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CLASSIFYING WEIGHTED PROJECTIVE SPACES

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Abstract. We obtain two classifications of complex weighted projective spaces, up to homeomorphism and up to homotopy equivalence. The first turns out to coincide with Al Amrani's classification up to isomorphism of algebraic varieties, and the second follows from our proof that the Mislin genus of any weighted projective space is rigid.

1. Introduction

Weighted projective spaces are the simplest projective toric varieties that exhibit orbifold singularities. They have been extensively investigated by algebraic geometers, but have attracted only fleeting attention from algebraic topologists since Kawasaki's pioneering work [8], in which he computed their integral cohomology rings. Subsequently, their K-theory was determined by Al Amrani [2], and the study of their KO-theory was initiated by Nishimura–Yosimura [11].

In toric geometry, weighted projective spaces are classified by their fans. Here, we give two classifications that are fundamental to algebraic topology: up to homeomorphism, and up to homotopy equivalence. We obtain the second as a consequence of the fact that the Mislin genus of a weighted projective space is rigid. Our results are stated below, following summaries of the definitions and notation.

A *weight vector* $\chi = (\chi_0, \dots, \chi_n)$ is a finite sequence of positive integers. It gives rise to a weighted action of S^1 on $S^{2n+1} \subset \mathbb{C}^{n+1}$,

$$g \cdot z = \big(g^{\chi_0} z_0, \dots, g^{\chi_n} z_n\big) \qquad \text{for } g \in S^1, z \in S^{2n+1}. \tag{1.1}$$

The quotient $S^{2n+1}/S^1\langle\chi\rangle$ is the weighted projective space $\mathbb{P}(\chi)$. Alternatively, $\mathbb{P}(\chi)$ may be defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the same weighted action of $\mathbb{C}^\times$; this exhibits $\mathbb{P}(\chi)$ as a complex projective variety.

Scaling the weight vector χ leads to isomorphic weighted projective spaces $\mathbb{P}(\chi)$ and $\mathbb{P}(m\chi)$, for any integer $m \geq 1$. Moreover, if all weights except, say, χ_0 are divisible by some prime p, then the map

$$\mathbb{P}(\chi) \to \mathbb{P}(\chi_0, \chi_1/p, \dots, \chi_n/p), \quad \big[z_0 : \cdots : z_n\big] \mapsto \big[z_0^p : z_1 : \cdots : z_n\big] \tag{1.2}$$

is an isomorphism as well, *cf.* [5, §5.7]. This leads to the notion of *normalized weights*: a weight vector χ is normalized if for any prime p at least two weights in χ are not divisible by p. Any weight vector can be transformed to a unique normalized vector by repeated application of scaling and (1.2). Consequently, two weighted projective spaces are isomorphic as algebraic varieties and homeomorphic as topological spaces if they have the same normalized weights, up to order. We

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prove that the converse is also true. In particular, we recover Al Amrani's classification up to isomorphism of algebraic varieties [1, §8.1].

Theorem 1.1. *The following are equivalent for any weight vectors χ and χ':*

1. *The normalizations of χ and χ' are the same, up to order.*
2. *$\mathbb{P}(\chi)$ and $\mathbb{P}(\chi')$ are isomorphic as algebraic varieties.*
3. *$\mathbb{P}(\chi)$ and $\mathbb{P}(\chi')$ are homeomorphic.*

For any prime p, the *p-content* ${}_p\chi$ of χ is the vector made up of the highest powers of p dividing the individual weights. For example, ${}_2(1,2,3,4) = (1,2,1,4)$. Let χ and χ' be two normalized weight vectors. It follows from Kawasaki's result that the cohomology rings $H^*(\mathbb{P}(\chi);\mathbb{Z})$ and $H^*(\mathbb{P}(\chi');\mathbb{Z})$ are isomorphic if and only if, for all primes p, the p-contents ${}_p\chi$ and ${}_p\chi'$ are the same up to order. The same phenomenon can be observed in K-theory and KO-theory. In fact, no cohomology theory can tell such spaces apart:

Theorem 1.2. *Two weighted projective spaces are homotopy equivalent if and only if for all primes p, the p-contents of their normalized weights are the same, up to order.*

The torus $T = (S^1)^{n+1}/S^1\langle\chi\rangle \cong (S^1)^n$ and its complexification $T_{\mathbb{C}}$ act on $\mathbb{P}(\chi)$ in a canonical way, and the resulting equivariant homotopy type is a finer invariant. As shown in [3, Thm. 5.1], the equivariant cohomology ring $H_T^*(\mathbb{P}(\chi);\mathbb{Z})$ determines the normalized weights up to order.

