_{i_1}}_{\text{alternating types}}, i=1:n$.

Figure 2.1: Quasidefinite $n \times n$ matrix polynomials $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ of degree m and their subclasses. A subclass **A** pointing to a subclass **B** with a solid line (dotted line) and property “C” means that the subclass **A** with the property “C” is exactly (is contained in) the subclass **B**.

Definition 2.2.1 (positive type/ negative type) Let $P(\lambda)$ be a Hermitian matrix polynomial. A finite real eigenvalue λ_0 of P is of positive type (negative type) if $x^*P'(\lambda_0)x > 0$ ($x^*P'(\lambda_0)x < 0$) for all nonzero $x \in \ker P(\lambda_0)$, respectively.

Thus for an eigenvalue λ_0 of positive type (negative type), the graph of the scalar polynomial $x^*P(\lambda)x$ for any nonzero $x \in \ker P(\lambda_0)$ crosses the x -axis at λ_0 with a positive slope (negative slope). Note that simple eigenvalues are either of positive type or of negative type since for nonzero $x \in \ker P(\lambda_0)$, $x^*P'(\lambda_0)x \neq 0$ [6, Thm. 3.2]. This does not necessarily hold for semisimple eigenvalues: for example the pencil

$$L(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}$$

has a semisimple¹ eigenvalue $\lambda_0 = a$ with corresponding eigenvectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and it is easily seen that $e_1^*L'(a)e_1 = 1$ and $e_2^*L'(a)e_2 = -1$. The eigenvalue $\lambda_0 = a$ is of mixed type. Note that $x^*L'(\lambda_0)x = 0$ for $x = e_1 + e_2$.

Definition 2.2.2 (definite type/mixed type) A finite real eigenvalue of a Hermitian matrix polynomial is of definite type if it is either of positive type or of negative type. It is of mixed type otherwise.

If λ_0 is a real eigenvalue of P of mixed type then there exist $x, y \in \ker P(\lambda_0)$ such that $x^*P'(\lambda_0)x > 0$ and $y^*P'(\lambda_0)y < 0$. But $x + \alpha y \in \ker P(\lambda_0)$, $\alpha \in \mathbb{C}$ and clearly $(x + \alpha y)^*P'(\lambda_0)(x + \alpha y) = 0$ for some nonzero α (see the previous example).

Lemma 2.2.3 A finite real eigenvalue λ_0 of a Hermitian matrix polynomial is of definite type if and only if $x^*P'(\lambda_0)x \neq 0$ for all nonzero $x \in \ker P(\lambda_0)$.

As shown in [12, Lem. 2.1], eigenvalues of definite type are necessarily semisimple. Indeed, if λ_0 is not semisimple then there is an eigenvector x and a generalized eigenvector y such that $P(\lambda_0)y + P'(\lambda_0)x = 0$. Multiplying on the left by x^* yields

$$x^*(P(\lambda_0)y + P'(\lambda_0)x) = x^*P'(\lambda_0)x = 0.$$

Hence λ_0 is of mixed type.

¹An eigenvalue of a matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ is semisimple if it appears only in 1×1 Jordan blocks in a Jordan form for P [31].

We will need a notion of eigenvalue type at infinity. To this end use the reversal of P denoted by $\text{rev}P$, which is obtained by reversing the order of the coefficient matrices of P :

$$\text{rev}P(\lambda) = \lambda^m P(1/\lambda) = \lambda^m A_0 + \lambda^{m-1} A_1 + \cdots + \lambda A_{m-1} + A_m. \quad (2.1)$$

Note that λ_0 is an eigenvalue of $P(\lambda)$ if and only if $1/\lambda_0$ is an eigenvalue of $\text{rev}P(\lambda)$ with 0 and ∞ regarded as reciprocals. Easy manipulations show that when $\lambda_0 \neq 0$ the equation $(m/\lambda_0)x^*P(\lambda_0)x = 0$ can be rewritten as

$$x^*P'(\lambda_0)x = -\lambda_0^{m-2}x^*(\text{rev}P)'(1/\lambda_0)x. \quad (2.2)$$

This suggests the following definition.

Definition 2.2.4 (type at ∞) *The type of $\lambda_0 = \infty$ as an eigenvalue of a Hermitian matrix polynomial P is given by the type of $1/\lambda_0 = 0$ as an eigenvalue of $-\text{rev}P$. In other words, $\lambda_0 = \infty$ is of positive type if $x^*A_{m-1}x < 0$ for every nonzero $x \in \ker \text{rev}P(0)$ and of negative type if $x^*A_{m-1}x > 0$ for every nonzero $x \in \ker \text{rev}P(0)$.*

The notion of eigenvalue type is connected with the more general notion of sign characteristic of a Hermitian matrix polynomial with nonsingular leading term [30], [31]. When all the eigenvalues λ_j of P are real, finite and of definite type, the sign characteristic, for a given ordering $\lambda_1, \dots, \lambda_{mn}$, is a set of signs $\{\varepsilon_j\}_{j=1}^{mn}$ with $\varepsilon_j = \text{sign}(x_j^*P'(\lambda_j)x_j)$, where x_j is an eigenvector corresponding to λ_j .

We will show in Sections 2.3 and 2.4 that the sign characteristic of definite pencils, overdamped and gyroscopically stabilized quadratics, and hyperbolic and definite polynomials has a particular distribution over the extended real line. Indeed the eigenvalues of these matrix polynomials belong to disjoint intervals, each interval containing eigenvalues of a single type. We say that an interval \mathcal{I} of \mathbb{R} is of positive (negative) type for a matrix polynomial P if every $\lambda \in \sigma(P) \cap \mathcal{I}$ is of positive (negative) type. The interval \mathcal{I} is of definite type if every $\lambda \in \sigma(P) \cap \mathcal{I}$ is of definite type. We also use the wording “ ε type” to denote positive type for $\varepsilon > 0$ and negative type for $\varepsilon < 0$.

2.2.2 Motivation

Eigenvalue problems $Ax = \lambda x$, with Hermitian A , have many desirable properties which lead to a variety of special algorithms. Here we consider what can be regarded as the closest analogues of this class of problems for the generalized eigenvalue problem $L(\lambda)x = 0$, with $L(\lambda) = \lambda A - B$, $A = A^*$, $B = B^*$, and for the polynomial eigenvalue problem $P(\lambda)x = 0$, with Hermitian $P(\lambda)$ of degree m . Namely, the classes of definite, definitizable, hyperbolic, quasihyperbolic, overdamped and gyroscopically stabilized eigenproblems [25], [38], [51], [59], [72], [90]. A property common to all these problems is that the eigenvalues are all real and of definite type.

The interest in matrix polynomials with real eigenvalues of definite type comes from systems of differential equations with constant coefficients of the form

$$\sum_{j=0}^m i^j A_j \frac{d^j u}{dt^j} = 0, \quad t \in \mathbb{R}, \quad (2.3)$$

where $i = \sqrt{-1}$, $A_j = A_j^* \in \mathbb{C}^{n \times n}$, $j = 0:m$, and A_m nonsingular. It is known [30, Thm. 13.1.1] that the general solution of (2.3) is given by

$$u(t) = [0 \quad \cdots \quad 0 \quad I_n] e^{-it\mathcal{C}} u_0, \quad (2.4)$$

where \mathcal{C} is as in (1.8) and $u_0 \in \mathbb{C}^{nm}$ is arbitrary. The solutions (2.4) are bounded on the half line $[0, \infty)$ if and only if \mathcal{C} , or equivalently $P(\lambda)$, has all its eigenvalues real and semisimple, and these solutions remain bounded under small perturbations of the matrix coefficients A_j of $P(\lambda)$ if and only if the eigenvalues of P are real and of definite type [30, Thm. 13.2.1].

The results presented in this chapter are useful in the solution of the inverse problem of constructing quasidefinite matrix polynomials and their subclasses from given spectral data, as will be shown in Chapter 4.

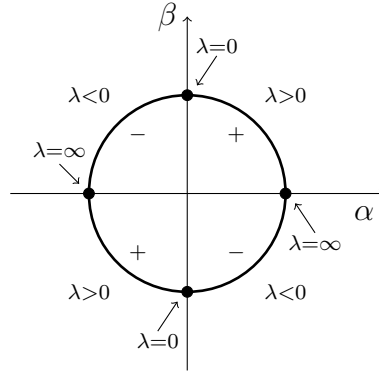


Figure 2.2: Correspondence between λ and (α, β) .

2.2.3 Homogeneous rotation

We will use the homogenous forms of the matrix polynomial $P(\lambda)$ in (1.1) and the pencil $L(\lambda) = \lambda A - B$, which are given by

$$P(\alpha, \beta) = \sum_{j=0}^m \alpha^j \beta^{m-j} A_j, \quad L(\alpha, \beta) = \alpha A - \beta B.$$

This form is particularly useful when A_m or A is singular or indefinite. An eigenvalue λ is identified with any pair $(\alpha, \beta) \neq (0, 0)$ for which $\lambda = \alpha/\beta$. Note that $P(0, \beta) = \beta^m A_0$ so that $\lambda = 0$ represented by $(0, \beta)$ is an eigenvalue of P if and only if A_0 is singular. Similarly, $\lambda = \infty$ represented by $(\alpha, 0)$ is an eigenvalue of P if and only if A_m is singular. Without loss of generality we can take $\alpha^2 + \beta^2 = 1$. We then have a direct correspondence between eigenvalues on the extended real line $\mathbb{R} \cup \{\infty\}$ and the unit circle (see Figure 2.2). Note the two copies of $\mathbb{R} \cup \{\infty\}$, represented by the upper semicircle and the lower semicircle.

The matrix polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ is obtained from $P(\alpha, \beta)$ by homogenous rotation if

$$G \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =: \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \quad c, s \in \mathbb{R}, \quad c^2 + s^2 = 1 \quad (2.5)$$

and

$$P(\alpha, \beta) = \sum_{j=0}^m (c\tilde{\alpha} - s\tilde{\beta})^j (s\tilde{\alpha} + c\tilde{\beta})^{m-j} A_j =: \sum_{j=0}^m \tilde{\alpha}^j \tilde{\beta}^{m-j} \tilde{A}_j =: \tilde{P}(\tilde{\alpha}, \tilde{\beta}). \quad (2.6)$$

Homogeneous rotations can be seen as an analogue of translations of λ in the non-homogeneous case: one common feature is that they both preserve the eigenvectors. Indeed P and \tilde{P} have the same eigenvectors but the corresponding eigenvalues are rotated. On using $P(\alpha, \beta) = \tilde{P}(\tilde{\alpha}, \tilde{\beta})$, the binomial expansion theorem leads to an expression for each \tilde{A}_j . In particular we find that

$$\begin{aligned}\tilde{A}_m &= P(c, s), \\ \tilde{A}_{m-1} &= \sum_{j=0}^m (-jc^{j-1}s^{m+1-j} + (m-j)c^{j+1}s^{m-j-1})A_j, \\ \tilde{A}_0 &= P(-s, c).\end{aligned}\tag{2.7}$$

We will use homogeneous rotations to transform a polynomial P with singular or indefinite leading coefficient A_m to a polynomial \tilde{P} with nonsingular or positive definite leading coefficient $\tilde{A}_m = P(c, s)$, which we can do provided that a pair (c, s) on the unit circle is known such that $\det(P(c, s)) \neq 0$ or $P(c, s) > 0$, respectively (see Example 2.2.5).

Example 2.2.5 The pencil

$$L(\lambda) = \lambda \text{diag}(1, 1, -1) - \text{diag}(2, 3, -5) =: \lambda A - B$$

has indefinite leading coefficient matrix A . Note that for $\mu = 4$, $L(\mu) = \text{diag}(2, 1, 1) > 0$. We homogeneously rotate L into \tilde{L} so that μ corresponds to ∞ . This is achieved by taking $c = \mu/\sqrt{\mu^2 + 1}$ and $s = 1/\sqrt{\mu^2 + 1}$ in (2.5). Then G rotates $L(\lambda)$ into $\tilde{L}(\tilde{\lambda}) =: \tilde{\lambda}\tilde{A} - \tilde{B}$, where $\tilde{A} = L(\mu)/\sqrt{\mu^2 + 1} > 0$. Note that L has eigenvalues 2 and 3 of positive type and eigenvalue 5 of negative type. These eigenvalues are rotated to 4.5, 13 and -21 , respectively, all of positive type since \tilde{A} is positive definite.

Example 2.2.5 shows that homogeneous rotation does not preserve the eigenvalue types, but as the next lemma shows it always preserves definite type. Now, we study the effects of homogeneous rotation and linearization on the eigenvalue types. To avoid ambiguity, P'_λ denotes the first derivative of P with respect to the variable λ .

Lemma 2.2.6 *Let \tilde{P} of degree m be obtained from P by homogeneous rotation (2.5). Let the real numbers $\lambda_0 = \frac{\alpha_0}{\beta_0}$ and $\tilde{\lambda}_0 = \frac{\tilde{\alpha}_0}{\tilde{\beta}_0}$ with $\begin{bmatrix} \tilde{\alpha}_0 \\ \tilde{\beta}_0 \end{bmatrix} = G \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}$ be eigenvalues of P*

and \tilde{P} , respectively, with corresponding eigenvector x .

(i) If λ_0 and $\tilde{\lambda}_0$ are both real and finite then $c - \lambda_0 s \neq 0$ and

$$x^* P'_\lambda(\lambda_0) x = (c - \lambda_0 s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0) x.$$

(ii) If λ_0 is real and finite and $\tilde{\lambda}_0 = \infty$ then $s \neq 0$ and

$$x^* P'_\lambda(\lambda_0) x = s^{2-m} x^* (-(\text{rev}\tilde{P})'_{\tilde{\lambda}}(0)) x.$$

(iii) If $\lambda_0 = \infty$ and $\tilde{\lambda}_0$ is real and finite then $s \neq 0$ and

$$x^* (-(\text{rev}P)'_\lambda(0)) x = (-s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0) x.$$

Proof. (i) Note that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = G \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ where $(\alpha, \beta) \neq (0, 0)$ and $(\tilde{\alpha}, \tilde{\beta}) \neq (0, 0)$ represent λ and $\tilde{\lambda}$ respectively, that is $\lambda = \alpha/\beta$ and $\tilde{\lambda} = \tilde{\alpha}/\tilde{\beta}$. When λ and $\tilde{\lambda}$ are finite, $\beta \neq 0$ and $\tilde{\beta} = \beta c - \alpha s \neq 0$ hence $c - \lambda s \neq 0$. It follows from (2.6) that $\beta^m P(\lambda) = \tilde{\beta}^m \tilde{P}(\tilde{\lambda}) = (\beta c - \alpha s)^m \tilde{P}(\tilde{\lambda})$ so that

$$P(\lambda) = (c - \lambda s)^m \tilde{P}(g(\lambda)), \quad g(\lambda) = \frac{\lambda c + s}{c - \lambda s} = \tilde{\lambda}. \quad (2.8)$$

Differentiating with respect to λ yields

$$P'_\lambda(\lambda) = -ms(c - \lambda s)^{m-1} \tilde{P}(g(\lambda)) + (c - \lambda s)^{m-2} \tilde{P}'_{\tilde{\lambda}}(g(\lambda)). \quad (2.9)$$

Multiplying (2.9) on the left by x^* and on the right by x , evaluating at λ_0 and using $\tilde{\lambda}_0 = g(\lambda_0)$ and $\tilde{P}(\tilde{\lambda}_0)x = 0$ yield the desired result.

(ii) When λ_0 is finite, $\beta_0 \neq 0$ and $\tilde{\lambda}_0$ infinite implies that $\tilde{\beta}_0 = \beta_0 c - \alpha_0 s = 0$, that is, $c = s\lambda_0$ with $s \neq 0$ since $c^2 + s^2 = 1$. Using (2.7) we obtain

$$s^{-m} \tilde{A}_{m-1} = \sum_{j=0}^m (-j\lambda_0^{j-1} + (m-j)\lambda_0^{j+1}) A_j. \quad (2.10)$$

Multiplying (2.10) on the left by x^* and on the right by x we find that

- if $\lambda_0 \neq 0$ then

$$\begin{aligned} s^{-m}x^*\tilde{A}_{m-1}x &= -x^*P'_\lambda(\lambda_0)x + \lambda_0^m x^* \left(A_{m-1} + \cdots + \frac{m-1}{\lambda_0^{m-2}}A_1 + \frac{m}{\lambda_0^{m-1}}A_0 \right) x \\ &= -x^*P'_\lambda(\lambda_0)x + \lambda_0^m x^*(\text{rev}P)'_\lambda(1/\lambda_0)x \\ &= -x^*P'_\lambda(\lambda_0)x - \lambda_0^2 x^*P'_\lambda(\lambda_0)x, \end{aligned}$$

where we used (2.2) for the last equality. The relation in (ii) follows on noting that $1 + \lambda_0^2 = s^{-2}$ and $x^*((\text{rev}\tilde{P})'_\lambda(0))x = x^*\tilde{A}_{m-1}x$.

- if $\lambda_0 = 0$ then $c = 0, s = \pm 1$ and

$$s^{-m}x^*\tilde{A}_{m-1}x = -x^*A_1x$$

which is the relation in (ii) since $s = \pm 1$.

(iii) Suppose that G rotates $\lambda_0 = \infty$ to a finite eigenvalue $\tilde{\lambda}_0$ then G^{-1} rotates $\tilde{\lambda}_0$ to $\lambda_0 = \infty$ and we can apply (ii) to obtain the desired result. ■

2.2.4 Hermitian linearizations

Higham et al. in [37] show that

$$\begin{aligned} \mathbb{H}(P) &:= \{ L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) : A^* = A, B^* = B \} \\ &= \left\{ \sum_{j=1}^m v_j L_j(\lambda), v \in \mathbb{R}^m \right\}, \end{aligned}$$

where $L_j(\lambda) = \lambda B_j - B_{j-1}$ and the matrices B_j are a direct sum of block Hankel matrices (see [37, (3.6)–(3.8)]). The pencil $\lambda B_j - B_{j-1} \in \mathbb{H}(P)$ with vector $v = e_j$, where e_j denotes the j th column of the identity matrix, is referred to as the j th basis pencil. When A_m is nonsingular,

$$\lambda B_m - B_{m-1} := \lambda \begin{bmatrix} & & & & A_m \\ & & & \ddots & A_{m-1} \\ & & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & A_2 \\ A_m & A_{m-1} & \cdots & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} & & & & A_m \\ & & & \ddots & A_{m-1} \\ & \ddots & \ddots & \ddots & \vdots \\ A_m & A_{m-1} & \cdots & A_2 & \\ & & & & -A_0 \end{bmatrix} \quad (2.11)$$

is always a linearization of P by the Eigenvalue Exclusion Theorem, Theorem 1.2.2. Note that $\mathbb{H}(P) \subset \mathbb{DL}(P)$.

Any linearization $L(\lambda) \in \mathbb{H}(P)$ with vector v has the property that x is a right eigenvector of P associated to the eigenvalue λ if and only if $\Lambda \otimes x$ (if λ is finite) or $e_1 \otimes x$ (if $\lambda = \infty$) is a right eigenvector for L with eigenvalue λ .

The following result relates the type of a real eigenvalue λ of P to its type as an eigenvalue of a linearization $L \in \mathbb{H}(P)$ and shows that linearizations in $\mathbb{H}(P)$ preserve definite type.

Lemma 2.2.7 *Let (λ_0, x) be an eigenpair of P with $\lambda_0 \in \mathbb{R}$ and let $L(\lambda) \in \mathbb{H}(P)$ with vector v be a linearization of P . Then,*

$$z^* L'(\lambda_0) z = \mathfrak{p}(\lambda_0; v) \cdot x^* P'(\lambda_0) x, \quad (2.12)$$

where $z = \Lambda_0 \otimes x$. Hence a real eigenvalue λ_0 of L is of definite type if and only if λ_0 as an eigenvalue of P is of definite type. Moreover, if $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ with A_m nonsingular then $\lambda B_m - B_{m-1}$ in (2.11) is a linearization of P that preserves the type of the real eigenvalues.

Proof. How to obtain (2.12) can be found in [39, Section 3]. Now if $L \in \mathbb{H}(P)$ is a linearization of P then by Theorem 1.2.2, $\mathfrak{p}(\lambda_0; v) \neq 0$. Hence $z^* L'(\lambda_0) z \neq 0$ if and only if $x^* P'(\lambda_0) x \neq 0$. The pencil in (2.11) is in $\mathbb{H}(P)$ with vector $v = e_m$ so $\mathfrak{p}(\lambda_0; e_m) = 1$. It is a linearization of P when A_m is nonsingular. ■

2.3 Definite matrix polynomials

The class of definite matrix polynomials (defined in Section 2.3.4) has recently been introduced and investigated by Higham, Mackey and Tisseur [38]. It includes definite pencils, hyperbolic matrix polynomials, and overdamped quadratics. We review these subclasses in the following way: for each subclass we provide a list of equivalent properties, named consistently according to

(P1) the distribution of the eigenvalue type on the real line,

(P2) certain definiteness properties,

(P3) the roots of the scalar equations $x^*P(\lambda)x = 0$ (or $x^*L(\lambda)x = 0$ for pencils).

Each subclass has extra equivalent properties listed, either because the property is commonly used to define the subclass or because the property is relevant to the rest of the chapter. We do not claim to provide a full list of characterizations.

2.3.1 Definite pencils

We start with definite pencils, whose occurrence is frequent in applications in science and engineering (see [21, Chap. 9], [46] for examples).

Definition 2.3.1 (definite pencils) *An $n \times n$ Hermitian pencil is definite if it satisfies any one (and hence all) of the equivalent properties of Theorem 2.3.2.*

Theorem 2.3.2 *For an $n \times n$ Hermitian pencil $L(\lambda) = \lambda A - B$ the following are equivalent:*

- (P1) $\sigma(L) \subset \mathbb{R} \cup \{\infty\}$ with all eigenvalues of definite type and where the eigenvalues of positive type are separated from the eigenvalues of negative type (see Figure 2.3).
- (P2) $L(\mu)$ is a definite matrix for some $\mu \in \mathbb{R} \cup \{\infty\}$, or equivalently $L(\alpha, \beta) > 0$ for some (α, β) on the unit circle.
- (P3) For every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^*L(\lambda)x = 0$ has exactly one zero in $\mathbb{R} \cup \{\infty\}$.
- (P4) $(x^*Ax, x^*Bx) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.
- (D) There exists a nonsingular $X \in \mathbb{C}^{n \times n}$ such that

$$X^*L(\lambda)X = \begin{bmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{bmatrix},$$

where $L_+(\lambda) = \lambda D_+ - J_+$ and $L_-(\lambda) = \lambda D_- - J_-$ are real diagonal pencils, such that $[\lambda_{\min}(L_+), \lambda_{\max}(L_+)] \cap [\lambda_{\min}(L_-), \lambda_{\max}(L_-)] = \emptyset$, D_+ has nonnegative entries, D_- has nonpositive entries and if $(D_+)_{ii} = 0$ then $(J_+)_{ii} > 0$ or if $(D_-)_{ii} = 0$ then $(J_-)_{ii} < 0$.

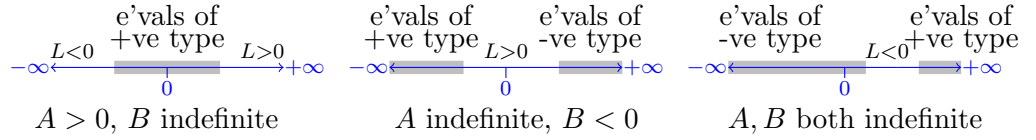


Figure 2.3: Distribution of the eigenvalue types for three types of definite pencils $L(\lambda) = \lambda A - B$. On the shaded intervals L is indefinite.

Proof. The proof of (P2) \Leftrightarrow (P4) can be found in [88, Thm. 6.1.18] and the equivalence (P3) \Leftrightarrow (P4) is immediate. We show that (P2) \Rightarrow (P1) \Rightarrow (D) \Rightarrow (P2).

(P2) \Rightarrow (P1): Suppose $L(c, s) > 0$ for some $c, s \in \mathbb{R}$, $c^2 + s^2 = 1$. If $s = 0$ then A is definite so that all the eigenvalues belong to one interval of either positive type if $A > 0$ or negative type if $A < 0$ since for all eigenpairs (λ, x) , $x^*L'(\lambda)x = x^*Ax$ is either positive or negative. Assume without loss of generality that $s > 0$ and homogeneously rotate L into $\tilde{L}(\tilde{\lambda}) = \tilde{\lambda}\tilde{A} - \tilde{B}$ as in Example 2.2.5 so that $\tilde{A} > 0$. Hence all the eigenvalues of \tilde{L} are real and of positive type. Let λ_j be an eigenvalue of L rotated to $\tilde{\lambda}_j$. By Lemma 2.2.6, their types are related by

$$x^*\tilde{L}'_{\tilde{\lambda}}(\tilde{\lambda}_j)x = \begin{cases} (c - \lambda_j s)x^*L'_{\lambda}(\lambda_j)x & \text{if } \lambda_j \text{ is finite,} \\ -sx^*(-(\text{rev}L)'_{\lambda}(0))x & \text{if } \lambda_j = \infty. \end{cases} \quad (2.13)$$

Note that

$$c - \lambda_j s = \det \begin{bmatrix} c & \lambda_j \\ s & 1 \end{bmatrix} \quad (2.14)$$

and the sign of these determinants is positive for any $\lambda_j = (\lambda_j, 1)$ that lies counterclockwise from (c, s) , and negative for any that lies clockwise from (c, s) ; see Figure 2.4. Hence it follows from (2.13) that eigenvalues of L lying clockwise from (c, s) (including $+\infty$) are of negative type and eigenvalues of L lying counterclockwise from (c, s) are of positive type. Also, there is a gap between the two types because c/s is not an eigenvalue of L .

(P1) \Rightarrow (D): Recall that a Hermitian pencil is diagonalizable by congruence if and only if its eigenvalues belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple [56]. Since eigenvalues of definite type are semisimple, there exists X nonsingular such that $X^*L(\lambda)X = \lambda D - J$, with D and J both real and diagonal. Their diagonal entries can be reordered so that $D = D_+ \oplus D_-$ and $J = J_+ \oplus J_-$, where the eigenvalues of $\lambda D_+ - J_+ = L_+(\lambda)$

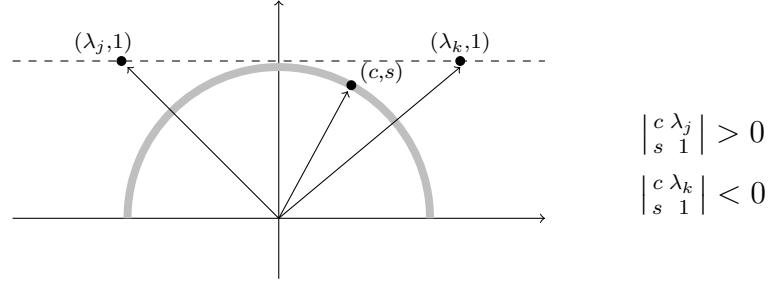


Figure 2.4: Pictorial representation of (2.14).

are of positive type and that of $\lambda D_- - J_- = L_-(\lambda)$ are of negative type. If A is singular then one of D_+ or D_- (but not both otherwise $\lambda = \infty$ would be of mixed type) must be singular. Hence D_+ and $-D_-$ have nonnegative entries. Each zero entry on D corresponds to an infinite eigenvalue. By Definition 2.2.4 when $(D_+)_{ii} = 0$ we must have $(J_+)_{ii} > 0$ for $\lambda = \infty$ to be of positive type and when $(D_-)_{ii} = 0$ then $(J_-)_{ii} < 0$. Finally because the eigenvalues of positive type are separated from the eigenvalues of negative type, the intersection between $[\lambda_{\min}(L_+), \lambda_{\max}(L_+)]$ and $[\lambda_{\min}(L_-), \lambda_{\max}(L_-)]$ must be empty.

