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Evatt, G.W. and Moriarty, J. and Johnson, P.V and Duck, P.W.

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Regulating Industries under Exogenous Uncertainty

G.W. Evatt · J. Moriarty ·
P.V. Johnson · P.W. Duck

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Abstract We present a quantitative method to find jointly optimal strategies for an industry regulator and a firm, who operate under exogenous uncertainty. The firm controls its operating policy in order to maximize its expected future profits, whilst taking account of regulatory fines. The regulator aims to control the probability that the firm enters a given undesirable state, such as ceasing production, by imposing a fine which is as low as possible, while achieving the required reduction in probability. The exogenous uncertainty is modeled using a stochastic differential equation, and we show this implies that the firm's behavior can be solved via the Hamilton-Jacobi-Bellman equation, and the regulatory fine can be obtained via the Feynman-Kac formula. We discuss both analytic and numerical solution methods. Our results are illustrated for a security of supply problem for vaccine production where future production costs are uncertain and, using empirical data, for an abandonment problem in a gold mining operation where future commodity prices are uncertain. The method determines the level of fine which establishes a Nash equilibrium in these nonzero-sum games, under uncertainty.

Keywords Regulation, Uncertainty, Nash Equilibrium, Early Termination, Mining, Vaccine Supply
JEL Classification G38, C58, Q32

1 Introduction

The quantitative mathematical analysis of industry regulation offers a principled basis for policymaking (Brock and Carpenter, 2007; Heyes, 2000; Povel, 1999). Such analysis is made more challenging when account is taken of economic uncertainty

G.W. Evatt, J. Moriarty, P.V. Johnson & P.W. Duck
School of Mathematics, Oxford Road, University of Manchester, Manchester UK M13 9PL
Tel.: +44 (0)161 275 5830
Fax: +44 (0)161 275 5819
E-mail: geoffrey.evatt@manchester.ac.uk

(Morgan et al., 1990; Brennan and Schwartz, 1982) and of uncertainty over firms' operating policies (Ruhl, 2005; Camerer, 1999; Sunstein, 1997). A cautionary example is provided by the banking crisis of 2008, in which the entwined nature of multiple parties' interests clouded analysis of the systemic risks (May and Arinaminpathy, 2010). In this paper we aim to contribute to the understanding of such economic systems: in particular, we present a quantitative method to solve a model of a regulated industry, which takes account of both exogenous uncertainty and the intertwined interests of a profit maximizing firm and a regulator.

In highly controlled markets such as the public utility industries, the regulator of an industry may impose direct control on the operating strategy of a firm, even though the company operates under exogenous uncertainty (Roseta-Palma and Xepapadeas, 2004). As an example, the UK water regulator OFWAT restricts the number of times a privatized water utility operator can impose a domestic hosepipe ban (Arnell, 1998) - despite uncertainty over the level and pattern of rainfall, and the fact that additional bans may be optimal for the operator. In other markets, direct regulatory control is either impossible or undesirable. An example is vaccine production in the US, where demand levels and costs can be highly uncertain (Danzon et al., 2005). Indeed, security of vaccine supply is highly important, since the effect on public health of vaccine shortages can be significant (Helms et al., 2005): during the period 2000 to 2005, one-third of all childhood vaccine shortages in the US are estimated to have been caused by vaccine manufacturers deciding to cease production due to unfavorable economic conditions (Hinman et al., 2006); we explore a variant of this problem in Section 3. However, a regulator can exert indirect influence by changing a firms economic parameters (Sappington, 2005), in particular by the use of financial incentives such as fines (Helms et al., 2005).

Motivated by such examples, this paper makes three core economic assumptions: firstly, that the future profits of a firm depend on the values taken by an exogenous stochastic process, such as a commodity price, level of demand or labor market costs; secondly, that the industry under review has a regulator, who wishes to reduce the probability of a certain undesirable *event* by appropriately controlling the size of a fine; and thirdly, that firms follow operating policies which maximize the expected future profit from their operation, taking account of fines. Under these assumptions we show how to derive partial differential equations which characterize this *event probability* and we describe fast and accurate numerical algorithms to determine the optimal level of fine.

The methodology in this paper is closely related to that of Evatt et al. (2011) in which it was described how various expected values in a model of resource extraction, including the probability of mine abandonment, may be derived. We presented analytical solutions, and numerical methods which build upon a recent algorithm first used in a real options context by Chen and Forsyth (2007). This current paper builds upon the mathematical structure of Evatt et al. (2011), to demonstrate a broader application in regulation; whilst the quantities determined in Evatt et al. (2011) inform the choice of operating policy by the mining firm, this paper shows how they also lend themselves to a broader assessment of risk. For instance, the decision to abandon a large mining project can have significant undesirable implications for the surrounding environment and economy (Otto, 2010; Veiga et al., 2001). A mining industry regulator may therefore seek to reduce the probability of abandonment to a target level. As our method models the intertwined interests of mining firm and industry regulator quantitatively, it may be used to efficiently choose the required level of abandonment fine; such an example is explored further in Section 4.

A regulator may have several quantitative targets which protect the security of supply (Helm, 2002), such as limits on the probability of undersupply, or target variance of supply levels. In this paper, we focus on controlling the probability of cessation of production, which we refer to as *abandonment*. We assume a target level for the probability of abandonment is given, although we do not specify how this target level should be set, since this is often a political decision (Holt, 2005) and is clearly outside the scope of this paper. In a non-public policy context, many business-to-business contracts already incorporate an early termination fee (Bates and Lemmon, 2003), whose primary objective is financial (Sharp et al., 2008). This fee can mitigate any additional financial costs associated with early termination, as it makes the counterparty less inclined to terminate due to the higher cost (Williamson, 1985).