Let ${}_p\chi^*$ be the p-content of χ, ordered as an increasing sequence. By Theorem 1.2, $\mathbb{P}(\chi)$ is homotopy equivalent to $\mathbb{P}(\chi^*)$, where the weights in the product χ^* of the ${}_p\chi^*$ form a *divisor chain*, in the sense that each divides the next. The space $\mathbb{P}(\chi^*)$ is particularly easy to work with because

$$* = \mathbb{P}(\chi_n^*),\ \ \mathbb{P}(\chi_{n-1}^*,\chi_n^*)\setminus\mathbb{P}(\chi_n^*),\ \ \ldots,\ \ \mathbb{P}(\chi_0^*,\ldots,\chi_n^*)\setminus\mathbb{P}(\chi_1^*,\ldots,\chi_n^*) \tag{1.3}$$

is a cell decomposition of $\mathbb{P}(\chi^*)$ (see Remark 3.2 below), and

$$* = \mathbb{P}(\chi_0^*) \subset \mathbb{P}(\chi_0^*,\chi_1^*) \subset \cdots \subset \mathbb{P}(\chi_0^*,\ldots,\chi_{n-1}^*) \subset \mathbb{P}(\chi_0^*,\ldots,\chi_n^*) \tag{1.4}$$

displays $\mathbb{P}(\chi^*)$ as an iterated Thom space [4, Cor. 3.8].

The *Mislin genus* of a weighted projective space $\mathbb{P}(\chi)$ is the set of all homotopy classes of simply connected CW complexes Y of finite type such that for all primes p the p-localizations of Y and $\mathbb{P}(\chi)$ are homotopy equivalent. The Mislin genus of a space is rigid if it contains only the class of the space itself.

Theorem 1.3. *The Mislin genus of any weighted projective space is rigid.*

For $\mathbb{CP}^n$, this has been established by McGibbon [9, Thm. 4.2 (ii)].

In Section 2 we review Kawasaki's results on which our work is based. Theorem 1.1 is proved in Section 3, and Theorems 1.2 and 1.3 in Section 4.

2. Kawasaki's results

From now on, $\chi = (\chi_0,\ldots,\chi_n)$ always denotes a normalized a weight vector, and cohomology is taken with integer coefficients unless otherwise stated. In order to make Kawasaki's description of $H^*(\mathbb{P}(\chi))$ explicit, we recall two of his definitions. For $0 \le i \le n$ we set

$$l_i = l_i(\chi) = \prod_{p \text{ prime}} r_{n-i+1}(\chi;p)\cdots r_n(\chi;p), \tag{2.1}$$

where $r_0(\chi;p) \le \cdots \le r_n(\chi;p)$ are the p-contents of the weights in increasing order. We also consider the map

$$\varphi = \varphi_\chi \colon \mathbb{CP}^n \to \mathbb{P}(\chi), \quad [z_0 : \cdots : z_n] \mapsto [z_0^{\chi_0} : \cdots : z_n^{\chi_n}]. \tag{2.2}$$

Theorem 2.1 ([8, Thm. 1])**.** *Additively, $H^*(\mathbb{P}(\chi)) \cong H^*(\mathbb{CP}^n)$. Furthermore, there exist generators $\xi_i \in H^{2i}(\mathbb{P}(\chi))$ and $\eta \in H^2(\mathbb{CP}^n)$ such that $\varphi^*(\xi_i) = l_i \eta^i$ for $0 \le i \le n$; the multiplicative structure is specified by*

$$\xi_i \xi_j = \frac{l_i l_j}{l_{i+j}} \, \xi_{i+j}$$

in $H^{2(i+j)}(\mathbb{P}(\chi))$, for $0 \le i+j \le n$.