(D) \Rightarrow (P2): It follows from (D) that

$$\begin{aligned} L_+(\mu) < 0 \text{ for } \mu < \lambda_{\min}(L_+), & \quad L_+(\mu) > 0 \text{ for } \mu > \lambda_{\max}(L_+) \text{ if } D_+ \text{ is nonsingular,} \\ L_-(\mu) > 0 \text{ for } \mu < \lambda_{\min}(L_-), & \quad L_-(\mu) < 0 \text{ for } \mu > \lambda_{\max}(L_-) \text{ if } D_- \text{ is nonsingular.} \end{aligned}$$

Hence

- (i) if L_- is void then $L(\mu) < 0$ for $\mu < \lambda_{\min}(L_+)$,
- (ii) if L_+ is void then $L(\mu) > 0$ for $\mu < \lambda_{\min}(L_-)$,
- (iii) if $\lambda_{\max}(L_+) < \lambda_{\min}(L_-)$ then $L(\mu) > 0$ for $\lambda_{\max}(L_+) < \mu < \lambda_{\min}(L_-)$,
- (iv) if $\lambda_{\max}(L_-) < \lambda_{\min}(L_+)$ then $L(\mu) < 0$ for $\lambda_{\max}(L_-) < \mu < \lambda_{\min}(L_+)$,

which completes the proof. ■

Characterizations (P2) and (P4) in Theorem 2.3.2 are commonly used as definitions of definite pencils. In (P2), $\mu = \infty$ is allowed and $L(\infty)$ definite means that A is definite. Note that (P4) is equivalent to saying that 0 is not in the field of values

of $A + iB$ or that the *Crawford number*

$$\gamma(A, B) = \min_{\substack{z \in \mathbb{C}^n \\ z^* z = 1}} \sqrt{(z^* A z)^2 + (z^* B z)^2} \quad (2.15)$$

is strictly positive. Finally we remark that in property (D) all the eigenvalues of L_+ are of positive type and those of L_- are of negative type.

Pencils $L(\lambda) = \lambda A - B$ with $A > 0$ have computational advantages: the eigenvalues can be computed by methods that exploit the definiteness of A [22]. When A and B are both indefinite, characterization (P1) offers an easy way to check definiteness, but it is computationally unattractive since it requires all the eigenpairs. As an alternative, the recently improved arc algorithm of Crawford and Moon [19], [34] efficiently detects whether $\lambda A - B$ is definite and determines μ such that $L(\mu) > 0$ at the cost of just a few Cholesky factorizations. The pencil can then be rotated to a pencil with positive definite leading term as in Example 2.2.5. An earlier level set algorithm of Higham, Tisseur, and Van Dooren [40, Alg. 2.3] is more expensive as it requires the solution of a complex QEP with a zero linear term for the eigenvalues and, possibly, eigenvectors.

2.3.2 Hyperbolic matrix polynomials

Hyperbolic matrix polynomials generalize definite pencils $\lambda A - B$ with $A > 0$.

Definition 2.3.3 (hyperbolic matrix polynomial) *A Hermitian matrix polynomial is hyperbolic if it satisfies any one (and hence all) of the equivalent properties of Theorem 2.3.4.*

Theorem 2.3.4 *For an $n \times n$ Hermitian matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ the following are equivalent:*

(P1) *All eigenvalues of P are real and finite, of definite type, and such that*

$$\underbrace{\lambda_{mn} \leq \dots \leq \lambda_{(m-1)n+1}}_{(-1)^{m-1} \text{ type}} < \dots < \underbrace{\lambda_{2n} \leq \dots \leq \lambda_{n+1}}_{\text{negative type}} < \underbrace{\lambda_n \leq \dots \leq \lambda_1}_{\text{positive type}}$$

where $\sigma(P) = \{\lambda_j\}_{j=1}^{mn}$ and “ $(-1)^{m-1}$ type” denotes positive type for odd m and negative type for even m .

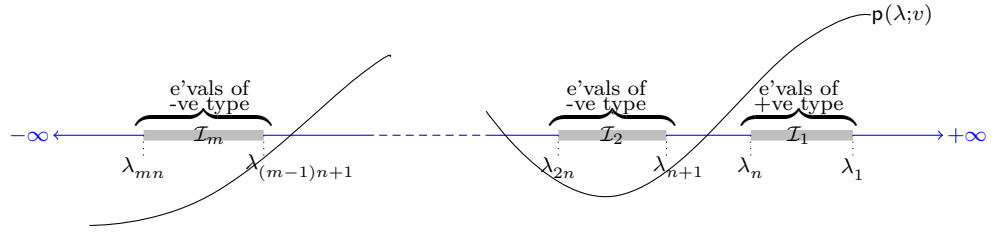


Figure 2.5: Distribution of eigenvalue types of $n \times n$ hyperbolic polynomials of even degree m .

(P2) *There exist $\mu_j \in \mathbb{R} \cup \{\infty\}$ such that*

$$(-1)^j P(\mu_j) > 0, \quad j = 0:m-1, \quad \infty = \mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}.$$

(P3) *$A_m > 0$ and for every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^*P(\lambda)x = 0$ has m distinct real and finite zeros.*

(L) *P has a definite linearization $L(\lambda) \in \mathbb{H}(P)$ with vector $v \in \mathbb{R}^m$, where $v_1 \neq 0$, such that $L(\infty) > 0$ if $v_1 > 0$ and $L(\infty) < 0$ if $v_1 < 0$.*

Proof. That (P2) \Leftrightarrow (P3) and (P2) \Rightarrow (P1) is due to Markus [72, Section 31]. We show that (P1) \Rightarrow (L) and (L) \Rightarrow (P3).

(P1) \Rightarrow (L): If $m = 1$ and $L \in \mathbb{H}(P)$ then $L(\lambda) = vP(\lambda)$ and L is a linearization if $v \neq 0$. By property (P1) of Theorem 2.3.2, $P(\lambda)$ is a definite pencil. Since all the eigenvalues are of positive type, property (D) of Theorem 2.3.2 implies that the leading coefficient of P is positive definite, i.e., $P(\infty) > 0$. Hence property (L) holds.

Now assume that $m > 1$. Let $v \in \mathbb{R}^m$ be such that the roots μ_j of the v -polynomial in (1.10) satisfy $\lambda_{jn} > \mu_j > \lambda_{j(n+1)}$, $j = 1:m-1$. Then by Theorem 1.2.2, $L(\lambda) \in \mathbb{H}(P)$ with vector v is a linearization of P . By construction, $\mathbf{p}(x; v) = v_1 \prod_{j=1}^{m-1} (x - \mu_j)$ with $v_1 \neq 0$ since all roots of $\mathbf{p}(x; v)$ are finite and $\text{sign}(\mathbf{p}(\lambda_k; v)) = (-1)^{j-1} \text{sign}(v_1)$ for $(j-1)n+1 \leq k \leq jn$, $j = 1:m-1$ (see Figure 2.5). By Lemma 2.2.7 we have that for each eigenpair (λ_k, x_k) of P , $z_k^* L'(\lambda_k) z_k = \mathbf{p}(\lambda_k; v) \cdot x_k^* P'(\lambda_k) x_k$, where $z_k = [\lambda_k^{m-1}, \lambda_k^{m-2}, \dots, 1]^T \otimes x_k$ is an eigenvector of L with eigenvalue λ_k . Hence all eigenvalues of L are of positive type when $v_1 > 0$ and of negative type when $v_1 < 0$. Now properties (P1) and (D) of Theorem 2.3.2 imply that L is a definite pencil with $L(\infty) > 0$ if $v_1 > 0$ and $L(\infty) < 0$ if $v_1 < 0$.

(L) \Rightarrow (P3): If P has a definite linearization then by [38, Thm. 4.1], there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has m distinct zeros in $\mathbb{R} \cup \{\infty\}$. By [38, Thm. 4.2], $L(\infty)$ definite implies $P(\infty)$ definite and [38, (4.7)] shows that $L(1, 0)$ is congruent to a block diagonal form whose $(1, 1)$ block is $v_1P(1, 0)$. Now if $v_1 > 0$ and $L(1, 0) > 0$ or $v_1 < 0$ and $L(1, 0) < 0$ then $A_m = P(1, 0) > 0$. Hence $x^*A_mx \neq 0$ and $x^*P(\lambda)x = 0$ has m distinct real finite zeros. ■

Property (P3) is used by Gohberg, Lancaster, and Rodman [30, Section 13.4] as the definition of hyperbolicity. Characterization (P1) is stated for the quadratic case and without proof in [48, Section 9] and [59].

It is shown in [80, Cor. 2] that for a Hermitian triple (A, B, C) ,

$$(x^*Ax, x^*Bx, x^*Cx) \neq 0$$

for all nonzero $x \in \mathbb{C}^n$ if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha A + \beta B + \gamma C > 0$. So in view of (P2) in Theorem 2.3.4, the natural extension to quadratics of property (P4) for definite pencils in Theorem 2.3.2, i.e., $(x^*A_2x, x^*A_1x, x^*A_0x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$, is a necessary condition for hyperbolicity (since $P(\mu) > 0$ for sufficiently large μ), but it is not sufficient: the quadratic $Q(\lambda) = \lambda^2 I_n + \lambda I_n + I_n$ is not hyperbolic since it has nonreal eigenvalues but $(x^*x, x^*x, x^*x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.

As for definite pencils, property (P1) is easy to check once all the eigenpairs of P are known. Guo and Lancaster's algorithm [35] tests for hyperbolicity by computing the eigenvalues of the QEP. More efficient approaches to check whether a quadratic matrix polynomial is hyperbolic or not include Guo, Higham and Tisseur's quadratically convergent matrix iteration based on cyclic reduction [33] or Guo, Higham and Tisseur's adaptation of the arc algorithm [34, Section 4.1]. For matrix polynomials of arbitrary degree, Niendorf and Voss [78] propose an algorithm that checks for hyperbolicity and which is based on a minmax and maxmin characterization of the eigenvalues.

Let $Q(\lambda)$ be a Hermitian quadratic matrix polynomial with $n \times n$ coefficient matrices and let $\widehat{Q}(\lambda)$ be the matrix polynomial obtained by deleting the last row

and column of $Q(\lambda)$. In what follows we characterize hyperbolic quadratic matrix polynomials according to the distribution of the eigenvalues of both Q and \widehat{Q} . Then we recall a recent result by Azizov et al. [9, Thm. 5.1] which implies that hyperbolic quadratics are always *tridiagonalizable*. We make use of this fact later in Chapter 4.

Theorem 2.3.5 *Let $Q(\lambda)$ be a Hermitian quadratic $n \times n$ matrix polynomial with positive definite leading coefficient and let $\mu_{2n-2} \leq \dots \leq \mu_1$, $\lambda_{2n} \leq \dots \leq \lambda_1$ be the eigenvalues of $\widehat{Q}(\lambda)$ and $Q(\lambda)$, respectively. Then $Q(\lambda)$ is hyperbolic if and only if $\widehat{Q}(\zeta) < 0$ for some $\zeta \in (\lambda_{n+1}, \lambda_n)$ and*

1. $\lambda_{j+1} \leq \mu_j \leq \lambda_j$, for $j = 1:n - 1$,
2. $\lambda_{j+2} \leq \mu_j \leq \lambda_{j+1}$, for $j = n:2n - 2$,
3. $\lambda_{n+1} < \lambda_n$.

Proof. (\Rightarrow): If Q is hyperbolic then [79, Lem. 7.1, Thm. 7.3] show that \widehat{Q} is hyperbolic with $\widehat{Q}(\zeta) < 0$ for some $\zeta \in (\lambda_{n+1}, \lambda_n)$. Veselić in [91] shows that the eigenvalues of Q and \widehat{Q} are distributed as in the statement of the theorem.

(\Leftarrow): Let $\widehat{Q}(\zeta)$ be negative definite for some $\zeta \in (\lambda_{n+1}, \lambda_n)$ and let the eigenvalues of Q and \widehat{Q} be distributed as in the statement of the theorem. The inclusion principle [42, Thm. 4.3.15] implies that the $n - 1$ smallest eigenvalues of the constant matrix $Q(\zeta)$ are negative. But $\det(Q(\zeta)) = c \prod_{j=1}^{2n} (\zeta - \lambda_j)$, where c is a positive constant since the leading coefficient matrix of Q is positive definite. Since $\zeta \in (\lambda_{n+1}, \lambda_n)$, the sign of $\det(Q(\zeta))$ is $(-1)^n$. Thus, the largest eigenvalue of $Q(\zeta)$ is negative and $Q(\zeta) < 0$ which is equivalent to saying that Q is hyperbolic, see Theorem 2.3.4. ■



Figure 2.6: The eigenvalues interlacing of a 3×3 quadratic matrix polynomial Q and its leading 2×2 submatrix \widehat{Q}

Towards extending Theorem 2.3.5 to higher degrees, let $P(\lambda)$ be a hyperbolic matrix polynomial of the form (1.1). Let $\lambda_1(x) > \lambda_2(x) > \dots > \lambda_m(x)$ be the roots of $x^*P(\lambda)x$ for some nonzero $x \in \mathbb{C}^n$ and recall the following result.

Theorem 2.3.6 [38] *A hyperbolic $P(\lambda)$ that is $n \times n$ of degree $m \geq 2$ with eigenvalues*

$$\lambda_{mn} \leq \cdots \leq \lambda_1,$$

has the properties displayed in the following diagram, where $\mathcal{I}_i = \{\lambda_i(x) : x \in \mathbb{C}^n, \|x\|_2 = 1\}$ and $P(\lambda)$ is indefinite on these intervals.

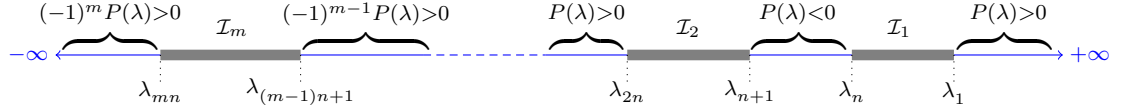


Figure 2.7: Definiteness diagram

A direct consequence is what follows.

Corollary 2.3.7 *If an $n \times n$ matrix polynomial $P(\lambda)$ of degree $m \geq 2$ is hyperbolic with eigenvalues $\lambda_{mn} \leq \cdots \leq \lambda_1$, then for each j , $1 \leq j \leq m$,*

$$\Pi_{(-1)^{j-1}}(P(\mu)) = jn - i, \quad i = 1 + (j-1)n : jn, \quad (2.16)$$

for all $\mu \in (\lambda_{i+1}, \lambda_i)$, $i = 1 : mn - 1$, where $\Pi_+(P(\mu))$ ($\Pi_-(P(\mu))$) is the number of positive (negative) eigenvalues of the matrix $P(\mu)$.

Next, we show that the result in [91] extends to hyperbolic matrix polynomials of any degree.

Lemma 2.3.8 *If an $n \times n$ matrix polynomial $P(\lambda)$ of degree $m \geq 2$ is hyperbolic and $\widehat{P}(\lambda)$ is the matrix polynomial obtained by deleting the last row and column of $P(\lambda)$, then $\widehat{P}(\lambda)$ is hyperbolic with eigenvalues $\mu_{m(n-1)} \leq \cdots \leq \mu_1$, for which*

$$\lambda_{i+j} \leq \mu_i \leq \lambda_{i+j-1}, \quad i = (j-1)(n-1) + 1 : j(n-1), \quad j = 1 : m, \quad (2.17)$$

where $\lambda_{mn} \leq \cdots \leq \lambda_1$, are the eigenvalues of $P(\lambda)$.

Proof. The proof is exactly the same as the proof of the special case of hyperbolic quadratics [91]. In fact, the first part of that proof can be avoided using the

definiteness diagram which proves (2.16). Now, let

$$P(\lambda) = \begin{bmatrix} \widehat{P}(\lambda) & P_{12}(\lambda) \\ P_{12}^*(\lambda) & p_{22}(\lambda) \end{bmatrix}, \quad (2.18)$$

be a partition of $P(\lambda)$, then Markus' characterization of hyperbolicity (i.e., (P2) \Leftrightarrow (P3) in Theorem 2.3.4) implies that $\widehat{P}(\lambda)$ is hyperbolic too. To prove (2.17) for $j = 1$, suppose on the contrary that $\mu_i > \lambda_i$ or $\mu_i < \lambda_{i+1}$ for some $i = 1:n-1$. Assume first that there is a $\lambda \notin \sigma(\widehat{P})$, with $\mu_i < \lambda < \lambda_{i+1}$. Now

$$P(\lambda) = W \begin{bmatrix} \widehat{P}(\lambda) & 0 \\ 0 & p_{22}(\lambda) - P_{12}^*(\lambda)\widehat{P}(\lambda)^{-1}P_{12}(\lambda) \end{bmatrix} W^*,$$

with

$$W = \begin{bmatrix} I_{n-1} & 0 \\ P_{12}^*(\lambda)\widehat{P}(\lambda)^{-1} & 1 \end{bmatrix}.$$

By Sylvester's law of inertia,

$$\Pi_+(\widehat{P}(\lambda)) \leq \Pi_+(P(\lambda)) \leq \Pi_+(\widehat{P}(\lambda)) + 1. \quad (2.19)$$

So, $\mu_i < \lambda < \lambda_{i+1}$ would imply

$$\Pi_+(\widehat{P}(\lambda)) \geq n - i, \quad \Pi_+(P(\lambda)) \leq n - i - 1,$$

which contradicts the first inequality in (2.19). Similarly, $\lambda_i < \lambda < \mu_i$ implies

$$\Pi_+(P(\lambda)) \geq n - i + 1, \quad \Pi_+(\widehat{P}(\lambda)) \leq n - i - 1,$$

which contradicts the second inequality in (2.19). This proves (2.17) for $j = 1$. The proof of (2.17) is completely analogous for $j > 1$. ■

Now, we are in position to extend Theorem 2.3.5 for degrees greater than two.

Theorem 2.3.9 *Let $P(\lambda)$ be a Hermitian $n \times n$ matrix polynomial of degree $m \geq 2$ with positive definite leading coefficient and let $\mu_{m(n-1)} \leq \dots \leq \mu_1$, $\lambda_{mn} \leq \dots \leq \lambda_1$ be the eigenvalues of $\widehat{P}(\lambda)$ and $P(\lambda)$, respectively. Then $P(\lambda)$ is hyperbolic if and*

only if there exist $\zeta_i \in \mathbb{R}$ such that $(-1)^i \widehat{P}(\zeta_i) > 0$, $\zeta_i \in (\lambda_{i+1}, \lambda_i)$, $i = 1:m-1$ and (2.17) holds.

Proof. (\Rightarrow): If P is hyperbolic then Markus characterization of hyperbolicity and partition (2.18) show that \widehat{P} is hyperbolic with $(-1)^i \widehat{P}(\zeta_i) > 0$, $\zeta_i \in (\lambda_{i+1}, \lambda_i)$, $i = 1:m-1$. The eigenvalues interlacing follows from Lemma 2.3.8.

(\Leftarrow): A direct extension of the proof of Theorem 2.3.5. To illustrate, consider the cubic case: Let $\widehat{P}(\zeta_1)$ ($\widehat{P}(\zeta_2)$) be negative definite (positive definite) for some $\zeta_1 \in (\lambda_{n+1}, \lambda_n)$ ($\zeta_2 \in (\lambda_{2n+1}, \lambda_{2n})$). Let the eigenvalues of P and \widehat{P} be distributed as in the statement of the theorem. The inclusion principle [42, Thm. 4.3.15] implies that the $n-1$ smallest (largest) eigenvalues of the constant matrix $P(\zeta_1)$ ($P(\zeta_2)$) are negative (positive). But $\det(P(\zeta_1)) = c \prod_{i=1}^{3n} (\zeta_1 - \lambda_i)$ ($\det(P(\zeta_2)) = c \prod_{i=1}^{3n} (\zeta_2 - \lambda_i)$), where c is a positive constant since the leading coefficient matrix of P is positive definite. As $\zeta_1 \in (\lambda_{n+1}, \lambda_n)$ ($\zeta_2 \in (\lambda_{2n+1}, \lambda_{2n})$), the sign of $\det(P(\zeta_1))$ ($\det(P(\zeta_2))$) is $(-1)^n$ ($(-1)^{2n}$). Thus, the largest (smallest) eigenvalue of $P(\zeta_1)$ ($P(\zeta_2)$) is negative (positive) and $P(\zeta_1) < 0$ ($P(\zeta_1) > 0$) which is equivalent to saying that P is hyperbolic. ■

The following result helps in solving a particular quadratic Hyperbolic IPEP.

Theorem 2.3.10 [9, Thm. 5.1] *Given two sets of real numbers $\{\lambda_j\}_{j=1}^{2n}$ and $\{\mu_j\}_{j=1}^{2n-2}$, $n \geq 2$, that satisfy (1) – (3) of Theorem 2.3.5, there exist $n \times n$ tridiagonal symmetric matrices A_1 and A_0 such that $\sigma(Q) = \{\lambda_j\}_{j=1}^{2n}$ where $Q(\lambda) = \lambda^2 I_n + \lambda A_1 + A_0$ and $\sigma(\widehat{Q}) = \{\mu_j\}_{j=1}^{2n-2}$. Moreover, Q is hyperbolic.*

Unfortunately, the proof of the previous result does not extend easily to higher degrees.

2.3.3 Overdamped quadratic matrix polynomials

Overdamped quadratics form a subclass of hyperbolic quadratics. They arise in overdamped systems in structural mechanics [25], [50, Section 7.6] and are defined as follows.

Definition 2.3.11 (overdamped quadratic) *A quadratic matrix polynomial is overdamped if it satisfies any one (and hence all) of the equivalent properties of Theorem 2.3.12.*

Theorem 2.3.12 *For a Hermitian quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ the following are equivalent:*

(P1) *All eigenvalues of Q are real, finite, nonpositive and distributed in two disjoint closed intervals, the left-most containing n eigenvalues of negative type and the right-most containing n eigenvalues of positive type.*

(P2) *$A_2 > 0$, $A_1 > 0$, $A_0 \geq 0$, and $Q(\mu) < 0$ for some $\mu < 0$.*

(P3) *$A_2 > 0$ and for every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^* Q(\lambda)x = 0$ has 2 distinct real and finite nonpositive zeros.*

(O) *$A_2 > 0$, $A_1 > 0$, $A_0 \geq 0$ and*

$$(x^* A_1 x)^2 > 4(x^* A_2 x)(x^* A_0 x) \quad \text{for all nonzero } x \in \mathbb{C}^n. \quad (2.20)$$

Proof. The equivalent characterizations of overdamping (P2), (P3) and (O) can be found in [33, Thm. 2.5]. Note that (P1) \Leftrightarrow (P3) follows from (P1) \Leftrightarrow (P3) in Theorem 2.3.4 to which is added the extra constraint that all the eigenvalues be nonpositive. ■

Note that property (O) is usually taken as the definition of overdamped quadratics. If equality is allowed in (2.20) for some nonzero x then the quadratic is said to be weakly overdamped. Its $2n$ eigenvalues are real and when ordered, $\lambda_n = \lambda_{n+1}$ with partial multiplicities² at most 2. Hence λ_n is either of mixed type or if it is not then the property that the eigenvalues are distributed in two disjoint intervals, each interval containing exactly n eigenvalues of one type, is lost.

Converting hyperbolic quadratics to overdamped ones

Hyperbolic quadratic matrix polynomials can be shifted to be overdamped. This shifting idea was discussed in [33], [40], [50]. Let $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ be Hermitian with $A_2 > 0$ and assume that Q is hyperbolic. Now with $\theta \geq \lambda_1 = \lambda_{\max}(Q)$, $\alpha = \lambda - \theta \leq 0$ for all $\lambda \in \sigma(Q)$ and

²The partial multiplicities of an eigenvalue of Q are the sizes of the Jordan blocks in which it appears in a Jordan form for Q [31].

$$\begin{aligned}
Q(\lambda) &= Q(\alpha + \theta) \\
&= \alpha^2 A_2 + \alpha(A_1 + 2\theta A_2) + (A_0 + \theta A_1 + \theta^2 A_2) \\
&= \alpha^2 A_{2_\theta} + \alpha A_{1_\theta} + A_{0_\theta} \\
&= \alpha^2 A_2 + \alpha Q'(\theta) + Q(\theta) \\
&=: Q_\theta(\alpha)
\end{aligned}$$

is overdamped. Thus, quadratic overdamped matrix polynomials are shifted hyperbolic ones and vice versa. In their effective algorithm of detecting and solving hyperbolic QEPs, Guo, Higham and Tisseur [33] use this shifting idea to test hyperbolicity via testing overdamping and they emphasize how large shifts of the spectrum, which can be obtained whenever one uses an upper bound to the modulus of the largest eigenvalue, can be. This is easily observed because we may have a case like $|\lambda_n| \gg |\lambda_1|$. Such a large shift could lead to slow convergence of their iterations.

2.3.4 Definiteness of matrix polynomials

Hyperbolic pencils $L(\lambda) = \lambda A - B$ are definite since their coefficient matrices are Hermitian with $A > 0$. However definite pairs are not necessarily hyperbolic since A and B can both be indefinite. By relaxing the requirement of definiteness of the leading coefficient, Higham, Mackey, and Tisseur [38] introduced a new class of Hermitian matrix polynomials, the definite matrix polynomials, that extends the notion of hyperbolicity and is consistent with the definition of definite pencils. Definite matrix polynomials that are not hyperbolic arise in acoustic fluid-structure interaction problems [38].

Definition 2.3.13 (definite matrix polynomial) *A Hermitian matrix polynomial is definite if it satisfies any one (and hence all) of the equivalent properties of Theorem 2.3.14.*

Theorem 2.3.14 *For an $n \times n$ Hermitian matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ of degree m the following are equivalent:*

(P1) All eigenvalues of P are real, of definite type and such that

$$\underbrace{\lambda_{mn} \leq \cdots \leq \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\varepsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn+p} \leq \cdots \leq \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\varepsilon \text{ type, } 1 \leq j \leq m-1}} < \cdots < \underbrace{\lambda_p \leq \cdots \leq \lambda_1}_{\substack{p \text{ eigenvalues} \\ \text{of } -\varepsilon \text{ type}}} \leq \infty,$$

with $0 \leq p < n$, where $\sigma(P) = \{\lambda_j\}_{j=1}^{mn}$, $\varepsilon = \pm 1$ and “ $\alpha\varepsilon$ type” denotes positive type when $\alpha\varepsilon > 0$ and negative type otherwise.

