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Delay reconstruction for multiprobe signals

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Abstract

A physical system governed by low-dimensional dynamics may be described completely with just a few measurements. Once one has such a description, any further measurements are redundant—one ought to be able to determine the results from what one already knows. Here we apply this idea to multivariate time series; we use the signal in one of the channels to build a model of the underlying system, then use the model to predict all the other channels. We demonstrate the method on a signal from a fluid-mechanical experiment, then discuss the implications for signal compression and for the secrecy of messages masked by chaotic noise.

1 Introduction

Nonlinear systems can produce complex behaviour from simple dynamics—keen pursuit of genuine physical manifestations of this truism has produced a circle of new, geometric techniques for the analysis of time series. The main idea is that a single, univariate time series can contain enough information to to build a faithful model of the underlying system—a model good enough for, say, short term prediction [5, 3, 6], or the design of control systems [10]. Here we will show how to use a model built from one channel of a multivariate signal to reproduce the signals in all the other channels.

Figure 1 summarises the usual approach: one imagines that the signal is a sequence of measurements taken from a deterministic system that evolves according to some differential equation

$$\frac{dx}{dt} = F(x), \tag{1}$$

where x is a point in the (perhaps unobservable) state space of the system, $\mathcal{M}$; at each instant $x(t)$ provides a complete description of the system. Supposing our multivariate signal to have k channels, it appears here as the value of some k-component, vector-valued function, $v : \mathcal{M} \rightarrow \mathbf{R}^k$, recorded at points spaced evenly in time along a trajectory $x(t)$ of (1). For the moment we will suppress the dependence on the phase point $x(t)$ and will measure time in units of the sampling interval so that the time series is

$$\begin{aligned} v(0) &= (v_1(0), v_2(0), \cdots, v_k(0)), \\ v(1) &= (v_1(1), v_2(1), \cdots, v_k(1)), \\ &\vdots \end{aligned}$$

Our first goal is to turn this series into a geometric object with properties similar to those of $\mathcal{M}$. Concentrating on just a single channel, say the j-th one, we choose a set of delays $\tau_1, \tau_2, \cdots \tau_n$ and use them to build vectors $\mathbf{V}_j(t) \in \mathbf{R}^n$,

$$\mathbf{V}_j(t) = (v_j(t+\tau_1), v_j(t+\tau_2), \ldots, v_j(t+\tau_n)). \tag{2}$$

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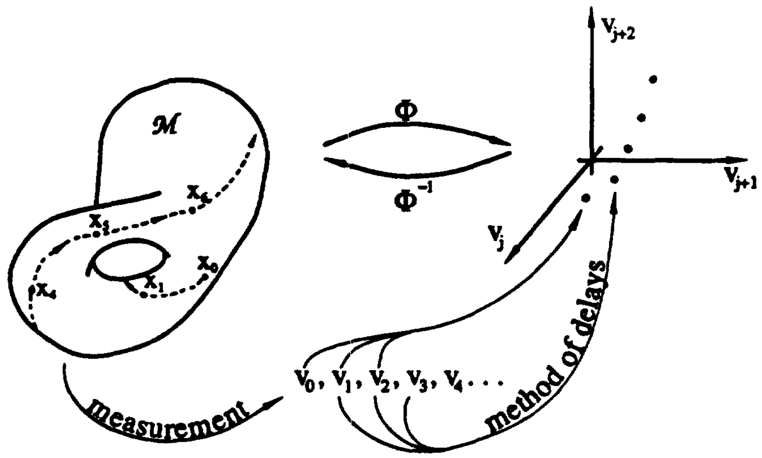

Figure 1: ***A schematic diagram of state space reconstruction using the method of delays. For a thorough treatment see, for example, [7, 13, 12, 4].***

Barring an unfortunate choice of observable v_j, delays τ_m or embedding dimension n, a theorem of Takens states that the resulting points lie on a set diffeomorphic to $\mathcal{M}$ [13].