Kawasaki also determined the cohomology of the generalized lens space $L(k;\chi) = S^{2n+1}/\mathbb{Z}_k\langle\chi\rangle$, where in this case χ describes the weights of the k-th roots of unity. The answer depends on the augmented weight vector $(\chi, k) = (\chi_0, \dots, \chi_n, k)$.

Theorem 2.2 ([8, Thm. 2])**.** *The non-zero cohomology groups of $L = L(k;\chi)$ are $H^0(L) \cong H^{2n+1}(L) \cong \mathbb{Z}$ and $H^{2i}(L) \cong \mathbb{Z}_q$ for $1 \le i \le n$, where $q = l_i(\chi,k)/l_i(\chi)$.*

3. Classification up to homeomorphism

Consider a point $z \in \mathbb{P}(\chi)$. Let I and J be the subsets of $\{0, \dots, n\}$ corresponding to the zero and non-zero homogeneous coordinates of z, respectively, and let $q = \gcd\{\chi_i : i \in J\}$. Also, let $U_I = \{[z_0 : \cdots : z_n] : z_i \ne 0 \text{ for } i \notin I\}$, and write $\chi_I \in \mathbb{Z}^I$ for the weights indexed by I.

Lemma 3.1 (*cf.* [5, §5.15])**.** *There is an isomorphism of algebraic varieties*

$$U_I \cong (\mathbb{C}^\times)^{|J|-1} \times \mathbb{C}^I / \, \mathbb{Z}_q\langle\chi_I\rangle,$$

sending z to a point of the form $(\tilde{z}, 0)$.

Observe that $\mathbb{C}^I/\mathbb{Z}_q\langle\chi_I\rangle$ is the unbounded cone over $L(q;\chi_I)$.

Proof. The weight vector χ_J determines a morphism $\mathbb{C}^\times \to (\mathbb{C}^\times)^J$ with kernel $\mathbb{Z}_q$. Let T' be its image and $T'' \cong (\mathbb{C}^\times)^{|J|-1}$ a torus complement. Then

$$U_I = \big((\mathbb{C}^\times)^J \times \mathbb{C}^I\big) \, / \, \mathbb{C}^\times\langle\chi\rangle = \big(T'' \times T' \times \mathbb{C}^I\big) \, / \, \mathbb{C}^\times\langle\chi\rangle = T'' \times \mathbb{C}^I / \, \mathbb{Z}_q\langle\chi_I\rangle. \quad \square$$

Remark 3.2. If $\chi_0 = 1$ and $z = [1 : 0 : \cdots : 0]$, then $U_I \cong \mathbb{C}^n$. If the weights form a divisor chain, we have $\mathbb{P}(\chi) \setminus U_I = \mathbb{P}(\chi_1, \dots, \chi_n) = \mathbb{P}(1, \chi_2/\chi_1, \dots, \chi_n/\chi_1)$, hence we inductively get a decomposition of $\mathbb{P}(\chi)$ into $n+1$ cells $*$, $\mathbb{C}$, $\mathbb{C}^2$, …, $\mathbb{C}^n$.

Lemma 3.3. $H^{2n-1}\big(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}\big) \cong \mathbb{Z}_q$.

Proof. Set $X = (\mathbb{C}^\times)^{|J|-1}$, $Y = \mathbb{C}^I/\mathbb{Z}_q\langle\chi_I\rangle$ and $m = |I| - 1$. Excision, Lemma 3.1 and the Künneth formula for relative cohomology imply

$$\begin{aligned} H^*\big(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}\big) &\cong H^*\big(U_I, U_I \setminus \{z\}\big) \\ &\cong H^*(X \times Y, (X \setminus \{\tilde{z}\}) \times Y \cup X \times (Y \setminus \{0\})) \\ &\cong H^*(X, X \setminus \{\tilde{z}\}) \otimes H^*(Y, Y \setminus \{0\}). \end{aligned}$$