(P2) There exist $\mu_j \in \mathbb{R} \cup \{\infty\}$ with $\mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}$ ($\mu_0 = \infty$ being possible) such that $P(\mu_0), P(\mu_1), \dots, P(\mu_{m-1})$ are definite matrices with alternating parity.

(P3) There exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that the matrix $P(\mu)$ is definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has m distinct zeros in $\mathbb{R} \cup \{\infty\}$.

(L) P has a definite linearization $L(\lambda) \in \mathbb{H}(P)$.

Note that when P is a definite matrix polynomial with A_m definite one of P or $-P$ is hyperbolic and by Theorem 2.3.4, $p = 0$ in property (P1) of Theorem 2.3.14.

Proof. The characterizations (P2), (P3) and (L) and their equivalence can be found in [38, Thms. 2.6 and 4.1].

(P1) \Rightarrow (P3): Suppose neither P nor $-P$ is hyperbolic, i.e. $p \neq 0$. Let μ be such that $\lambda_{p+1} < \mu < \lambda_p$. Then homogeneously rotate P into \tilde{P} so that μ corresponds to $\tilde{\mu} = \infty$. The rotation moves the p largest eigenvalues of P to the $n - p$ smallest ones to form a single group of n eigenvalues (see Figures 2.2 and 2.4) which, by Lemma 2.2.6, are all of $(-1)^{m-1}\varepsilon$ type. The types of the remaining $m - 1$ groups of n eigenvalues remain unchanged. Hence by property (P1) of Theorem 2.3.4, \tilde{P} or $-\tilde{P}$ is hyperbolic. By property (P3) of Theorem 2.3.4, $x^*\tilde{P}(\tilde{\lambda})x = 0$ has real distinct roots for all nonzero $x \in \mathbb{C}^n$ and therefore $x^*P(\lambda)x = 0$ has distinct roots in $\mathbb{R} \cup \{\infty\}$ ($x^*A_mx = 0$ is possible). Also by [38, Lem. 2.1], $P(\mu)$ is definite.

(P3) \Rightarrow (P1): Homogeneously rotate P into \tilde{P} so that μ corresponds to $\tilde{\mu} = \infty$. Then $\tilde{P}(\infty) = \tilde{A}_m$ is definite, say $\tilde{A}_m > 0$. Now if $x^*P(\lambda)x = 0$ has distinct roots in $\mathbb{R} \cup \{\infty\}$ then $x^*\tilde{P}(\tilde{\lambda})x = 0$ has real distinct roots (and no infinite root since $\tilde{A}_m > 0$).

Thus $\tilde{P}(\tilde{\lambda})$ is hyperbolic. Then property (P1) for P follows from property (P1) of Theorem 2.3.4, Lemma 2.2.6 and Figure 2.4. ■

The following result follows from (P2) in Theorem 2.3.14 and from counting sign changes in eigenvalues of the matrix $P(\mu)$ (see [38, Thm. 2.4 and its proof]).

Theorem 2.3.15 *For a definite matrix polynomial $P(\lambda)$ of degree m with eigenvalues as in property (P1) of Theorem 2.3.14, let*

$$\mathcal{I}_j = (\lambda_{jn+p+1}, \lambda_{jn+p}), \quad j = 1: m - 1$$

and

$$\mathcal{I}_0 = \begin{cases} (\lambda_{p+1}, \lambda_p) & \text{if } p \neq 0, \\ (\lambda_1, +\infty) & \text{if } p = 0, \end{cases} \quad \mathcal{I}_m = \begin{cases} \emptyset & \text{if } p \neq 0, \\ (-\infty, \lambda_{mn}) & \text{if } p = 0. \end{cases}$$

Then $P(\mu)$ is definite for any $\mu \in \mathcal{I}_j$, $j = 0: m$ and if $\mu_j \in \mathcal{I}_j$, $\mu_{j+1} \in \mathcal{I}_{j+1}$ then $P(\mu_j)$ and $P(\mu_{j+1})$ have opposite parity.

Niendorf and Voss's algorithm [78] can be used to detect whether a Hermitian matrix polynomial is definite or not. For definite polynomials it also returns the μ_j of property (P2) so that a definite linearization can be built as shown in [38, Thm. 4.2].

2.4 Quasidefinite matrix polynomials

We have just seen that definite matrix polynomials are characterized by the fact that all their eigenvalues are real and of definite type and with a particular distribution of the eigenvalue types. We now consider a wider class of Hermitian matrix polynomials with real eigenvalues of definite type for which no assumption is made on the distribution of the eigenvalues types.

2.4.1 Definitizable pencils

Definite pencils form only a small subclass of Hermitian pencils with real and semisimple eigenvalues. We now consider a larger subclass of such pencils.

Definition 2.4.1 (definitizable pencils) *A Hermitian pencil $\lambda A - B$ is definitizable if it satisfies any one (and hence all) of the equivalent properties of Theorem 2.4.2.*

Theorem 2.4.2 *For an $n \times n$ Hermitian pencil $L(\lambda) = \lambda A - B$ the following are equivalent:*

- (P1) *All the eigenvalues of L are real, finite, and of definite type.*
- (P2) *A is nonsingular and there exists a real polynomial q such that $Aq(A^{-1}B) > 0$.*
- (P3) *A is nonsingular and the scalar equation $x^*L(\lambda)x = 0$ has one zero in \mathbb{R} for all eigenvectors $x \in \mathbb{C}^n$ of L .*
- (P4) *A is nonsingular and $(x^*Ax, x^*Bx, \dots, x^*A(A^{-1}B)^{n-1}x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.*
- (D) *There exists a nonsingular $X \in \mathbb{C}^{n \times n}$ such that*

$$X^*AX = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}, \quad X^*BX = \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix}, \quad (2.21)$$

where $J_+ \in \mathbb{R}^{k \times k}$ and $J_- \in \mathbb{R}^{(n-k) \times (n-k)}$ are diagonal and $\sigma(J_+) \cap \sigma(J_-) = \emptyset$.

Proof. The equivalence of the characterizations (P1), (P2) and (P4) can be found in [59, Thm. 1.3]. (D) \Rightarrow (P3) is immediate. We show that (P3) \Rightarrow (P1) and (P1) \Rightarrow (D).

(P3) \Rightarrow (P1) Suppose one eigenvalue is not real or is of mixed type then by [59, Lem. 2.2] there exists a corresponding eigenvector x such that $x^*Ax = 0$ and hence (P3) does not hold.

(P1) \Rightarrow (D) A is nonsingular since all eigenvalues are finite and $\lambda A - B$ is simultaneously diagonalizable by congruence since all the eigenvalues are real and semisimple. Hence there exists X nonsingular such that $X^*(\lambda A - B)X = \lambda D - J$ is real diagonal. Since D is nonsingular, we can choose X such that $D = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}$ and partition J as $\begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix}$ conformably with D . Then the property that all the eigenvalues are of definite type implies that $\sigma(J_+) \cap \sigma(J_-) = \emptyset$. ■

Lancaster and Ye [59] define definitizable pencils by property (P2) and add the adverb “*strongly*” to definitizable to emphasize the strict inequality in (P2). Note

that the real polynomial q in (P2) is not unique and that J_+ in (2.21) contains the eigenvalues of positive type and J_- those of negative type. Now if $L(\lambda) = \lambda A - B$ with nonsingular A is definite then by property (P2) of Theorem 2.3.2, there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that the matrix $L(\mu)$ is definite. Then $Aq(A^{-1}B) > 0$ for $q(x) = -\varepsilon$ if $\mu = \infty$ and $q(x) = \varepsilon(x - \mu)$ otherwise, where $\varepsilon = 1$ if $L(\mu) < 0$ and $\varepsilon = -1$ if $L(\mu) > 0$. Hence definite pencils with nonsingular leading coefficient matrix are definitizable.

Though not necessarily computationally efficient, property (P1) provides an easy way to check whether a Hermitian pencil $\lambda A - B$ is definite or definitizable or none of these.

As a by-product of the proof of [59, Thm. 1.3], a real polynomial q of minimal degree such that $Aq(A^{-1}B) > 0$ can easily be constructed once all the eigenvalues of a definitizable pencil $\lambda A - B$ are known together with their types, as shown in the next theorem. The knowledge of q can be useful when constructing conjugate gradient iterations for solving saddle point problems [63].

Theorem 2.4.3 *For an $n \times n$ definitizable pencil $\lambda A - B$ with eigenvalues $\lambda_n \leq \dots \leq \lambda_1$, let $k_j, j = 1: \ell - 1$ be the set of increasing integers such that*

$$\underbrace{\lambda_n \leq \dots \leq \lambda_{k_{\ell-1}+1}}_{\substack{n - k_{\ell-1} \text{ eigenvalues} \\ \text{of } (-1)^{\ell-1}\varepsilon \text{ type}}} < \dots < \underbrace{\lambda_{k_{j+1}} \leq \dots \leq \lambda_{k_j+1}}_{\substack{k_{j+1} - k_j \text{ eigenvalues} \\ \text{of } (-1)^j\varepsilon \text{ type}}} < \dots < \underbrace{\lambda_{k_1} \leq \dots \leq \lambda_1}_{\substack{k_1 \text{ eigenvalues} \\ \text{of } \varepsilon \text{ type}}} .$$

Then $p(x) = \varepsilon \prod_{j=1}^{\ell-1} (x - \mu_j)$ with $\lambda_{k_{j+1}} < \mu_j < \lambda_{k_j}$ is a real polynomial of minimal degree $\ell - 1$ such that $Ap(A^{-1}B) > 0$.

Example 2.4.4 The pencils

$$L_1(\lambda) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

have the same eigenvalues, i.e., $\sigma(L_1) = \sigma(L_2) = \{1, 2, 3\}$. Both pencils are definitizable because the eigenvalues are real, distinct and hence of definite type but only L_2 is definite since the eigenvalue 3, of negative type, is separated from the eigenvalues

1 and 2 of positive type as in the right-most depiction of Figure 2.3. Furthermore, by Theorem 2.3.15, $L_2(\mu) > 0$ for all $\mu \in (2, 3)$. Also, by Theorem 2.4.3 any polynomial of the form $p(x) = (x - \mu_1)(x - \mu_2)$ with $\mu_1 \in (2, 3)$ and $\mu_2 \in (1, 2)$ satisfies $A_1 p(A_1^{-1} B_1) > 0$, where $L_1(\lambda) =: \lambda A_1 - B_1$.

2.4.2 Quasihyperbolic matrix polynomials

The notion of definitizable pencils extends to matrix polynomials of degree higher than one.

Definition 2.4.5 (quasihyperbolic matrix polynomial) *A Hermitian matrix polynomial is quasihyperbolic if it satisfies either (and hence both) of the equivalent properties of Theorem 2.4.6.*

Theorem 2.4.6 *For an $n \times n$ Hermitian matrix polynomial $P(\lambda)$ the following are equivalent:*

(P1) *All the eigenvalues of P are real, finite and of definite type.*

(L) *Any linearization $L(\lambda) \in \mathbb{H}(P)$ is definitizable.*

Proof. We note that this result was proved in [53, Thm. 7.1] for a particular linearization in $\mathbb{H}(P)$.

A matrix polynomial has only finite eigenvalues if and only if its leading coefficient matrix is nonsingular. Let $L \in \mathbb{H}(P)$ be a linearization of P . Then P has only finite eigenvalues if and only if L has only finite eigenvalues or equivalently L has nonsingular leading matrix coefficient. Moreover $\sigma(P) \subseteq \mathbb{R}$ if and only if $\sigma(L) \subseteq \mathbb{R}$. By Lemma 2.2.7, the eigenvalues of P are of definite type if and only if those of L are of definite type. Hence by (P1) of Theorem 2.4.2, (P1) is equivalent to (L). ■

There is no obvious extension of properties (P2) and (P4) of Theorem 2.4.2 to quasihyperbolic matrix polynomials at the nonlinear level but by property (L) of Theorem 2.4.6 and property (P2) of Theorem 2.4.2, we have that P is quasihyperbolic if and only if there exists a real polynomial q such that $B_m q(B_m^{-1} B_{m-1}) > 0$, where $\lambda B_m - B_{m-1}$ is the $mn \times mn$ pencil (2.11). Property (P3) of Theorem 2.4.2 extends to quadratic matrix polynomials but not to higher degrees as shown by the next theorem and the following example.

Theorem 2.4.7 *Let P be a Hermitian matrix polynomial of degree m with nonsingular leading matrix coefficient. If the scalar equation $x^*P(\lambda)x = 0$ has m real distinct zeros for every eigenvector x of P then P is quasihyperbolic. The converse is also true when $m \leq 2$.*

Proof. Distinct real roots of $x^*P(\lambda)x = 0$ for all eigenvectors x of P implies that $\sigma(P) \subset \mathbb{R}$ and $x^*P'(\lambda_0)x \neq 0$ for each eigenvalue $\lambda_0 \in \sigma(P)$. Hence all eigenvalues are real, finite and of definite type, so P is quasihyperbolic by Theorem 2.4.6.

The converse is clearly true for linear P (see Theorem 2.4.2). Now for quadratic P , suppose the scalar quadratic $x^*P(\lambda)x = 0$, where x is an eigenvector, has a real double root. Then this double root is necessarily an eigenvalue of P , say λ_0 associated with x , and since it is a double root, $x^*P'(\lambda_0)x = 0$, so that λ_0 is of mixed type. Hence P is not quasihyperbolic. ■

Here is an example to show that the converse of Theorem 2.4.7 does not hold for polynomials of degree 3. The cubic polynomial

$$P(\lambda) = \lambda^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda^2 \begin{bmatrix} 9 & 0 \\ 0 & -6 \end{bmatrix} + \lambda \begin{bmatrix} -10 & 0 \\ 0 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix}$$

is quasihyperbolic. Any nonzero $x \in \mathbb{C}^2$ is an eigenvector associated with the eigenvalue $\lambda = 1$. It is easily checked that with $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the scalar polynomial $x^*P(\lambda)x$ has one real root and two complex conjugate roots.

We remark that a definite pencil $\lambda A - B$ is not necessarily definitizable because A can be singular, and a definite matrix polynomial is not necessarily quasihyperbolic because its leading term can be singular. We therefore extend the definition of quasihyperbolic matrix polynomials to allow singular leading term and call such polynomials quasidefinite.

Definition 2.4.8 *A Hermitian matrix polynomial $P(\lambda)$ is quasidefinite if $\sigma(P) \subset \mathbb{R} \cup \{\infty\}$ and each eigenvalue is of definite type.*

Take a quasidefinite matrix polynomial P . Since P is regular, there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is nonsingular. Homogeneously rotate P into \tilde{P} so that μ corresponds to ∞ and $\tilde{A}_m = \tilde{P}(\mu)$ is nonsingular. Then by Lemma 2.2.6 the eigenvalues of \tilde{P} are all of definite type and \tilde{P} is quasihyperbolic. Hence any quasidefinite matrix

polynomial is a “homogeneously rotated” quasihyperbolic one. Note that amongst the properties (P1), (P2) and (P3) we started with in Section 2.3, only a property of type (P1) remains for quasihyperbolic and quasidefinite matrix polynomials.

2.4.3 Gyroscopically stabilized systems

Quadratic matrix polynomials associated with gyroscopic systems have the form

$$G(\lambda) = \lambda^2 M + \lambda C + K,$$

where M, K are Hermitian and C is skew-Hermitian [90]. As $G(\lambda)^* = G(-\bar{\lambda})$, the spectrum of $G(\lambda)$ is symmetric with respect to the imaginary axis. The quadratic $G(\lambda)$ is not Hermitian but

$$Q(\lambda) = -G(-i\lambda) = \lambda^2 M + \lambda(iC) - K =: \lambda^2 A_2 + \lambda A_1 + A_0$$

is. The gyroscopic system is said to be weakly stable if all the eigenvalues of G lie on the imaginary axis or equivalently, if the eigenvalues of Q are all real. The following definition appears in [12]. For a Hermitian B we write $|B| = (B^2)^{1/2}$, where the square root is the principal square root [36, Prob. 1.27].

Definition 2.4.9 *A Hermitian $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ is gyroscopically stabilized if $A_2 > 0$, $A_0 > 0$ and A_1 is indefinite and nonsingular with $|A_1| > \mu A_2 + \mu^{-1} A_0$ for some positive μ .*

Barkwell, Lancaster, and Markus [12] prove that gyroscopically stabilized quadratics have real eigenvalues of definite type that belong to at most four distinct intervals of alternating types, with the number of eigenvalues in each interval depending on the number of positive eigenvalues p of A_1 (see Figure 2.8). Hence gyroscopically stabilized quadratics are quasihyperbolic. They are overdamped when $A_1 > 0$.

Note that the eigenvalue type distribution in Figure 2.8 is just a property of gyroscopically stabilized quadratics.

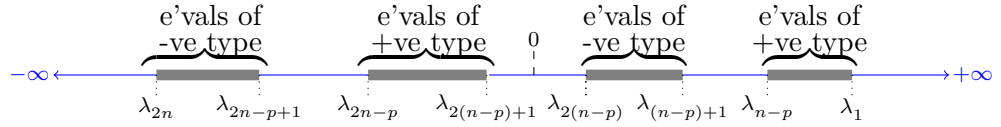


Figure 2.8: Eigenvalue distribution of gyroscopically stabilized quadratics $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where p is the number of positive eigenvalues of A_1 .

Example 2.4.10 Let $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ with

$$A_2 = I_3, \quad A_1 = \begin{bmatrix} 1.7122 & -0.0865 & 0 \\ -0.0865 & 1.3770 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0.7227 & -0.0865 & 0 \\ -0.0865 & 0.4164 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Note that $A_2, A_0 > 0$ and A_1 is indefinite and nonsingular. Since $\lambda^2 A_2 + \lambda |A_1| + A_0$ is not overdamped, $Q(\lambda)$ is not gyroscopically stabilized while its eigenvalues are distributed as in Figure 2.8 with $p = 2$.

2.5 Diagonalizable quasidefinite matrix polynomials

Recall that a Hermitian pencil is diagonalizable by congruence if and only if its eigenvalues belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple [56], a property shared by both definite and definitizable pencils. We now investigate how this property extends to (quasi)hyperbolic and definite matrix polynomials, thereby extending the simultaneous diagonalization property (D) in Theorems 2.3.2 and 2.4.2.

Matrix polynomials cannot in general be simultaneously diagonalized by a strict equivalence transformation. However, Lancaster and Zaballa [61] have recently characterized a class of quadratic matrix polynomials that can be diagonalized by applying strict equivalence transformations or congruences to a linearization of the quadratic while preserving the structure of the linearization. Along the same line, we identify amongst all quasidefinite matrix polynomials of arbitrary degree those that can be diagonalized by a congruence transformation applied to a Hermitian linearization L of the matrix polynomial P while maintaining the block structure of L . In particular, we show that all hyperbolic matrix polynomials can be transformed to a diagonal

form in this way.

Two matrix polynomials are *isospectral* if they have the same eigenvalues with the same partial multiplicities. If furthermore they share the same sign characteristic then these two matrix polynomials are *strictly isospectral* [54]. For example any linearization $L(\lambda) \in \mathbb{H}(P)$ is isospectral to P but not necessarily strictly isospectral as shown by Lemma 2.2.7.

In [28] the authors focus on the theory and practice of diagonalizing $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ by the application of simple λ -dependent transformations which they call *filters*. This can be done without using linearizations. They show that, in general, if systems

$$Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0, \quad \tilde{Q}(\lambda) = \lambda^2 \tilde{A}_2 + \lambda \tilde{A}_1 + \tilde{A}_0$$

are isospectral, then there exist pencils $F(\lambda)$ and $\tilde{F}(\lambda)$ such that

$$\tilde{F}(\lambda)Q(\lambda) = \tilde{Q}(\lambda)F(\lambda).$$

The interest, then, is in finding filters for which $\tilde{Q}(\lambda)$ is diagonal, which is useful in engineering and control theory as there are mechanisms admitting physical implementation of such filters.

Now suppose that two $n \times n$ quasihyperbolic matrix polynomials P_1 and P_2 of degree $m > 1$ are strictly isospectral. Let L_{1_m} and L_{2_m} be the m th basis pencils of $\mathbb{H}(P_1)$ and $\mathbb{H}(P_2)$, respectively (see (2.11)). By Lemma 2.2.7 L_{1_m} and L_{2_m} are strictly isospectral and by Theorem 2.4.6 they are also definitizable. It follows from property (D) of Theorem 2.4.2, that there exist nonsingular matrices $X_1, X_2 \in \mathbb{C}^{nm \times nm}$ such that

$$X_1 L_{1_m}(\lambda) X_1^* = \lambda \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} - \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix} = X_2 L_{2_m}(\lambda) X_2^*.$$

The matrix $X_2^{-1} X_1$ defines a structure preserving congruence [54], [61], [89] since it preserves the block structure of $L_{1_m}(\lambda)$: $(X_2^{-1} X_1) L_{1_m}(\lambda) (X_2^{-1} X_1)^* = L_{2_m}(\lambda)$. Thus if there exists a diagonal quasihyperbolic matrix polynomial $D(\lambda)$ of degree m strictly isospectral to $P(\lambda)$ then there exists a congruence transformation that preserves the block structure of the m th basis pencils of $\mathbb{H}(P)$ but also diagonalizes each block. If such structure preserving congruence exists then P is said to be diagonalizable.

In what follows $D(\lambda)$ has the form

$$D(\lambda) = \text{diag}(d_1(\lambda), \dots, d_n(\lambda)), \quad (2.22a)$$

$$d_i(\lambda) = \delta_i(\lambda - \lambda_{i_m}) \cdots (\lambda - \lambda_{i_2})(\lambda - \lambda_{i_1}), \quad \delta_i \neq 0, \quad i = 1:n \quad (2.22b)$$

with $\bigcup_{i=1}^n \{i_1, \dots, i_m\} = \{1, \dots, mn\}$ and $\delta_i = \pm 1$. The scalars $\lambda_{i_j}, j = 1:m, i = 1:n$ are the eigenvalues of $D(\lambda)$ and $P(\lambda)$ and are therefore real. We assume that they are ordered as

$$\lambda_{mn} \leq \cdots \leq \lambda_{j+1} \leq \lambda_j \leq \cdots \leq \lambda_1.$$

Theorem 2.5.1 *An $n \times n$ quasihyperbolic matrix polynomial of degree m with eigenvalues $\lambda_{mn} \leq \cdots \leq \lambda_1$ is strictly isospectral to an $n \times n$ diagonal matrix polynomial of degree m if and only if there is a grouping of its eigenvalues into n subsets of m distinct eigenvalues $\{\lambda_{i_j} : j = 1:m\}_{i=1}^n$ such that with the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1:n$, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type, where λ_{i_1} is of δ_i type.*

Proof. Let $P(\lambda)$ denote the $n \times n$ quasihyperbolic matrix polynomial of degree m .

(\Rightarrow) Suppose $P(\lambda)$ is strictly isospectral to an $n \times n$ diagonal matrix $D(\lambda)$ of degree m as in (2.22). The scalar polynomials $d_i(\lambda)$ must have distinct roots since otherwise $0 = d_i'(\lambda_{i_j}) = e_i^* D'(\lambda_{i_j}) e_i$ for some eigenvalue λ_{i_j} , which implies that λ_{i_j} is not of definite type, a contradiction. Here e_i , the i th column of I_n , is a corresponding eigenvector. Consider the grouping $\{\lambda_{i_j}, j = 1:m\}_{i=1}^n$ of the eigenvalues. With the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1:n$, it is easily seen that this grouping must be such that, in each group, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type and the sign of δ_i is determined by the type of λ_{i_1} .

(\Leftarrow) Let $\{\lambda_{i_j}, j = 1:m\}_{i=1}^n$ be a grouping of the eigenvalues of P into n subsets of m distinct eigenvalues, such that with the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1:n$, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type, where λ_{i_1} is of δ_i type. Let $D(\lambda)$ and $d_i(\lambda)$ be as in (2.22). Then by construction $D(\lambda)$ is quasihyperbolic and its eigenvalues and their types are the same as the eigenvalues of P and their types. Hence D is strictly isospectral to P . ■

Example 2.5.2 If there is a 2×2 cubic quasihyperbolic matrix polynomial $P(\lambda)$ with real eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$ and associated types $\{+, -, +, +, +, -\}$,

where $+$ means positive type and $-$ denotes negative type, then this polynomial is not diagonalizable by structure preserving congruence because there is no sorting of the eigenvalues into two groups of three distinct eigenvalues, which when ordered have alternating types. Note that if the sign characteristic had been $\{+, -, +, +, -, +\}$ then P would have been diagonalizable by structure preserving congruence.

Quasihyperbolic matrix polynomials of degree m strictly isospectral to diagonal matrix polynomials of degree m form a new subclass of Hermitian polynomials with eigenvalues all real and of definite type. Note that $n \times n$ Hermitian quasihyperbolic quadratics have n eigenvalues of positive type and n eigenvalues of negative type [29, Thm. 1.3]. So there is always a sorting of the eigenvalues into n groups of two distinct eigenvalues with opposite types. By Theorem 2.3.4, the eigenvalues of an $n \times n$ hyperbolic matrix polynomial of degree m are distributed in m disjoint intervals each of which contains n eigenvalues and, the types of the intervals alternate. So we can always sort the eigenvalues in n subsets of m distinct eigenvalues, which when ordered have alternating types. Hence by Theorem 2.5.1, quasihyperbolic quadratics and hyperbolic matrix polynomials of arbitrary degree, say m , are strictly isospectral to diagonal matrix polynomials of degree m . This result also applies to quasidefinite quadratic matrix polynomials and definite matrix polynomials.

Corollary 2.5.3 (a) *A quasidefinite quadratic matrix polynomial is always strictly isospectral to a quasidefinite diagonal quadratic matrix polynomial.*

(b) *A definite matrix polynomial is always strictly isospectral to a definite diagonal matrix polynomial.*

Proof. The proofs of (a) and (b) are similar so we just provide that for (b). A definite matrix polynomial P of degree m is a homogeneously rotated hyperbolic matrix polynomial \tilde{P} of degree m . From the comments preceding Corollary 2.5.3, \tilde{P} is strictly isospectral to a diagonal matrix polynomial \tilde{D} of degree m . Applying back the homogeneous rotation to \tilde{D} produces a diagonal matrix polynomial of degree m , which is strictly isospectral to P . ■

The following result is a direct consequence of [9, Thm. 5.1].

Corollary 2.5.4 *Every $n \times n$ hyperbolic quadratic matrix polynomial is strictly isospectral to a real $n \times n$ tridiagonal hyperbolic matrix polynomial.*

Applying [1, Thm. 6.1] for $2n$, if $Q(\lambda)$ is a given $n \times n$ quasihyperbolic quadratic matrix polynomial, we can construct an $n \times n$ tridiagonal symmetric quadratic matrix polynomial, say Q_{tri} , for which $\sigma(Q_{tri}) = \sigma(Q)$ by assigning $2n - 2$ distinct real numbers to be the eigenvalues of $\widehat{Q}_{tri}(\lambda)$, which is the leading $(n - 1) \times (n - 1)$ submatrix of $Q_{tri}(\lambda)$, such that $\sigma(Q_{tri}) \cap \sigma(\widehat{Q}_{tri})$ is empty.

