

*Smoothing non-smooth systems with the moving
average transformation*

James, Hook and Broomhead, Dave

2011

MIMS EPrint: **2011.58**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

Smoothing non-smooth systems with the Moving Average Transformation

James Hook, David Broomhead - School of Mathematics, University of Manchester

Abstract

We present a novel, systematic procedure for smoothing non-smooth dynamical systems. In particular we introduce the Moving Average Transformation which can be thought of as a change of variables which transforms discontinuous systems into *dynamically equivalent* continuous systems and continuous but non-differentiable systems into dynamically equivalent differentiable systems. This smoothing gives us a new way to compute the stability properties of a non-smooth systems and provides a new theoretical link between smooth and non-smooth systems. The dynamics and algebraic structure of systems obtained by transforming typical non-smooth systems are investigated.

1 Introduction

Discontinuities in non-smooth models of physical systems are approximations of highly non-linear smooth processes that take place on very short time scales. The Newtonian impact of a ball bouncing on the ground is discontinuous in velocity but in reality the ball and ground are compliant, elastic bodies and discontinuities in velocity are physically impossible! Introducing compliance gives us a continuous but non-smooth system. The dynamics are non-smooth because there is only a force between the ball and ground when they are in contact, but this too is a false assumption. The electric fields of the atoms in the ball and ground are continually interacting in a smooth way it is just that the forces produced by these interactions are only significant during the short impacting phase of the dynamics.

Of course we approximate these fast processes by non-smooth discontinuities to obtain simpler, more manageable models of physical processes. Ironically the piecewise smooth nature of the resulting models can make them more difficult to study. For example non-differentiability makes it impossible to apply many standard numerical techniques to these systems [1, 2, 3, 4, 5]. For a discontinuous system like the bouncing ball the topology of the state space and any attractors it contains is broken up by jumps in the evolution so that an attractor will typically appear to be comprised of several disconnected parts, if we patch these together by connecting jump take offs and landings we arrive at a new topological space which can tell us much more about the systems dynamics than the topology of the original disconnected object.

There is therefore some motivation for transforming non-smooth systems (which as outlined are approximations of smooth systems) back into smooth systems. Several authors have tried a fairly direct approach, adding a small region where fast, smooth dynamics replace the non-smooth discontinuity in a process called regularization [5]. Whilst this doesn't necessarily mean introducing more layers of physically realistic modeling, since the extra components can be chosen to be as simple as possible, it still feels like a step in the wrong direction. The non-smooth discontinuities are a simplification and we should be able to exploit this trade off in realism by studying more mathematically appealing tractable systems.

In addition to these approximated physical processes there is a huge catalogue of non-smooth systems in control theory where digital switching between different modes really should be thought of as being intrinsically non-smooth [6]. Abstract non-smooth systems are also of great interest mathematically as they can generically exhibit bifurcation structures which would be impossible or of high codimension in the space of smooth systems [7]. Although it is not natural to think of one of these systems as the limit of some smooth system it can still be advantageous to smooth them in some way as all the problems associated with the analysis of non-smooth physical systems are also present here.

In this paper we introduce the Moving Average Transformation which can be thought of as a change of variables which transforms discontinuous systems into continuous systems and continuous but non-differentiable systems into differentiable systems. For a non-smooth system with evolution $\phi_t : X \rightarrow X$ the transformation is defined by

$$\Phi(x) = \int_{-1}^0 \phi_\tau(x) d\tau \quad (1)$$

Since the evolution of the system is automatically incorporated into the transformation it is possible to systematically obtain smoothed systems using this technique. The definition of the transformation is reviewed in 1.2.

The transformation provides an explicit link between a non-smooth system and its smoothed *Dynamically Equivalent* counterpart. This enables us to better understand topological aspects of the dynamics and apply standard numerical techniques that rely on differentiability.

Crucially the technique is totally systematic, introduces no extraneous dynamics and provides a clear equivalence between the original and transformed (smoothed) system.

Although dynamical equivalence guarantees that all topological invariants other than continuity are preserved under the transformation and any standard smooth technique can be applied to the smoothed systems this paper focuses on orbit stability. Small perturbations to an orbit in a non-smooth system evolve in a discontinuous way making stability analysis very complicated even in simple systems [3, 8]. In section 3 we show how the Saltation matrices which capture singular stability features of non-smooth discontinuities are naturally integrated into our smoothed systems. Their discontinuous action is essentially spread out over a non-zero time interval of the flow.

In section 2 we demonstrate a novel way to calculate the stability of a periodic orbit in a non-smooth system. We start with a non-smooth system that contains a

periodic orbit which we want to analyze. First we use the transformation to obtain a dynamically equivalent differentiable system. This smoothed system contains a periodic orbit corresponding to the orbit in the original system, we can then calculate the stability of this smoothed orbit in the usual smooth way, by integrating the derivative of the flow. The dynamical equivalence guarantees that the Lyapunov spectrum of orbits in the original and smoothed systems are the same. In section 4 we use the same idea, that the stability characteristics of the original and transformed systems are the same, to compute the Lyapunov exponent of a chaotic non-smooth system from a time-series recording. In principal this technique could be applied to experimental data.

This paper is organized as follows. In the remainder of section 1 we introduce the moving average transformation and provide a simple example illustrating the sort of construction/analysis we hope to achieve using it. In section 2 we present a full and explicit example of applying the transformation twice to a discontinuous system. In section 3 we apply the transformation to the normal forms of some generic non-smooth discontinuities, this section provides most of the theory in the paper and justifies our claim that the transformation smooths non-smooth systems. Finally in section 4 we smooth a more complicated system with a numerically implemented procedure based on a time-series reconstruction.

1.1 Simple example of smoothing

Before we formally introduce our method for smoothing we shall first by way of motivation, present a very simple example. Although the method here is quite ad hoc it illustrates the desired relationship between the original and smooth systems as well as the necessary properties of the transformation between them.

Consider a unit mass attached to a spring with unit stiffness along with a wall where the mass undergoes Newtonian impacts positioned at $x = 0$ where the spring is at its natural length. We assume that the spring is light and linear and that there is no friction so that away from the wall the dynamics are governed by the linear differential equation

$$\frac{d}{dt} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad (2)$$

When the mass hits the wall there is an impact with restitution c so that whenever we reach the set $In = \{(x, \dot{x}) \in \mathbb{R}^2 : x = 0, \dot{x} < 0\}$ we instantaneously apply the map $R(x, \dot{x}) = (x, -c\dot{x})$ which maps In to $Out = \{(x, \dot{x}) \in \mathbb{R}^2 : x = 0, \dot{x} > 0\}$. Since we are mapped instantaneously from In to Out the state space of the system is given by $M = \{(x, \dot{x}) \in \mathbb{R}^2 : x \geq 0\} / \{(x, \dot{x}) \in \mathbb{R}^2 : x = 0, \dot{x} < 0\}$.

The system then evolves by flowing according to (2) until hitting In then mapping to Out and flowing according to (2) again... Define $\phi_t : M \mapsto M$ to be the time t evolution map that takes a point in the state space forward in time t seconds.

We seek a smoothing transformation T which can be thought of as a change of variables with the property that the transformed system $\varphi_t = T \circ \phi_t \circ T^{-1}$ is smooth.

Claim The transformation $T : M \mapsto \{(y_1, y_2) \in \mathbb{R}^2\}$ defined in polar coordinates by

$$T(\theta, r) = (2\theta, rc^{\frac{\theta}{\pi}}) \quad (3)$$

provides an equivalence between our discontinuous system and the linear system evolved by the ODE

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\pi} \log c & 2 \\ -2 & \frac{1}{\pi} \log c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (4)$$

defined on \mathbb{R}^2 equipped with the Euclidian metric.