Since X is a manifold of dimension $2(n-m+1)$, we find

$$H^{2n-1}\big(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}\big) \cong H^{2m+1}\big(Y, Y \setminus \{0\}\big) \cong \tilde{H}^{2m}(L(q;\chi_I)).$$

If $m = 0$, then $q = 1$ because χ is normalized, and the claim holds. Otherwise, Theorem 2.2 gives $H^{2m}(L(q;\chi_I)) \cong \mathbb{Z}_{q'}$, where the p-content of q' is given by

$$p\text{-content of } \frac{l_m(\chi_I, q)}{l_m(\chi_I)} = \prod_{i=1}^{m} \frac{r_{m+2-i}(\chi_I, q; p)}{r_{m+1-i}(\chi_I; p)}. \tag{3.1}$$

We have to show $q' = q$, which means that q' and q have the same p-content for all p. This is clearly true if q is not divisible by p. Otherwise, χ_I inherits from the normalized weight vector χ two weights not divisible by p. (Recall that q is the gcd of the weights appearing in χ, but not in χ_I.) Hence, $r_1(\chi; p) = 1$, and the numerator of (3.1) differs from the denominator by the p-content of q. This finishes the proof. □

Proof of Theorem 1.1. By the remarks preceding the theorem, we only have to prove the implication (3) ⇒ (1). In order to do so, we show how to read off the normalized weights from topological invariants of a weighted projective space $\mathbb{P}(\chi)$. For $z \in \mathbb{P}(\chi)$, let $q'(z)$ be the order of the finite group $H^{2n-1}\big(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}\big)$. Lemma 3.3 implies that for all $d \geq 1$ the space

$$X(d) = \big\{ z \in \mathbb{P}(\chi) : d \mid q'(z) \big\}$$

is again a weighted projective space or empty. In fact,

$$X(d) = \big\{ [z_0 : \cdots : z_n] \in \mathbb{P}(\chi) : z_i = 0 \text{ if } d \nmid \chi_i \big\}$$

because d divides $q'(z) = q$ if and only if it divides χ_i for all i such that $z_i \neq 0$. For each d, the dimension of $X(d)$ (which equals the degree of the highest non-vanishing cohomology group) therefore tells us the number of weights divisible by d. This determines the normalized weights completely up to order. □

4. Classification up to homotopy equivalence

This section relies heavily on the theory of localization and homotopy pullbacks. We refer readers to [7], especially Chapter II, for all background information.

Throughout the section, every unlocalized space is a simply connected CW complex of finite type. A map $f\colon X \to Y$ is therefore a homotopy equivalence (written $X \simeq Y$) if and only if it induces an isomorphism $H^*(f)$ of integral cohomology; in this case, f^{-1} denotes a homotopy inverse for f.

Given any set $\mathcal{P}$ of primes, the algebraic localization of $\mathbb{Z}$ is denoted by $\mathbb{Z}_{\mathcal{P}}$, and the homotopy theoretic localization of f by $f_{\mathcal{P}}\colon X_{\mathcal{P}} \to Y_{\mathcal{P}}$. Each $X_{\mathcal{P}}$ is unique up to homotopy equivalence, and admits a homotopy class of localization maps $l_{\mathcal{P}}\colon X \to X_{\mathcal{P}}$. If $\mathcal{P} = \{p\}$, then $f_{\mathcal{P}}$ is abbreviated to $f_p\colon X_p \to Y_p$; if $\mathcal{P} = \emptyset$, then localization is *rationalization* and abbreviated to $f_0\colon X_0 \to Y_0$. It follows from the definitions that $H^*(l_{\mathcal{P}}; \mathbb{Z}_{\mathcal{P}})$ is an isomorphism, and that $f\colon X_{\mathcal{P}} \to Y_{\mathcal{P}}$ is a homotopy equivalence if and only if $H^*(f; \mathbb{Z}_{\mathcal{P}})$ is an isomorphism.