2.6 Summary and further discussion

In this chapter we have described the diverse subclasses of Hermitian matrix polynomials with real eigenvalues of definite type scattered in the literature in a consistent way. We have pointed out their differences and similarities in particular with respect to the distribution of their eigenvalue types on the extended real line. This latter property allows to produce the diagram in Figure 2.1. Among these subclasses we have identified a new class of diagonalizable matrix polynomials. One of the main tools used in this study is the Hermitian m th basis pencil $\lambda B_m - B_{m-1}$ in (2.11) which preserves the eigenvalue type. For this particular linearization, as shown in Lemma 2.2.7, the v -polynomial equals 1. This may explain why this linearization is the most frequent one appearing in the literature concerning Hermitian matrix polynomials. Though, other Hermitian linearizations of quasidefinite matrix polynomials possess the property of preserving the eigenvalue type. For example, Fiedler-like Hermitian linearizations defined in [7] have this property. We realized this non-obvious fact during Fernando De Terán's visit to Manchester in February–May 2010. He generously shared a lemma showing that if y and x are, respectively, left and right eigenvectors of a given Fiedler-like linearization F of P associated with an eigenvalue λ , then

$$y^* F'(\lambda) x = w^* P'(\lambda) v,$$

where w and v are, respectively, left and right eigenvectors of P associated with λ . Thus F preserves the eigenvalue type.

Chapter 3

Standard Triples of Structured Matrix Polynomials

3.1 Introduction

Standard and Jordan triples for matrix polynomials were introduced and developed by Gohberg, Lancaster and Rodman (see for example [29], [30], [31]). Jordan triples extend to matrix polynomials of degree m

$$P(\lambda) = \sum_{j=0}^m \lambda^j A_j, \quad A_j \in \mathbb{F}^{n \times n}, \quad \det(A_m) \neq 0, \quad (3.1)$$

the notion of Jordan pair (X, J) for a single matrix $A \in \mathbb{C}^{n \times n}$, where $X \in \mathbb{C}^{n \times n}$ is nonsingular, J is a Jordan canonical form for A , and $A = XJX^{-1}$. The matrix X in a Jordan triple (X, J, Y) for $P(\lambda)$ is $n \times mn$, and as for the single matrix case, it contains the right eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix $J \in \mathbb{C}^{mn \times mn}$ is in Jordan canonical form, displaying the elementary divisors of $P(\lambda)$, and the matrix $Y \in \mathbb{C}^{mn \times n}$ plays the role of X^{-1} for a single matrix, i.e., the columns of Y^* determine left eigenvectors and generalized eigenvectors of $P(\lambda)$. A Jordan triple is a particular standard triple (U, T, V) in which the matrix T is in canonical form. Precise definitions of standard and Jordan triples are given in Section 3.2.2.

Our objective in this chapter is to study the standard and Jordan triples of structured matrix polynomials $P(\lambda)$ of the types listed in Table 1.1. Indeed the eigen-triples' pairing of such matrix polynomials given in Table 3.1 indicates that when all

Table 3.1: Eigentriple pairings (if any) for $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ with structure $\mathcal{S} \in \mathbb{S}$. When $\mathbb{F} = \mathbb{R}$, the eigenvalues of T -even and T -odd $P(\lambda)$ occur in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$ and those of T -palindromic and T -antipalindromic $P(\lambda)$ occur in quadruples $(\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda})$.

Herm	Symm	*-even, *-odd	T -even, T -odd	*-(anti)palindromic	T -(anti)palindromic
(λ, x, y)	(λ, x, \bar{x})	(λ, x, y)	(λ, x, y)	(λ, x, y)	(λ, x, y)
$(\bar{\lambda}, y, x)$		$(-\bar{\lambda}, y, x)$	$(-\lambda, \bar{y}, \bar{x})$	$(\frac{1}{\bar{\lambda}}, y, x)$	$(\frac{1}{\lambda}, \bar{y}, \bar{x})$

eigenvalues are semisimple, there exists a nonsingular matrix S that depends on the structure and connects Y with X^* . For example, consider a 2×2 Hermitian quadratic with Jordan form $J = \text{diag}(\lambda_1, \lambda_2, \lambda, \bar{\lambda})$ and corresponding right eigenvector matrix $X = [x_1 \ x_2 \ x \ y]$, where λ_1, λ_2 are real and λ is nonreal. The first column of Table 3.1 implies that $Y^* = [\bar{s}_1 x_1 \ \bar{s}_2 x_2 \ \bar{s}_4 y \ \bar{s}_3 x] = XS^*$ for some nonsingular matrix S of the form

$$S = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & s_4 \\ 0 & 0 & s_3 & 0 \end{bmatrix}.$$

The structure of standard and Jordan triples are well understood for Hermitian matrix polynomials [29], [30] and more recently real symmetric matrix polynomials [18], [60]. With no assumption on the sizes of the Jordan blocks, GLR [29] show that if (X, J, Y) is a Jordan triple for a Hermitian matrix polynomial then $Y = SX^*$ for some nonsingular $mn \times mn$ matrix S such that $S = S^*$ and $JS = (JS)^*$. We show in Section 3.3 that results of this type also hold for the structures in \mathbb{S} , where

$$\mathbb{S} = \{\text{Hermitian, symmetric, *-even, *-odd, } T\text{-even, } T\text{-odd,} \\ \text{*palindromic, *-antipalindromic, } T\text{-palindromic, } T\text{-antipalindromic}\}. \quad (3.2)$$

For $\mathcal{S} \in \mathbb{S}$, we introduce the notion of \mathcal{S} -structured standard triples. With the exception of T -(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we show that $P(\lambda)$ has structure \mathcal{S} if and only if $P(\lambda)$ admits an \mathcal{S} -structured standard triple, and that for any $P(\lambda)$ with structure \mathcal{S} , all standard triples for $P(\lambda)$ are \mathcal{S} -structured. Finally, we study in Section 3.4 the special case of \mathcal{S} -structured Jordan triples.

Two features of this work are (a) a distinction, when necessary, between triples and matrix polynomials defined over the complex (\mathbb{C}) or real (\mathbb{R}) fields, and (b) a unified presentation of the results, except in Section 3.5, where we provide explicit expressions for the S -matrix of \mathcal{S} -structured Jordan triples, which are structure-dependent.

3.2 Preliminaries

The set of all matrix polynomials with coefficient matrices in $\mathbb{F}^{n \times n}$ is denoted by $\mathcal{P}(\mathbb{F}^n)$. When the polynomials are structured with structure \mathcal{S} , the corresponding space is denoted by $\mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ (see Table 1.1). Throughout this chapter we assume that $P(\lambda)$ has a nonsingular leading coefficient matrix as in (3.1).

Recall that linearizations play a major role in the development of the theory of matrix polynomials and note that both $P(\lambda)$ and its linearization $L(\lambda)$ share the same Jordan form. The construction of linearizations that respect the structure of a given matrix polynomial has been an important area of research in the last decade (see for example [8], [24], [37], [69]). We give in Section 3.2.1 explicit expressions for the structured linearizations, which are important for the derivation of some results in Section 3.3 and all the results in Section 3.4

3.2.1 Structured linearizations

Let $P(\lambda) = \sum_{j=0}^m \lambda^j A_j \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ with $\det(A_m) \neq 0$ and $\mathcal{S} \in \mathbb{S}$. Recall that the vector space of pencils $\mathbb{L}_1(P)$ in (1.9) provides a rich source for structured linearizations. It is shown in [37], [65], [69] that for some $v \in \mathbb{F}^m$ satisfying the additional constraint

- (i) $v \in \mathbb{R}^m$ if $\mathcal{S} = \text{Hermitian}$,
- (ii) $v = \Sigma_m v$ if $\mathcal{S} \in \{T\text{-even}, T\text{-odd}\}$ or $v = \Sigma_m \bar{v}$ if $\mathcal{S} \in \{*\text{-even}, *\text{-odd}\}$,
- (iii) $v = F_m v$ if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ or $v = F_m \bar{v}$ if $\mathcal{S} \in \{*\text{-palindromic}, *\text{-antipalindromic}\}$,

where

$$\Sigma_m = \text{diag}((-1)^{m-1}, \dots, (-1)^0), \quad F_m = \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}_{m \times m},$$

there exists a unique pencil $\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} \in \mathbb{L}_1(P)$ with structure $\mathcal{S} \in \mathbb{S}$. This pencil is a linearization of $P(\lambda)$ if the roots of the \mathbf{v} -polynomial are not eigenvalues of P , see Theorem 1.2.2. The vector $v = e_m$ is an admissible vector for $\mathcal{S} \in \{\text{Hermitian, symmetric, } \star\text{-even, } \star\text{-odd}\}$ since $e_m \in \mathbb{R}^m$ and $\Sigma_m e_m = e_m$. Also, the roots of $\mathbf{p}(x; e_m)$ are all equal to ∞ and since $\det(A_m) \neq 0$ then $\infty \notin \sigma(P)$. Hence the structured pencils $\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} \in \mathbb{L}_1(P)$ with vector e_m are linearizations of P . They are given by

$$\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda\mathcal{A}(1) + \mathcal{B}(1) & \text{when } \mathcal{S} \in \{\text{Hermitian, symmetric}\}, \\ \lambda\mathcal{A}(-1) + \mathcal{B}(-1) & \text{when } \mathcal{S} \in \{\star\text{-even, } \star\text{-odd}\}, \end{cases} \quad (3.3)$$

where

$$\mathcal{A}(\varepsilon) = \begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1}A_m \\ \vdots & & \ddots & \varepsilon^{m-2}A_{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \varepsilon^0 A_m & \varepsilon^0 A_{m-1} & \cdots & \varepsilon^0 A_1 \end{bmatrix},$$

and

$$\mathcal{B}(\varepsilon) = - \begin{bmatrix} 0 & \cdots & 0 & \varepsilon^{m-1}A_m & 0 \\ \vdots & \ddots & \varepsilon^{m-2}A_m & \varepsilon^{m-2}A_{m-1} & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \varepsilon A_m & \varepsilon A_{m-1} & \cdots & \varepsilon A_2 & 0 \\ 0 & \cdots & \cdots & 0 & -A_0 \end{bmatrix}.$$

Note that for $\mathcal{S} \in \{\text{Hermitian, symmetric}\}$, $\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \lambda B_m - B_{m-1}$ that is given in (2.11) and for \star -(anti)palindromic $P(\lambda)$, we have $0 \notin \sigma(P)$ since $\infty \notin \sigma(P)$. When $m = 2k + 1$, $v = e_{k+1}$ satisfies $v = F_m v = F_m \bar{v}$ and $0, \infty$ are the only roots of the \mathbf{v} -polynomial. The corresponding \star -(anti)palindromic pencils in $\mathbb{L}_1(P)$, given by

$$\lambda\mathcal{A}_\mathcal{S} + \mathcal{B}_\mathcal{S} = \begin{cases} \lambda\mathcal{A}^{odd} + (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-palindromic with } m = 2k + 1, \\ \lambda\mathcal{A}^{odd} - (\mathcal{A}^{odd})^\star & \text{when } \mathcal{S} = \star\text{-antipalindromic with } m = 2k + 1, \end{cases} \quad (3.4)$$

where

$$\mathcal{A}^{odd} = \begin{bmatrix} \mathcal{A}_{11}^{odd} & \mathcal{A}_{12}^{odd} \\ \mathcal{A}_{21}^{odd} & \mathcal{A}_{22}^{odd} \end{bmatrix}, \quad (3.5)$$

with $\mathcal{A}_{11}^{odd} = (\mathcal{A}_{22}^{odd})^T = 0_{nk \times n(k+1)}$ and

$$\mathcal{A}_{12}^{odd} = \begin{bmatrix} -A_m^* & 0 & \cdots & 0 \\ -A_{m-1}^* & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -A_{k+2}^* & \cdots & -A_{m-1}^* & -A_m^* \end{bmatrix}, \quad \mathcal{A}_{21}^{odd} = \begin{bmatrix} A_m & A_{m-1} & \cdots & A_{k+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{m-1} \\ 0 & \cdots & 0 & A_m \end{bmatrix},$$

are linearizations of $P(\lambda)$.

For \star -(anti)palindromic polynomials of even degree $m = 2k$, the simplest nonzero vector v satisfying $F_m v = v$ when $\star = T$ or $F_m v = \bar{v}$ when $\star = *$ is of the form

$$v = [0 \quad \cdots \quad 0 \quad z \quad z^* \quad 0 \quad \cdots \quad 0]^T,$$

where z and z^* are in positions k and $k + 1$, respectively. The corresponding \star -(anti)palindromic pencil in $\mathbb{L}_1(P)$ is a linearization of $P(\lambda)$ if $-z/z^*$ is not an eigenvalue of P and is given by

$$\lambda \mathcal{A}_S + \mathcal{B}_S = \begin{cases} \lambda \mathcal{A}_-^{even}(z) + (\mathcal{A}_-^{even}(z))^* & \text{when } \mathcal{S} = \star\text{-palindromic with } m = 2k, \\ \lambda \mathcal{A}_-^{even}(z) - (\mathcal{A}_-^{even}(z))^* & \text{where } \mathcal{S} = \star\text{-antipalindromic with } m = 2k, \end{cases} \quad (3.6)$$

where

$$\mathcal{A}_-^{even}(z) = \begin{bmatrix} \mathcal{A}_{11}^{even}(z) & \mathcal{A}_{12}^{even}(z) \\ \mathcal{A}_{21}^{even}(z) & \mathcal{A}_{22}^{even}(z) \end{bmatrix},$$

with

$$\mathcal{A}_{11}^{even}(z) = z \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ A_m & A_{m-1} & \cdots & A_{k+1} \end{bmatrix}, \quad \mathcal{A}_{22}^{even}(z) = z \begin{bmatrix} A_{k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m-1} & 0 & \cdots & 0 \\ A_m & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathcal{A}_{12}^{even}(z) = - \begin{bmatrix} z^* A_0 & z A_0 & 0 & \cdots & \cdots & 0 \\ z^* A_1 & z^* A_0 + z A_1 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ z^* A_{k-2} & z^* A_{k-2} + z A_{k-1} & \cdots & z^* A_1 + z A_2 & z^* A_0 + z A_1 & z A_0 \\ -z A_k + z^* A_{k-1} & z^* A_{k-2} & \cdots & \cdots & z^* A_1 & z^* A_0 \end{bmatrix},$$

$$\mathcal{A}_{21}^{even}(z) = \begin{bmatrix} z^\star A_m & zA_m + z^\star A_{m-1} & zA_{m-1} + z^\star A_{m-2} & \dots & \dots & zA_{k+2} + z^\star A_{k+1} \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & z^\star A_m & zA_m + z^\star A_{m-1} \\ 0 & \dots & \dots & \dots & 0 & z^\star A_m \end{bmatrix}.$$

Note that when $\star = *$, we can always pick a $z \in \mathbb{F}$ such that $-z/z^\star \notin \sigma(P)$. But when $\star = T$, $-z/z^\star = -1$ so if $-1 \in \sigma(P)$, the corresponding \star -(anti)palindromic pencil in $\mathbb{L}_1(P)$ is not a linearization of $P(\lambda)$. In fact it is shown in [69] that some T -(anti)palindromic matrix polynomials of even degree do not have T -(anti)palindromic linearizations. Instead, we allow a linearization with ‘‘anti’’ structure: palindromic becomes antipalindromic and vice versa. For this, let $v = [0 \ \dots \ 0 \ 1 \ -1 \ 0 \ \dots \ 0]^T$ satisfying $v = -F_m v$. If $P(\lambda)$ is T -palindromic then there is a unique T -antipalindromic pencil in $\mathbb{L}_1(P)$ with vector v . Similarly if $P(\lambda)$ is T -antipalindromic then there is unique T -palindromic pencil in $\mathbb{L}_1(P)$ with vector v . Such pencils are linearizations of P if $1 \notin \sigma(P)$ and are given by

$$\lambda \mathcal{A}_S + \mathcal{B}_S = \begin{cases} \lambda \mathcal{A}_+^{even} - (\mathcal{A}_+^{even})^T & \text{when } \mathcal{S} = T\text{-palindromic with } m = 2k, \\ \lambda \mathcal{A}_+^{even} + (\mathcal{A}_+^{even})^T & \text{when } \mathcal{S} = T\text{-antipalindromic with } m = 2k, \end{cases} \quad (3.7)$$

where $\mathcal{A}_+^{even}(z)$ has a block structure similar to that of $\mathcal{A}_-^{even}(z)$ in (3.6) with z replaced by 1 and z^\star replaced by -1 . In particular, when $m = 2$,

$$\mathcal{A}_+^{even} = \begin{bmatrix} A_2 & A_1 + A_0 \\ -A_2 & A_2 \end{bmatrix}.$$

The next result, useful later, shows that the linearizations (3.3)–(3.7) share a property.

Lemma 3.2.1 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_S(\mathbb{F}^n)$ with nonsingular leading coefficient. If $\lambda \mathcal{A}_S + \mathcal{B}_S$ is a structured linearization of $P(\lambda)$ as in (3.3)–(3.7) then $\mathcal{C} = -\mathcal{A}_S^{-1} \mathcal{B}_S$, where \mathcal{C} is the companion form of $P(\lambda)$ given in (1.8).*

Proof. Some easy calculations show that $-\mathcal{A}_S \mathcal{C} = \mathcal{B}_S$. ■

Hence, with the exception of T -(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, the companion form of $P(\lambda)$ can be factorized as $\mathcal{C} = -\mathcal{A}_S^{-1} \mathcal{B}_S$, where $\lambda \mathcal{A}_S + \mathcal{B}_S = \mathcal{A}_S(\lambda I - \mathcal{C})$ is a structured linearization

of $P(\lambda)$.

3.2.2 Standard triples

Recall that (U, T) is an (m, n) -standard pair over \mathbb{F} if $T \in \mathbb{F}^{mn \times mn}$ and $U \in \mathbb{F}^{n \times mn}$ are such that

$$Q = Q(U, T) := \begin{bmatrix} UT^{m-1} \\ \vdots \\ UT \\ U \end{bmatrix} \quad (3.8)$$

is nonsingular [62, Def. 2.1]. The triple (U, T, V) forms an (m, n) -standard triple over \mathbb{F} if (U, T) is an (m, n) -standard pair over \mathbb{F} and $V \in \mathbb{F}^{mn \times n}$ is such that $UT^{m-1}V$ is nonsingular and, if $m \geq 2$,

$$UT^j V = 0, \quad j = 0: m-2, \quad (3.9)$$

or equivalently,

$$QV = e_1 \otimes N \quad (3.10)$$

for some nonsingular $n \times n$ matrix N , and where e_1 is the first column of the $m \times m$ identity matrix [62, Def. 2.3]. Note that the definitions of standard pairs and triples make no reference to matrix polynomials.

An (m, n) -standard pair (U, T) over \mathbb{F} is a *standard pair for* $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ if

$$A_m UT^m + A_{m-1} UT^{m-1} + \cdots + A_1 UT + A_0 U = 0, \quad (3.11)$$

[31, p. 46]. A standard triple (U, T, V) is a *standard triple for* $P(\lambda)$ if (3.11) holds and $A_m = (UT^{m-1}V)^{-1}$. Any $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient admits a standard triple. For example, it is easy to check that

$$(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}) \quad (3.12)$$

with \mathcal{C} as in (1.8) is a standard triple for $P(\lambda)$.

Let $U_i \in \mathbb{F}^{n \times mn}$, $T_i \in \mathbb{F}^{mn \times mn}$ and $V_i \in \mathbb{F}^{mn \times n}$, $i = 1, 2$. Then (U_1, T_1, V_1) is

similar to (U_2, T_2, V_2) if there exists a nonsingular $G \in \mathbb{F}^{mn \times mn}$ such that

$$U_2 = U_1 G, \quad T_2 = G^{-1} T_1 G, \quad V_2 = G^{-1} V_1.$$

Moreover if (U_1, T_1, V_1) is a standard triple so is (U_2, T_2, V_2) [30, Prop. 12.1.3]. Note that if (U, T, V) is a standard triple for $P(\lambda)$ then

$$(e_m^T \otimes I_n)Q = U, \quad Q^{-1}CQ = T, \quad e_1 \otimes A_m^{-1} = QV,$$

with Q as in (3.8). Hence any standard triple (U, T, V) for $P(\lambda)$ is similar to $(e_m^T \otimes I_n, C, e_1 \otimes A_m^{-1})$. Note that because T is similar to C , $\lambda I - T$ is a linearization of $P(\lambda)$ and $\sigma(P) = \sigma(T)$. The following result [30, Thm. 12.1.4] will be useful.

Lemma 3.2.2 *Let $U \in \mathbb{F}^{n \times mn}$, $T \in \mathbb{F}^{mn \times mn}$, $V \in \mathbb{F}^{mn \times n}$ and let $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ be of degree m with nonsingular leading coefficient. Then (U, T, V) is a standard triple for $P(\lambda)$ if and only if $P(\lambda)^{-1} = U(\lambda I - T)^{-1}V$ for $\lambda \in \mathbb{C} \setminus \sigma(P)$.*

A Jordan triple (X, J, Y) over \mathbb{F} for $P(\lambda)$ is a standard triple for $P(\lambda)$ for which the matrix J is in Jordan form or real Jordan form if $\mathbb{F} = \mathbb{R}$. By (3.11) and [31, Prop. 2.1], we have that $\sum_{j=0}^m A_j X J^j = 0$ and $\sum_{j=0}^m J^j Y A_j = 0$. The columns of X and Y^* determine right and left eigenvectors and generalized eigenvectors of $P(\lambda)$. The matrix J is the Jordan form of the companion form C of $P(\lambda)$. In a similar way that a Jordan pair (X, J) with X nonsingular generates $A - \lambda I$ via $A = X J X^{-1}$, $I = X X^{-1}$, an (m, n) -Jordan triple (X, J, Y) over \mathbb{F} generates a matrix polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j \in \mathcal{P}(\mathbb{F}^n)$ uniquely, where

$$A_m = (X J^{m-1} Y)^{-1}, \tag{3.13}$$

and for $1 \leq j \leq m$,

$$A_{m-j} = -A_m \sum_{i=m-j+1}^m X J^{i+j-1} Y A_i = - \sum_{i=m-j+1}^m A_i X J^{i+j-1} Y A_m. \tag{3.14}$$

These formulae follow directly from [58, Thm. 14.7.1] on using [58, Thm. 14.2.5], and a similar argument to that in the proof of [49, Thm. 1] can be used to prove the uniqueness of $P(\lambda)$ (see also [62, Thm. 2.4]).

3.3 \mathcal{S} -structured standard triples

We now consider standard triples in the context of structured matrix polynomials. We start by listing two assumptions used in our analysis. Let $\mathcal{S} \in \mathbb{S}$, $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ have degree m with nonsingular leading coefficient and let $T \in \mathbb{F}^{mn \times mn}$.

Assumption (a): if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $P(\lambda)$ has degree $m = 2k$ then either $-1 \notin \sigma(P)$ or $1 \notin \sigma(P)$.

Assumption (b): if $\mathcal{S} \in \{T\text{-palindromic}, T\text{-antipalindromic}\}$ and $m = 2k$ then either $-1 \notin \sigma(T)$ or $1 \notin \sigma(T)$.

These two assumptions are equivalent when $\lambda I - T$ is a linearization of $P(\lambda)$. For some T satisfying assumption (b) we define $u_{\mathcal{S}}(T), t_{\mathcal{S}}(T), v_{\mathcal{S}}(T)$ as in Table 3.2. Note that assumption (b) ensures the existence of $\alpha \in \mathbb{F}$ such that $\alpha^{\star}\alpha = 1$ and $-\alpha \notin \sigma(T)$. Also, for \star -(anti)palindromic structures, the eigenvalues of T come in pairs $(\lambda, \lambda^{-\star})$. Hence $0 \notin \sigma(T)$ since $\infty \notin \sigma(T)$ and $T^{-\star}$ is well defined.

Before stating our main result in Theorem 3.3.5, we provide a few lemmas and introduce the notion of \mathcal{S} -structured standard triple. The first lemma of this section extends to all structures in \mathbb{S} a result in [58, Cor. 14.2.1] for Hermitian structure.

Table 3.2: Definition of $u_{\mathcal{S}}(T), t_{\mathcal{S}}(T), v_{\mathcal{S}}(T)$ for some $T \in \mathbb{F}^{mn \times mn}$ satisfying assumption (b), where α is some scalar in \mathbb{F} such that $\alpha^{\star}\alpha = 1$ and $-\alpha \notin \sigma(T)$.

Structure \mathcal{S}	$u_{\mathcal{S}}(T)$	$t_{\mathcal{S}}(T)$	$v_{\mathcal{S}}(T)$
Hermitian/symmetric	I	T^{\star}	I
\star -even	$-I$	$-T^{\star}$	I
\star -odd	I	$-T^{\star}$	I
\star -palindromic, $m = 2k + 1$	$-T^{\star(k-1)}$	$T^{-\star}$	$T^{\star k}$
\star -palindromic, $m = 2k$	$-T^{\star(k-1)}(I + \alpha T^{\star})^{-1}$	$T^{-\star}$	$(I + \alpha T^{\star})T^{\star(k-1)}$
\star -antipalindromic, $m = 2k + 1$	$T^{\star(k-1)}$	$T^{-\star}$	$T^{\star k}$
\star -antipalindromic $m = 2k$	$T^{\star(k-1)}(I + \alpha T^{\star})^{-1}$	$T^{-\star}$	$(I + \alpha T^{\star})T^{\star(k-1)}$

Lemma 3.3.1 *Let (U, T, V) be an (m, n) -standard triple for $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient and let $\mathcal{S} \in \mathbb{S}$. Assume that T satisfies assumption (b). Then $P(\lambda)$ has structure \mathcal{S} if and only if $(V^{\star}u_{\mathcal{S}}(T), t_{\mathcal{S}}(T), v_{\mathcal{S}}(T)U^{\star})$ is a standard triple for $P(\lambda)$.*

Proof. The proof for $\mathcal{S} \in \{\text{Hermitian, symmetric, } \star\text{-even, } \star\text{-odd}\}$ is easy to obtain using the resolvent form for $P(\lambda)$ given in Lemma 3.2.2 and the definition of the structures in Table 1.1. See also [58, Cor. 14.2.1] for the Hermitian structure.