The problem of controlling event probabilities in regulated industries has been considered previously in related contexts. In financial regulation, the Value at Risk measure limits the probability of losses above a given size in a portfolio due to market movements (Duffie and Pan, 1997). It does not, however, aim to address behavioural considerations in the construction of financial portfolios, nor the setting of appropriate regulatory fines. Optimal control with probabilistic constraints has been investigated in non-regulatory contexts, such as control engineering (Kandukuri and Boyd, 2002) and operations research (White, 1974), although these studies have only considered the objectives of a single party. The regulatory problem considered in this paper involves the strategies of two participants: the firm must choose an operating strategy which maximises returns, while the regulator chooses the level of a fine, taking account of the firm's operating strategy, in order to match the abandonment probability to the target level. The solutions presented in this paper are therefore Nash equilibria for a nonzero-sum game (Starr and Ho, 1969).

The paper is organized as follows. Details of the mathematical methods are presented in Section 2. In Section 3 the method is applied to a regulator wishing to control the security of supply of a vaccine. A more complex, data-driven example is presented in Section 4, where a mining industry regulator wishes to reduce the probability of total abandonment of a gold mining project. We conclude by discussing further potential applications in Section 5.

2 Optimal Control with Probabilistic Constraints

In this section we present the mathematical details of the method, in a manner intended to be sufficiently general to be employed in a range of regulatory contexts. The method is a one-dimensional search over the level K of the fine for abandonment, and may be summarized as follows:

1. Find the optimal control strategy for the current value of K
2. Find the abandonment probability for this control strategy
3. Update K and return to step 1

In considering the firm's optimal strategy, the essential mathematical tool is the Hamilton-Jacobi-Bellman (HJB) equation, which links controlled Itô diffusions and partial differential equations (PDEs). A controlled Itô diffusion in $\mathbb{R}^n$ takes the form

$$dX_t = dX_t^u = b(X_t, u_t)dt + \sigma(X_t, u_t)dB_t, \tag{1}$$

where b is a function taking values in $\mathbb{R}^n$ representing an instantaneous drift, σ is the instantaneous volatility function, taking values in $\mathbb{R}^{n\times m}$, and $(B_t)_{t\geq 0}$ is a Wiener process in $\mathbb{R}^m$. In our setting, the process $(X_t)_{t\geq 0}$ represents the time evolution of the economic state of a firm, driven by the noise process B which represents random fluctuations in, for example, commodity or labor prices or demand levels. We assume that the firm will abandon production when its economic state ceases to be favorable, which corresponds to the first time at which X_t leaves a predetermined set H. We denote this *abandonment time* by ν. The function u represents the firm's operating strategy (specified for each possible economic state X_t), which is assumed to be fixed at the outset. At each time t, the value u_t depends on the economic state X_t and may only take values in the admissible set U. For further background on stochastic optimal control we refer the reader to Øksendal (2003).

Given an operating strategy u, we express the firm's future profits using a running profit function g, discounted at the rate of interest r. Let T be the time at which the operating license expires. In addition to the running profit, the firm also experiences a final cashflow $h(X_\nu)$; if $\nu < T$ then the firm has abandoned early, and so $h(X_\nu)$ includes the fine. We define the *performance function* w^u to be the firm's expected total profit, net of any fine:

$$w^u(x) = E_x\left[\int_0^\nu e^{-\hat{r}z}g(X_z, u_z)dz + e^{-\hat{r}\nu}h(X_\nu)\right], \tag{2}$$

where E_x denotes the expected value when $X_0 = x \in \mathbb{R}^n$. We remove the endogenous behavioral uncertainty due to the firm's choice of operating strategy u by assuming that there exists an optimal strategy u^* which maximizes the value of the performance function $w^u(x)$. This value is thus given by the function V^*, where

$$V^*(x) = w^{u^*}(x) \tag{3}$$

The HJB equation gives that V^* is the solution to the PDE

$$\begin{aligned} \sup_{v\in U}\{LV(x) + g(x,v) - \hat{r}V(x)\} = 0 \quad \text{in} \quad H \\ \lim_{x\to y} V(x) = -h(y,v) \quad \text{for} \quad y \in \partial H \end{aligned} \tag{4}$$

where

$$L \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \tag{5}$$

and $[a_{ij}] = \frac{1}{2}\sigma\sigma'$, and H is the set of all economic states which itself is often part of the solution to the optimal control. A time derivative is indeed present in (4), as time forms part of the stochastic process (1): $dt = 1.dt + 0.dB$.

In order to study the effect of varying the level of the fine K, we specify the following form for h:

$$h(X_\nu, u_\nu) = l(X_\nu, u_\nu) + K I_{\nu<T} \tag{6}$$

where $I_{\nu<T}$ equals 1 if the firm abandons early (ie., if $\nu < T$) and equals 0 otherwise. Note that in equation (6), the fine is not inflation-linked; for longer horizons T it may be appropriate to inflation-link the fine, which corresponds to premultiplying K by a factor $e^{\hat{r}\nu}$. To make the dependence on K explicit, let us write $V^* = V^*(x, K)$ and $u^* = u^*(K)$. Our three steps are now:

1. We first solve (4) to obtain $V^*(x, K)$ and $u^*(K)$. This provides the optimal operating strategy $u^*(K)$ under the model, given that the fine is set at level K.
2. The firm is now assumed to follow this operating strategy and, under this assumption, its economic state $X^{u^*(K)}$ is an (uncontrolled) Itô diffusion. As a result the quantity of interest to the regulator, namely the abandonment probability $P(x) = P(x, K)$ for a firm with initial economic state x, is then found as described in Evatt et al. (2011) by solving a form of the Feynman-Kac formula:

$$\begin{aligned} LP(x) &= 0 \quad \text{in} \quad H \\ \lim_{x\to y} P(x) &= I_{t_y<T} \quad \text{for} \quad y \in \partial H \end{aligned} \tag{7}$$

   where t_y is the value of the time co-ordinate at the point y.
3. We now vary K (which in turn varies u^*) until the abandonment probability is reduced to the level Y required by the regulator. Let K^* be this optimal fine, so that

$$P(x, K^*) = Y. \tag{8}$$

By construction of this three step process, once the condition (8) is met we are in a form of nonzero sum Nash equilibrium (Starr and Ho, 1969).

It is worth noting that, depending on the application, the rate of interest $\hat{r}$ above (which is used to find V) may be different from the one used to solve the Feynman-Kac formula (7) (and hence to find K). If the economic uncertainty is the price process of a traded commodity, then the market prices effectively determine a risk-adjusted rate of interest. In order to avoid the possibility of arbitrage, it is this rate that must be used to calculate V. In contrast, when calculating the abandonment probability the regulator is free to leave the probability undiscounted. Alternatively, a bespoke discount rate may be specified for P, in order to place greater weight on early abandonment and correspondingly less weight on later abandonment.

2.1 Feasibility

The fine K^* obtained above is optimal, in the sense that a lower fine would not achieve the desired reduction in abandonment probability and a higher fine would have a negative societal impact by discouraging the purchase of new licenses. However, the optimal fine may be sufficiently high so that no rational firm would buy a new license. The optimal *feasible* fine, $K_f^*(x)$, is therefore

$$K_f^*(x) = \min\{K_f(x), K^*(x)\}, \tag{9}$$

where $K_f(x)$ is the maximum fine the firm can afford; further discussion on this is given by Lear and Maxwell (1998), and the practical implementation of fines (such as collecting performance bonds) is discussed by Holt (2005).

2.2 Notation

The notation used in this section is consistent with that generally used in probability theory, in the sense of Øksendal (2003). Yet as can be seen, there are three different symbols related to the stochastic process: the set of all possible processes X, the individual point in the process X_t^u and the quantity post-averaging x. The notation serves a purpose in deriving the theoretical basis. But given its purpose has been satisfied, we now intentionally relax some of the notational rigor in order for an interdisciplinary reader to more easily move their way through the remaining sections. To be consistent with some of the key work on real options theory and quantitative finance (such as Dixit and Pindyck (1994) and Wilmott et al. (1995)) as far as possible we use just the capital letter to denote the stochastic variable.

3 Improving security of vaccine supply

In this section, motivated by Hinman et al. (2006) and Helms et al. (2005), we investigate the problem of increasing the security of supply of a vaccine. We employ a simple real options model, in which a firm is contracted to supply vaccines at a fixed quantity and price, and is exposed to both fixed and uncertain input costs. The only control available to the firm is the early termination of the contract, whilst the industry regulator may wish to use a fine to control the probability of termination. Termination fines are commonly used (Bates and Lemmon, 2003), and the method of Section 2 provides a quantitative method for setting the level of fine, taking account of the profit maximizing behavior of the firm.

Our model in this section is the following. The firm must deliver a fixed number q of doses each year for T years, which are to be sold at an agreed sale price of s_c per dose. The firm's costs include a fixed amount of ϵ per year and a variable amount S which is uncertain and is assumed to follow a geometric Brownian motion

$$dS = \mu S dt + \sigma S dB \tag{10}$$

where μ is the percentage drift and σ is the percentage volatility. The firm's control strategy is simply to terminate the contract when S rises to a predetermined level $S^*(t)$. This level may be time-dependent (although we will often suppress the time parameter for notational convenience), and takes account of all termination costs. With the notation of Section 2, the firm exerts no control until the termination time

ν, and so the choice of strategy reduces to the choice of a termination surface S^*. We supply our own plausible parameter values characterizing the uncertainty and where possible, we use parameter values consistent with those given in Hinman et al. (2006):

$$\mu = 2.5\%\,\text{yr}^{-1}, \quad \sigma = 0.3\,\text{yr}^{-1/2}, \quad r = 2\%\,\text{yr}^{-1}, \quad s_c = \$30\,\text{U}^{-1},$$
$$T = 5\,\text{yr}, \quad I = \$100\text{M}, \quad q = 5\text{M U yr}^{-1}, \quad \epsilon = \$1.3\text{M yr}^{-1} \tag{11}$$

where I is the initial investment and we suppose that the current level of variable input costs is $S = \$10 \;\; \text{U}^{-1}$, where U refers to a single dose. We suppose that the regulator's target is to reduce the termination probability by 20% relative to the baseline level when there is no fine, that is:

$$P(\nu < T|K = K^*) = Y = \frac{4P(\nu < T|K = 0)}{5}. \tag{12}$$

3.1 Regulator problem: Termination probability

In the notation of Section 2 we have $n = 2$ and $m = 1$. Since the firm exerts no control before the termination time, we do not model a control variable u_t and (1) reduces to

$$X_t = \begin{bmatrix} S_t \\ t \end{bmatrix}, \; b(X_t, u_t) = \begin{bmatrix} \mu S \\ 1 \end{bmatrix}, \; \sigma(X_t) = \begin{bmatrix} \sigma S \\ 0 \end{bmatrix}, \tag{13}$$

where the economic state X_t takes values in the region H bounded above (in its first coordinate) by the surface S^*. Inserting this in equation (7), we have the equation governing the probability of contract termination:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} = 0,$$

$$\begin{aligned} P = 0 \quad &\text{when} \quad t = T, \\ P = 1 \quad &\text{on} \quad S = S^*, \\ P \to 0 \quad &\text{as} \quad S \to 0. \end{aligned} \tag{14}$$

The solution to this equation depends on the firm's control strategy, which is given by the termination surface S^* obtained in Section (3.2) below.