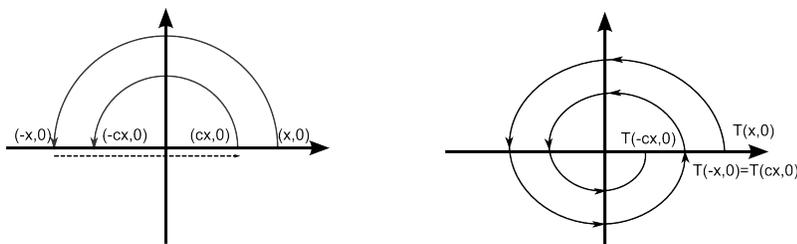


Figure 1: An orbit to the original system (Left) and its image under T (right).

Proof This rests quite heavily on our definition of equivalent! Clearly the two systems are not topologically equivalent since continuity is a topological invariant.

Definition If ϕ is the evolution of a non-smooth system which includes some jumps that are applied instantaneously on reaching a set Σ then we say that the transformation T provides a *Dynamical Equivalence* between ϕ and φ via $\varphi_t = T \circ \phi_t \circ T^{-1}$ if

- T is continuous
- T^{-1} exists and is continuous everywhere except $T(\Sigma)$
- $\frac{\delta T}{\delta x}$ exists and is non-zero everywhere

This is a relaxation of the usual definition for topological equivalence where T^{-1} is required to be continuous everywhere. Clearly our claim holds for this notion of equivalence.

Definition We define a *Cheat Metric* d on a discontinuous system to be a continuous metric which identifies jump take offs and landings so that $d(s, R(s)) = 0$ for all $s \in \Sigma$. To define d everywhere else we could just measure the length of the shortest path between two points but allow the path to jump over the discontinues in the same way as ϕ .

Viewed with the cheat metric all systems have continuous orbits, hence the cheat. This means that the original system with Euclidian metric is not topologically equivalent to the original system with cheat metric which in tern is topologically equivalent to the smoothed system with Euclidian metric.

The second condition for dynamical equivalence that T^{-1} exists and is continuous everywhere except $T(\Sigma)$ is equivalent to the condition that T^{-1} is continuous w.r.t a cheat metric. Therefore dynamically equivalent systems are topologically equivalent when viewed with a cheat metric.

The relationship between our original discontinuous system and the dynamically equivalent smooth system is quite subtle. If we equip M with the Euclidian metric then T^{-1} is discontinuous so the two systems are not topologically equivalent. However the evolutions are still interchangeable via $\varphi_\tau = T \circ \phi_\tau \circ T^{-1}$ so we can use the new system to describe the original. Since our new system is linear it is easy to solve and we can therefore obtain a neat expression for the evolution

$$\phi_t = T^{-1} \circ \frac{t}{\pi} \log c \begin{pmatrix} \sin 2t & \cos 2t \\ -\cos 2t & \sin 2t \end{pmatrix} \circ T \quad (5)$$

Complications caused by the discontinuity have all been captured in the transformation T . Since the new system is differentiable we can also calculate the stability of the fixed point at the origin which is determined by the real part of (4)'s eigenvalue pair $\lambda_\pm = \frac{1}{\pi} \log c \pm 2i$. This agrees with the standard non-smooth analysis using the saltation matrix formulation [2].

We shall see that the moving average transformation has the same properties as T and can therefore be used to systematically obtain continuous systems on \mathbb{R}^n dynamically equivalent to any discontinuous system we like. Moreover the transformation can be used to obtain differentiable systems topologically equivalent to continuous but non differentiable systems.

1.2 The moving average transformation

Suppose that ϕ_t is the evolution operator of some possibly discontinuous or non differentiable dynamical system in a state space M which we assume is embedded in \mathbb{R}^n . The moving average transformation is then defined as a map $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ with

$$\Phi(x) = \int_{-1}^0 \phi_\tau(x) d\tau \quad (6)$$

So that $\Phi(x)_i$ is the time averaged value of x 's i th component over the last second of evolution. The averaging period of one second is totally arbitrary and from a theoretical point of view irrelevant, integrating back in time is also not important in fact anything of the form

$$\Upsilon(x) = \int_a^b \phi_\tau(x) d\tau \quad (7)$$

with $a < b$ will work in more or less the same way. We choose 1 for convenience and integrate backwards so that when applied to time-series we have something that makes sense from a signal processing point of view. For a discussion of applications to time-series and the averaging periods numerical importance see section 4.

In order for $\varphi_t = \Phi \circ \phi_t \circ \Phi^{-1}$ to be a dynamical equivalence between our original system and that obtained through the transformation we require that Φ is continuous and differentiable with non-zero derivative and an inverse that is continuous everywhere except the image of the jump set Σ .

The differentiability of Φ is a sufficient condition for our procedure to work (although we shall see that sliding systems which violate this condition can still be treated in this way). For a smooth flow the derivative is defined by considering the

evolution of small perturbations to $\phi_{-1}(x)$ over one second. These perturbations obey the non-autonomous linear ODE

$$\frac{dz}{d\tau} = \frac{\delta^2 \phi_{\tau-1}(x)}{\delta \tau \delta x} z \quad (8)$$

Where the RHS is an $n \times n$ matrix of partial derivatives multiplying the n vector state. This equation has evolution operator $J(\tau)$ which is a τ dependent matrix. The derivative of Φ is then given by

$$\frac{\delta \Phi}{\delta x} = \int_0^1 J(\tau) d\tau J(1)^{-1} \quad (9)$$

If on the other hand the one second backwards time orbit of x encounters a non-smooth discontinuity like a jump or switch at $\phi_{-t^*}(x)$ with saltation S (see 2.1) then the derivative is given by

$$\frac{\delta \Phi}{\delta x} = \int_0^{1-t^*} J_1(\tau) d\tau [J_1(1-t^*) S J_2(t^*)]^{-1} + \int_0^{t^*} J_2(\tau) d\tau J_2(t^*)^{-1} \quad (10)$$

Where $J_1(\tau)$ is the evolution associated with perturbations about $\phi_{-1}(x)$ and $J_2(\tau)$ perturbations about $\phi_{-t^*}(x)$.

It is important to note that we do not need to know that saltation matrices to construct Φ , the construction and application of the transformation does not rely on any higher non-smooth theory. As shown in section 2 we can compute Φ by directly integrating along orbits, if we then differentiate our expression for Φ the saltation matrices naturally emerge in the analysis.

This is reviewed in section 3. That the derivative is non-zero is not guaranteed but can generically (in the topological sense) be remedied by taking a delay vector as in section 2 [9]. All that remains is to show that Φ^{-1} is continuous over discontinuities which is simple. If x and $F(x)$ are a preimage/image pair of points under a discontinuity in the system then $\phi_{-\epsilon}(x) = \phi_{-\epsilon}[F(x)]$ so the averages will agree.

Therefore we can apply Φ to a discontinuous system to obtain a dynamically equivalent continuous system. But we can actually go one better than that. Since the transformation is able to encode information from the saltation matrices it is also able to transform continuous but non-differentiable systems (so ODEs with discontinuous RHSs) into differentiable systems (ODEs with continuous RHSs). Thus if we begin with a discontinuous system and apply the filter twice we obtain a continuous but non-differentiable system followed by a differentiable system.

It is important to note that for every non-smooth event in the original system there will be two smoother but still non-smooth events in the transformed system one corresponding to the head of the averaging period crossing a discontinuity and another for the tail.

Definition The *Head Discontinuity* of a smoothed system corresponds to the head of the averaging period crossing the discontinuity in our original (untransformed) system. So that for example a switch at Σ in the original system gives rise to a head discontinuity in the smoothed system at $\Phi(\Sigma)$.

Definition The *Tail Discontinuity* of a smoothed system corresponds to the tail of the averaging period crossing the discontinuity in our original (untransformed) system. So that for example a switch at Σ in the original system gives rise to a tail discontinuity in the smoothed system at $\Phi \circ \phi_1(\Sigma)$.