Now suppose that $P(\lambda)$ is \star -palindromic. Since any standard triple for $P(\lambda)$ is similar to $(e_m^T \otimes I_n, \mathcal{C}, e_1 \otimes A_m^{-1}) =: (\tilde{U}, \mathcal{C}, \tilde{V})$, it suffices to show that this standard triple is similar to $(\tilde{V}^\star u_{\mathcal{S}}(\mathcal{C}), \mathcal{C}^{-\star}, v_{\mathcal{S}}(\mathcal{C}) \tilde{U}^\star)$. We need to consider three cases:

- (i) $m = 2k + 1$. The pencil $\lambda \mathcal{A}^{odd} + (\mathcal{A}^{odd})^\star$ with \mathcal{A}^{odd} as in (3.5) is a linearization of $P(\lambda)$. By Lemma 3.2.1, $\mathcal{C} = -(\mathcal{A}^{odd})^{-1}(\mathcal{A}^{odd})^\star$. So if we let $G^{-1} = \mathcal{A}^{odd}$ then

$$\tilde{V}^\star u_{\mathcal{S}}(\mathcal{C}) = -(e_1 \otimes A_m^{-1})^\star (\mathcal{C}^\star)^{k-1} = \tilde{U}G, \quad G^{-1}\mathcal{C}G = \mathcal{C}^{-\star} = t_{\mathcal{S}}(\mathcal{C}),$$

and

$$G^{-1}\tilde{V} = G^{-1}(e_1 \otimes A_m^{-1}) = e_{2k} \otimes I = (\mathcal{C}^\star)^k (e_m \otimes I) = v_{\mathcal{S}}(\mathcal{C}) \tilde{U}^\star.$$

- (ii) $m = 2k$, $\star = T$ and $-1 \in \sigma(T)$. From assumption (b) it follows that $1 \notin \sigma(T)$ so we can take $\alpha = -1$ in the definition of $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$. The pencil $\lambda \mathcal{A}_+^{even} - (\mathcal{A}_+^{even})^\star$ with \mathcal{A}_+^{even} as in (3.7) is a linearization of $P(\lambda)$. By Lemma 3.2.1, $\mathcal{C} = -(\mathcal{A}_+^{even})^{-1}(\mathcal{A}_+^{even})^\star$. If we let $G^{-1} = \mathcal{A}_+^{even}$ then as in (i), $G^{-1}\mathcal{C}G = t_{\mathcal{S}}(\mathcal{C})$. Also,

$$\begin{aligned} v_{\mathcal{S}}(\mathcal{C}) \tilde{U}^T &= (I - \mathcal{C}^T) \mathcal{C}^{T(k-1)} (e_m \otimes I_n) \\ &= e_{k+1} \otimes I - e_k \otimes I \\ &= -G^{-1}(e_1 \otimes I) A_m^{-1} = -G^{-1}\tilde{V}. \end{aligned}$$

From $\tilde{V} = -Gv_{\mathcal{S}}(\mathcal{C})\tilde{U}^T$ it follows that $\tilde{V}^T = -\tilde{U}\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T$ so that

$$\begin{aligned} \tilde{V}^T u_{\mathcal{S}}(\mathcal{C}) &= \tilde{U}\mathcal{C}^{(k-1)}(I - \mathcal{C})G^T \mathcal{C}^{T(k-1)}(I - \mathcal{C}^T)^{-1} \\ &= \tilde{U}\mathcal{C}^{(k-1)}(I - \mathcal{C})\mathcal{C}^{(1-k)}G^T(I - \mathcal{C}^T)^{-1} \\ &= -\tilde{U}G(I - \mathcal{C}^T)(I - \mathcal{C}^T)^{-1} = -\tilde{U}G, \end{aligned}$$

where we used $G^T \mathcal{C}^{T(k-1)} G^{-T} = \mathcal{C}^{-(k-1)}$ and $\mathcal{C}G^T = -G$.

- (iii) $m = 2k$, $\star = *, T$ and $-1 \notin \sigma(T)$. In this case Lemma 3.2.1 says that $\mathcal{C} =$

$-(\mathcal{A}_-^{even}(z))^{-1}(\mathcal{A}_-^{even}(z))^\star$ with $\mathcal{A}_-^{even}(z)$ as in (3.6). The proof is similar to that in (ii) with $\alpha = z/z^\star$.

Conversely, suppose that (U, T, V) and $(V^\star u_{\mathcal{S}}(T), T^{-\star}, v_{\mathcal{S}}(T)U^\star)$ are standard triples for $P(\lambda)$. Assumption (b) guarantees the existence of $\alpha \in \mathbb{F}$ such that $\alpha^\star \alpha = 1$ and $I + \alpha T^\star$ is nonsingular. Hence $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$ are well defined. Using the resolvent form for $P(\lambda)$, $P(\lambda)^{-1} = U(\lambda I - T)^{-1}V$, we obtain

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = \lambda^{-m}(U(\lambda^{-\star}I - T)^{-1}V)^\star = \lambda^{1-m}V^\star(I - \lambda T^\star)^{-1}U^\star.$$

If $\rho(\lambda T^\star) < 1$ then

$$(I - \lambda T^\star)^{-1} = I + \lambda T^\star + \lambda^2 T^{\star 2} + \dots \quad (3.15)$$

Using (3.15) and the fact that $V^\star T^{\star j} U^\star = 0$, $j = 0: m-2$ (see (3.9)), we obtain

$$\begin{aligned} \lambda^{-m}(P(\lambda^{-\star}))^{-\star} &= V^\star T^{\star(m-1)}(I + \lambda T^\star + \lambda^2 T^{\star 2} + \dots)U^\star \\ &= V^\star T^{\star(k-1)}(I - \lambda T^\star)^{-1}T^{\star(m-k)}U^\star \\ &= -V^\star T^{\star(k-1)}(\lambda I - T^{-\star})^{-1}T^{\star(m-k-1)}U^\star \end{aligned} \quad (3.16)$$

for all $|\lambda| < (\rho(T))^{-1}$. When $m = 2k + 1$, (3.16) reads as

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = V^\star u_{\mathcal{S}}(T)(\lambda I - T^{-\star})^{-1}v_{\mathcal{S}}(T)U^\star = P(\lambda)^{-1}.$$

Note that $(\lambda I - T^{-\star})^{-1}$ commutes with $T^{\star k-1}$, $(I + \alpha T^\star)$ and $(I + \alpha T^\star)^{-1}$ so when $m = 2k$, (3.16) can be rewritten as

$$\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = V^\star u_{\mathcal{S}}(T)(\lambda I - T^{-\star})^{-1}v_{\mathcal{S}}(T)U^\star = P(\lambda)^{-1}$$

for all $|\lambda| < (\rho(T))^{-1}$. Since $\lambda^{-m}(P(\lambda^{-\star}))^{-\star} = P(\lambda)^{-1}$ holds for many values of λ , $P(\lambda) = \lambda^m P^\star(\lambda^{-1})$ for all λ , that is, $P(\lambda)$ is \star -palindromic.

The results for the \star -antipalindromic structure are proved in a similar way. ■

In the proof of Lemma 3.3.1 we use the fact that if (U, T, V) is a standard triple for a structured $P(\lambda)$ then there exists a nonsingular S such that

$$US = V^\star u_{\mathcal{S}}(T), \quad S^{-1}TS = t_{\mathcal{S}}(T), \quad S^{-1}V = v_{\mathcal{S}}(T)U^\star. \quad (3.17)$$

These relations imply certain properties of S , which we use in our definition of \mathcal{S} -structured standard triples.

Definition 3.3.2 (\mathcal{S} -structured standard triple) *Let $\mathcal{S} \in \mathbb{S}$. An (m, n) -standard triple (U, T, V) with T satisfying assumption (b) is said to be \mathcal{S} -structured if $V = Sv_{\mathcal{S}}(T)U^\star$ for some nonsingular $S \in \mathbb{F}^{mn \times mn}$ having the following properties:*

- $S = S^\star, TS = (TS)^\star$ when $\mathcal{S} \in \{\text{Hermitian, symmetric}\}$,
- $S = -S^\star, TS = (TS)^\star$ when $\mathcal{S} = \star\text{-even}$,
- $S = S^\star, TS = -(TS)^\star$ when $\mathcal{S} = \star\text{-odd}$,
- $TS^\star = -S$ when $\mathcal{S} = \star\text{-palindromic}$ and $m = 2k + 1$ or $TS^\star = -\alpha S$ when $\mathcal{S} = \star\text{-palindromic}$ and $m = 2k$,
- $TS^\star = S$ when $\mathcal{S} = \star\text{-antipalindromic}$ and $m = 2k + 1$ or $TS^\star = \alpha S$ when $\mathcal{S} = \star\text{-antipalindromic}$ and $m = 2k$,

for some $\alpha \in \mathbb{F}$ such that $\alpha^\star\alpha = 1$ and $-\alpha \notin \sigma(T)$.

We refer to the matrix S in Definition 3.3.2 as the S -matrix of the \mathcal{S} -structured standard triple (U, T, V) . We point out that Hermitian and symmetric structured standard triples are called self-adjoint standard triples in the literature (see for example [30, p. 244]). For \star -(anti)palindromic structures, the matrix T in Definition 3.3.2 is S^{-1} -unitary, that is, $T^\star S^{-1}T = S^{-1}$, and with additional constraints on T 's structure, Lancaster, Prells and Rodman refer to (U, T, V) as a *unitary standard triple* [55, Def. 4]. Hence a unitary standard triple is \mathcal{S} -structured but the converse is not true in general.

Our definition of \mathcal{S} -structured standard triples is justified by the next lemma.

Lemma 3.3.3 *Let $\mathcal{S} \in \mathbb{S}$. An (m, n) -standard triple (U, T, V) with T satisfying assumption (b) is \mathcal{S} -structured if and only if it is similar to $(V^\star u_{\mathcal{S}}(T), t_{\mathcal{S}}(T), v_{\mathcal{S}}(T)U^\star)$.*

Proof. The proof of this lemma appears in [60, Thm. 3.4] for the symmetric structure and the proof there extends easily to structures $\mathcal{S} \in \{\text{Hermitian}, \star\text{-even}, \star\text{-odd}\}$.

Suppose $\mathcal{S}=\star$ -palindromic (the proof for \star -antipalindromic structure is similar and so we omit it). If $m = 2k + 1$ and (U, T, V) is \mathcal{S} -structured, then there exists S nonsingular such that $TS^\star = -S$ and $V = ST^{\star k}U^\star$. Hence $S^{-1}TS = T^{-\star}$ and

$$US = V^\star S^{-\star} T^{-k} S = -V^\star S^{-\star} T^{-(k-1)} S^\star = -V^\star T^{\star(k-1)}.$$

Hence (U, T, V) is similar to $(-V^\star T^{\star k-1}, T^{-\star}, T^{\star m-k-1} U^\star)$.

When $m = 2k$, $TS^\star = -\alpha S$ and $V = S(I + \alpha T^\star)T^{\star(k-1)}U^\star$ for some $\alpha \in \mathbb{F}$ such that $\alpha^\star \alpha = 1$ and $-\alpha \notin \sigma(T)$. The first equality implies that $S^{-1}TS = T^{-\star}$ while the first and second equality yield

$$\begin{aligned} US &= V^\star S^{-\star} T^{-(k-1)} (I + \alpha^\star T)^{-1} S \\ &= V^\star T^{\star(k-1)} S^{-\star} (I + \alpha^\star T)^{-1} S \\ &= -V^\star T^{\star(k-1)} (I + \alpha T^\star)^{-1} = V^\star u_{\mathcal{S}}(T). \end{aligned}$$

Conversely, if (U, T, V) is similar to $(V^\star u_{\mathcal{S}}(T), T^{-\star}, v_{\mathcal{S}}(T)U^\star)$ then there exists S nonsingular such that (3.17) holds with $t_{\mathcal{S}}(T) = T^{-\star}$. It remains to show that $TS^\star = -S$ when $m = 2k + 1$ and $TS^\star = -\alpha S$ when $m = 2k$. If $m = 2k + 1$, the first and last equalities in (3.17) imply that $V = -ST^{\star k}S^{-\star}T^{k-1}V$ and since V has full rank, $-ST^{\star k}S^{-\star}T^{k-1} = I$, which on using the second equality in (3.17) yields $TS^\star = -S$. When $m = 2k$, the first and last equalities in (3.17) and the fact that V has full rank imply that

$$\begin{aligned} &-ST^{\star(k-1)}(I + \alpha T^\star)S^{-\star}(I + \alpha^\star T)^{-1}T^{(k-1)} = I \\ \Leftrightarrow &-T^{-(k-1)}(I + \alpha T^{-1})SS^{-\star}(I + \alpha^\star T)^{-1}T^{(k-1)} = I \\ \Leftrightarrow &-T^{-(k-1)}SS^{-\star}(\alpha T^{-1})T^{(k-1)} = I \\ \Leftrightarrow &-\alpha SS^{-\star} = T^{(k-1)}T^{-(k-2)} = T, \end{aligned}$$

which completes the proof. ■

The next lemma shows that any standard triple that is similar to an \mathcal{S} -structured standard triple is itself \mathcal{S} -structured.

Lemma 3.3.4 *Let (U, T, V) be a standard triple similar to (U_1, T_1, V_1) , that is,*

$$(U_1, T_1, V_1) = (UG, G^{-1}TG, G^{-1}V)$$

for some nonsingular matrix G . Let $\mathcal{S} \in \mathbb{S}$ and assume that T satisfies assumption (b). If (U, T, V) is \mathcal{S} -structured with matrix S then (U_1, T_1, V_1) is \mathcal{S} -structured with matrix $S_1 = G^{-1}SG^{-\star}$.

Proof. It is easy to check that $V_1 = S_1 v_{\mathcal{S}}(T_1) U_1^{\star}$ and since the properties of S are preserved by \star -congruence, (U_1, T_1, V_1) is \mathcal{S} -structured with matrix S_1 . ■

We can now state our main result, which is a direct consequence of Lemma 3.3.1, Lemma 3.3.3 and Lemma 3.3.4. It extends a result for Hermitian structure [30, Thm. 12.2.2] to all structures in \mathbb{S} .

Theorem 3.3.5 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ with nonsingular leading coefficient satisfying assumption (a). Then $P(\lambda)$ has structure \mathcal{S} if and only if $P(\lambda)$ admits an \mathcal{S} -structured standard triple, in which case every standard triple for $P(\lambda)$ is \mathcal{S} -structured.*

The structure of the matrix S in an \mathcal{S} -structured standard triple is uniquely determined by the triple, as shown by the next result.

Proposition 3.3.6 *Let $\mathcal{S} \in \mathbb{S}$ and (U, T, V) be an \mathcal{S} -structured standard triple with matrix S . Then*

$$S = Q(U, T)^{-1} Q(V^{\star} u_{\mathcal{S}}(T), t_{\mathcal{S}}(T)).$$

with $Q(U, T)$ as in (3.8).

Proof. Using Definition 3.3.2 we check that $Q(U, T)S = Q(V^{\star} u_{\mathcal{S}}(T), t_{\mathcal{S}}(T))$. ■

The matrix S is also easy to construct when the matrix coefficients of $P(\lambda)$ are known.

Proposition 3.3.7 *Let $\mathcal{S} \in \mathbb{S}$ and $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ of degree m with nonsingular leading coefficient be satisfying assumption (a). If (U, T) is a standard pair for $P(\lambda)$ then $(U, T, S v_{\mathcal{S}}(T) U^{\star})$ is an \mathcal{S} -structured standard triple for $P(\lambda)$ with matrix S given*

by

$$S^{-1} = \begin{cases} Q^T \mathcal{A}_+^{even} Q & \text{if } P \text{ is } T\text{-}(anti)\text{palindromic, } m = 2k \text{ and } -1 \in \sigma(P), \\ z^{-\star} Q^{\star} \mathcal{A}_-^{even}(z) Q & \text{if } P \text{ is } \star\text{-}(anti)\text{palindromic, } m = 2k, -z/z^{\star} \notin \sigma(P), \\ Q^{\star} \mathcal{A}_S Q & \text{otherwise,} \end{cases}$$

where $Q := Q(U, T)$ is as in (3.8), and \mathcal{A}_S , $\mathcal{A}_-^{even}(z)$ and \mathcal{A}_+^{even} are as in (3.3)–(3.7).

Proof. We first show that the matrix S in the proposition has the properties listed in Definition 3.3.2. Note that under assumption (a), $P(\lambda)$ has a structured linearization $\lambda \mathcal{A}_S + \mathcal{B}_S$, which is one of (3.3)–(3.7). The pair (Q, T) is a standard pair for $\lambda \mathcal{A}_S + \mathcal{B}_S$, and hence

$$Q^{\star} \mathcal{A}_S Q T = -Q^{\star} \mathcal{B}_S Q \Leftrightarrow Q^{\star} \mathcal{B}_S Q = -z^{\star} S^{-1} T,$$

where $z = 1$ except when $\mathcal{A}_S = \mathcal{A}_-^{even}(z)$, in which case z is such that $-z/z^{\star} \notin \sigma(P)$. Since \star -congruence preserves any structure in \mathbb{S} , the pencil

$$Q^{\star} (\lambda \mathcal{A}_S + \mathcal{B}_S) Q = z^{\star} (\lambda S^{-1} - S^{-1} T)$$

has the same structure as $\lambda \mathcal{A}_S + \mathcal{B}_S$, and hence S satisfies the appropriate properties.

It remains to show that V in (3.10), for which (U, T, V) is a standard triple for $P(\lambda) \in \mathcal{P}_S(\mathbb{F}^n)$, has the form $V = S v_S(T) U^{\star}$. For $\mathcal{S} \in \{\text{Hermitian, symmetric, } \star\text{-even, } \star\text{-odd}\}$, $v_S(T) = I$. Since $U^{\star} = Q^{\star}(e_m \otimes I_n)$ and $S^{-1} = Q^{\star} \mathcal{A}_S Q$, we find that

$$S U^{\star} = Q^{-1} \mathcal{A}_S^{-1} Q^{-\star} Q^{\star}(e_m \otimes I_n) = Q^{-1} \mathcal{A}_S^{-1}(e_m \otimes I_n).$$

From the block structure of \mathcal{A}_S in (3.3) we see that $\mathcal{A}_S(e_1 \otimes I_n) = (e_m \otimes I) A_m$, or equivalently, $\mathcal{A}_S^{-1}(e_m \otimes I) = (e_1 \otimes I_n) A_m^{-1}$ since \mathcal{A}_S and A_m are both nonsingular. Hence $S U^{\star} = Q^{-1}(e_1 \otimes I_n) A_m^{-1} = V$.

When $P(\lambda)$ is \star -(anti)palindromic of odd degree then the definition of V in (3.10), the expression for S in the proposition and the structure of $\mathcal{A}_S = \mathcal{A}^{odd}$ in (3.4) yield

$$S^{-1} V = Q^{\star} \mathcal{A}_S Q Q^{-1}(e_1 \otimes I_n) A_m^{-1} = T^{\star k} U^{\star}.$$

For a \star -palindromic P of even degree, we have shown that $T S^{\star} = -\alpha S$, where $\alpha = -z/z^{\star}$. From the definition of Q in (3.8), $T^{\star(k-1)} U^{\star} = Q^{\star}(e_{k+1} \otimes I_n)$. Hence on

using the definition of V in (3.10), the expression for S in the proposition and the structure of $\mathcal{A}_S = \mathcal{A}_-^{even}(z)$ in (3.6) we have that

$$\begin{aligned} QS(I + \alpha T^\star)T^{\star(k-1)}U^\star &= Q(S - S^\star)Q^\star(e_{k+1} \otimes I_n) \\ &= z^\star(\mathcal{A}_-^{even}(z))^{-1}(e_{k+1} \otimes I_n) - z(\mathcal{A}_-^{even}(z))^{-\star}(e_{k+1} \otimes I_n). \end{aligned}$$

From the definition of $\mathcal{A}_-^{even}(z)$ in (3.6), we find that

$$\mathcal{A}_-^{even}(z) \begin{bmatrix} z^{-1}A_m^{-1} - z^{m-1}P(z, -z^\star)^{-1} \\ (-z)^{m-2}z^\star P(z, -z^\star)^{-1} \\ (-z)^{m-3}z^{\star 2}P(z, -z^\star)^{-1} \\ \vdots \\ (-z)^0 z^{\star(m-1)}P(z, -z^\star)^{-1} \end{bmatrix} = -\alpha^{-1}e_{k+1} \otimes I_n,$$

$$(\mathcal{A}_-^{even}(z))^\star \begin{bmatrix} -z^{m-1}(z^\star)^0 P(z, -z^\star)^{-1} \\ (-z)^{m-2}z^\star P(z, -z^\star)^{-1} \\ (-z)^{m-3}z^{\star 2}P(z, -z^\star)^{-1} \\ \vdots \\ (-z)^0 z^{\star(m-1)}P(z, -z^\star)^{-1} \end{bmatrix} = e_{k+1} \otimes I_n,$$

where $P(z, -z^\star) = \sum_{j=0}^m z^j (-z^\star)^{m-j} A_j$. Hence,

$$z^\star(\mathcal{A}_-^{even}(z))^{-1}(e_{k+1} \otimes I_n) - z(\mathcal{A}_-^{even}(z))^{-\star}(e_{k+1} \otimes I_n) = (e_1 \otimes I_n)A_m^{-1},$$

that is, $S(I + \alpha T^\star)T^{\star(k-1)}U^\star = Q^{-1}(e_1 \otimes I_n)A_m^{-1} = V$. The proof for \star -antipalindromic P is along the same lines. ■

It follows from Theorem 3.3.5 and Proposition 3.3.7 that if $P(\lambda)$ has structure \mathcal{S} then the standard triple (3.12) is \mathcal{S} -structured with matrix $S = \mathcal{A}_S^{-1}$ except when $\mathcal{A}_S = \mathcal{A}_-^{even}(z)$ in which case $S = z^\star(\mathcal{A}_-^{even}(z))^{-1}$.

3.4 \mathcal{S} -structured Jordan triples

We now explain how to obtain explicit expressions for the Jordan matrix and S -matrix of \mathcal{S} -structured Jordan triples $(X, J, S_J v_{\mathcal{S}}(J)X^\star)$ for $P(\lambda) \in \mathcal{P}_S(\mathbb{F}^n)$. We note that the matrix S_J displays the sign characteristic of $P(\lambda)$. Indeed, if $(U, T, S_T v_{\mathcal{S}}(T)U^\star)$

is a standard triple for $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$, then the sign characteristic of $P(\lambda)$ can be defined as the sign characteristic of the pair (T, S_T^{-1}) , which is a list of signs, with a sign (+1 or -1) attached to each partial multiplicity of

- real eigenvalues of Hermitian or real symmetric matrix polynomials,
- purely imaginary eigenvalues of *-even, *-odd, real T -even and real T -odd matrix polynomials, and
- eigenvalues with unit modulus of *(anti)palindromic and real T -(anti)palindromic matrix polynomials.

These signs can be read off the canonical decomposition of $\lambda S_T^{-1} - S_T^{-1}T$ via \star -congruence (see [30, Section 12.4] for Hermitian structure). Note that the definition of the sign characteristic for $P(\lambda)$ is independent of the choice of standard triple. Indeed if $(U_i, T_i, S_{T_i} v_{\mathcal{S}}(T_i) U_i^{\star})$, $i = 1, 2$ are \mathcal{S} -structured standard triples for $P(\lambda)$, then by Lemma 3.3.4 there exists a nonsingular G such that $T_2 = G^{-1}T_1G$ and $S_{T_2} = G^{-1}S_{T_1}G^{-\star}$. Hence, $\lambda S_{T_2}^{-1} - S_{T_2}^{-1}T_2 = G^{\star}(\lambda S_{T_1}^{-1} - S_{T_1}^{-1}T_1)G$, that is, the pencils $\lambda S_{T_i}^{-1} - S_{T_i}^{-1}T_i$, $i = 1, 2$ are \star -congruent. They share the same canonical decomposition via \star -congruence and therefore the same sign characteristic.

We know that the triple $((e_m^T \otimes I_n), \mathcal{C}, (e_1 \otimes A_m^{-1}))$ is a standard triple for $P(\lambda)$ and by Theorem 3.3.5, it is \mathcal{S} -structured with S -matrix as in Proposition 3.3.7 with $Q = I_{mn}$, say $S_{\mathcal{C}} = z^{\star} \mathcal{A}_{\mathcal{S}}^{-1}$. Hence, on using Lemma 3.2.1, we find that

$$\lambda S_{\mathcal{C}}^{-1} - S_{\mathcal{C}}^{-1} \mathcal{C} = \lambda z^{-\star} \mathcal{A}_{\mathcal{S}} + z^{-\star} \mathcal{B}_{\mathcal{S}},$$

where $\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ is a structured linearization of $P(\lambda)$ as in (3.3)–(3.7), and $z = 1$ except when $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_{-}^{even}(z)$, in which case $z \in \mathbb{F}$ is chosen such that $-z/z^{\star} \notin \sigma(P)$. So what we need is a canonical decomposition of $\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}}$ via \star -congruence,

$$Z^{\star}(\lambda \mathcal{A}_{\mathcal{S}} + \mathcal{B}_{\mathcal{S}})Z = \lambda(Z^{\star} \mathcal{A}_{\mathcal{S}}Z) - (Z^{\star} \mathcal{A}_{\mathcal{S}}Z)(Z^{-1} \mathcal{C}Z) = z^{\star}(\lambda S_J^{-1} - S_J^{-1}J),$$

where $J = Z^{-1} \mathcal{C}Z$ is the Jordan form of \mathcal{C} . Fortunately, such decompositions are available in the literature for all the structures in \mathbb{S} . We use these canonical decompositions to provide explicit expressions for J and S_J in Section 3.5. These expressions

show that S_J and J have the same block structure and that we can read the sign characteristic of $P(\lambda)$ from certain diagonal blocks of S_J .

3.5 Explicit expressions for J and S_J

Using the canonical decompositions of structured pencils via \star -congruences, we provide in this section an explicit expression for the Jordan matrix and S -matrix of \mathcal{S} -structured Jordan triples $(X, J, S_J v_{\mathcal{S}}(J) X^*)$ of $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ for each $\mathcal{S} \in \mathbb{S}$. We assume that $P(\lambda)$ is of degree m with nonsingular leading coefficient matrix. To facilitate the description of J and S_J , we introduce the matrices $F_1 = E_1 = [1]$ and for integers $k > 1$

$$E_k = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & & \ddots & \\ & & 1 & \\ (-1)^{k-1} & & & \end{bmatrix}_{k \times k} = (-1)^{k-1} E_k^T, \quad F_k = \begin{bmatrix} & & & 1 \\ & & \ddots & \\ & & 1 & \end{bmatrix}_{k \times k}.$$

We denote by

$$J_{\ell_k}(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{\ell_k \times \ell_k}, \quad (3.18)$$

the Jordan block of size ℓ_k associated with λ_k , and by

$$K_{2m_k}(\lambda_k, \bar{\lambda}_k) = K_{2m_k}(\Lambda_k) = \begin{bmatrix} \Lambda_k & I & & \\ & \Lambda_k & \ddots & \\ & & \ddots & I \\ & & & \Lambda_k \end{bmatrix} \in \mathbb{R}^{2m_k \times 2m_k}, \quad \Lambda_k = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}, \quad (3.19)$$

the $2m_k \times 2m_k$ real Jordan block associated with the pair of complex conjugate eigenvalues $(\lambda_k, \bar{\lambda}_k)$, where $\lambda_k = \alpha_k + i\beta_k$ with $\alpha_k, \beta_k \in \mathbb{R}$, $\beta_k \neq 0$. We use the notation $\bigoplus_{j=1}^r F_j$ to denote the direct sum of the matrices F_1, \dots, F_r .