We note that if the contract is held in perpetuity, and if the final cashflow h is constant in time, we obtain a time-stationary version of the problem. These assumptions are commonly used in real options studies (Dixit and Pindyck, 1994), and make closed form solutions available by removing the time dimension from the problem. However, in the setting of this paper, the effect of discounting a constant fine reduces its effect so greatly that such perpetual solutions are uninformative. In the following we therefore explore solutions to finite time horizon problems, using robust numerical algorithms.

3.2 Operator problem: termination surface

We now obtain the optimal termination surface S^*, as a function of the fine level K. The surface $S^*(K)$ is both an input to the calculation in section (3.1), and the optimal operating strategy for the firm under this model. The firm's running profit is modeled as the sum of the sales, variable costs and fixed costs:

$$g(S) = q(s_c - S) - \epsilon \tag{15}$$

and the function l from (6) in section 4 represents the capital cost of closure, C. Inserting the price process (10) and cashflows (15) into (4), we can write the equation governing the valuation as[1]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - rV + d(s_c - S) - \epsilon = 0$$

$$\begin{aligned} V = 0 \quad &\text{when} \quad t = T, \\ V = \tfrac{q s_c - \epsilon}{r}\left(1 - e^{-r(T-t)}\right) \quad &\text{when} \quad S = 0, \\ V = -C - K \quad &\text{on} \quad S = S^*. \end{aligned} \tag{16}$$

To maximize V, the firm must choose the surface S^* optimally. This may be achieved using the smooth pasting principle as fully detailed by Dixit and Pindyck (1994). In the case of this abandonment problem, in which there are no future cash flows once abandonment has been completed, the condition is given by,

$$\frac{\partial V}{\partial S} = 0 \quad \text{on} \quad S = S^*. \tag{17}$$

3.3 Numerical Approach

The boundary $S^*(K)$ found in Section (3.2) may now be input to the calculation in Section (3.1), to obtain the abandonment probability. We denote this probability $P(K, S, t)$, again to indicate dependence on K. Given the current variable production cost S, it is now a matter of a one-dimensional search to find the minimum value of K such that $P(K, S, t) = Y$.

The precise numerical scheme we shall use to solve for the operational control determined via equation (16), is a projected successive over relaxation (PSOR) method, which is an accurate procedure for solving such free boundary problems in quantitative finance. The solution to the regulators problem for determining the probability, (14), does not involve determining a free boundary, as this boundary is purely an input. As such, one can use a standard implicit finite-difference scheme. Both of these schemes are explained and detailed in Wilmott et al. (1995).

[1] It is straightforward to show that when the running profit function g is linear, and when a continuous interval $[a, b]$ of choices is available for U, then the resulting optimal strategy is of a 'bang-bang' nature, where the optimal control always lies within a finite set. In these cases, although the HJB equation holds, it simplifies to solving Feynman-Kac equations on overlapping domains with free boundaries (Øksendal, 2003). When the running profit is nonlinear this is no longer true, and the full HJB equation must be solved.

3.4 Results

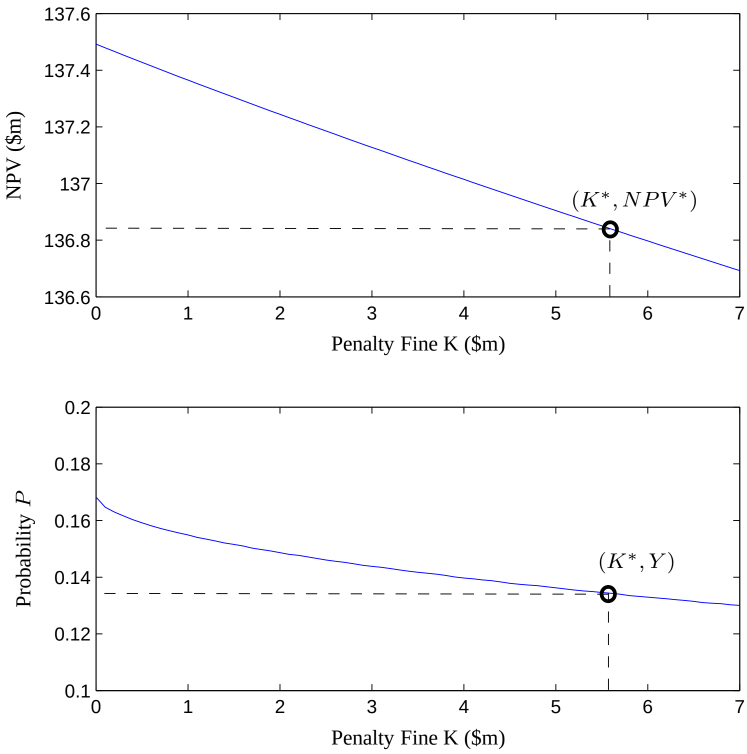

Fig. 1 The top graph shows the firm's NPV for each termination fine K, and the lower graph shows the firm's probability of early termination (the quantity of interest to the regulator) for each K. The fine K^* achieving a 20% reduction in termination probability is shown in the lower graph (where the curve meets the horizontal dashed line), and the corresponding NPV is given in the top graph.