We shall see that for a jump or switch (3.1,3.2) the head and tail discontinuities are characteristically the same but that for slides and grazes (3.3,3.4,3.5,3.6) they are qualitatively different.

2 Explicit example

In this section we will explicitly obtain an algebraic form for the moving average transformation on a simple discontinuous system and apply it to obtain a dynamically equivalent continuous non-differentiable system. We then repeat the procedure to obtain an equivalent differentiable system. We shall then compute the stability of a periodic orbit in the differentiable system by integrating the derivative of the flow which we compare with a standard analysis of the discontinuous system.

Like in the motivating example our transformation will join jump take offs and landings in such a way that the new system is differentiable and dynamically equivalent to the original.

It should become clear through this section that the process of applying the filter and obtaining an induced flow is really intended as an exercise for computers rather than human beings! The various stages of solving differential equations, integrating solutions and inverting maps makes it difficult to find a non-trivial example which is still analytically tractable. The hope is that this example gives the reader a general idea of what happens and how. More complicated systems should be dealt with numerically either by the same direct method used here or with the time-series reconstruction approach outlined in section 4. Our system evolves according to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ ay \end{pmatrix} \quad (11)$$

along with the rule that at $\{(x, y) \in \mathbb{R}^2 : x = \pi\}$ we apply the map $R(x, y) = (x, by)$. Note that $\{(x, y) \in \mathbb{R}^2 : y = 0, x \in [0, \pi)\}$ is a periodic orbit independent of the real parameters a and b , see fig 2,a,i. Before we proceed with the moving average transformation we will review the calculation of saltation matrices and compute the stability of the systems periodic orbit with the standard non-smooth approach.

2.1 Calculation of saltation matrices

For a smooth flow $\dot{x} = F(x)$ the jacobian which governs the evolution of small perturbations to an orbit $\{\phi_t(x) : t \in \mathbb{R}_+\}$ is defined as the solution to the linear non-autonomous ODE

$$\dot{J}(t) = \frac{\delta^2 \phi_\tau(x')}{\delta x' \delta \tau} \Big|_{\phi_t(x)} J(t) \quad (12)$$

which evolves continuously. Non-smooth discontinuities give rise to discontinuities in the evolution of these small perturbations and therefore the jacobian. We can capture

these discontinuities in a linear way with a saltation matrix which is constructed for a jump as follows. Suppose that the flow is an ODE $\dot{x} = F(x)$ until reaching a set Σ where we instantaneously apply the map R then switch to the ODE $\dot{x} = G(x)$. For an orbit that hits Σ at a point s it suffices to assume that the flow is locally constant. Now take the following perturbations about s

- $a_1 = s - \epsilon F(s)$
- a_i to form a basis for the linearization of Σ at s

which after ϵ seconds are mapped by the flow to the following perturbations about $R(s) + \epsilon G(R(s))$

- $b_1 = -\epsilon G[R(s)]$
- $b_i = \frac{\delta R}{\delta x}|_s a_i + \epsilon G[R(s)]$

This transformation can be described uniquely by a linear map S , the saltation matrix. See Fig 2,b. For a switch or slide the analysis is exactly the same except that we set $R = I$.

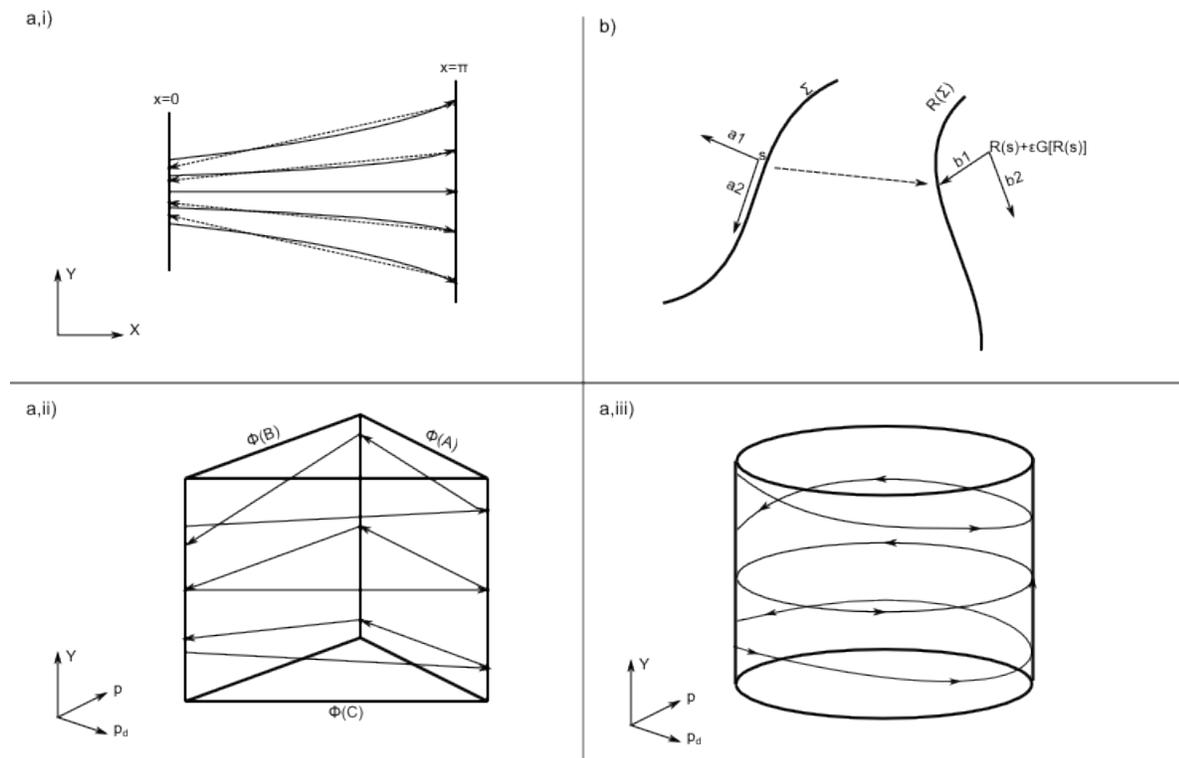


Figure 2: a,i) Orbits to original discontinuous system. ii) Continuous once transformed system. iii) Differentiable twice transformed system. b) Configuration of perturbations for saltation matrix calculation.

2.2 Stability of discontinuous system

The evolution of small perturbations along one period of our systems periodic orbit is given by $SJ(\pi)$ where $J(\pi)$ is the jacobian evaluated along the flow from $(0, 0)$ to $(\pi, 0)$ in the usual way and S is the saltation matrix associated with the jump at $(\pi, 0)$. The jacobian is evolved by

$$\dot{J}(t) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} J(t) \quad (13)$$

so that

$$J(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{a\pi} \end{pmatrix} \quad (14)$$

The saltation matrix is given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad (15)$$

The stability of the orbit is determined by the eigenvalue $be^{a\pi}$ which corresponds to the Lyapunov exponent $a + \frac{\log b}{\pi}$.