Note that there are restrictions on the Jordan structure of P . For instance, a regular $n \times n$ matrix polynomial cannot have more than n elementary divisors

associated with the same eigenvalue [64]. Also, the elementary divisors have certain pairing, which depends on the structure $\mathcal{S} \in \mathbb{S}$ and the eigenvalue. Hence we describe for each $\mathcal{S} \in \mathbb{S}$ the elementary divisors arising from $P(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$ and then provide an expression for J and S_J .

3.5.1 Hermitian structure

Suppose $P(\lambda)$ is Hermitian with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1:r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1:s$,

with ℓ_j, m_j such that $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^s m_j = mn$. It follows from [56, Thm. 6.1] that

$$J = \bigoplus_{j=1}^r J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^s (J_{m_j}(\bar{\mu}_j) \oplus J_{m_j}(\mu_j)), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^r \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^s F_{2m_j}.$$

Here $\{\varepsilon_1, \dots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the real eigenvalues λ_j , $j = 1:r$ of $P(\lambda)$. We easily check that $S_J = S_J^*$ and $JS_J = (JS_J)^*$.

3.5.2 Real symmetric structure

Suppose $P(\lambda)$ is real symmetric with

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1:r$, and
- s pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1:s$,

with ℓ_j, m_j such that $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^s m_j = mn$. On using [56, Thm. 9.2] we find that

$$J = \bigoplus_{j=1}^r J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^s K_{2m_j}(\mu_j, \bar{\mu}_j), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^r \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^s F_{2m_j},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the real eigenvalues of $P(\lambda)$. Note that $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

3.5.3 Complex symmetric structure

Suppose $P(\lambda)$ is complex symmetric with q elementary divisors $(\lambda - \lambda_j)^{m_j}$, $\lambda_j \in \mathbb{C}$, $j = 1:q$, with m_j such that $\sum_{j=1}^q m_j = mn$. Then [83, Prop. 4.3] leads to

$$J = \bigoplus_{j=1}^q J_{m_j}(\lambda_j), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^q F_{m_j},$$

which satisfy $S_J = S_J^T$ and $JS_J = (JS_J)^T$.

3.5.4 *-even structure

Suppose $P(\lambda)$ is *-even with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1:r$, and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1:s$,

with ℓ_j, m_j such that $\sum_{j=1}^r \ell_j + 2\sum_{j=1}^s m_j = mn$. With the change of eigenvalue parameter $\lambda = -i\mu$, the *-even linearization of $P(\lambda)$, $\lambda\mathcal{A}_S + \mathcal{B}_S = \mu(-i\mathcal{A}_S) + \mathcal{B}_S$ becomes a Hermitian pencil in μ . Using Section 3.5.1 we obtain that

$$J = -i \left(\bigoplus_{j=1}^r J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^s (J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)) \right),$$

$$S_J = -i \left(\bigoplus_{j=1}^r \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^s F_{2m_j} \right).$$

Here $\{\varepsilon_1, \dots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$. Note that $S_J = -S_J^*$ and $JS_J = (JS_J)^*$.

3.5.5 Real T -even structure

Suppose $P(\lambda)$ is real T -even with (see [66])

- t zero elementary divisors λ^{n_j} with n_j even, $j = 1:t$,
- r pairs of real elementary divisors $(\lambda + \alpha_j)^{p_j}$, $(\lambda - \alpha_j)^{p_j}$ with p_j odd if $\alpha_j = 0$, $j = 1:r$,

- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1: s$, and
- q quadruples of nonreal and non-purely imaginary elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \bar{\mu}_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1: q$,

with n_j, p_j, k_j, m_j such that $\sum_{j=1}^t n_j + 2\sum_{j=1}^r p_j + 2\sum_{j=1}^s k_j + 4\sum_{j=1}^q m_j = mn$. Using [57, Thm. 16.1], we find that

$$J = \bigoplus_{j=1}^t J_{n_j}(0) \oplus \bigoplus_{j=1}^r (J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T) \\ \oplus \bigoplus_{j=1}^s K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^q (K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T),$$

$$S_J = \bigoplus_{j=1}^t \varepsilon_j E_{n_j} \oplus \bigoplus_{j=1}^r \begin{bmatrix} 0 & -I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^s \varepsilon_j (E_{k_j} \otimes E_2^{k_j}) \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0 & -I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and zero eigenvalues of even partial multiplicities (see [77]). We easily check that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

3.5.6 Complex T -even structure

Let $\lambda_j \in \mathbb{C} \setminus \{0\}$ and suppose $P(\lambda)$ is complex T -even with (see [66])

- t zero elementary divisors λ^{m_j} with m_j even, $j = 1: t$, and
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda + \lambda_j)^{k_j}$ with k_j odd if $\lambda_j = 0$, $j = 1: q$,

with m_j, k_j such that $\sum_{j=1}^t m_j + 2\sum_{j=1}^q k_j = mn$. Then, by [83, Prop. 4.7 (b)], we obtain that

$$J = \bigoplus_{j=1}^t J_{m_j}(0) \oplus \bigoplus_{j=1}^q (J_{k_j}(\lambda_j) \oplus J_{k_j}(-\lambda_j)), \\ S_J = \bigoplus_{j=1}^t \begin{bmatrix} 0 & -F_{\frac{1}{2}m_j} \\ F_{\frac{1}{2}m_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0 & -F_{k_j} \\ F_{k_j} & 0 \end{bmatrix}.$$

Note that $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

3.5.7 *-odd structure

Suppose $P(\lambda)$ is *-odd with

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1:r$, and
- s pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1:s$,

with ℓ_j, m_j such that $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^s m_j = mn$. Note that for the *-odd linearization $\lambda\mathcal{A}_S + \mathcal{B}_S$ of $P(\lambda)$ in (3.3), the pencil $i(\lambda\mathcal{A}_S + \mathcal{B}_S)$ is *-even and the structure for S_J and J follows from Section 3.5.4. We find that

$$J = -i \left(\bigoplus_{j=1}^r J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^s (J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)) \right),$$

$$S_J = S_J^{-1} = \bigoplus_{j=1}^r \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^s F_{2m_j},$$

which satisfy $S_J = S_J^*$ and $JS_J = -(JS_J)^*$. Here $\{\varepsilon_1, \dots, \varepsilon_r\}$ with $\varepsilon_j = \pm 1$ is the sign characteristic associated with the zero and purely imaginary eigenvalues of $P(\lambda)$.

3.5.8 Real T -odd structure

Suppose $P(\lambda)$ is real T -odd with (see [66])

- t zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1:t$,
- r pairs of real elementary divisors $(\lambda + \alpha_j)^{p_j}$, $(\lambda - \alpha_j)^{p_j}$ with p_j even if $\alpha_j = 0$, $j = 1:r$,
- s pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1:s$, and
- q quadruples of nonreal and non-purely imaginary elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \bar{\mu}_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$ $j = 1:q$,

with ℓ_j, p_j, k_j, m_j such that $\sum_{j=1}^t \ell_j + 2 \sum_{j=1}^r p_j + 2 \sum_{j=1}^s k_j + 4 \sum_{j=1}^q m_j = mn$. On using [57, Thm. 17.1] we find that

$$J = \bigoplus_{j=1}^t J_{\ell_j}(0) \oplus \bigoplus_{j=1}^r (J_{p_j}(\alpha_j) \oplus -J_{p_j}(\alpha_j)^T) \\ \oplus \bigoplus_{j=1}^s K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^q (K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T),$$

$$S_J = S_J^{-1} = \bigoplus_{j=1}^t \varepsilon_j E_{\ell_j} \oplus \bigoplus_{j=1}^r \begin{bmatrix} 0 & I_{p_j} \\ I_{p_j} & 0 \end{bmatrix} \oplus \bigoplus_{j=1}^s \varepsilon_j (E_{k_j} \otimes E_2^{k_j-1}) \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0 & I_{2m_j} \\ I_{2m_j} & 0 \end{bmatrix},$$

where the scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the purely imaginary eigenvalues and the zero eigenvalues with odd partial multiplicities. We easily check that $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

3.5.9 Complex T -odd structure

Let $\lambda_j \in \mathbb{C} \setminus \{0\}$ and suppose $P(\lambda)$ is complex T -odd with (see [66])

- s zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1:s$, and
- q pairs of elementary divisors $(\lambda + \lambda_j)^{k_j}$, $(\lambda - \lambda_j)^{k_j}$ with k_j even if $\lambda_j = 0$, $j = 1:q$,

with ℓ_j, k_j such that $\sum_{j=1}^s \ell_j + 2 \sum_{j=1}^q k_j = mn$. It follows from [83, Prop. 4.7 (b)] that

$$J = \bigoplus_{j=1}^s J_{\ell_j}(0) \oplus \bigoplus_{j=1}^q (-J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j)), \quad S_J = S_J^{-1} = \bigoplus_{j=1}^s E_{\ell_j} \oplus \bigoplus_{j=1}^q F_{2k_j}.$$

Clearly, $S_J = S_J^T$ and $JS_J = -(JS_J)^T$.

Notice the difference between the zero elementary divisors associated with T -even and T -odd pencils (see [66, Cor. 4.3]).

3.5.10 $*$ -(anti)palindromic structure

Suppose $P(\lambda)$ is complex $*$ -palindromic with $-1 \notin \sigma(P)$ and (see [68])

- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\bar{\lambda}_j)^{k_j}$ with $\lambda_j \in \mathbb{C} \setminus \{0\}$, $|\lambda_j| \neq 1$, $j = 1:q$,

- t elementary divisors $(\lambda - \lambda_j)^{2\ell_j+1}$ with $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1$, $j = 1:t$, and
- s elementary divisors $(\lambda - \lambda_j)^{2m_j}$ with $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, $j = 1:s$,

with k_j, ℓ_j, m_j such that $2 \sum_{j=1}^q k_j + \sum_{j=1}^t (2\ell_j + 1) + 2 \sum_{j=1}^s m_j = mn$. Then using either [86, Thm. 5] or [87, Section 2.2.2] we find that

$$J = -S_J S_J^{-*}$$

with

$$S_J = \bigoplus_{j=1}^q \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix} \oplus \bigoplus_{j=1}^t \varepsilon_j \begin{bmatrix} 0 & 0 & F_{\ell_j} J_{\ell_j}(-\lambda_j) \\ 0 & (-\lambda_j)^{1/2} & e_1^T \\ F_{\ell_j} & 0 & 0 \end{bmatrix} \\ \oplus \bigoplus_{j=1}^s \varepsilon_j \begin{bmatrix} 0_{m_j} & F_{m_j} J_{m_j}(-\lambda_j) \\ F_{m_j} & e_1 e_1^T \end{bmatrix}$$

has the above elementary divisors. The scalars $\varepsilon_j = \pm 1$ form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ (see [55]).

For the $*$ -antipalindromic structure, $J = S_J S_J^{-*}$ with S_J as above but with $-\lambda_j$ replaced by λ_j .

3.5.11 Real T -(anti)palindromic structure

Suppose $P(\lambda)$ is real T -palindromic with $-1 \notin \sigma(P)$, $\lambda_j \in \mathbb{C} \setminus \{0\}$, and (see [68])

- r pairs of real elementary divisors $(\lambda - \lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with $\lambda_j \in \mathbb{R}$, $|\lambda_j| \neq 1$, $j = 1:r$,
- q quadruples of nonreal elementary divisors $(\lambda - \lambda_j)^{n_j}, (\lambda - \bar{\lambda}_j)^{n_j}, (\lambda - 1/\lambda_j)^{n_j}, (\lambda - 1/\bar{\lambda}_j)^{n_j}$ with $|\lambda_j| \neq 1$, $j = 1:q$,
- s elementary divisors $(\lambda - 1)^{2m_j}$, $j = 1:s$,
- t pairs of elementary divisors $(\lambda - 1)^{2\ell_j+1}, (\lambda - 1)^{2\ell_j+1}$ $j = 1:t$,
- u pairs of elementary divisors $(\lambda - \lambda_j)^{\ell'_j}, (\lambda - \bar{\lambda}_j)^{\ell'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, ℓ'_j odd, $j = 1:u$, and

- p pairs of elementary divisors $(\lambda - \lambda_j)^{m'_j}$, $(\lambda - \bar{\lambda}_j)^{m'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, m'_j even, $j = 1:p$.

We have that $2 \sum_{j=1}^r k_j + 4 \sum_{j=1}^q n_j + 2 \sum_{j=1}^s m_j + 2 \sum_{j=1}^t (2\ell_j + 1) + 2 \sum_{j=1}^u \ell'_j + 2 \sum_{j=1}^p m'_j = mn$.

On using [87, Thm. 2.8] we find that $J = -S_J S_J^{-T}$ has the above list of elementary divisors, where

$$\begin{aligned}
 S_J = & \bigoplus_{j=1}^r \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix} \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0_{2n_j} & K_{2n_j}(-\Lambda_j) \\ F_{n_j} \otimes I_2 & 0_{2n_j} \end{bmatrix} \\
 & \oplus \bigoplus_{j=1}^s \begin{bmatrix} 0 & F_{m_j} J_{m_j}(-1) \\ F_{m_j} & 0 \end{bmatrix} \\
 & \oplus \bigoplus_{j=1}^t \varepsilon_j \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \oplus \bigoplus_{j=1}^t \varepsilon_j \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(-1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \\
 & \oplus \bigoplus_{j=1}^u \varepsilon_j \begin{bmatrix} 0_{\ell'_j-1} & 0 & K_{\ell'_j-1}(-\Lambda_j) \\ 0 & (-\Lambda_j)^{\frac{1}{2}} & e_1^T \otimes I_2 \\ F_{(\ell'_j-1)/2} \otimes I_2 & 0 & 0_{\ell'_j-1} \end{bmatrix} \oplus \bigoplus_{j=1}^p \varepsilon_j \begin{bmatrix} 0_{m'_j} & K_{m'_j}(-\Lambda_j) \\ F_{\frac{1}{2}m'_j} \otimes I_2 & e_1 e_1^T \otimes I_2 \end{bmatrix}.
 \end{aligned}$$

Here $(-\Lambda_j)^{\frac{1}{2}}$ is the principal square root of $-\Lambda_j$. The scalars ε_j are signs ± 1 and form the sign characteristic associated with the eigenvalues of unit modulus of $P(\lambda)$ except the eigenvalues 1 with even partial multiplicities (see [55]).

For the T -antipalindromic $P(\lambda)$, $J = S_J S_J^{-T}$ where S_J is as above but with $-\lambda_j, -1, -\Lambda_j$ replaced by $\lambda_j, 1, \Lambda_j$, respectively.

3.5.12 Complex T -(anti)palindromic structure

Suppose $P(\lambda)$ is complex T -palindromic with $-1 \notin \sigma(P)$ and (see [68])

- t elementary divisors $(\lambda - 1)^{m_j}$ with m_j even, $j = 1:t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\lambda_j)^{k_j}$ with k_j odd when $\lambda_j = 1$, $\lambda_j \neq 0$, $j = 1:q$,

with m_j, k_j such that $\sum_{j=1}^t m_j + 2 \sum_{j=1}^q k_j = mn$. On using either [86, Thm. 1] or [87, Thm. 2.6], we find that with

$$S_J = \bigoplus_{j=1}^t \begin{bmatrix} 0_{m_j/2} & F_{m_j/2} J_{m_j/2}(-1) \\ F_{m_j/2} & e_1 e_1^T \end{bmatrix} \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(-\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix}$$

the matrix $J = -S_J S_J^{-T}$ has the above elementary divisors.

Now if $P(\lambda)$ is complex T -antipalindromic with $-1 \notin \sigma(P)$ and (see [68])

- t elementary divisors $(\lambda - 1)^{\ell_j}$ with ℓ_j odd, $j = 1:t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with k_j even if $\lambda_j = 1$, $j = 1:q$,

with ℓ_j, k_j such that $\sum_{j=1}^t \ell_j + 2 \sum_{j=1}^q k_j = mn$. On using [87, Thm. 2.6], we find that the matrix $J = S_J S_J^{-T}$ with

$$S_J = \bigoplus_{j=1}^t \begin{bmatrix} 0_{\ell_j} & 0 & F_{\ell_j} J_{\ell_j}(1) \\ 0 & 1 & e_1^T \\ F_{\ell_j} & 0 & 0_{\ell_j} \end{bmatrix} \oplus \bigoplus_{j=1}^q \begin{bmatrix} 0_{k_j} & F_{k_j} J_{k_j}(\lambda_j) \\ F_{k_j} & 0_{k_j} \end{bmatrix}$$

has the above elementary divisors.

Note that J in Section 3.5.10—Section 3.5.12 is “almost” in Jordan canonical form.

3.6 Applications

The theory developed in this chapter simplifies the structured inverse quadratic eigenvalue problem having the form that given a structure \mathcal{S} and $2n$ eigenvalues with their algebraic, geometric and partial multiplicities plus the attached sign characteristic when necessary, construct the corresponding $Q(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. We illustrate with some examples now and offer a deeper discussion in the next chapter.

Example 3.6.1 (complex symmetric) If $J = J_2(1+i) \oplus J_2(2-i)$ is a Jordan matrix containing the eigenvalues of a complex symmetric quadratic matrix polynomial, then we can use Section 3.5.3 to construct $S_J = F_2 \oplus F_2$. A solution X for

$XS_JX^T = 0$ is

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Thus

$$A_2 = (XJS_JX^T)^{-1} = \begin{bmatrix} 0 & -0.2 - 0.4i \\ -0.2 - 0.4i & 0.24 - 0.32i \end{bmatrix},$$

$$A_1 = -A_2XJ^2S_JX^TA_2 = \begin{bmatrix} 0 & 0.6 + 1.2i \\ 0.6 + 1.2i & -0.72 + 0.96i \end{bmatrix},$$

and

$$A_0 = -A_2(XJ^2S_JX^TA_1 + XJ^3S_JX^TA_2) = \begin{bmatrix} 0 & -0.2 - 1.4i \\ -0.2 - 1.4i & 0.04 - 0.72i \end{bmatrix},$$

define a complex symmetric $Q(\lambda) = \lambda^2A_2 + \lambda A_1 + A_0$ with the prescribed spectrum.

Example 3.6.2 (real T -even) If $J = J_1(1) \oplus -J_1(1)^T \oplus K_2(1+i, 1-i) \oplus -K_2(1+i, 1-i)^T \oplus K_2(i, -i)$ is a Jordan matrix containing the eigenvalues of a real T -even quadratic matrix polynomial where ε_1 associated with $i, -i$ equals 1. Then we use Section 3.5.5 to construct

$$S_J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

A solution for $XS_JX^T = 0$ is

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Thus

$$A_2 = (XJS_JX^T)^{-1} = \begin{bmatrix} -0.2 & -0.2 & -0.4 & -0.6 \\ -0.2 & -0.2 & 0.6 & 0.4 \\ -0.4 & 0.6 & -0.8 & -1.2 \\ -0.6 & 0.4 & -1.2 & -0.8 \end{bmatrix},$$

$$A_1 = -A_2XJ^2S_JX^TA_2 = \begin{bmatrix} 0 & -0.6 & 1 & 0.2 \\ 0.6 & 0 & 0.2 & 1 \\ -1 & -0.2 & 0 & -2.6 \\ -0.2 & -1 & 2.6 & 0 \end{bmatrix},$$

and

$$A_0 = -A_2(XJ^2S_JX^TA_1 + XJ^3S_JX^TA_2) = \begin{bmatrix} 0.2 & 0.8 & -0.6 & 0.4 \\ 0.8 & 0.2 & -0.4 & 0.6 \\ -0.6 & -0.4 & 1.8 & -1.2 \\ 0.4 & 0.6 & -1.2 & 1.8 \end{bmatrix},$$

define a real T -even $Q(\lambda) = \lambda^2A_2 + \lambda A_1 + A_0$ with the prescribed spectrum.

Example 3.6.3 (real or complex \star -palindromic) If $2, 1/2, i, -i$ are the simple eigenvalues of either a real or a complex quadratic \star -palindromic matrix polynomial where $\varepsilon_{1,2} = 1$. Then we can use Sections 3.5.10, 3.5.12 and 3.5.11 to construct complex $Q_1(\lambda), Q_2(\lambda)$ and a real $Q_3(\lambda)$ with the prescribed eigenvalues and the desirable structure. For example,

$$Q_1(\lambda) = \lambda^2A_2 + \lambda A_1 + A_2^* :=$$

$$\lambda^2 \begin{bmatrix} 0.2879 & -0.1111 - 0.1768i \\ -0.1111 + 0.1768i & 0.2879 \end{bmatrix}$$

$$+ \lambda \begin{bmatrix} -0.2778 & 0.2778 \\ 0.2778 & -0.2778 \end{bmatrix} + A_2^*,$$

$$Q_2(\lambda) = \lambda^2A_2 + \lambda A_1 + A_2^T :=$$

$$\lambda^2 \begin{bmatrix} 0 & 0.1333 - 0.2667i \\ -0.1333 - 0.2667i & 0 \end{bmatrix}$$

$$+ \lambda \begin{bmatrix} 0 & 0.2000 + 0.2667i \\ 0.2000 + 0.2667i & 0 \end{bmatrix} + A_2^T,$$

and

$$Q_3(\lambda) = \lambda^2 \begin{bmatrix} 0.3536 & 0.1667 \\ -0.6667 & 0.3143 \end{bmatrix} + \lambda \begin{bmatrix} -0.8839 & 0 \\ 0 & -0.7857 \end{bmatrix} + \begin{bmatrix} 0.3536 & 0.1667 \\ -0.6667 & 0.3143 \end{bmatrix}^T.$$

3.7 Concluding remarks

The results in this chapter represent a first step towards the solution of the SIPEP: given a list of admissible elementary divisors for the structure, and possibly, corresponding right eigenvectors and generalized eigenvectors, construct a structured matrix polynomial having these elementary divisors and eigenvectors/generalized eigenvectors. Indeed, using the results in Sections 3.3 and 3.4 we show in Chapter 4 how to construct an \mathcal{S} -structured $(2, n)$ -Jordan triple (X, J, Y) from a given list of $2n$ prescribed eigenvalues and n linearly independent eigenvectors and generalized eigenvectors, and use the fact that an \mathcal{S} -structured $(2, n)$ -Jordan triple defines a unique structured quadratic $Q(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$.

Also, in the next chapter, we provide a detailed treatment of the associated quadratic and cubic quasidefinite IPEP using self-adjoint Jordan triples.

Finally, we note that standard triples have been useful to describe structure preserving transformations (SPTs) for matrix polynomials, and in particular quadratic matrix polynomials [28]. We believe that the notion of \mathcal{S} -structured standard triples will further our understanding of SPTs for structured matrix polynomials.

Chapter 4

Structured Inverse Polynomial Eigenvalue Problems

4.1 Introduction

The IPEP is to determine the coefficient matrices of $P(\lambda)$ from its prescribed spectral data (i.e., eigenvalues and eigenvectors). Standard, generalized and quadratic inverse eigenvalue problems have been widely investigated, see [15], [16], [17], [20], [27] and the references therein. For example, suppose that a $2n \times 2n$ Jordan matrix J is given together with an $n \times 2n$ matrix X whose columns contain the corresponding eigenvectors and generalized eigenvectors. Then if $\begin{bmatrix} XJ \\ X \end{bmatrix}$ is nonsingular, the $n \times n$ quadratic $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ defined by

$$A_2 = I_n, \quad [A_1 \quad A_0] = -XJ^2 \begin{bmatrix} XJ \\ X \end{bmatrix}^{-1} \quad (4.1)$$

has the prescribed Jordan form J and eigenvector matrix X [31, p. 3].

In applications, the coefficients of $P(\lambda)$ often have a specific structure such as for example symmetry or skew-symmetry, which corresponds to the physics of the problem. This leads to SIPEPs. Note that apart from A_2 , the other coefficient matrices in (4.1) have no particular structure in general. The inverse problem for Hermitian and symmetric quadratics has been the topic of many papers such as [18], [48], [49] to cite just a few.

The interest in this chapter is in constructing, from spectral data, complex and

real quadratics with structures \mathcal{S} listed in Table 1.1 as well as generating quadratic and cubic quasidefinite matrix polynomials displayed in Figure 2.1.

Our solution process makes use of the recent concept of \mathcal{S} -structured Jordan triples introduced in Chapter 3. The structure of $P(\lambda)$ imposes certain constraints on the Jordan form of \mathcal{S} -structured Jordan triples. For example, the spectrum of Hermitian matrix polynomials is symmetric with respect to the real line. So for each $\mathcal{S} \in \mathbb{S}$, we carefully describe in Section 4.3 the structure of the Jordan matrix, or equivalently, the list of elementary divisors admissible for the structure \mathcal{S} of a quadratic $Q(\lambda) \in \mathcal{P}_{\mathcal{S}}(\mathbb{F}^n)$. Given such a list, we show how to construct families of \mathcal{S} -structured Jordan triples that generates structured quadratics having the given list of elementary divisors. Our construction shows that a solution to the quadratic inverse problem also exists when half of the eigenvectors and generalized eigenvectors are also provided. Since the solution process we provide allows freedom on how to choose the sign characteristic, which is an invariant for certain structured λ -matrices, an appropriate choice for the sign characteristic will facilitate generating quasihyperbolic quadratics and cubics.

We also show how to use Theorem 2.3.10 along with [81, Alg. 2.1] to generate tridiagonal hyperbolic quadratics.

Some of our constructions form the basis of a MATLAB function, `gen_hyper2`, in the NLEVP collection [13].

4.2 Preliminaries

Throughout this chapter, $J \in \mathbb{F}^{mn \times mn}$ satisfies Assumption (b) of Section 3.3.

Recall that an $\ell \times \ell$ Jordan matrix over \mathbb{C} is a block diagonal matrix displaying the algebraic, geometric and partial multiplicities of the eigenvalues, that is,

$$J = \bigoplus_{i=1}^s \bigoplus_{j=1}^{m_i} J_{\ell_{ij}}(\lambda_i) \in \mathbb{C}^{\ell \times \ell}, \quad \lambda_i \neq \lambda_k \text{ if } i \neq k, \quad (4.2)$$

where

$$J_{\ell_{ij}}(\lambda_i) := \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{\ell_{ij} \times \ell_{ij}}$$

is a Jordan block of size ℓ_{ij} associated with λ_i . To each $J_{\ell_{ij}}(\lambda_i)$ corresponds an elementary divisor $(\lambda - \lambda_i)^{\ell_{ij}}$ of $\lambda I - J$, or respectively, $P(\lambda)$. The integer ℓ_{ij} is a partial multiplicity of λ_i and λ_i can have several partial multiplicities. The integer m_i , which is the number of partial multiplicities associated with λ_i is called the geometric multiplicity of λ_i . The algebraic multiplicity of λ_i is the sum of its partial multiplicities. Note that $\sum_{i=1}^s \sum_{j=1}^{m_i} \ell_{ij} = \ell$.