The *homotopy pullback* of a diagram $X \to Z \leftarrow Y$ is well-defined up to homotopy equivalence. It may be constructed by replacing either map with a fibration, and pulling it back along the other in the standard fashion; homotopic defining maps therefore lead to homotopy equivalent pullbacks. If $\mathcal{P}$ and $\mathcal{Q}$ are disjoint sets of primes, then the homotopy pullback of $X_{\mathcal{Q}} \to X_0 \leftarrow X_{\mathcal{P}}$ is $X_{\mathcal{P}\cup\mathcal{Q}}$, for any X, *cf.* [10, Prop. 2.9.3] or [7, proof of Thm. 7.13].

Lemma 4.1. *Given two disjoint sets $\mathcal{P}$ and $\mathcal{Q}$ of primes, assume that $f\colon X_{\mathcal{P}} \to Y_{\mathcal{P}}$ and $g\colon X_{\mathcal{Q}} \to Y_{\mathcal{Q}}$ are homotopy equivalences, and let $h = f_0^{-1} g_0$; then $Y_{\mathcal{P}\cup\mathcal{Q}}$ is the homotopy pullback of the diagram*

$$X_{\mathcal{Q}} \xrightarrow{hl_0} X_0 \xleftarrow{l_0} X_{\mathcal{P}}\,. \tag{4.1}$$

Proof. The vertical maps in the homotopy commutative ladder

$$\begin{array}{ccccc} X_{\mathcal{Q}} & \xrightarrow{hl_0} & X_0 & \xleftarrow{l_0} & X_{\mathcal{P}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Y_{\mathcal{Q}} & \xrightarrow[l_0]{} & Y_0 & \xleftarrow[l_0]{} & Y_{\mathcal{P}} \end{array} \tag{4.2}$$

are homotopy equivalences, and the homotopy pullback of the lower row is $Y_{\mathcal{P}\cup\mathcal{Q}}$. For strictly commuting such diagrams, the homotopy pullbacks of both rows are homotopy equivalent by the Mayer–Vietoris sequence, *cf.* [7, p. 95]. By passing through the diagram $X_{\mathcal{Q}} \to Y_0 \leftarrow X_{\mathcal{P}}$ in two different ways, we can replace the ladder (4.2) by a strictly commutative version, and the claim follows. □

In Theorem 2.1 we selected a generator $\xi_1 \in H^2(\mathbb{P}(\chi)) \cong \mathbb{Z}$, whose localization in $H^2(\mathbb{P}(\chi)_{\mathcal{P}}; \mathbb{Z}_{\mathcal{P}}) \cong \mathbb{Z}_{\mathcal{P}}$ is necessarily a generator for any set $\mathcal{P}$ of primes. We define the *degree* $\deg(h)$ of a self-map $h\colon \mathbb{P}(\chi)_{\mathcal{P}} \to \mathbb{P}(\chi)_{\mathcal{P}}$ by $H^*(h;\mathbb{Z}_{\mathcal{P}})(\xi_1) = \deg(h)\,\xi_1$. This determines a multiplicative map

$$\deg\colon [\mathbb{P}(\chi)_{\mathcal{P}}, \mathbb{P}(\chi)_{\mathcal{P}}] \to \mathbb{Z}_{\mathcal{P}}. \tag{4.3}$$

Proposition 4.2.

(1) *A self-map of $\mathbb{P}(\chi)_{\mathcal{P}}$ is a homotopy equivalence if and only if its degree is a unit in $\mathbb{Z}_{\mathcal{P}}$.*
(2) *The degree map* (4.3) *is surjective.*
(3) *If $\mathcal{P}$ contains no divisor of any χ_j, then the degree map is a bijection.*

Proof. Let f be any self-map of $\mathbb{P}(\chi)_{\mathcal{P}}$, and assume that it has degree a. By Theorem 2.1, $H^*(f;\mathbb{Z}_{\mathcal{P}})$ induces multiplication by a^k on $H^{2k}(\mathbb{P}(\chi)_{\mathcal{P}};\mathbb{Z}_{\mathcal{P}}) \cong \mathbb{Z}_{\mathcal{P}}$, for every $1 \le k \le n$. If f is a homotopy equivalence, then $a \in \mathbb{Z}_{\mathcal{P}}$ is a unit and f^{-1} has degree a^{-1}. Conversely, if $\deg(f)$ is a unit, then $H^*(f;\mathbb{Z}_{\mathcal{P}})$ is an isomorphism, and f is a homotopy equivalence. Thus (1) holds.