2.3 Once transformed system

The moving average transformation

$$\Phi(x) = \int_{-1}^0 \phi_t(x) dt \quad (16)$$

is piecewise smooth on two regions $A = \{0 \leq x \leq 1\}$ and $B = \{1 \leq x \leq \pi\}$. In A there is a jump in the 1-second backwards time flow so that

$$\begin{aligned} \Phi \begin{pmatrix} x \\ y \end{pmatrix} &= \int_{-x}^0 \begin{pmatrix} x+t \\ ye^{at} \end{pmatrix} dt + \int_{-1}^{-x} \begin{pmatrix} \pi+x+t \\ \frac{ye^{at}}{b} \end{pmatrix} dt \\ &= \begin{pmatrix} x(1-\pi) + \pi - \frac{1}{2} \\ \frac{y}{a} [1 - \frac{e^{-a}}{b} + e^{-ax}(\frac{1}{b} - 1)] \end{pmatrix} \end{aligned} \quad (17)$$

In B there is no jump in the 1-second backwards time flow so

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \int_{-1}^0 \begin{pmatrix} x+t \\ ye^{at} \end{pmatrix} dt = \begin{pmatrix} x - \frac{1}{2} \\ \frac{y}{a}(1 - e^{-a}) \end{pmatrix} \quad (18)$$

If we now set $(p, q) = \Phi(x, y)$ and attempt to obtain the induced flow of (p, q) we run into some difficulties. In particular Φ is not injective so that the new system will require a hybrid formulation (that is an extra discrete variable to keep track of which branch of Φ^{-1} we are using). This situation is easily resolved however by making use of a delay vector. We add one delay with the map

$$D \begin{pmatrix} p \\ q \end{pmatrix} = \left(\begin{pmatrix} p \\ q \end{pmatrix} \quad \varphi_{-1} \begin{pmatrix} p \\ q \end{pmatrix} \right) \quad (19)$$

So that the image of $D \circ \Phi$ is now a 2-dimensional manifold in \mathbb{R}^4 . The transformation plus delay is an injection so that the new system admits a valid dynamical system description. Conveniently the 2-dimensional manifold can be recovered after the following projection Π in the sense that Π^{-1} has a bijective branch which maps the image of $\Pi \circ D \circ \Phi$ to the image of $D \circ \Phi$. The projection is given by

$$\Pi \begin{pmatrix} p \\ q \\ p_D \\ q_D \end{pmatrix} = \begin{pmatrix} p \\ q \\ p_D \\ q_D \end{pmatrix} \quad (20)$$

where p_D and q_D are the delayed co-ordinates. This means that we only need to take a delay in the p variable to obtain a valid dynamical systems description and we can therefore represent everything in \mathbb{R}^3 . Defining the composition $\Phi_D = \Pi \circ D \circ \Phi$ we have

$$\Phi_D \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} x(1-\pi) + \pi - \frac{1}{2} \\ \frac{y}{a} [1 - \frac{e^{-1}}{b} + e^{-ax}(\frac{1}{b} - 1)] \\ \pi + x - \frac{3}{2} \end{pmatrix} & \text{in } A \\ \begin{pmatrix} x - \frac{1}{2} \\ \frac{y}{a}(1 - e^{-a}) \\ (x-1)(1-\pi) + \pi - \frac{1}{2} \end{pmatrix} & \text{in } B \\ \begin{pmatrix} x - \frac{1}{2} \\ \frac{y}{a}(1 - e^{-a}) \\ x - \frac{3}{2} \end{pmatrix} & \text{in } C \end{cases} \quad (21)$$

Where A is still $\{0 \leq x \leq 1\}$, B is now $\{1 \leq x \leq 2\}$ and $C = \{2 \leq x \leq \pi\}$. Differentiating (21) w.r.t. time using (11) then expressing in terms of (p, q, p_D) we obtain the discontinuous RHS for the ODE governing the now continuous transformed system

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{p}_D \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 - \pi \\ \frac{aq(1 - \frac{e^{-a}}{b})}{1 - \frac{e^{-1}}{b} + e^{-a} \frac{p + \frac{1}{2} - \pi}{1 - \pi} (\frac{1}{b} - 1)} \\ 1 \end{pmatrix} & \text{in } \Phi_D(A) \\ \begin{pmatrix} 1 \\ aq \\ 1 - \pi \end{pmatrix} & \text{in } \Phi_D(B) \\ \begin{pmatrix} 1 \\ aq \\ 1 \end{pmatrix} & \text{in } \Phi_D(C) \end{cases} \quad (22)$$

Where $\Phi_D(A), \Phi_D(B), \Phi_D(C)$ form the sides of an extruded triangle, see fig 2,a,ii.

2.4 Twice transformed system

Successive applications of the moving average transformation smooths the system further, so a twice transformed discontinuous system will be differentiable. If the once transformed system's evolution operator is φ , as opposed to the original discontinuous

systems operator ϕ then the transformation of the once transformed system is given by

$$\begin{pmatrix} P \\ Q \\ P_D \end{pmatrix} = \Phi \begin{pmatrix} p \\ q \\ p_D \end{pmatrix} = \int_{-1}^0 \varphi_t \begin{pmatrix} p \\ q \\ p_D \end{pmatrix} dt = \underbrace{\Phi \circ \Pi \circ D \circ \Phi}_{\text{underbraced}} \begin{pmatrix} x \\ y \end{pmatrix} \quad (23)$$

Now $\Pi \circ D$ commutes with Φ so the underbraced term equals $\Pi \circ D \circ \Phi^2$ and

$$\begin{pmatrix} P \\ Q \\ P_D \end{pmatrix} = \Pi \circ D \int_{-1}^0 \int_{-1}^0 \phi_{t_1+t_2} \begin{pmatrix} x \\ y \end{pmatrix} dt_1 dt_2 = \Pi \circ D \int_{-2}^0 \phi_t \begin{pmatrix} x \\ y \end{pmatrix} h(t) dt \quad (24)$$

where

$$h(t) = \begin{cases} -t & \text{for } 0 \leq t \leq 1 \\ 2+t & \text{for } 1 \leq t \leq 2 \end{cases} \quad (25)$$

So that twice transforming the discontinuous system is equivalent to once applying the smoothing transformation

$$\Phi^2(x) = \Psi(x) = \int_{-2}^0 \phi_t(x) h(t) dt \quad (26)$$

which when composed with $\Pi \circ D$ is piecewise smooth on 4 regions depending on the 3-second backwards time flow. As before depending on which of these regions we are in the backwards integrals are divided up into up to three separate parts all of which can easily be evaluated. In this way we obtain

$$\Psi_D \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{-\pi}{2}x^2 + x + \pi - 1 \\ \frac{y}{a^2b}[b - 2e^{-a} + e^{-2a} + e^{-ax}(1-b)(1+ax)] \end{pmatrix} & \text{in } A \\ \begin{pmatrix} \pi + x - 2 \\ \frac{\pi}{2}x^2 + x(1-2\pi) + 2\pi - 1 \\ \frac{y}{a^2b}[b + e^{-2a} - 2be^{-a} + e^{-ax}[-1 + 2a + b - 2ab + x(ab-a)]] \end{pmatrix} & \text{in } B \\ \begin{pmatrix} x - 1 \\ \frac{y}{a^2}(1 + e^{-2a} - 2e^{-a}) \\ \frac{\pi}{2}x^2 + x(1-3\pi) + \frac{9\pi}{2} - 2 \end{pmatrix} & \text{in } C \\ \begin{pmatrix} x - 1 \\ \frac{y}{a^2}(1 + e^{-2a} - 2e^{-a}) \\ x - 2 \end{pmatrix} & \text{in } D \end{cases} \quad (27)$$

where $A = \{0 \leq x \leq 1\}$ $B = \{1 \leq x \leq 2\}$ $C = \{2 \leq x \leq 3\}$ and $D = \{3 \leq x \leq \pi\}$. Again we differentiate (27) using (11) then express in terms of (P, Q, P_D) to obtain the Continuous RHS for the ODE governing the now differentiable dynamics of the

twice transformed system.