An $\ell \times \ell$ Jordan matrix J over \mathbb{R} is a block diagonal matrix

$$J = \bigoplus_{i=1}^r \bigoplus_{j=1}^{m_i} J_{\ell_{ij}}(\lambda_i) \bigoplus_{i=1}^t \bigoplus_{j=1}^{m'_i} K_{2\ell'_{ij}}(\mu_i, \bar{\mu}_i) \in \mathbb{C}^{\ell \times \ell}, \quad \lambda_i \neq \lambda_k, \mu_i \neq \mu_k \text{ if } i \neq k, \quad (4.3)$$

where $\lambda_i \in \mathbb{R}$, $i = 1:r$ and $\mu_i \in \mathbb{C}$ with $\text{Im}(\mu_i) > 0$, $i = 1:t$, and

$$K_{2\ell'_{ij}}(\mu_i, \bar{\mu}_i) = \begin{bmatrix} \Lambda_i & I_2 & & \\ & \Lambda_i & \ddots & \\ & & \ddots & I_2 \\ & & & \Lambda_i \end{bmatrix} \in \mathbb{R}^{2\ell'_{ij} \times 2\ell'_{ij}}, \quad \Lambda_i = \begin{bmatrix} \text{Re}(\mu_i) & \text{Im}(\mu_i) \\ -\text{Im}(\mu_i) & \text{Re}(\mu_i) \end{bmatrix} \in \mathbb{R}^{2 \times 2},$$

is a real Jordan block associated with $(\mu_i, \bar{\mu}_i)$. To each $K_{2\ell'_{ij}}(\mu_i, \bar{\mu}_i)$ corresponds a pair of elementary divisors $(\lambda - \mu_i)^{\ell'_{ij}}$, $(\lambda - \bar{\mu}_i)^{\ell'_{ij}}$ of $\lambda I - J$. Note that $\sum_{i=1}^r \sum_{j=1}^{m_i} \ell_{ij} + 2 \sum_{i=1}^t \sum_{j=1}^{m'_i} \ell'_{ij} = \ell$.

A quadratic $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 \in \mathcal{P}(\mathbb{F}^n)$ with A_2 nonsingular has $2n$ finite eigenvalues so its Jordan form J over \mathbb{F} is $2n \times 2n$. Note that the geometric multiplicity of λ_i is the number of linearly independent eigenvectors of $Q(\lambda)$ corresponding to λ_i and since $Q(\lambda)$ is $n \times n$, we have that m_i and m'_i in (4.2) and (4.3) cannot be larger than n . Recall that, as a direct consequence of Theorem 3.3.5, if (X, J, Y) is an \mathcal{S} -structured Jordan triple then the matrices in (3.13)–(3.14) with $m = 2$ define a unique structured quadratic.

If $Q(\lambda)$ is a quadratic matrix polynomial with structure $\mathcal{S} \in \mathbb{S}$ and \mathcal{S} -structured Jordan triple $(X, J, Sv_{\mathcal{S}}(J)X^*)$ then the sign characteristic is defined as the sign

characteristic of the pair (J, S^{-1}) (see Section 3.4). This is a list of signs, with a sign (+1 or -1). These signs can be read off the block diagonal entries of S . They form an invariant for $Q(\lambda)$ and they play an essential role in the construction of Hermitian and symmetric quadratics with special properties (see Chapter 2 and Section 4.3.1).

4.3 Building \mathcal{S} -structured Jordan triples

As explained in the introduction, our main objective is to construct structured quadratics, both real and complex, from given spectral data. To be more specific, we solve the following problem:

(P) *Given a structure $\mathcal{S} \in \mathbb{S}$ and a list of elementary divisors*

$$\mathcal{L} = \{(\lambda - \lambda_i)^{\ell_{ij}}, i = 1:s, j = 1:m_i \leq n, \lambda_i \neq \lambda_k \text{ if } i \neq k, \sum_{i=1}^s \sum_{j=1}^{m_i} \ell_{ij} = 2n\}, \quad (4.4)$$

admissible for the structure, construct an $n \times n$ quadratic $Q(\lambda)$ with structure \mathcal{S} having the prescribed spectral data.

One way to solve problem **(P)** is to construct an \mathcal{S} -structured Jordan triple $(X, J, S_J v_{\mathcal{S}}(J)X^*)$, where J displays the given list \mathcal{L} of elementary divisors. To do so, we use the explicit expressions for J and S_J provided for each $\mathcal{S} \in \mathbb{S}$ in Section 3.5. Note that the structure of J imposes constraints on the structure of the list \mathcal{L} . The main difficulty lies in constructing $X \in \mathbb{F}^{n \times 2n}$ such that

$$X S_J v_{\mathcal{S}}(J) X^* = 0 \quad (4.5)$$

and

$$\det \begin{bmatrix} XJ \\ X \end{bmatrix} \neq 0, \quad \det(X J S_J v_{\mathcal{S}}(J) X^*) \neq 0. \quad (4.6)$$

Note that the latter constraints are more likely to hold than the former constraint. We consider each structure separately.

4.3.1 Hermitian and real symmetric quadratics

It follows from [71] that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r real elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $j = 1:r$, and
- t pairs of nonreal conjugate elementary divisors $(\lambda - \mu_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1:t$,

with ℓ_j, m_j such that $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^t m_j = 2n$. The corresponding complex and real Jordan matrices are given by Section 3.5.1, Section 3.5.2

$$J = \bigoplus_{j=1}^r J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^t (J_{m_j}(\bar{\mu}_j) \oplus J_{m_j}(\mu_j)), \quad J = \bigoplus_{j=1}^r J_{\ell_j}(\lambda_j) \oplus \bigoplus_{j=1}^t K_{2m_j}(\mu_j, \bar{\mu}_j),$$

respectively, and

$$S_J = S_J^{-1} = \bigoplus_{j=1}^r \varepsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^t F_{2m_j},$$

where the $\varepsilon_j = \pm 1$, $j = 1:r$ define the sign characteristic associated with the real eigenvalues λ_j and must be such that [31, Prop. 10.12]

$$\text{sig}(S_J) = \sum_{j=1}^r \frac{1}{2} (1 - (-1)^{\ell_j}) \varepsilon_j = 0, \quad (4.7)$$

where $\text{sig}(S_J)$ is the signature of S_J , that is, the difference between the number of positive eigenvalues and the number of negative eigenvalues of S_J . We easily check that $S_J = S_J^*$ and $JS_J = (JS_J)^*$.

Then on using (4.7), we see that there exists an orthogonal matrix $W \in \mathbb{F}^{2n \times 2n}$ such that $W^T S_J W = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$. If we let $\tilde{X} = XW = [\tilde{X}_1 \quad \tilde{X}_2]$ with $\tilde{X}_1 \in \mathbb{F}^{n \times n}$ of full rank and write $\tilde{X}_2 = \tilde{X}_1 \Theta$ for some unitary $\Theta \in \mathbb{F}^{n \times n}$ then

$$X S_J v_S(J) X^* = \tilde{X}_1 \tilde{X}_1^* - \tilde{X}_2 \tilde{X}_2^* = 0,$$

Hence we can build X as

$$X = \tilde{X} W^T = [\tilde{X}_1 \quad \tilde{X}_1 \Theta] W^T, \quad (4.8)$$

which is completely determined by $\tilde{X}_1, \Theta \in \mathbb{F}^{n \times n}$ with Θ unitary. Note that the conditions in (4.6) hold for almost all \tilde{X}_1 and Θ .

In the case where some of the $2n$ given eigenvalues are infinite, we start by applying a homogeneous rotation $G := \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ to the given spectrum as in (2.5) so that the rotated eigenvalues are all finite (this can be done by setting c and s so that c/s is

not one of the given eigenvalues). Then we build a $(2, n)$ -self-adjoint Jordan triple and the unique Hermitian/symmetric quadratic $\tilde{Q}(\tilde{\lambda}) = \tilde{\lambda}^2 \tilde{A}_2 + \tilde{\lambda} \tilde{A}_1 + \tilde{A}_0$ that it generates. Next, we homogeneously rotate back $\tilde{Q}(\tilde{\lambda})$ into $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where $A_2 = \tilde{Q}(c, s)$ is singular, $A_1 = -2cs\tilde{A}_2 + (c^2 - s^2)\tilde{A}_1 + 2cs\tilde{A}_0$ and $A_0 = \tilde{Q}(-s, c)$.

Example 4.3.1 Consider the symmetric structure and $\mathcal{L} = \{(\lambda - 2)^3, (\lambda - 2), (\lambda - i), (\lambda + i)\}$ with sign characteristic $\{+1, -1\}$, i.e., a ‘+’ sign associated with $(\lambda - 2)^3$ and a ‘-’ sign associated with $(\lambda - 2)$, so that (4.7) holds with $(\ell_1, \varepsilon_1) = (3, 1)$ and $(\ell_2, \varepsilon_2) = (1, -1)$. Then

$$J = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix} \oplus [2] \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S_J = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \oplus (-1)[1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With the choice $\tilde{X}_1 = \Theta = I_3$, we obtain

$$X = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

We easily check that $\begin{bmatrix} XJ \\ X \end{bmatrix}$ is nonsingular. The second and third columns of X are generalized eigenvectors and for matrix polynomials, they can be equal to the zero vector. Then (3.13)–(3.14) with $Y = S_J X^T$ and $m = 2$ yield

$$A_2 = \frac{1}{4} \begin{bmatrix} -5 & \sqrt{2} & -2\sqrt{2} \\ \sqrt{2} & -2 & 0 \\ -2\sqrt{2} & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 5 & -\sqrt{2} & \frac{3}{2\sqrt{2}} \\ -\sqrt{2} & 1 & \frac{1}{2} \\ \frac{3}{2\sqrt{2}} & \frac{1}{2} & -1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -5 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ \sqrt{2} & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & -\frac{1}{2} \end{bmatrix}.$$

Note that our construction of Hermitian and real symmetric quadratics extends the approach in [18], [48] which solves Problem **(P)** for real symmetric quadratics under the assumption that \mathcal{L} consists entirely of linear elementary divisors, i.e., $\ell_{ij} = 1$ for all $i = 1:s, j = 1:m_i$ (or equivalently the eigenvalues are all semisimple). Lancaster in [48] writes

$$X = [X_{r_1} \quad X_{r_2} \quad X_c \quad \bar{X}_c],$$

where X_c is an $n \times (n - r)$ matrix corresponding to complex eigenvalues in the upper half plane, X_{r_1} (X_{r_2}) is a real $n \times r$ matrix corresponding to the real eigenvalues for which $\varepsilon_j = 1$ ($\varepsilon_j = -1$), and $0 < r < n$. Assuming X_{r_1} and X_{r_2} are given together with \mathcal{L} ,

$$XS_JX^* = 0 \Leftrightarrow X_cX_c^T + \overline{X_cX_c^T} = -X_{r_1}X_{r_1}^T + X_{r_2}X_{r_2}^T.$$

Thus, $X_cX_c^T = R_1 - iR$ where $R_1 = \frac{1}{2}(-X_{r_1}X_{r_1}^T + X_{r_2}X_{r_2}^T)$ and R is a real symmetric matrix which is chosen so that $R_1 - iR$ has rank $n - r$. Takagi's factorization is then the tool to recover X_c , see [48, App. A] or [42, Section 4.4.4].

Generating quasidefinite quadratics

Recall that an eigenvalue λ_0 is of definite type if its partial multiplicities are all equal to 1 and the sign (in the sign characteristic) attached to each partial multiplicity is the same (either positive or negative) and a definite type eigenvalue with positive(negative) sign is said to be of positive(negative) type. Note that the eigenvalue 2 in Example 4.3.1 is not of definite type.

The task of constructing an appropriate self-adjoint Jordan triple defined by the three matrices X , J and S_J to produce quasidefinite quadratics follows directly from Section 4.3.1. So for a quasihyperbolic quadratic,

$$J = \text{diag}(\lambda_1, \dots, \lambda_{2n}), \quad S_J = \text{diag}(\varepsilon_1, \dots, \varepsilon_{2n}),$$

with $\varepsilon_j = \pm 1$ such that $\varepsilon_j = \varepsilon_k$ if $\lambda_j = \lambda_k$ and $\sum_{j=1}^{2n} \varepsilon_j = 0$ (see (4.7)). The matrix X such that (X, J, S_JX^\star) forms an \mathcal{S} -structured Jordan triple can be constructed as in (4.8).

Example 4.3.2 Let $J = \text{diag}(-1, -2, -3, -4)$ and let $\tilde{X}_1 = \Theta = I_2$ in (4.8).

If $S_J = \text{diag}(1, -1, 1, -1)$ then $W = \begin{bmatrix} e_1 & e_3 & e_2 & e_4 \end{bmatrix}$, where e_j is the j th column of I_4 , $X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and on using (3.13)–(3.14) with $m = 2$ and $Y = S_JX^T$,

$$A_2 = I_2, \quad A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}, \quad (4.9)$$

define a quasihyperbolic quadratic.

Applying the homogeneous rotation $G = -\sqrt{2}/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ on J provides the quasidefinite $\tilde{Q}(\tilde{\lambda}) = \tilde{\lambda}^2 \tilde{A}_2 + \tilde{\lambda} \tilde{A}_1 + \tilde{A}_0$ with $\sigma(\tilde{Q}) = \{5/3, 2, 3, \infty\}$ where

$$\tilde{A}_2 = \frac{1}{2}(A_2 - A_1 + A_0) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\tilde{A}_1 = A_2 - A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -11 \end{bmatrix}, \quad \tilde{A}_0 = \frac{1}{2}(A_2 + A_1 + A_0) = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}.$$

If $S_J = \text{diag}(1, 1, -1, -1)$ then $W = I_4$ and (3.13)–(3.14) with $m = 2$ and $Y = S_J X^T$ yield

$$A_2 = \frac{1}{2}I_2, \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 3/2 & 0 \\ 0 & 4 \end{bmatrix}, \quad (4.10)$$

which defines a hyperbolic quadratic that is overdamped since all of its eigenvalues are negative. The two quadratics defined in (4.9) and (4.10) have the same eigenvalues but different properties. The quadratic defined in (4.9) cannot be linearized into a definite pencil whereas the quadratic defined in (4.10) can.

4.3.2 Complex symmetric quadratics

There are no additional constraints on the list \mathcal{L} of elementary divisors in (4.4). If \mathcal{L} consists of $(\lambda - \lambda_j)^{\ell_j}$, $\lambda_j \in \mathbb{C}$, $j = 1:t$ with $\sum_{j=1}^t \ell_j = 2n$ then the corresponding complex Jordan matrix is $J = \bigoplus_{j=1}^t J_{\ell_j}(\lambda_j)$. The matrix S_J , using Section 3.5.3, takes the form

$$S_J = S_J^{-1} = \bigoplus_{j=1}^t F_{\ell_j}.$$

Since $S_J = S_J^T \in \mathbb{R}^{2n \times 2n}$, $S_J^2 = I$, there exists an orthogonal matrix W such that $W^T S_J W = \begin{bmatrix} I_n & 0 \\ 0 & \Sigma^2 \end{bmatrix}$, where $\Sigma = \begin{bmatrix} I_{n-p} & 0 \\ 0 & iI_p \end{bmatrix}$ and $p \leq n$ depends on the number of odd partial multiplicities ℓ_j ($p = 0$ when all eigenvalues are semisimple). If we let $\tilde{X} = XW = [\tilde{X}_1 \quad \tilde{X}_2]$ with \tilde{X}_1 nonsingular and write $\tilde{X}_2 = i\tilde{X}_1 \Theta \Sigma$ for some complex orthogonal $\Theta \in \mathbb{C}^{n \times n}$ then since $v_S(J) = I$,

$$X S_J X^T = \tilde{X}_1 \tilde{X}_1^T + \tilde{X}_2 \Sigma^2 \tilde{X}_2^T = 0.$$

Hence we can build X as $X = \tilde{X}W^T = [\tilde{X}_1 \quad i\tilde{X}_1\Theta\Sigma]W^T$ which is completely determined by \tilde{X}_1 , Σ and Θ .

Example 4.3.3 Consider the complex symmetric structure and $\mathcal{L} = \{(\lambda - 2)^2, (\lambda - 2), (\lambda - i)\}$. Then

$$J = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \oplus [2] \oplus [i], \quad S_J = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus [1] \oplus [1].$$

With the choice $\tilde{X}_1 = \Theta = I_3$, we obtain

$$X = \begin{bmatrix} -\sqrt{2}i & \sqrt{2}i & -i & 0 \\ \sqrt{2} & \sqrt{2} & 0 & -i \end{bmatrix}.$$

Since $\begin{bmatrix} XJ \\ X \end{bmatrix}$ is nonsingular, (3.13)–(3.14) with $m = 2$ and $Y = S_J X^T$ yield

$$A_2 = \begin{bmatrix} -2.4 - 0.2i & 0.2 - 0.4i \\ 0.2 - 0.4i & 0.4 + 0.2i \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 8.6 + 0.8i & -0.8 + 0.6i \\ -0.8 + 0.6i & -0.6 - 0.8i \end{bmatrix}, \quad A_0 = \begin{bmatrix} -7.6 - 0.8i & 0.8 + 0.4i \\ 0.8 + 0.4i & -0.4 + 0.8i \end{bmatrix}.$$

4.3.3 *-even quadratics

It follows from [66] and Section 3.5.4 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1:r$, and
- t pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1:t$,

with $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^t m_j = 2n$. The corresponding complex Jordan matrix is

$$J = -i \left(\bigoplus_{j=1}^r J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^t (J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)) \right).$$

The matrix S_J given in Section 3.5.4 satisfies $S_J = -S_J^*$, $JS_J = (JS_J)^*$ and must be such that

$$\text{sig}(iS_J) = \sum_{j=1}^r \frac{1}{2}(1 - (-1)^{\ell_j})\varepsilon_j = 0. \quad (4.11)$$

Note that iS_J is real symmetric and on using (4.11) we see that there exists an orthogonal matrix W such that $W^T S_J W = -i \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$. As for the Hermitian case in Section 4.3.1 we can choose X to be of the form

$$X = [\tilde{X}_1 \quad \tilde{X}_1 \Theta] W^T,$$

where $\tilde{X}_1, \Theta \in \mathbb{C}^{n \times n}$ with \tilde{X}_1 nonsingular and Θ unitary.

Example 4.3.4 Consider the complex *-even structure and $\mathcal{L} = \{\lambda, (\lambda - 2i), (\lambda - i)^2, (\lambda - (1 + i)), (\lambda - (-1 + i))\}$ with sign characteristic $\{-1, +1, -1\}$ so that (4.11) holds with $(\ell_1, \varepsilon_1) = (1, -1)$, $(\ell_2, \varepsilon_2) = (1, 1)$, $(\ell_3, \varepsilon_3) = (2, -1)$. Then

$$J = -i \left([0] \oplus [-2] \oplus \begin{bmatrix} -1 & 1 \\ & -1 \end{bmatrix} \oplus [-1 - i] \oplus [i - 1] \right),$$

$$S_J = -i \left((-1)[1] \oplus [1] \oplus (-1) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right).$$

With the choice $\tilde{X}_1 = \Theta = I_3$, we obtain

$$X = \begin{bmatrix} 0 & 0 & \sqrt{2}i & -\sqrt{2}i & -\sqrt{2}i & \sqrt{2}i \\ -1 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & -i & -\sqrt{2}i & -\sqrt{2}i & 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} XJ \\ X \end{bmatrix}$ is nonsingular. Now, (3.13)–(3.14) with $m = 2$ and $Y = S_J X^*$ yield

$$A_2 = \begin{bmatrix} -1.5 & 1.5 & -0.5 \\ 1.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 5i & -2i & 2i \\ -2i & -i & -i \\ 2i & -i & -i \end{bmatrix}, \quad A_0 = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

4.3.4 Real T -even quadratics

It follows from [66, Thm. 4.2] and Section 3.5.5 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r zero elementary divisors λ^{n_j} with n_j even, $j = 1:r$,
- t pairs of real elementary divisors $(\lambda + \alpha_j)^{\ell_j}$, $(\lambda - \alpha_j)^{\ell_j}$ with ℓ_j odd if $\alpha_j = 0$, $j = 1:t$,
- q pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1:q$,
- p quadruples of nonreal and non-purely imaginary elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \bar{\mu}_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$, $j = 1:p$,

with $\sum_{j=1}^r n_j + 2 \sum_{j=1}^t \ell_j + 2 \sum_{j=1}^q k_j + 4 \sum_{j=1}^p m_j = 2n$. The corresponding real Jordan matrix is

$$J = \bigoplus_{j=1}^r J_{n_j}(0) \oplus \bigoplus_{j=1}^t (J_{\ell_j}(\alpha_j) \oplus -J_{\ell_j}(\alpha_j)^T) \\ \oplus \bigoplus_{j=1}^q K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^p (K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T).$$

The matrix S_J given in Section 3.5.5 satisfies $S_J = -S_J^T$ and $JS_J = (JS_J)^T$.

Now, from the Takagi factorization of real skew-symmetric matrices (see [42, Section 4.4, problems 25-26]) and the form of S_J , we notice that there exists an orthogonal matrix W such that $W^T S_J W = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. If we let $\tilde{X} = XW = [\tilde{X}_1 \quad \tilde{X}_2]$ with $\tilde{X}_1 \in \mathbb{R}^{n \times n}$ of full rank and write $\tilde{X}_2 = \tilde{X}_1 H$ for some real symmetric matrix H then

$$X S_J X^T = -\tilde{X}_2 \tilde{X}_1^T + \tilde{X}_1 \tilde{X}_2^T = -\tilde{X}_1 H \tilde{X}_1^T + \tilde{X}_1 H \tilde{X}_1^T = 0.$$

Hence we can build X as $X = \tilde{X} W^T = [\tilde{X}_1 \quad \tilde{X}_1 H] W^T$ for some $\tilde{X}_1, H \in \mathbb{R}^{n \times n}$ with H symmetric.

Note that H can be the zero matrix. A simple 1×1 example is $Q(\lambda) = 1 - \lambda^2$ which is built by

$$J = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad S_J = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad W = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{-1-i}{2} \end{bmatrix},$$

with $\tilde{X}_1 = 1$ and $H = 0$.

4.3.5 Complex T -even quadratics

It follows from [66, Thm. 4.2] and Section 3.5.6 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r zero elementary divisors λ^{m_j} with m_j even, $j = 1:r$, and
- t pairs of elementary divisors $(\lambda - \lambda_j)^{\ell_j}$, $(\lambda + \lambda_j)^{\ell_j}$ with ℓ_j odd if $\lambda_j = 0$, $j = 1:t$,

with $\sum_{j=1}^r m_j + 2 \sum_{j=1}^t \ell_j = 2n$. The corresponding complex Jordan matrix is

$$J = \bigoplus_{j=1}^r J_{m_j}(0) \oplus \bigoplus_{j=1}^t (J_{\ell_j}(\lambda_j) \oplus J_{\ell_j}(-\lambda_j)).$$

From the structure of the matrix S_J that is given in Section 3.5.6, there exists an orthogonal matrix W such that $W^T S_J W = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. The rest of the construction is similar to that for real T -even quadratics in Section 4.3.4, except that H must now be complex symmetric.

4.3.6 *-odd quadratics

It follows from [66] and Section 3.5.7 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r purely imaginary (including 0) elementary divisors $(\lambda - i\beta_j)^{\ell_j}$, $j = 1:r$, and
- t pairs of nonzero and non-purely imaginary elementary divisors $(\lambda - i\mu_j)^{m_j}$, $(\lambda - i\bar{\mu}_j)^{m_j}$, $j = 1:t$,

with $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^t m_j = 2n$. The corresponding complex Jordan matrix is

$$J = -i \left(\bigoplus_{j=1}^r J_{\ell_j}(-\beta_j) \oplus \bigoplus_{j=1}^t (J_{m_j}(-\bar{\mu}_j) \oplus J_{m_j}(-\mu_j)) \right).$$

The matrix S_J given in Section 3.5.7 satisfies $S_J = S_J^{-1} = S_J^*$, $JS_J = -(JS_J)^*$ and must be such that

$$\text{sig}(S_J) = \sum_{j=1}^r \frac{1}{2} (1 - (-1)^{\ell_j}) \varepsilon_j = 0.$$

The construction of \mathcal{S} -structured Jordan triples follows that of Hermitian quadratics in Section 4.3.1. In this case, for the leading matrix coefficient of $Q(\lambda)$ to be nonsingular, n must be even.

4.3.7 Real T -odd quadratics

It follows from [66, Thm. 4.2] and Section 3.5.8 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1:r$,
- t pairs of real elementary divisors $(\lambda + \alpha_j)^{n_j}$, $(\lambda - \alpha_j)^{n_j}$ with n_j even if $\alpha_j = 0$, $j = 1:t$,
- q pairs of purely imaginary elementary divisors $(\lambda + i\beta_j)^{k_j}$, $(\lambda - i\beta_j)^{k_j}$ with $\beta_j > 0$, $j = 1:q$, and
- p quadruples of nonreal and non-purely elementary divisors $(\lambda + \mu_j)^{m_j}$, $(\lambda - \mu_j)^{m_j}$, $(\lambda + \bar{\mu}_j)^{m_j}$, $(\lambda - \bar{\mu}_j)^{m_j}$ $j = 1:p$.

with $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^t n_j + 2 \sum_{j=1}^q k_j + 4 \sum_{j=1}^p m_j = 2n$. The corresponding real Jordan matrix is

$$\begin{aligned}
 J = & \bigoplus_{j=1}^r J_{\ell_j}(0) \oplus \bigoplus_{j=1}^t (J_{n_j}(\alpha_j) \oplus -J_{n_j}(\alpha_j)^T) \\
 & \oplus \bigoplus_{j=1}^q K_{2k_j}(i\beta_j, -i\beta_j) \oplus \bigoplus_{j=1}^p (K_{2m_j}(\mu_j, \bar{\mu}_j) \oplus -K_{2m_j}(\mu_j, \bar{\mu}_j)^T).
 \end{aligned}$$

The matrix S_J given in Section 3.5.8 satisfies $S_J = S_J^{-1} = S_J^T$ and $S_J^2 = I$. The construction of \mathcal{S} -structured Jordan triples follows that of real symmetric quadratics in Section 4.3.1.

Note that for $\mathcal{S} \in \{\text{real } T\text{-even}\}$ ($\mathcal{S} \in \{\text{real } T\text{-odd}\}$), $S_J (JS_J)$ is a skew-symmetric matrix and thus $\text{sig}(iS_J) = 0$ ($\text{sig}(iJS_J) = 0$). Using the form of S_J for real T -even (real T -odd) in Section 3.5, $\text{sig}(iS_J) = 0$ ($\text{sig}(iJS_J) = 0$) does not impose any constraint on ε_j 's.

4.3.8 Complex T -odd quadratics

It follows from [66, Thm. 4.2] and Section 3.5.9 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r zero elementary divisors λ^{ℓ_j} with ℓ_j odd, $j = 1:r$, and
- t pairs of elementary divisors $(\lambda - \lambda_j)^{m_j}$, $(\lambda + \lambda_j)^{m_j}$ with m_j even if $\lambda_j = 0$, $j = 1:t$,

with $\sum_{j=1}^r \ell_j + 2 \sum_{j=1}^t m_j = 2n$. The corresponding complex Jordan matrix is

$$J = \bigoplus_{j=1}^r J_{\ell_j}(0) \oplus \bigoplus_{j=1}^t (-J_{m_j}(\lambda_j) \oplus J_{m_j}(\lambda_j)).$$

The matrix S_J given in Section 3.5.9 satisfies $S_J = S_J^{-1} = -S_J^T$ and $JS_J = (JS_J)^T$.