Fix a positive integer a, and define the self-map $h\colon \mathbb{P}(\chi) \to \mathbb{P}(\chi)$ by raising each homogeneous coordinate to the power a; in particular, write $h'\colon \mathbb{CP}^n \to \mathbb{CP}^n$ for the standard case. Thus h and h' commute with φ, leading to the commutative diagram

$$\begin{array}{ccc} H^*(\mathbb{P}(\chi)) & \xrightarrow{H^*(\varphi)} & H^*(\mathbb{CP}^n) \\ {\scriptstyle H^*(h)}\downarrow & & \downarrow{\scriptstyle H^*(h')} \\ H^*(\mathbb{P}(\chi)) & \xrightarrow[H^*(\varphi)]{} & H^*(\mathbb{CP}^n) \end{array}.$$

Since $H^2(h')$ is multiplication by a, it follows that $\deg(h) = a$. But every element $c \in \mathbb{Z}_{\mathcal{P}}$ may be written as a quotient $c = b/a$ of integers, where $a \in \mathbb{Z}_{\mathcal{P}}$ is a positive unit. Then (2) follows from (1), together with the observations that complex conjugation on a single coordinate has degree -1 and constant self-maps have degree 0.

If $\mathcal{P}$ contains no divisor of any weight, then $\varphi_{\mathcal{P}}\colon \mathbb{CP}^n_{\mathcal{P}} \to \mathbb{P}(\chi)_{\mathcal{P}}$ is a homotopy equivalence by Theorem 2.1. To prove (3), it therefore suffices to consider maps $h_1, h_2\colon \mathbb{CP}^n_{\mathcal{P}} \to \mathbb{CP}^n_{\mathcal{P}}$ of equal degree. Then their restrictions to the 2-skeleton $S^2_{\mathcal{P}} \subset \mathbb{CP}^n_{\mathcal{P}}$ are homotopic, because $[S^2_{\mathcal{P}}, \mathbb{CP}^n_{\mathcal{P}}] \cong \pi_2(\mathbb{CP}^n_{\mathcal{P}}) \cong \mathbb{Z}_{\mathcal{P}}$. The obstructions to extending a homotopy over $\mathbb{CP}^n_{\mathcal{P}}$ lie in the groups $H^k(\mathbb{CP}^n_{\mathcal{P}}; \pi_k(\mathbb{CP}^n_{\mathcal{P}}))$ for $k > 2$, *cf.* [6, Ex. 4.24]. Furthermore, $\mathbb{CP}^n_{\mathcal{P}}$ is $2n$-dimensional and $\pi_k(\mathbb{CP}^n_{\mathcal{P}}) = 0$ whenever $2 < k \leq 2n$, so these obstruction groups vanish. Thus h_1 and h_2 are homotopic, deg is injective, and (3) follows. □

Part (3) of Proposition 4.2 is well-known for $\mathbb{CP}^n$ [9, Thm. 2.2].

Proof of Theorem 1.3. Let Y be an element of the Mislin genus of $\mathbb{P}(\chi)$. Since $H^*(Y) \cong H^*(\mathbb{P}(\chi))$ as abelian groups, Y is homotopy equivalent to a CW complex of dimension $2n$, *cf.* [6, Prop. 4C.1]. Moreover, $H^*(\mathbb{P}(\chi); \mathbb{Q})$ is multiplicatively generated by a single element of degree 2, therefore so is $H^*(Y; \mathbb{Q})$. A generator of $H^2(Y)$ determines a map $e\colon Y \to K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ such that $H^2(e)$ is an isomorphism. Up to homotopy, e factors through the $2n$-skeleton $\mathbb{CP}^n \subset \mathbb{CP}^\infty$, and its corestriction $f\colon Y \to \mathbb{CP}^n$ is a rational homotopy equivalence. By Theorem 2.1, $\varphi\colon \mathbb{CP}^n \to \mathbb{P}(\chi)$ is also a rational equivalence.