$$\begin{pmatrix} \dot{P} \\ \dot{Q} \\ \dot{P}_D \end{pmatrix} = \begin{cases} \begin{pmatrix} -\pi P_D + \pi^2 - 2\pi + 1 \\ aQ + \frac{Qa^2 e^{-a \frac{P+P_D+3-\frac{5\pi}{2}}{2-\pi}} (\frac{P+P_D+3-\frac{5\pi}{2}}{2-\pi} - 2)(1-b)}{b+e^{-2a}-2be^{-a}+e^{-a \frac{P+P_D+3-\frac{5\pi}{2}}{2-\pi}} [-1+2a+b-2ab+\frac{P+P_D+3-\frac{5\pi}{2}}{2-\pi}(ab-a)]} \end{pmatrix} & \text{in } \Psi_D(A) \\ \begin{pmatrix} 1 \\ aQ + \frac{\frac{\pi}{2-\pi}(P+P_D+3-\frac{5\pi}{2})+1-2\pi}{a^2 Q e^{-a \frac{P_D+P+3-\frac{5\pi}{2}}{2-\pi}} (b-1)(2-\frac{P_D+P+3-\frac{5\pi}{2}}{2-\pi})} \\ b+e^{-a(1-2b)}+e^{-a \frac{P_D+P+3-\frac{5\pi}{2}}{2-\pi}} [-1+b+(ab-a)(\frac{P_D+P+3-\frac{5\pi}{2}}{2-\pi}-2)] \end{pmatrix} & \text{in } \Psi_D(B) \\ \begin{pmatrix} 1 \\ aQ \\ \pi P + 1 - 2\pi \end{pmatrix} & \text{in } \Psi_D(C) \\ \begin{pmatrix} 1 \\ aQ \\ 1 \end{pmatrix} & \text{in } \Psi_D(D) \end{cases} \quad (28)$$

where $S = \Psi_D(A \cup B \cup C \cup D)$ forms a differentiable cylinder whose cross section is formed from three parabolas and one line segment, see fig 2,a,iii.

2.5 Stability of differentiable system

Now we have a differentiable flow on a cylinder S we can calculate the stability of the periodic orbit in the usual way, by integrating the derivative. This could be a little tricky since the flow we are interested in lives on a 2-dimensional sub-manifold of \mathbb{R}^3 so we would normally have to compute the tangent space at each point on the orbit then integrate the derivative of the flow restricted to these tangencies. However the form of (28) gives us a perfectly good differentiable extension of the vector field onto \mathbb{R}^3 we can therefore integrate the whole derivative along the orbit and pick of the eigenvalues corresponding to perturbations in S two of which will exist since it is a 2-dimensional invariant set.

Perturbations about the point to the periodic orbit evolve according to the ODE

$$\begin{pmatrix} \delta \dot{P} \\ \delta \dot{Q} \\ \delta \dot{P}_D \end{pmatrix} = A(t) \begin{pmatrix} \delta P \\ \delta Q \\ \delta P_D \end{pmatrix} \quad (29)$$

where differentiating (28) gives

$$A(t) = \begin{cases} \begin{pmatrix} 0 & 0 & -\pi \\ 0 & a + \frac{a^2 b(b-1)e^{-at}}{b-2e^{-a}+e^{-at}(at+1)(1-b)} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } 0 \leq t \leq 1 \\ \begin{pmatrix} \frac{-\pi}{\pi-2} & 0 & \frac{-\pi}{\pi-2} \\ 0 & a + \frac{a^2 e^{-at}(b-1)(2-t)}{b+e^{-2a}-2be^{-a}+e^{-at}[-a+2a+b-2ab+t(ab-a)]} & 0 \\ \frac{\pi}{\pi-2} & 0 & \frac{\pi}{\pi-2} \end{pmatrix} & \text{for } 1 \leq t \leq 2 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ \pi & 0 & 0 \end{pmatrix} & \text{for } 2 \leq t \leq 3 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } 3 \leq t \leq \pi \end{cases} \quad (30)$$

Perturbations in P and P_D evolve in a piecewise autonomous way so we can separate and solve their evolution easily. The time- t map for this evolution is given by

$$J_{P,P_D} = \begin{cases} \begin{pmatrix} 1 & -\pi t \\ 0 & 1 \end{pmatrix} & \text{for } 0 \leq t \leq 1 \\ \begin{pmatrix} 1 - \frac{\pi t}{\pi-2} & \frac{-\pi t}{\pi-2} \\ \frac{\pi t}{\pi-2} & 1 + \frac{\pi t}{\pi-2} \end{pmatrix} \begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix} & \text{for } 1 \leq t \leq 2 \\ \begin{pmatrix} 1 & 0 \\ \pi t & 1 \end{pmatrix} \begin{pmatrix} \frac{-2}{\pi-2} & \frac{\pi}{\pi-2} \\ \frac{\pi}{\pi-2} & \frac{-\pi^2+2\pi-2}{\pi-2} \end{pmatrix} & \text{for } 2 \leq t \leq 3 \\ \begin{pmatrix} \frac{-2}{\pi-2} & \frac{\pi}{\pi-2} \\ \frac{-\pi}{\pi-2} & \frac{2\pi-2}{\pi-2} \end{pmatrix} & \text{for } 3 \leq t \leq \pi \end{cases} \quad (31)$$

The final matrix has an eigenvalue of 1 corresponding to the eigenvector $(1, 1)$ which is tangent to the cylinder S at $\Psi_D(0, 0)$. It can be verified that the evolution of this perturbation is always in the tangent space of S .

The evolution of perturbations in the Q co-ordinate can now be posed and solved as a 1-dimensional non-autonomous linear ODE the time t evolution of which is given by

$$J_Q = \begin{cases} \frac{e^{at}[b-2e^{-a}+e^{-2a}+e^{-ax}(1-b)(1+at)]}{1-2e^{-a}+e^{-2a}} & \text{for } 0 \leq t \leq 1 \\ \frac{e^{at}[b+e^{-2a}-2be^{-a}+e^{-ax}[-1+2a+b-2ab+x(ab-a)]]}{1-2e^{-a}+e^{-2a}} & \text{for } 1 \leq t \leq 2 \\ be^{at} & \text{for } 2 \leq t \leq \pi \end{cases} \quad (32)$$

which has the stability determining eigenvalue of $be^{a\pi}$ agreeing with the original system as claimed.

2.6 Remarks

We began with a discontinuous system with evolution operator ϕ and constructed a differentiable map Ψ_D and a differentiable system whose evolution operator we shall denote ψ . By construction the two systems are dynamically equivalent and the

following commutation holds

$$\phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \Psi_D^{-1} \circ \psi_t \circ \Psi_D \begin{pmatrix} x \\ y \end{pmatrix} \quad (33)$$

which means that via the smoothing transformation Ψ_D we can substitute one flow for the other.

Suppose we have some time series data obtained from the discontinuous system

$$F(t) = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \phi_t \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \quad (34)$$

for $t \in [0, T]$ and we apply the smoothing filter to the series

$$G(t) = \int_{-2}^0 F(t + \tau) h(\tau) d\tau \quad (35)$$

to obtain a new time series $\{G(t) : t \in [2, T]\}$. This time series is identical to the series we would record if instead of measuring the discontinuous system along some orbit we measured the (smoothed) differentiable system along the corresponding smoothed orbit.

$$G(t) = \left\langle \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \int_{-2}^0 \phi_{t+\tau} \begin{pmatrix} x \\ y \end{pmatrix} h(\tau) d\tau \right\rangle = \left\langle \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \psi_t \Psi_D \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \quad (36)$$

In fact a stronger result holds. Generally we would not just measure $ax(t) + by(t)$ for our time-series but some possibly unknown smooth non-linear function of the state $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose that $\{F(t) = f \circ \phi_t(x) : t \in [0, T]\}$ is a time-series obtained from the discontinuous dynamical system then

$$G(t) = \int_{-1}^0 F(t + \tau) d\tau = g \circ \varphi_t \circ \Phi_D(x) \quad (37)$$

for some continuous function g where Φ_D is the moving average transformation (applied just once) with delay and φ_t is the induced flow of the transformed system. To prove this we just set $g(p) = \Phi^f \circ \Phi_D^{-1}(p)$ where

$$\Phi^f \begin{pmatrix} x \\ y \end{pmatrix} = \int_{-1}^0 f \circ \phi_t \begin{pmatrix} x \\ y \end{pmatrix} dt \quad (38)$$

Now Φ^f is continuous and Φ_D^{-1} is continuous everywhere except the transformation of the discontinuity surface $\Phi_D(\Sigma)$ where $\Phi_D^{-1}\Phi_D(\sigma)^- = \sigma$ and $\Phi_D^{-1}\Phi_D(\sigma)^+ = R(\sigma)$ but $\Phi^f(\sigma) = \Phi^f[R(\sigma)]$ so g is continuous everywhere.