Consequently, there exists an orthogonal matrix W such that $W^T S_J W = \sigma \begin{bmatrix} I_n & 0 \\ 0 & \Sigma^2 \end{bmatrix}$, where $\sigma = \pm 1$ and $\Sigma = \begin{bmatrix} I_{n-p} & 0 \\ 0 & iI_p \end{bmatrix}$ and $p \leq n$. The construction of X is then analogous to that for complex symmetric quadratics in Section 4.3.2.

4.3.9 \star -(anti)-palindromic quadratics

For simplicity, we assume that $-1 \notin \sigma(Q)$. It follows from [68, Thm. 7.10] and Section 3.5.10 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- r elementary divisors $(\lambda - \lambda_j)^{2\ell_j+1}$ with $\lambda_j \in \mathbb{C}$ such that $|\lambda_j| = 1$, $j = 1:r$,
- t elementary divisors $(\lambda - \lambda_j)^{2m_j}$ with $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, $j = 1:t$, and
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}$, $(\lambda - 1/\bar{\lambda}_j)^{k_j}$ with $\lambda_j \in \mathbb{C} \setminus \{0\}$, $|\lambda_j| \neq 1$, $j = 1:q$,

with $\sum_{j=1}^r (2\ell_j+1) + 2 \sum_{j=1}^t m_j + 2 \sum_{j=1}^q k_j = 2n$. The corresponding complex Jordan-like matrix is $J = -S_J S_J^{-\star}$ with S_J as in Section 3.5.10. The sign characteristic associated with the eigenvalues of unit modulus of $Q(\lambda)$ must satisfy a constraint similar to (4.11). However, we will not go through the details here as we can use the correspondence via Cayley transformations between \star -even/odd quadratics and \star -(anti)-palindromic quadratics as described in [65, Chap. 6]. The effect of Cayley transformations on the sign characteristic is left for future investigation.

Consider for example using the complex *-even case to handle the *-palindromic one. Assume that the list of elementary divisors of a complex *-palindromic quadratic matrix polynomial $Q(\lambda)$ is given as above. Since $-1 \notin \sigma(Q)$, we can apply the Cayley transformation

$$\lambda_j \mapsto \rho_j := \frac{\lambda_j - 1}{\lambda_j + 1} \quad (4.12)$$

on the given list \mathcal{L} to obtain $\tilde{\mathcal{L}}$. The transformation (4.12) maps 1 to zero, other eigenvalues of unit modulus are mapped to the imaginary axis. The pairs $(\lambda_j, 1/\bar{\lambda}_j)$ for which $\lambda_j \neq 0$ and $|\lambda_j| \neq 1$ are mapped to $(\rho_j, -\bar{\rho}_j)$, $\rho_j \neq 0$. Now, we construct J and S_J according to Section 4.3.3 to build a complex *-even $\tilde{Q}(\lambda)$ that corresponds to $\tilde{\mathcal{L}}$. Finally, $Q(\lambda) = (\lambda + 1)^2 \tilde{Q}(\frac{\lambda-1}{\lambda+1})$ is a complex *-palindromic quadratic matrix polynomial with elementary divisor list \mathcal{L} .

Example 4.3.5 Consider the complex *-palindromic structure and

$$\mathcal{L} = \left\{ (\lambda - 1), \left(\lambda - \frac{4i - 3}{5} \right), (\lambda - i)^2, (\lambda - (2i - 1)), \left(\lambda - \frac{1}{2i - 1} \right) \right\}.$$

The transformation (4.12) maps \mathcal{L} to

$$\tilde{\mathcal{L}} = \{ \lambda, (\lambda - 2i), (\lambda - i)^2, (\lambda - (1 + i)), (\lambda - (-1 + i)) \},$$

which, with sign characteristic $\{-1, +1, -1\}$, leads to the quadratic $\tilde{Q}(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ of Example 4.3.4. Now,

$$Q(\lambda) = \lambda^2 (A_2 + A_1 + A_0) + \lambda (-2A_2 + 2A_0) + (A_2 - A_1 + A_0)$$

is *-palindromic with elementary divisor list \mathcal{L} .

Note that concerning the case where $-1 \in \sigma(Q)$ but $1 \notin \sigma(Q)$, the admissible list of elementary divisors together with the corresponding matrices J and S_J can be derived similarly from [68] and [87, Section 2.2.2]. The construction of Q then, using a suitable Cayley transformation, is along the same lines as the case where $-1 \notin \sigma(Q)$.

4.3.10 Real T -(anti)-palindromic quadratics

We assume that $-1 \notin \sigma(Q)$. It follows from [68, Thm. 7.6] and Section 3.5.11 that the list of elementary divisors \mathcal{L} in (4.4) is made up of

- p pairs of real elementary divisors $(\lambda - \lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with $\lambda_j \in \mathbb{R} \setminus \{0\}$, $|\lambda_j| \neq 1$, $j = 1:p$,
- q quadruples of nonreal elementary divisors $(\lambda - \lambda_j)^{n_j}, (\lambda - \bar{\lambda}_j)^{n_j}, (\lambda - 1/\lambda_j)^{n_j}, (\lambda - 1/\bar{\lambda}_j)^{n_j}$ with $|\lambda_j| \neq 1$, $\lambda_j \neq 0$, $j = 1:q$,
- v elementary divisors $(\lambda - 1)^{2m_j}$, $j = 1:v$,
- t pairs of elementary divisors $(\lambda - 1)^{2\ell_j+1}, (\lambda - 1)^{2\ell_j+1}$ $j = 1:t$,
- u pairs of elementary divisors $(\lambda - \lambda_j)^{\ell'_j}, (\lambda - \bar{\lambda}_j)^{\ell'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, ℓ'_j odd, $j = 1:u$,
- r pairs of elementary divisors $(\lambda - \lambda_j)^{m'_j}, (\lambda - \bar{\lambda}_j)^{m'_j}$ with $|\lambda_j| = 1$, $\lambda_j \neq 1$, m'_j even, $j = 1:r$,

with $2 \sum_{j=1}^p k_j + 4 \sum_{j=1}^q n_j + 2 \sum_{j=1}^v m_j + 2 \sum_{j=1}^t (2\ell_j + 1) + 2 \sum_{j=1}^u \ell'_j + 2 \sum_{j=1}^r m'_j = 2n$. The corresponding real Jordan-like matrix is $J = -S_J S_J^{-T}$ with S_J as in Section 3.5.11.

The rest follows from the case of real T -even/real T -odd quadratics as explained in Section 4.3.9.

4.3.11 Complex T -(anti)-palindromic quadratics

We assume that $-1 \notin \sigma(Q)$. It follows from [68, Thm. 7.6] and Section 3.5.12 that the list of elementary divisors \mathcal{L} in (4.4) for a complex T -palindromic $Q(\lambda)$ is made up of

- t elementary divisors $(\lambda - 1)^{m_j}$ with m_j even, $j = 1:t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with k_j odd when $\lambda_j = 1$, $j = 1:q$,

with $\sum_{j=1}^t m_j + 2 \sum_{j=1}^q k_j = 2n$. The corresponding complex Jordan-like matrix is $J = -S_J S_J^{-T}$ with S_J as in Section 3.5.12.

Now for a complex T -antipalindromic $Q(\lambda)$ with $-1 \notin \sigma(Q)$, see [68, Thm. 7.6], \mathcal{L} in (4.4) must be made of

- t elementary divisors $(\lambda - 1)^{\ell_j}$ with ℓ_j odd, $j = 1:t$,
- q pairs of elementary divisors $(\lambda - \lambda_j)^{k_j}, (\lambda - 1/\lambda_j)^{k_j}$ with k_j even if $\lambda_j = 1$, $j = 1:q$,

with $\sum_{j=1}^t \ell_j + 2 \sum_{j=1}^q k_j = 2n$. The corresponding complex Jordan-like matrix $J = -S_J S_J^{-T}$ with the corresponding S_J as in Section 3.5.12.

The rest follows from the case of complex T -even/complex T -odd quadratics as explained in Section 4.3.9.

4.4 Other structured inverse eigenvalue problems

Now, we consider the problems of building a cubic $P(\lambda)$ that belongs to some classes presented in Figure 2.1 and generating hyperbolic tridiagonal quadratics.

4.4.1 Generating quasidefinite cubics

By Theorem 3.3.5 and (3.13)–(3.14), an (m, n) -self-adjoint Jordan triple $(X, J, S_J X^*)$ generates a uniquely defined $P(\lambda) = \sum_{j=0}^m \lambda^j A_j \in \mathcal{P}_S(\mathbb{F}^n)$ where A_j , given by (3.13)–(3.14) for $j = 0:m$ with $Y = S_J X^*$, are Hermitian and [29, Thm. 1.3] shows that

$$\text{sig}(S_J) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ \text{sig}(A_m) & \text{if } m \text{ is odd.} \end{cases} \quad (4.13)$$

The columns of X determine the eigenvectors of a quasihyperbolic $P(\lambda)$. In order to build quasihyperbolic cubics, we want to construct an $n \times 3n$ matrix X , a $3n \times 3n$ real Jordan matrix J and a $3n \times 3n$ matrix S_J such that

$$\det \begin{bmatrix} X J^2 \\ X J \\ X \end{bmatrix} \neq 0, \quad \begin{cases} X S_J X^* = 0 \\ X J S_J X^* = 0 \end{cases}, \quad \det(X J^2 S_J X^*) \neq 0. \quad (4.14)$$

Theorem 4.4.1 (quasihyperbolic cubic) *For a given $n \times n$ matrix $S_1 = \text{diag}(\pm 1)$, a given $n \times n$ real diagonal matrix J_1 and a given $n \times n$ matrix X_1 with columns normalized such that $I_n - X_1 S_1 X_1^\star > 0$, let X_2 be the Hermitian positive definite square root of $I_n - X_1 S_1 X_1^\star$ and let X_3, J_3 be such that $X_3 J_3 X_3^\star = X_1 J_1 S_1 X_1^\star + X_2 J_2 X_2^\star$, where J_2 is diagonal with real entries chosen such that $X_1 J_1^2 S_1 X_1^\star + X_2 J_2^2 X_2^\star - X_3 J_3^2 X_3^\star$ is nonsingular. Define*

$$X = [X_1 \quad X_2 \quad X_3], \quad J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}, \quad S_J = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & -I_n \end{bmatrix}. \quad (4.15)$$

If J has pairwise distinct diagonal entries then $(X, J, S_J X^\star)$ is a $(3, n)$ -self-adjoint Jordan triple that generates a unique quasihyperbolic cubic matrix polynomial $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$. Moreover if $S_1 = I_n$ and J_1, J_2 are chosen so that

$$\lambda_{\max}(J_2) < \lambda_{\min}(J_3) \leq \lambda_{\max}(J_3) < \lambda_{\min}(J_1), \quad (4.16)$$

then $P(\lambda)$ is hyperbolic.

Proof. Note that since $S_J = S_J^\star$ and $J S_J = (J S_J)^\star$, $(X, J, S_J X^\star)$ is a $(3, n)$ -self-adjoint Jordan triple if the constraints in (4.14) hold. Note that we just have to check the first two constraints since the last one holds by assumption. The matrix X is clearly of full rank because X_2 is of full rank and, because J has distinct entries, the first constraint in (4.14) is satisfied. For the second constraint we have that

$$\begin{aligned} X S_J X^\star &= X_1 S_1 X_1^\star + X_2 X_2^\star - X_3 X_3^\star \\ &= X_1 S_1 X_1^\star + I_n - X_1 S_1 X_1^\star - I_n = 0, \\ X J S_J X^\star &= X_1 J_1 S_1 X_1^\star + X_2 J_2 X_2^\star - X_3 J_3 X_3^\star = 0. \end{aligned}$$

If $S_1 = I_n$ then by (4.13), $A_3 > 0$. If (4.16) holds then $J_2 < J_3 < J_1$ and hence all eigenvalues belong to three distinct intervals on \mathbb{R} , each containing n eigenvalues of one type and the eigenvalue type of each interval alternates in sign with the right most interval being of positive type. It follows from Theorem 2.3.4 that $P(\lambda)$ is hyperbolic. ■

Note that one can use Weyl's theorem [42, p. 181] together with [42, Thm. 4.5.9] to chose X_1, J_1, J_2 so that (4.16) is satisfied.

Example 4.4.2 For $n = 2$ the matrices

$$J_1 = \begin{bmatrix} 3.5 & 0 \\ 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -1.5 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_1 = I_2,$$

and

$$X_1 = \begin{bmatrix} -0.8819i & 0.4714i \\ 0.4714i & 0.8819i \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.4142 & 0 \\ 0 & 1.4142 \end{bmatrix},$$

satisfy the assumptions of Theorem 4.4.1 where $\star = T$ and thus, with $X_3 J_3 X_3^T$ being the eigendecomposition of $X_1 J_1 X_1^T + X_2 J_2 X_2^T$, define a $(3, 2)$ -self-adjoint Jordan triple that uniquely determines a hyperbolic cubic matrix polynomial whose coefficient matrices are given by (3.13)–(3.14) with $m = 3$ and $Y = S_J X^T$.

Once more, homogeneous rotations are the tool to generate quasidefinite cubics with a singular leading coefficient.

Note that, for any m , we may use Section 2.5 to generate (diagonal) hyperbolic matrix polynomials by choosing real eigenvalues that respect the characterization (P1) in Theorem 2.3.4. These can be used to generate infinitely many (full) hyperbolic matrix polynomials. The same applies for quasihyperbolics that are diagonalizable by structure preserving congruences.

4.4.2 Generating tridiagonal hyperbolic quadratics

Ram and Elhay in [81] solve an IPEP for symmetric tridiagonal quadratic matrix polynomials in the following settings: Given two sets of distinct complex numbers $\{\lambda_j\}_{j=1}^{2n}$ and $\{\mu_j\}_{j=1}^{2n-2}$, two tridiagonal $n \times n$ symmetric matrices A_1 and A_0 are determined such that

$$\sigma(Q) = \{\lambda_j\}_{j=1}^{2n}, \quad \sigma(\widehat{Q}) = \{\mu_j\}_{j=1}^{2n-2}, \quad (4.17)$$

where $Q(\lambda) = \lambda^2 I_n + \lambda A_1 + A_0$ and $\widehat{Q}(\lambda)$ is the matrix polynomial obtained by deleting the last row and column of $Q(\lambda)$. Using [81, Alg. 2.1], we can generate infinitely many

quasihyperbolic tridiagonal symmetric polynomials starting with $\{\lambda_j\}_{j=1}^{2n} \subset \mathbb{R}$. Note that all λ_j 's are of definite type as they are simple eigenvalues. We recall the following algorithms.

Algorithm 4.4.1 [81, Alg. 3.1]

Input A set $\{(z_j, y_j)\}_{j=1}^m$, where $m \geq 2$ and z_j distinct.

Output At most $2(m-1)$ solution sets $\{r, \{d_j\}_{j=0}^m\}$. Each set of d_j defines a polynomial $y(z) = \sum_{j=0}^m d_j z^j$ for which r is double zero and $y(z_j) = y_j$, $j = 1:m$.

The algorithm

(a) Solve the linear systems of equations

$$Vg = [y_1 \ y_2 \ \cdots \ y_m]^T, \quad Vh = [z_1^m \ z_2^m \ \cdots \ z_m^m]^T,$$

for the vectors g and h where

$$V = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{m-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_m & z_m^2 & \cdots & z_m^{m-1} \end{bmatrix}.$$

(b) Form the polynomials $\pi_1(z) = \sum_{j=1}^m g_j z^{j-1}$, $\pi_2(z) = -z^m + \sum_{j=1}^m h_j z^{j-1}$ and $\theta(z) = \pi_1(z)\pi_2'(z) - \pi_2(z)\pi_1'(z)$.

(c) Solve $\theta(z) = 0$.

(d) For each zero r of $\theta(z)$,

(i) Use $\pi_1(r) - d_m \pi_2(r) = 0$ to find d_m .

(ii) Use $d = g - d_m h$ to determine the vector $d = [d_0 \ d_1 \ \cdots \ d_{m-1}]^T$.

Algorithm 4.4.2 [81, Alg. 2.1]

Input Two sets of distinct eigenvalues $\{\lambda_k^{(n)}\}_{k=1}^{2n}$ and $\{\lambda_k^{(n-1)}\}_{k=1}^{2n-2}$.

Output Two symmetric tridiagonal matrices

$$A_1 = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & \alpha_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{bmatrix}, \quad A_0 = \begin{bmatrix} \gamma_1 & \delta_1 & 0 & \cdots & 0 \\ \delta_1 & \gamma_2 & \delta_2 & \cdots & 0 \\ 0 & \delta_2 & \gamma_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_n \end{bmatrix},$$

for which $\sigma(Q) := \sigma(\lambda^2 I_n + \lambda A_1 + A_0) = \{\lambda_k^{(n)}\}_{k=1}^n$ and $\sigma(\widehat{Q}) = \{\lambda_k^{(n-1)}\}_{k=1}^{2n-2}$.

The algorithm

(a) For $k = n, n-1, \dots, 2$,

(i) Compute $y_j = y(\lambda_j^{(k-1)}) = -\prod_{i=1}^{2k} (\lambda_j^{(k-1)} - \lambda_i^{(k)})$, $j = 1:2k-2$.

(ii) Choose one solution set $\{r, \{c_i\}_{i=0}^{2k-2}\}$ from the choices produced by applying Algorithm 4.4.1 to $\{(\lambda_j^{(k-1)}, y_j)\}_{j=1}^{2k-2}$ and define $p(\lambda) = \sum_{i=0}^{2k-2} c_i \lambda^i$.

(iii) Assign

$$\alpha_k = \sum_{i=1}^{2k-2} \lambda_i^{(k-1)} - \sum_{i=1}^{2k} \lambda_i^{(k)}, \quad \beta_{k-1} = \pm \sqrt{c_{2k-2}},$$

$$\gamma_k = \frac{\prod_{i=1}^{2k} (r - \lambda_i^{(k)})}{\prod_{i=1}^{2k-2} (r - \lambda_i^{(k-1)})} - r^2 - r\alpha_k, \quad \delta_{k-1} = -r\beta_{k-1}.$$

(iv) If $k > 2$,

(1) Determine the polynomial $q_{2k-4}(\lambda)$ of degree $2k-4$ by the synthetic division of $p(\lambda)$ with the quadratic factor $(\beta_{k-1}\lambda + \delta_{k-1})^2$.

(2) Find the zeros $\{\lambda_i^{(k-2)}\}_{i=1}^{2k-4}$ of q_{2k-4} .

(b) Assign

$$\alpha_1 = -(\lambda_1^{(1)} + \lambda_2^{(1)}), \quad \gamma_1 = \lambda_1^{(1)} \lambda_2^{(1)}.$$

Example 4.4.3 Given $\{\lambda_j\}_{j=1}^6 = \{-6.5, -5, -4, -2, -1.5, 3\}$, we may take $\{\mu_j\}_{j=1}^4 =$

$\{-3, -1, 0, 2\}$ and construct the following quasihyperbolic tridiagonal polynomial using Algorithm 4.4.1 and Algorithm 4.4.2.

$$Q(\lambda) = \lambda^2 I_3 + \lambda \begin{bmatrix} 4.0400 & 0.4834 & 0 \\ 0.4834 & -2.0400 & 18.7285 \\ 0 & 18.7285 & 14 \end{bmatrix} + \begin{bmatrix} 3.1312 & 1.0388 & 0 \\ 1.0388 & 0.3449 & -19.3303 \\ 0 & -19.3303 & 406.5050 \end{bmatrix},$$

with $\sigma(Q) = \{\lambda_j\}_{j=1}^6$ and $\sigma(\widehat{Q}) = \{\mu_j\}_{j=1}^4$.

Now, if the interest is in generating hyperbolic quadratic tridiagonal symmetric matrix polynomials then extra conditions must be imposed. Theorem 2.3.5 is useful in solving the SIPEP of generating hyperbolic quadratic tridiagonal matrix polynomials given two sets of real numbers $\sigma(Q) = \{\lambda_j\}_{j=1}^{2n}$ and $\sigma(\widehat{Q}) = \{\mu_j\}_{j=1}^{2n-2}$ that satisfy 1,2 and 3 of Theorem 2.3.5. This problem is always solvable, [9, Thm. 5.1].

Example 4.4.4 Given

$$\{\lambda_j\}_{j=1}^6 = \{-7, -5, -4, -3, -2, -0.5\}, \quad \{\mu_j\}_{j=1}^4 = \{-6, -4.5, -2.5, -1.5\},$$

we construct the following hyperbolic tridiagonal quadratic matrix polynomial which is consistent with the given data using Algorithm 4.4.1 and Algorithm 4.4.2.

$$\lambda^2 I_3 + \lambda \begin{bmatrix} 7.0484 & 1.2162 & 0 \\ 1.2162 & 7.4516 & 2.4121 \\ 0 & 2.4121 & 7 \end{bmatrix} + \begin{bmatrix} 9.89 & 3.9526 & 0 \\ 3.9526 & 11.8173 & 8.4171 \\ 0 & 8.4171 & 11.0684 \end{bmatrix}.$$

Note that, by Theorem 2.3.5, any matrix polynomial constructed from the data given in Example 4.4.3 cannot be hyperbolic. The tridiagonal matrices produced by Algorithm 4.4.2 are unique up to signs change along the subdiagonal, see (5) on [9, p. 35].

4.5 Concluding remarks

The hardest part in treating a SIPEP via building an associated \mathcal{S} -structured Jordan triple $(X, J, S_J v_{\mathcal{S}}(J)X^{\star})$ is to solve

$$XJ^j S_J v_{\mathcal{S}}(J)X^{\star} = 0, \quad j = 0:m-2, \quad (4.18)$$

for X . Equations (4.18) get more difficult to solve as m increases. We have shown how to solve these equations when $m = 2$ for structures $\mathcal{S} \in \mathbb{S}$, where

$$\begin{aligned} \mathbb{S} = \{ & \text{Hermitian, symmetric, *-even, *-odd, real } T\text{-even, real } T\text{-odd, complex } T\text{-even,} \\ & \text{complex } T\text{-odd, *-palindromic, *-antipalindromic, real } T\text{-palindromic,} \\ & \text{real } T\text{-antipalindromic, complex } T\text{-palindromic, complex } T\text{-antipalindromic} \}. \end{aligned}$$

We have discussed the related quasihyperbolic inverse eigenvalue problem for $m = 2, 3$. Building an \mathcal{S} -structured Jordan triple associated with an unknown matrix polynomial of degree greater than two given some spectral information is a challenge. Another challenge is to directly handle an IPEP when infinity is a prescribed eigenvalue. Approaches to solve (structured or unstructured) inverse quadratic eigenvalue problems without constructing standard triples are possible [64], [71].

Chapter 5

Conclusions and Future Work

The thesis provides an extensive study of a group of structured matrix polynomials that are associated with applications in science and engineering, in particular, vibration analysis in structural mechanics. Hermitian, symmetric real or complex, \star -alternating and \star -palindromic matrix polynomials have been investigated from different aspects. Hermitian matrix polynomials with only definite type eigenvalues in $\mathbb{R} \cup \{\infty\}$ are related to systems of differential equations with bounded solutions under small perturbations. We call such polynomials quasidefinite. The well known definite pencils and overdamped quadratics are just special cases of quasidefinite matrix polynomials. Other subclasses of quasidefinite matrix polynomials are hyperbolic, definite, quasihyperbolic and gyroscopically stabilized matrix polynomials. We have collected the quasidefinite matrix polynomials appearing in the literature and presented them in a unified framework, leading to a clear classification of these polynomials and their many subclasses, as shown in Figure 2.1. The main tools we have used to produce this diagram are homogeneous rotations of matrix polynomials and a new characterization of hyperbolic matrix polynomials that depends on the distribution of their eigenvalue types along the real line. This characterization itself relies mainly on the fact that a matrix polynomial is hyperbolic if and only if it has a definite linearization in $\mathbb{H}(P)$. Moreover, while studying quasihyperbolic matrix polynomials, we were able to spot a new class of diagonalizable matrix polynomials. Methods to determine whether a given matrix polynomial is overdamped, definite or hyperbolic have been highlighted.

We have developed a general theory of structured standard triples for structured

matrix polynomials by introducing the notion of \mathcal{S} -structured standard triples, which extends the known notion of self-adjoint standard triples linked to Hermitian matrix polynomials. With the exception of T -(anti)palindromic matrix polynomials of even degree with both -1 and 1 as eigenvalues, we have shown that $P(\lambda)$ has structure \mathcal{S} if and only if $P(\lambda)$ admits an \mathcal{S} -structured standard triple, and moreover that every standard triple of a matrix polynomial with structure \mathcal{S} is \mathcal{S} -structured. We have investigated the important special case of \mathcal{S} -structured Jordan triples. Indeed, it turns out that structured standard triples involve a parameter matrix, usually denoted by S as it is connected with the sign characteristic attached to real, purely imaginary eigenvalues or eigenvalues of modulus 1, according to the structure of the polynomial. The explicit form of S has been a topic of research in some recent papers that handle real symmetric and T -even quadratic matrix polynomials [18], [45]. However, we have shown that the explicit form of S , for any matrix polynomial with structure $\mathcal{S} \in \mathbb{S}$ of arbitrary degree, can be obtained easily by employing the existing literature on canonical forms of structured pencils.

Structured standard triples can be used to solve an SIPEP in the setting that some eigeninformation about the polynomial is given beforehand. The difficulty with this approach of treating the inverse problem comes from the conditions that the eigenvectors/generalized eigenvectors have to satisfy in order to define a structured standard triple. For structures in \mathbb{S} , we have dealt with a structured quadratic inverse eigenvalue problem and we have shown how to generate quasidefinite quadratic and cubic matrix polynomials by solving proper IPEPs. We are considering the possibility of having a MATLAB toolbox generating such structured quadratics and quasidefinite cubics. For more general structured matrix polynomials of higher degrees, the inverse problem is much more complicated. Another approach for handling structured quadratic inverse eigenvalue problems is the *quadratic realizability problem* (QRP) [64] which is to explicitly construct a specific $Q(\lambda)$ that realizes a given admissible list of elementary divisors. The solution of the QRP in [64] does not involve eigenvectors at all which allows more freedom and simplifies the inverse problem significantly. The quadratic matrix polynomials constructed in [64] are sparse and more specifically they consist of blocks that are lower anti-diagonal with very low (anti-)bandwidth. Structured version of the QRP, in particular Hermitian, is a topic of an ongoing research [71]. The structured version is more complicated due to the

required symmetries plus the role that the concept of sign characteristic plays.

There is a Cayley correspondence between \star -alternating and \star -palindromic matrix polynomials [65, Section 6.3]. In [2], we are working on a similar correspondence between Hermitian and \ast -palindromic matrix polynomials which leads to a clear definition of the eigenvalue type for eigenvalues of modulus 1 associated with a \ast -palindromic matrix polynomial of either even or odd degree. Consequently we can extend [55, Thm. 14], which provides stable boundedness conditions for the solutions of \ast -palindromic difference equations of even degree, to include the case of odd degree equations. The result we obtain is consistent with the one in [82] which handles the linear \ast -palindromic difference equations.

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