The solutions to (16) and (14), when using the parameters given by (11), are shown in Figure 1. The top graph shows how the optimal net present value (NPV $= V - I$) of the operation is dependent upon the termination fine K, and the lower graph shows how the resulting probability of termination varies. The objective of the regulator is to reduce the probability of early termination by 20%, and the fine which achieves this aim is indicated by the intersecting dashed lines in the lower graph: $K^* = \$5.6$M. The introduction of this fine leads to a reduction of the firm's NPV from $\$137.5$M to $\$136.82$M, a reduction in NPV of just 0.5%.

Figure 2 illustrates the effect of the optimal fine K^* on the optimal operating strategy of the firm, where the termination boundaries $S(0)$ (dashed line) and $S(K^*)$ (solid line) are shown. The difference between these two termination prices becomes

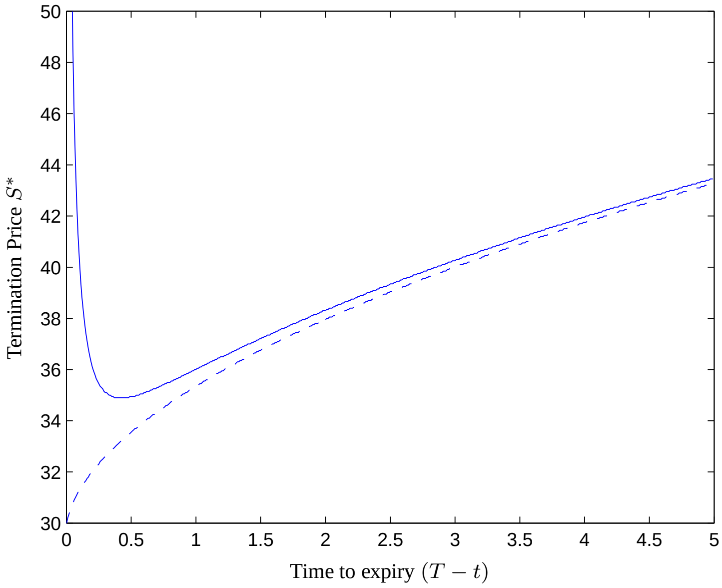

Fig. 2 Optimal termination prices for the manufacturer, viewed as the time to expiry is approached, for two different penalty fines. The continuous line is when the optimal termination fine is imposed upon the operator $K^* = \$5.6$M, and the dashed line shows the optimal strategy when no penalty fine is imposed $K = \$0$.

greater close to contract expiry. It is clear from Figure 2 that the reduction in termination probability is principally achieved by excluding termination decisions close to expiry of the operating license: this feature also explains the relatively low impact of the termination fine on the firm's NPV.

In Figure 3 we investigate the sensitivity of the results to the parameter values specified in (10). The top graph shows how, given the initial price level, the probability of early termination increases as the percentage volatility σ is increased. In the bottom graph we show how the probability of termination increases as the percentage drift μ increases. We note that the main effect is a translation of the curve, so that the objective of a 20% reduction in termination probability appears robust to uncertainty over the parameters μ and σ.

4 Increasing the societal benefit of an extraction project

In this section we consider the use of fines to reduce the probability that a mining project will be abandoned early. The societal and economic benefits from a mining project can be large, with increased investment in an area providing increased levels of employment and improved community resources. When extraction ceases, these benefits also cease, to the detriment of the community (Andrews-Speed et al., 2005); the abandonment of the extraction site may also carry an environmental cost. A re-

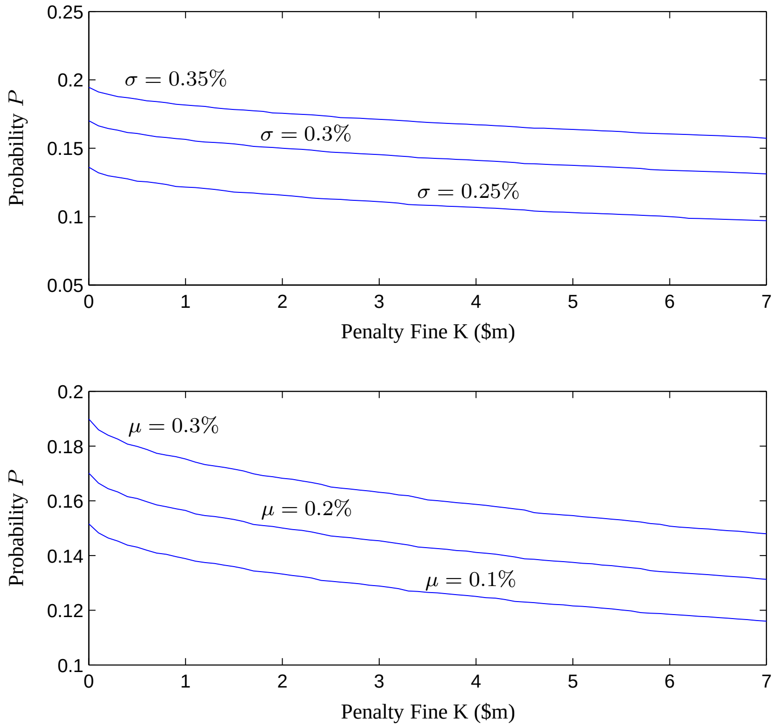

Fig. 3 The sensitivity of the probability of early termination, as one varies the parameters of the stochastic process (10). The top graph shows the sensitivity towards the volatility: $\sigma = 0.25, 0.3$ and 0.35 %. The bottom graph shows the sensitivity towards the drift: $\mu = 0.1, 0.2$ and 0.3 %. Unless being varied in the graph, all other parameters are given by (11).