Likewise if $\{F(t) = f \circ \phi_t(x) : t \in [0, T]\}$ is a time-series obtained from the discontinuous dynamical system then

$$G(t) = \int_{-2}^0 F(t + \tau) h(\tau) d\tau = g \circ \psi_t \circ \Psi_D(x) \quad (39)$$

for some differentiable function g where Ψ_D is the double moving average transformation with delay and ψ is the induced flow of the transformed system.

These observations mean we can apply standard time series techniques that rely on differentiability to non-smooth systems. First record some possibly discontinuous time series, smooth it with a single or double application of the moving average filter then treat it as a differentiable recording from the differentiable smoothed system.

This give us a novel way to compute the stability of the discontinuous systems periodic orbit. First record a time series $\{F(t) = f \circ \phi_t(x) : t \in [0, T]\}$ with some arbitrary differentiable function f . Now apply the smoothing filter

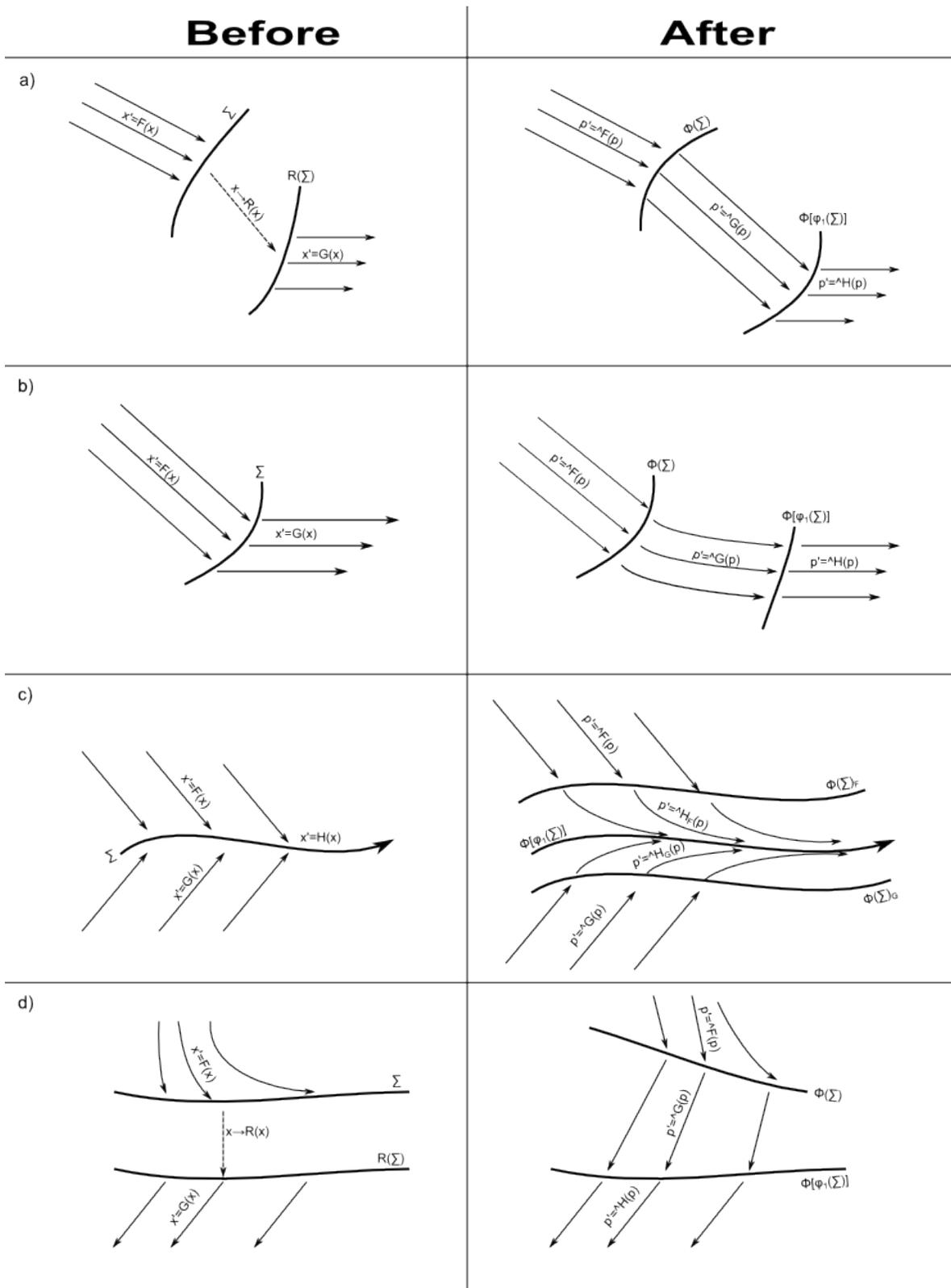
$$G(t) = \int_{-2}^0 F(t + \tau)h(\tau)dt \quad (40)$$

to obtain a new time series $\{G(t) = g \circ \psi_t \circ \Psi_D(x) : t \in [2, T]\}$ which we can think of as being obtained directly from the smoothed system by measuring with the differentiable function g . Standard numerical techniques enable us to calculate the lyupanov exponents of the smoothed system from this data [11] and since the stability of the discontinuous and smoothed systems are identical we have the lyupanov exponents of the original system.

On the flip-side we can also use these time-series reconstruction techniques to study the properties of the smoothed systems in the same way. Simulate the non-smooth system then smooth the data and treat it as coming directly from the smoothed system. In section 4 we use these ideas to investigate the dynamics of a more complex sliding oscillator.

3 Local analysis

Although it is difficult to obtain an explicit global description of the smoothed system $\varphi_t = \Phi \circ \phi_t \Phi^{-1}$ we can obtain local descriptions about some typical non-smooth discontinuities. By taking the piecewise linear normal form of a discontinuity we can formulate the smoothed system in the vicinity of the Φ image of the discontinuity and at the corresponding tail discontinuity. We shall see that the derivative of Φ captures the information in the saltation matrices and that this is then translated into the smoothed flow. The action of the filter is shown to transform discontinuous systems into continuous systems and continuous but non differentiable systems into differentiable systems. Since the analysis is quite involved we summarize our results in the following table, see fig 3.