Let $\mathcal{P} = \{p_1, \dots, p_s\}$ be the set of primes that must be inverted for $H^*(f)$ and $H^*(\varphi)$ to be isomorphisms, and $\mathcal{Q}$ its complement. Define $\mathcal{Q}_i = \mathcal{Q} \cup \{p_1, \dots, p_i\}$, and note that $\mathcal{Q}_s$ contains all primes. It therefore suffices to show that $Y_{\mathcal{Q}_i} \simeq \mathbb{P}(\chi)_{\mathcal{Q}_i}$ for every $0 \leq i \leq s$.

We proceed by induction on i; the base case $i = 0$ is valid because $\varphi_{\mathcal{Q}}$ and $f_{\mathcal{Q}}$ induce isomorphisms $H^*(\varphi_{\mathcal{Q}}; \mathbb{Z}_{\mathcal{Q}})$ and $H^*(f_{\mathcal{Q}}; \mathbb{Z}_{\mathcal{Q}})$, and are homotopy equivalences. So assume that $Y_{\mathcal{Q}_i} \simeq \mathbb{P}(\chi)_{\mathcal{Q}_i}$, and write $p = p_{i+1}$. By choice of Y there is a homotopy equivalence $Y_p \simeq \mathbb{P}(\chi)_p$, so Lemma 4.1 identifies $Y_{\mathcal{Q}_{i+1}}$ as the homotopy pullback of (4.1) for some homotopy equivalence $h\colon \mathbb{P}(\chi)_0 \to \mathbb{P}(\chi)_0$. Since $\deg(h) \in \mathbb{Z}_{\mathcal{Q}_{i+1}}$, it equals b/a for some units $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_{\mathcal{Q}_i}$.

By Proposition 4.2 (2) there are homotopy equivalences $f\colon \mathbb{P}(\chi)_p \to \mathbb{P}(\chi)_p$ and $g\colon \mathbb{P}(\chi)_{\mathcal{Q}_i} \to \mathbb{P}(\chi)_{\mathcal{Q}_i}$ of degrees a and b respectively. So by Proposition 4.2 (3), $h \simeq f_0^{-1} g_0$. Lemma 4.1 with $X = Y = \mathbb{P}(\chi)$ then implies that $Y_{\mathcal{Q}_{i+1}} \simeq \mathbb{P}(\chi)_{\mathcal{Q}_{i+1}}$, and completes the inductive step. □

Proof of Theorem 1.2. If χ and χ' have the same p-content up to order, then some permutation of homogeneous coordinates defines a homeomorphism $\mathbb{P}({}_p\chi) \cong \mathbb{P}({}_p\chi')$ for each prime p. This homeomorphism may be localized at p.

Moreover, $\varphi_\chi\colon \mathbb{CP}^n \to \mathbb{P}(\chi)$ factorizes as $\varphi_\chi = f\varphi_{{}_p\chi}$, where

$$f\colon \mathbb{P}({}_p\chi) \to \mathbb{P}(\chi), \quad [z_0 : \cdots : z_n] \mapsto [z_0^{\alpha(0)} : \cdots : z_n^{\alpha(n)}],$$

and $\alpha(j) = \chi_j / {}_p\chi_j$ for $0 \leq j \leq n$. Theorem 2.1 then implies that $H^*(f; \mathbb{Z}_p)$ is an isomorphism, and hence that f_p is a homotopy equivalence. So f_p and f_p^{-1} feature in the chain of maps

$$\mathbb{P}(\chi)_p \simeq \mathbb{P}({}_p\chi)_p \cong \mathbb{P}({}_p\chi')_p \simeq \mathbb{P}(\chi')_p$$

for any prime p, and the result follows from Theorem 1.3. □

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