cent World Bank report (Otto, 2010) discusses a legal framework within which local governments can ensure more sustainable resource extraction projects. Such considerations include making provisions for new forms of economic activity and employment post-extraction (Veiga et al., 2001). However, these provisions may not address the risk of early abandonment of the mining project due to unfavorable commodity prices. In the context of a regulated market, a suitable equilibrium must be sought between the regulator and mining firm (Kniesner and Leeth, 2004; Otto, 1997). If the equilibrium is achieved through a fine for abandonment, the regulator must avoid deterring this often vital corporate investment (Otto, 2010). We now show that this problem falls within the scope of Section 2, when the exogenous uncertainty S is the commodity price and the operating strategy u is the rate at which the commodity is extracted.

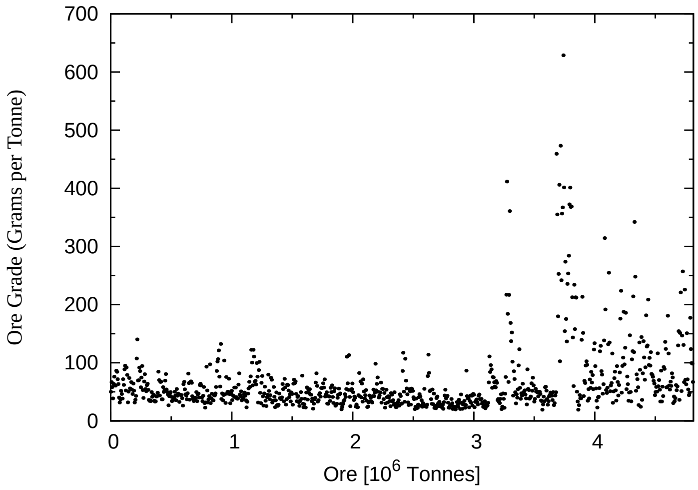

Fig. 4 The ore grade of a gold mine, ordered in a chronological sequence of extraction. The grade, G, is measured in grammes per tonne, and the x-axis denotes the amount of remaining ore, Q. This data was supplied by Gemcom Software International.

4.1 Model: the mining operation

In this example we use empirical data on the ore-grade quality G (grammes of gold per tonne of earth) from a real gold mining operation, whose data has been supplied by Gemcom Software International (a large mining solutions provider). The data is plotted in Figure 4, following an extraction schedule which is scheduled to last 4.9 years at constant extraction rate. In our model, the economic state of the mine is (S,t,Q), where Q is the volume of ore remaining in the mine. We assume that S follows a mean-reverting Cox-Ingersoll-Ross process, with stochastic dynamics

$$dS = \kappa(\mu - S)dt + \sigma\sqrt{S}dB, \tag{18}$$

where μ is the long-term average price of gold and κ is the speed of mean reversion. The diffusion (S,t,Q) is controlled by choosing the rate of extraction $q = -\frac{dQ}{dt}$. In the notation of Section 2 we have $n = 3$, $m = 1$ and

$$u_t = q_t,\ X_t = \begin{bmatrix} S_t \\ t \\ Q_t, \end{bmatrix},\ b(X_t,u_t) = \begin{bmatrix} \kappa(\mu - S) \\ 1 \\ -q_t \end{bmatrix},\ \sigma(X_t,u_t) = \begin{bmatrix} \sigma\sqrt{S} \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

We take a simple model in which q may have either the value q_1 or q_2, with $q_1 < q_2$; we will refer to state 1 as ‘normal operation’, and to state 2 as ‘expanded operation’. The capital cost of switching from state 1 to state 2 is C_e, and switching from state 2 to state 1 is not possible. The mine may be abandoned from either state, incurring a capital cost C_{1a} and C_{2a} from states 1 and 2 respectively. The firm’s running cost per unit time in state i is ϵ_i, so that the running cashflows for the mine are

$$g_i(S,t,Q) = q_i G(Q) S - \epsilon_i. \tag{20}$$

The operational strategy of any firm is assumed to consist of three price thresholds $S_e^*(t,Q)$, $S_{1a}^*(t,Q)$ and $S_{2a}^*(t,Q)$, all of which may depend on both the time t and the remaining quantity of ore Q. The threshold $S_e^*(t,Q)$ is the gold price at which the operation expands from state 1 to state 2, while $S_{1a}^*(t,Q)$ and $S_{2a}^*(t,Q)$ are the gold prices at which the project is abandoned, respectively from states 1 and 2.