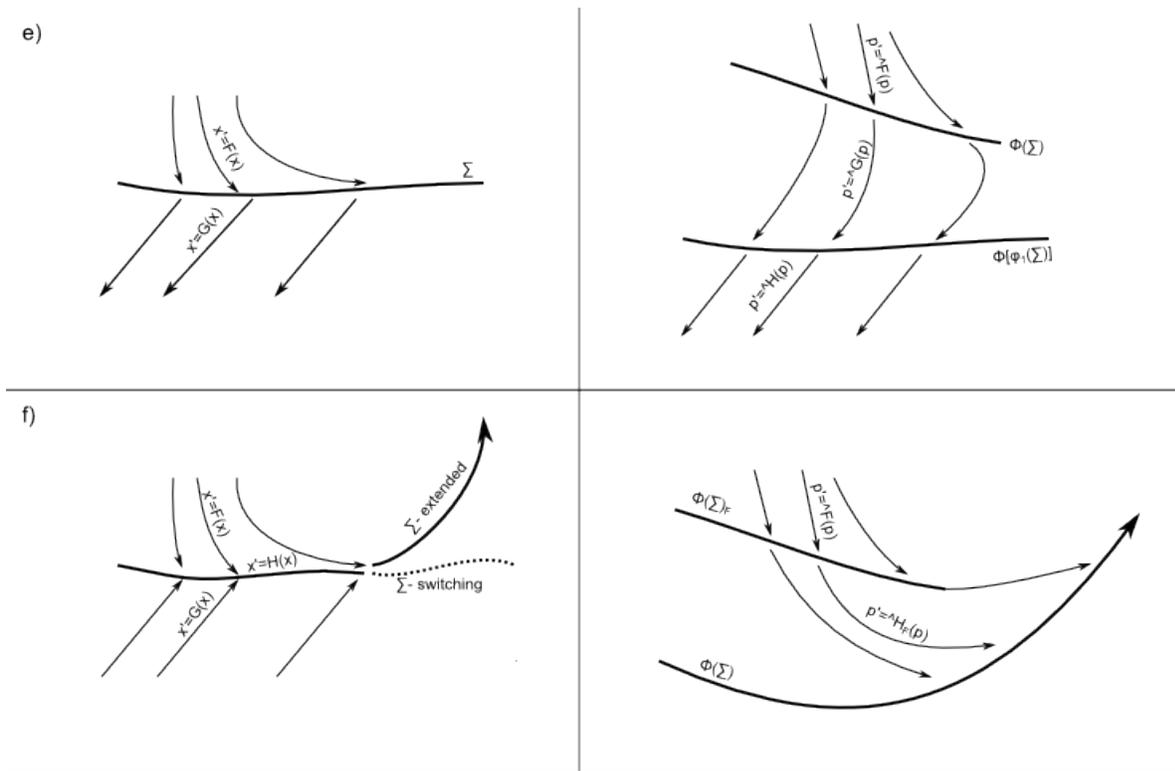


Figure 3: Non-smooth discontinuities and their smoothed counterparts. a) Discontinuous jump \rightarrow two non-differentiable switches, b) Non-differentiable switch \rightarrow two differentiable switches, c) Non-differentiable slide \rightarrow differentiable switch followed by differentiable slide, d) Grazing discontinuous jump \rightarrow grazing non-differentiable switch followed by non-differentiable switch, e) Grazing non-differentiable switch \rightarrow grazing differentiable switch followed by differentiable switch, f) Grazing non-differentiable slide \rightarrow grazing differentiable switch followed by differentiable slide.

3.0.1 Normal forms

We will consider 5 typical non-smooth discontinuities jumps, switches, slides, jumping grazes, switching grazes and sliding grazes. In each case we formulate the simplest and most general non-degenerate form.

For the jump we assume that a differentiable vector field flows into a differentiable set where we apply a differentiable map, then after applying the map we flow according to another differentiable vector field. We can therefore linearize everything in the vicinity of a point s on the discontinuity surface and its image after the jump $R(s)$, however this isn't the whole story since we also need information about the flow up to one second before the jump. In order to obtain an appropriate approximation (piecewise linear plus square root terms) of the transformed system at the image of the discontinuity $\Phi(s)$ we need to additionally know the one second backwards time position of the point $\phi_{-1}(s)$, the integrated jacobian J that maps perturbations about $\phi_{-1}(s)$ to perturbations about s and \tilde{J} the integral of the integrated jacobian along this orbit. It is worth noting that there is nothing to stop this one second backwards

time flow containing any number of non-smooth discontinuities, everything important to the transformed system in the vicinity of the image of the discontinuity is contained in the point $\phi_{-1}(s)$ and pair of matrices J, \tilde{J} which can in principal be calculated or just assumed to be generic as is the case here.

Likewise when we analyze the tail discontinuity, the smoothed system in the vicinity of $\Phi \circ \phi_{+1}(s)$ the image of a point one second after the discontinuity surface, we will also need to know the point on the discontinuity surface we are flowing from s , its image under the jump map $R(s)$, the integrated jacobian J which maps perturbations about $R(s)$ to perturbations about s and its integral \tilde{J} .

The remaining discontinuities are treated in much the same way with the normal form being outlined briefly at the beginning of each subsection along with some references for standard analysis and applications.

3.1 Jump

A jump is defined as follows. The flow is given by an ODE $\dot{x} = F(x)$ until reaching a set Σ where we instantaneously apply a map R , thereafter the flow is a second ODE $\dot{x} = G(x)$. Jumps are used to model Newtonian impacts [1] and arise as resets in some control systems [6] or when periodic variables are embedded as intervals rather than circles as in section 4. We will expand everything about a point $s \in \Sigma$ to obtain a piecewise linear description of the transformed system

$$\dot{p} = \frac{\delta x}{\delta t} \frac{\delta \Phi}{\delta x} \circ \Phi^{-1}(p) \quad (41)$$

about the point $\Phi(s)$.

3.1.1 Linearization of Φ before jump

We must differentiate Φ at s . By definition

$$\frac{\delta \Phi}{\delta x} \Big|_s = \lim_{|h| \rightarrow 0} \frac{\Phi(s+h) - \Phi(s)}{|h|} = \lim_{|h| \rightarrow 0} \frac{\int_{-1}^0 \phi_t(s+h) - \phi_t(s) dt}{|h|} \quad (42)$$

which can be expressed in terms of the jacobians integrated from $\phi_{-1}(s)$ so that the derivative is given by

$$\left[\int_0^1 J(t) dt \right] J(1)^{-1} = \tilde{J} J^{-1} \quad (43)$$

where J maps perturbations about $\phi_{-1}(s)$ to perturbations about s and \tilde{J} maps perturbations about $\phi_{-1}(s)$ to the integral of the perturbation evolving over one second. The linearization is given by

$$\Phi(x) = \Phi(s) + \tilde{J} J^{-1} (x - s) \quad (44)$$

3.1.2 Linearization of Φ after jump

The Linearization of Φ at $R(s)$ is a little more complicated. We might try to use the formula $\tilde{J}(SJ)^{-1}$ for the derivative since S^{-1} maps perturbations about $R(s)$ to perturbations about s . However this is not the whole story since

$$\Phi(x) = \Phi(s) + \tilde{J}(SJ)^{-1}[x - R(s)] \quad (45)$$

does not include any contribution to the integral from after the jump and instead contains a spurious contribution from the wrong side of the discontinuity surface. Figure 4 shows the configuration of s , $R(s)$, and x with the various components of $\Delta\Phi$.

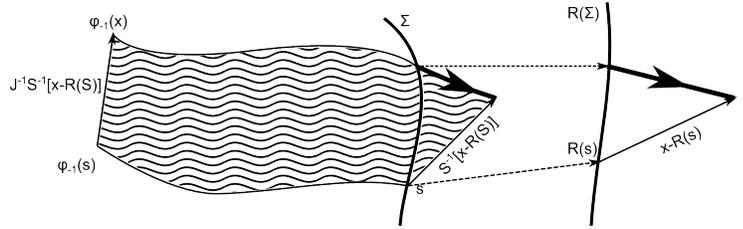


Figure 4: Hatched region represents jacobian approximation $\tilde{J}(SJ)^{-1}[x - R(s)]$ we need to remove the spurious contribution from the wrong side of Σ and add the contribution to Φ from after the jump (both bold).