This approach, the method of delays, works no matter which channel we choose; each yields a representation of $\mathcal{M}$. But $\mathcal{M}$ is the state space of the underlying physical system—to know one's position in it is to know everything there is to know about the system, even the values of the other observables. Just as we think of the components of our signal as values of functions $v_m : \mathcal{M} \rightarrow \mathbf{R}$, so we can also think of them as functions on the reconstructed object. By approximating these functions then we could build a model capable of recovering all k channels of information from an observation of just one.

The next section introduces the tools we need; it opens with a few more remarks about the method of delays, then introduces a way to approximate functions on the reconstructed $\mathcal{M}$. The third section applies these ideas to a real experimental signal and the last discusses possible uses for this method, including signal compression, the recovery of signals masked by chaotic noise and some very interesting recent work [2] which shows that a band-pass filtered version of a chaotic signal contains sufficient information to recover the whole, unfiltered signal.

2 Delays, diffeomorphisms and functions

The method of delays converts a time series into a cloud of points in the embedding space, $\mathbf{R}^n$. How is this set connected to the original system's state space? The careful answer is that each channel of the multiprobe signal gives rise to a diffeomorphism $\Phi_j : \mathcal{M} \rightarrow \mathbf{R}^n$,

$$\Phi_j(x_0) = \langle v_j(x(\tau_1)),\, v_j(x(\tau_2)), \ldots,\, v_j(x(\tau_n))\rangle. \tag{3}$$

Here $x_0 \in \mathcal{M}$ is a point in the system's original state space and $x(t)$ is the trajectory of (1) that passes through x_0 at $t = 0$. In other words, the reconstructed state space is the essentially the same as $\mathcal{M}$, differening from it only by a smooth change of coordinates.

2.1 approximating functions

The method of delays leaves us with a low dimensional model of our system's state space. Here we introduce an elegant scheme to approximate functions on the reconstructed $\mathcal{M}$. The method, first proposed by Martin Casdagli in [3], was originally intended for short term prediction.

Suppose then that we have some delay embedding Φ like the one in equation 3 and also, for each the points $x_0, x_1, x_2, \cdots$ along our system's trajectory, the value of some function $u : \mathcal{M} \to \mathbf{R}$. We want to find a function $f_u : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$f_u(\Phi(x_j)) = u(x_j);$$

that is, we want f_u to have the same value at the embedded point $\Phi(x_j)$ that u does at x_j. Casdagli suggests that we approximate f_u with a linear combination of radial basis functions, something of the form

$$f_u(\Phi(x)) = \sum_{\text{centres } \mathbf{c}_k} \lambda_k \phi(||\Phi(x) - \mathbf{c}_k||) \tag{4}$$

where the centres $\mathbf{c}_k \in \mathbf{R}^n$ are a collection of points in the embedding space, $||x||$ is the usual euclidean norm in $\mathbf{R}^n$, $\phi : \mathbf{R} \to \mathbf{R}$ is a function and the coefficients λ_k are chosen to minimize the mean square error. The advantage of this approach is that the λ_k appear linearly and so provided there are fewer centres than points in the series, they are given by an overdetermined linear system—if the trajectory has N points and there are K centres the equation is:

$$\begin{bmatrix} \phi(||\Phi(x_0) - \mathbf{c}_1||) & \cdots & \phi(||\Phi(x_0) - \mathbf{c}_K||) \\ \phi(||\Phi(x_1) - \mathbf{c}_1||) & \cdots & \phi(||\Phi(x_1) - \mathbf{c}_K||) \\ \vdots & & \vdots \\ \phi(||\Phi(x_{N-1}) - \mathbf{c}_1||) & \cdots & \phi(||\Phi(x_{N-1}) - \mathbf{c}_K||) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_K \end{bmatrix} = \begin{bmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{bmatrix}. \tag{5}$$

One can solve this in the least squares sense by doing a singular-valued decomposition (see, *e.g.* [9]).