The objective of the regulator is assumed to be the introduction of a fine for abandonment which halves the probability of abandonment (so as to improve the security of supply of economic benefit to the surrounding community), relative to the probability without a fine. The remaining parameter values of this gold mine extraction project are given by

$$\begin{gathered}\mu = \$24.4\text{gr}^{-}1, \quad \sigma = 25\%, \quad \kappa = 0.01yr^{-1}, \quad r = 8\%\, yr^{-1}, \\ I = \$50M, \quad q_1 = 1Mt\, yr^{-1}, \quad q_2 = 2Mt\, yr^{-1}, \quad \epsilon_1 = \epsilon_2 = \$yr^{-1}, \\ C_{1a} = \$10M, \quad C_{2a} = \$10M, \quad C_e = \$20M, \quad S_0 = \$25.7\text{gr}^{-1}. \end{gathered} \tag{21}$$

4.2 Abandonment probability

Inserting the controlled diffusion (19) into equation (7), the probability of abandonment is given by the coupled equations

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 P_1}{\partial S^2} - \frac{\partial P_1}{\partial \tau} - q_1\frac{\partial P_1}{\partial Q} + \kappa(\mu - S)\frac{\partial P_1}{\partial S} = 0,$$

$$\begin{gathered} P_1 = 1 \quad \text{on} \quad S = S_{1a}^*, \\ P_1 = 0 \quad \text{when} \quad \min\{Q,\tau\} = 0, \\ P_1 = P_2 \quad \text{on} \quad S = S_e^*, \end{gathered} \tag{22}$$

and

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 P_2}{\partial S^2} - \frac{\partial P_2}{\partial \tau} - q_2\frac{\partial P_2}{\partial Q} + \kappa(\mu - S)\frac{\partial P_2}{\partial S} = 0,$$

$$\begin{gathered} P_2 = 1 \quad \text{on} \quad S = S_{2a}^*, \\ P_2 = 0 \quad \text{when} \quad \min\{Q,\tau\} = 0, \\ P_2 \to 0 \quad \text{as} \quad S \to \infty, \end{gathered} \tag{23}$$

where $\tau = T - t$. These particular boundary conditions are mathematically analogous to hysteresis problem in physics solved by Freidlin et al. (2000). As in the previous section, the solution to the coupled equations (22)—(23) depends on the firm's control strategy. We now describe, from the point of view of a firm, the optimal choice for the three surfaces $S_{1a}^*(\tau,Q)$, $S_{2a}^*(\tau,Q)$ and $S_e^*(\tau,Q)$.

4.3 Operator

From the price process (18), controlled diffusion (19) and running cash flows (20), we can insert them into equation (4), to obtain the following PDE for the mines expected profit:

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 V_1}{\partial S^2} - \frac{\partial V_1}{\partial \tau} - q_1\frac{\partial V_1}{\partial Q} + \kappa(\mu - S)\frac{\partial V_1}{\partial S} - rV_1 + q_1 GS - \epsilon_1 = 0,$$

$$\begin{aligned} V_1 = 0 &\quad \text{when} \quad \min\{\tau, Q\} = 0, \\ V_1 = -C_{1a} - K &\quad \text{when} \quad S = S^*_{1a} \\ V_1 = V_2 - C_e &\quad \text{on} \quad S = S^*_e, \end{aligned} \tag{24}$$

and

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 V_2}{\partial S^2} - \frac{\partial V_2}{\partial \tau} - q_2\frac{\partial V_2}{\partial Q} + \kappa(\mu - S)\frac{\partial V_2}{\partial S} - rV_2 + q_2 GS - \epsilon_2 = 0,$$

$$\begin{aligned} V_2 = 0 &\quad \text{when} \quad \min\{\tau, Q\} = 0, \\ V_2 = -C_{2a} - K &\quad \text{when} \quad S = S^*_{2a} \\ V_2 \sim S &\quad \text{as} \quad S \to \infty. \end{aligned} \tag{25}$$

We refer to Brennan and Schwartz (1985) for the justification of these boundary conditions. In addition, for optimality to be obtained, we require that at each of the transitions the smooth pasting condition must hold:

$$\frac{\partial V}{\partial S} = 0 \quad \text{on} \quad S = \{S^*_e, S^*_{1a}, S^*_{2a}\}. \tag{26}$$

The boundaries $S^*_{1a}(\tau, Q)$, $S^*_{2a}(\tau, Q)$ and $S^*_e(\tau, Q)$, which again depend on K (although we have suppressed this for notational convenience), may be obtained from (24)—(25) as described in Evatt et al. (2011) and input to the calculation in section (4.2). A one-dimensional search is again sufficient to obtain the minimum value of K achieving the desired reduction in abandonment probability under the model.

4.4 Results

Since equations (24)—(25) are convection dominated in the Q variable, we obtain numerical results by the semi-Lagrangian method, as utilized by Chen and Forsyth (2007). In Figure 5 we show the probability of abandonment versus level of fine K (top graph), and the mining project's NPV versus K (bottom graph). The optimal level for K is $K^* =$ \$8.85M with an associated NPV of \$97.05M, which compares to \$97.8M when $K = 0$. The halving of abandonment probability is thus achieved in return for a reduction in NPV of \$0.75M, or 0.8%.The roughness of Figure 5 (top) is a consequence of roughness in the ore grade data as shown in Figure 4. Figure 6 shows the optimal thresholds for changing operational state, when no fine is imposed (top)

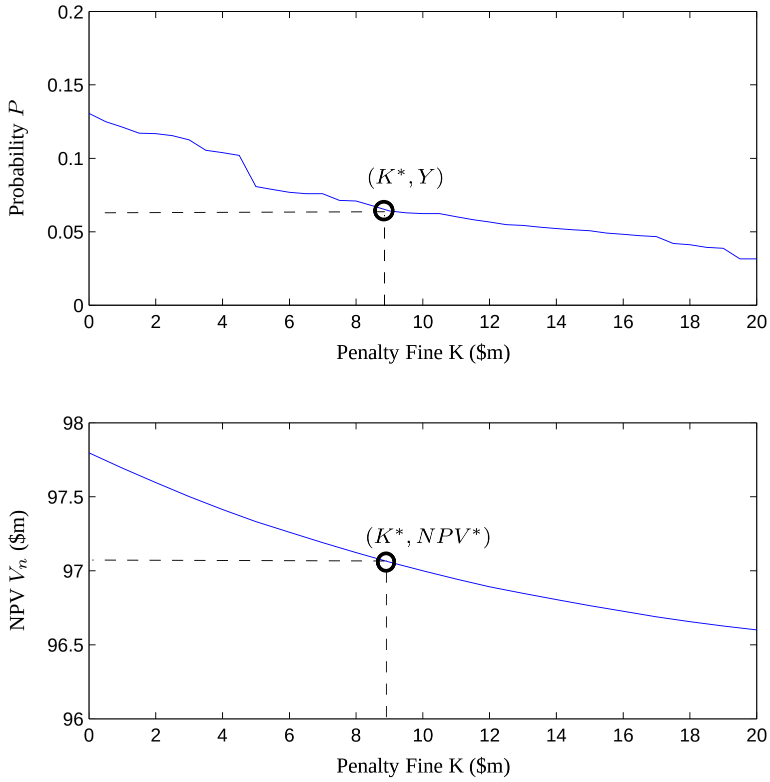

Fig. 5 The probability of abandonment (top graph) and NPV (bottom graph) of a gold mining project, plotted as a function of abandonment penalty fine. The dashed lines show where the regulators objective is met - which is to half the probability of abandonment - and the resulting optimal NPV. The results are plotted at $\$0.5M$ intervals, using the parameters values of (21).

and when the optimal fine $K = \$8.85M$ is imposed (bottom). The higher dashed line is the optimal price to expand operation S_e^*, the lower dotted line is the optimal price to abandon from the expanded state S_{a2}^* and the continuous line is the decision to abandon from normal operation S_{a1}^*. The most significant difference between the two operating strategies is found towards the end of extraction, as Q approaches 0.

5 Conclusion and discussion

We have presented a method for determining regulatory fines which establish a Nash equilibrium between an industry regulator and a firm, in the presence of exogenous uncertainty. The method uses the partial differential equations which govern both the firm's expected profit and the values of regulatory interest: it contrasts with scenario analysis (Postma and Liebl, 2005) by considering *all* possible future scenarios, and

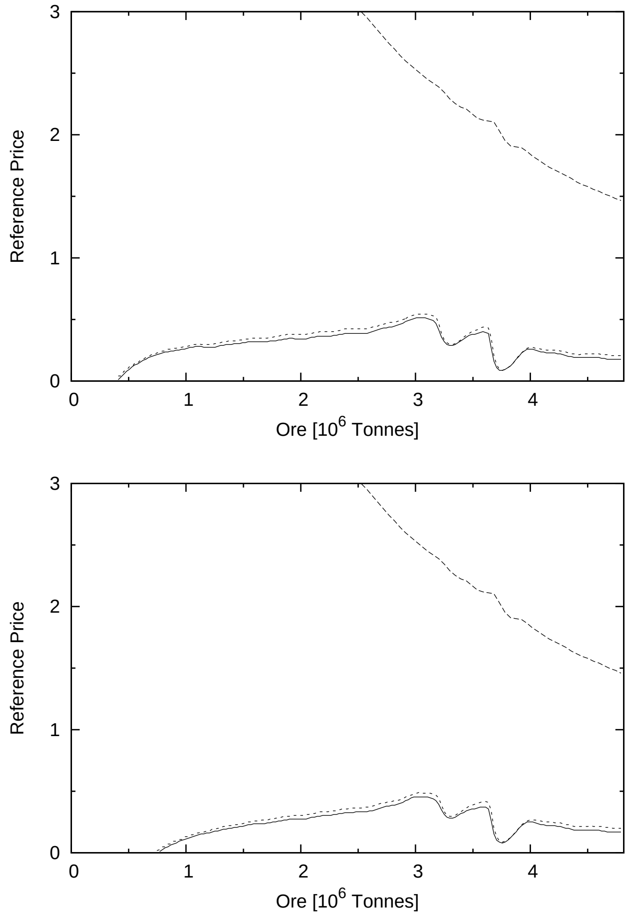

Fig. 6 Graphs showing the optimal prices for when to change operational state of a gold mine, for when no penalty fine is present (top graph) and for when the optimal penalty fine, $K = \$8.85M$, is imposed (bottom graph). In each figure, results are shown for the optimal expansion decision S_e^* (higher dashed line), the optimal abandonment decision from the expanded state S_{a2}^* (lower dashed line), and the optimal abandonment decision from the normal extraction state S_{a1}^* (lower continuous line). The graphs were calculated using the parameters values of (21).

is exact in that it does not require simulations. Our work is applicable whenever the evolution of a firm's economic state can be modeled by a controlled Itô diffusion, and its profits can be expressed in the form (2). We have given illustrative applications to two example regulatory scenarios.

A classic application of stochastic optimal control theory in finance is the portfolio allocation problem (Øksendal, 2003), in which the (often high dimensional) state of the portfolio is the controlled Itô diffusion of interest. However, when the state is described by a high dimensional controlled Itô diffusion, the numerical methods we have described are currently impractical. Indeed, quantitative analysis for financial services regulation, which takes into account the profit-maximizing behavior of financial agents, is currently a challenging area of major interest (VanHoose, 2007). For a current example of regulatory fines imposed on banks in order to control the probability of default, see Jimenez-Martin et al. (2009). With advances in numerical schemes and computational power (Xiu, 2009) however, analytic methods such as the one presented in this paper should become increasingly applicable to the regulation of financial services and other similarly complex industries.

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