3 Exploiting redundancy

Consider Figure 2. It shows an experiment (about which we will say more below) that produces 32 channels of data. Choose one channel, say channel 20, as a reference and use it, along with the method of delays, to build a model of the underlying dynamical system. Now consider a point on $\mathcal{M}$, say x_1, and the corresponding point in the reconstruction. It is easy to recover the value of the twentieth observeable from the reconstruction; it is just the first coordinate of the point $\Phi_{20}(x_1) = V_{20}(1) \in \mathbf{R}^n$. But one can also think of the observeable as a function on the reconstructed object, the function that picks off the first coordinate. Similarly, one can think of the measurement in channel twelve as another function on the reconstructed object. One can represent this function by listing its values on the embedded trajectory, building up a table of pairs

$$\begin{matrix} (\Phi_{20}(x_1),\ v_{12}(x_1)) \\ (\Phi_{20}(x_2),\ v_{12}(x_2)) \\ \vdots \end{matrix},$$

where the first entry is a point in the reconstruction and the second is a real number, the value measured in channel twelve at the same moment. Finally, one can interpolate the table of pairs with radial basis functions to get a smooth approximation to that measurement function which gives the value of the twelfth observeable. Similarly, one could use the reconstruction based on channel 20 to recover all the other signals.

3.1 an application

The apparatus in Figure 2 is a cartoon version of an experiment designed to simulate atmospheric circulation in the mid-latitudes. The annular tank rests on a turntable and rotates slowly, simulating

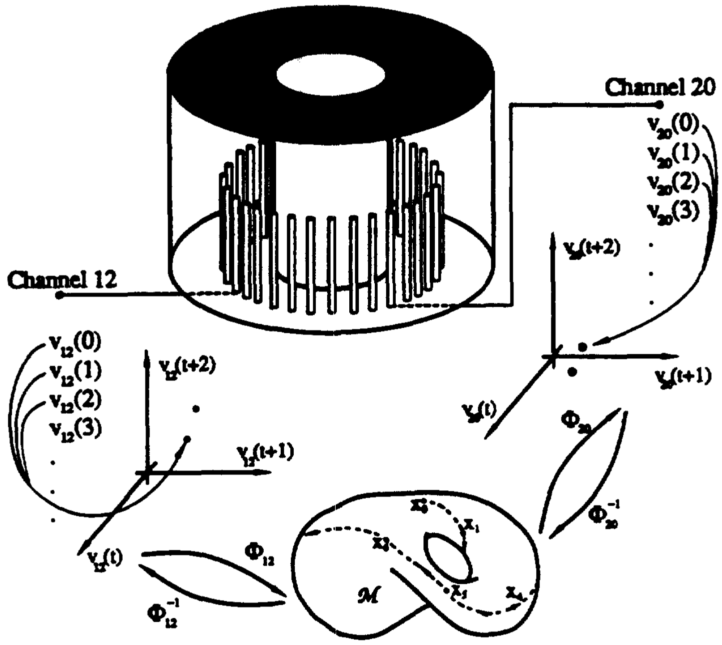

Figure 2: *A multiprobe version of Figure 1. Two signals from the same apparatus give rise to two different diffeomorphic images of $\mathcal{M}$, but a point in either reconstruction specifies the state of the system completely.*

Channel number	$\Delta\theta$	r.m.s. error	σ	(r.m.s. error)/σ
20	0°	0.0212	0.430	0.0492
21	11.25°	0.0752	0.435	0.173
22	22.5°	0.0834	0.430	0.194
24	45°	0.0877	0.399	0.220
28	90°	0.0760	0.357	0.213

Table 1: *A summary of results for several channels: $\Delta\theta$ is the angular separation between the given probe and the reference probe; σ is the r.m.s. value of the fluctuating part of each signal—it sets a scale against which to compare the errors.*

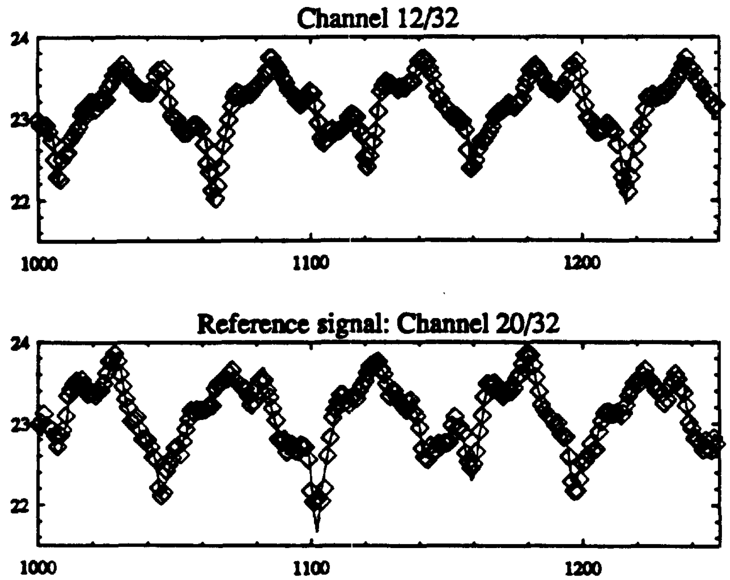

Figure 3: ***Short segments of the reference signal and of one of the approximated signals, illustrating the success of the reconstruction and approximation. The diamonds are our approximations; the solid curves are the actual measurements.***

the rotation of the earth. The inner wall is cooled to simulate the north pole while the outer while is heated to simulate the equator. A ring of 32 thermocouples located at mid-radius and mid-height in the tank provide the signals we analyse. This data, which was given us by Peter Read of Oxford, is discussed more thoroughly in his article [11].

We used the signal from the twentieth probe to reconstruct the state space of the system (dimension $n = 9$, delays $\tau_j = 6(j-1)$), then studied the first half the data (1000 points) to build approximators for several of the other signals. These were of the form (4), with 120 centres chosen randomly from the data and basis function

$$\phi(r) = e^{(r/r_0)^2}$$

where r_0 was half the range of the data.

Finally, we made an out-of-sample test of the approximators using the second half of the data (1000 points); Figure 3 shows the agreement between the approximators and the actual measurements for both the reference channel and one of the others while Table 1 summarizes the results for channels at various angular distances from the reference probe. Somewhat surprisingly, the quality of the approximation is similar for all channels but the reference probe. One can get better results by preprocessing the data to get a better embedding—see [1] for a good approach.

4 Conclusions

The most obvious application of these ideas is to signal compression. If we wished to transmit the signals discussed above our work shows that after an initial phase of model-construction we could send enough information to reproduce all 32 channels of the signal with only one channel's worth of data. A somewhat more abstract description of what we have done: "use information in one component of a signal from a deterministic system to determine all other signals," suggests other possibilities. Perhaps the most exciting is the prospect of recovering the whole of a chaotic signal from filtered versions. A recent preprint, [2], shows how to exploit this idea to recover band-limited signals from behind an arbitrarily strong chaotic screen. Their methods bear directly on the security of signals masked by chaotic noise.

Suppose, for example, that we were using a protocol suggested by Pecora and Carroll—they suggest that both the transmitter and the receiver should have identical copies of a chaotic dynamical system, say, two Lorenz attractors with the same parameters. A signal derived from the transmitter's system is used to drive, and synchronise, the receiver's system. After this synchronising preamble the actual message is transmitted as a weak signal masked by a stronger chaotic component. Since the receiver has an identical copy of the chaotic source, he recovers the message easily.

But the transmitter-receiver pair are just the sort of system treated in this paper; coupled together as they must be to achieve synchronisation, they are two parts of the same system. Close study of one component, the synchronising preamble, is sufficient to describe the whole assembly and so unmask the message. Although we have overstated the case for the methods presented here (the model-construction phase of this paper's approach would require observations on both the transmitter and the receiver), the problem is genuine—signals masked by chaotic noise may be recovered by anyone who can model the chaotic noise source. And one can build good models of a chaotic system from nothing more than a time series of measurements.

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