We can correct for these contributions by assuming that the flow is locally constant at s and $R(s)$. The contribution from after the jump is given by

$$\int_{-t^*}^0 (x + G[R(s)]t) dt \quad (46)$$

where t^* the length of time since the perturbation made the jump is given by

$$t^* = \frac{\langle x - R(s), n_2 \rangle}{\langle G[R(s)], n_2 \rangle} \quad (47)$$

where n_2 is normal to $R(\Sigma)$ at $R(s)$. The spurious contribution from the wrong side of the jump is given by

$$\int_{t^*}^0 (s + S^{-1}[x - R(s)] + F(s)t) dt \quad (48)$$

where t^* the length of time that the perturbation is taken to be on the wrong side of the jump given by

$$t^* = \frac{\langle S^{-1}[x - R(s)], n_1 \rangle}{\langle F(s), n_1 \rangle} \quad (49)$$

where n_1 is normal to Σ at s . It follows from the definition of the saltation matrix that the two expressions for t^* are the same. Finally the linearization is given by

$$\Phi(x) = \Phi[R(s)] + \tilde{J}(SJ)^{-1}[x - R(s)] + t^*[R(s) - s] \quad (50)$$

To invert the transformation we need to express t^* in terms of $\Phi(x)$. First we have

$$x = R(s) + SJ\tilde{J}^{-1}[\Phi(x) - \Phi[R(s)] - t^*(R[s] - s)] \quad (51)$$

and

$$t^* = \frac{\langle J\tilde{J}^{-1}[\Phi(x) - \Phi[R(s)] - t^*(R[s] - s)], n_1 \rangle}{\langle F(s), n_1 \rangle} \quad (52)$$

so that

$$t^* = \frac{\langle J\tilde{J}^{-1}(\Phi(x) - \Phi[R(s)]), n_1 \rangle}{\langle F(s) + J\tilde{J}^{-1}[R(s) - s], n_1 \rangle} \quad (53)$$

3.1.3 Transformed system

To obtain a piecewise linear description of the transformed system we need to be able to express \dot{p} as a function of p . This would first require the following expansion

$$\dot{p} = [F(s) + \frac{\delta F}{\delta x}|_s(x - s)][\frac{\delta \Phi}{\delta x}|_s + \frac{\delta \frac{\delta \Phi}{\delta x}|_{x'}}{\delta x'}|_s(x - s)] \quad (54)$$

which we could then use the linearization of Φ to put in terms of p rather than x . However we do not need to compute the complicated second derivative of Φ . Instead we use the following

$$\dot{p} = \lim_{h \rightarrow 0} \frac{p(t+h) - p(t)}{h} = \lim_{h \rightarrow 0} \frac{\int_{-1}^0 \phi_{\tau+h}(x) - \phi_{\tau}(x) dt}{h} = x - \phi_{-1}(x) \quad (55)$$

Which can easily be linearized so that before the jump we have

$$\dot{p} = x - \phi_{-1}(s) - J^{-1}(x - s) \quad (56)$$

and afterwards we have

$$\dot{p} = x - \phi_{-1}(s) - J^{-1}S^{-1}[x - R(s)] \quad (57)$$

Thus our transformed system first flows according to the ODE $\dot{p} = \hat{F}(p)$ whose linearization about $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (I - J^{-1})J\tilde{J}^{-1}[p - \Phi(s)] \quad (58)$$

Until we reach the set $\Phi(\Sigma)$ whose linearization about $\Phi(s)$ is given by

$$\{p : \langle J\tilde{J}^{-1}[p - \Phi(s)], n_1 \rangle = 0\} \quad (59)$$

where we instantaneously switch to the ODE $\dot{p} = \hat{G}(p)$ whose linearization about $\Phi(s)$ is given by

$$\begin{aligned} \dot{p} &= R(s) - \phi_{-1}(s) + (SJ - I)\tilde{J}^{-1}[p - \Phi(s)] \\ &+ \frac{\langle J\tilde{J}^{-1}(\Phi(x) - \Phi[R(s)]), n_1 \rangle}{\langle F(s) + J\tilde{J}^{-1}[R(s) - s], n_1 \rangle} (I - SJ)\tilde{J}^{-1}[R(s) - s] \end{aligned} \quad (60)$$

3.1.4 Tail discontinuity

There will be a second non-smooth event associated with the jump in the smoothed system. One second after crossing the jump the tail of the averaging period will cross the jump resulting in a second switch in the smooth system on the switching surface $\Phi \circ \phi_1(\Sigma)$. The analysis here is more or less the same as before. To see this consider the backwards evolution of the jumping system where we flow according to the ODE $\dot{x} = -G(x)$ until reaching the set $R(\Sigma)$ where we instantaneously apply the map R^{-1} then flow according to the second ODE $\dot{x} = -F(x)$. The transformation of this system in the vicinity of the jump will be an ODE of the form $\dot{x} = H(x)$ which defines the forewords system at the second switch by $\dot{x} = -H(x)$.

Since $\dot{x} = H(x)$ will be of the same form as the transformed system in 3.1.3 the reversed system at the tail discontinuity $\dot{x} = -H(x)$ will also be a non-differentiable switch consisting of a differentiable vector field flowing into a differentiable manifold at which point we switch to another differentiable vector field.

3.2 Switch

A switch is defined as follows. The dynamics are governed by an ODE $\dot{x} = F(x)$ which flows into a set Σ where we instantaneously switch to a second ODE $\dot{x} = G(x)$ and do not switch back. Switches are used to model discontinuous changes of force in mechanics [1] and discontinuous changes to control input in closed loop control systems [6]. We will linearize everything about a point $s \in \Sigma$ to obtain a piecewise linear description of the transformed system about the point $\Phi(s)$. The switch can therefore be thought of as a special case of the jump with $R = I$ the identity so can therefore go straight to the transformed system.

The flow is given by the ODE $\dot{p} = \tilde{F}(p)$ whose linearization at $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (J - I)\tilde{J}^{-1}[p - \Phi(s)] \quad (61)$$

Until we reach the set $\Phi(\Sigma)$ whose linearization at $\Phi(s)$ is given by

$$\{p : \langle J\tilde{J}^{-1}[p - \Phi(s)], n_1 \rangle = 0\} \quad (62)$$

where we switch to the ODE $\dot{p} = \hat{G}(p)$ whose linearization at $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (SJ - I)\tilde{J}^{-1}[p - \Phi(s)] \quad (63)$$

For $p \in \Phi(\Sigma)$ we have $SJ\tilde{J}^{-1}[p - \Phi(s)] = J\tilde{J}^{-1}[p - \Phi(s)]$ since $J\tilde{J}^{-1}[p - \Phi(s)]$ will lie on the original switching surface where the saltation matrix acts as the identity. Therefore the two expressions (61) and (63) agree on the switching surface and the system is differentiable.

Likewise the tail discontinuity is a differentiable switch, which we can prove in the same way we treated the jump in 3.1.4.

3.3 Slide

A slide is defined as follows. We have a switching surface Σ where the dynamics are governed by an ODE $\dot{x} = F(x)$ on one side of Σ and another ODE $\dot{x} = G(x)$ on the

other side of Σ . For $s \in \Sigma$ to be a sliding point we require $\langle F(s), n \rangle$ and $\langle G(s), n \rangle$ to have opposite signs where n is normal to Σ at s . So if we employed something like an Euler scheme then when we reach (and generically overshoot) Σ evolving according to $x_{n+1} = x_n + \tau F(x_n)$ we are forced to switch back and forth between the two RHSs trapped in a small neighborhood of Σ which collapses onto Σ as $\tau \rightarrow 0$.

In order to properly define the solution of the non smooth ODE we use the Filippov formalism in which we consider the RHS of our differential equation as being a well defined set valued function obtained by taking the convex hull of all possible values of the discontinuous vector field at each point

$$\dot{x} \in \begin{cases} \{F(x)\} & \text{for } \langle x, n \rangle > 0 \\ \{aF(x) + (1-a)G(x) : a \in [0, 1]\} & \text{for } \langle x, n \rangle = 0 \\ \{G(x)\} & \text{for } \langle x, n \rangle < 0 \end{cases} \quad (64)$$

where n is normal to Σ at the closest point in Σ to x . It can then be shown that the solution to this *Differential Inclusion* is well defined and satisfies the following ODE on the switching surface [10]

$$\dot{x} = H(x) = \frac{F^\perp(x)|G^\parallel(x)| + G^\perp(x)|F^\parallel(x)|}{|G^\parallel(x)| + |F^\parallel(x)|} \quad (65)$$

Where F^\parallel and F^\perp are F 's components parallel and perpendicular to n respectively - likewise for G^\parallel, G^\perp . Slides are used to model systems with static friction, see section 4 or [1, 2, 8].

3.3.1 Example

There are some interesting complications here so we will first look at a simple example. Our system flows according to the non-smooth ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ \begin{cases} -1 & \text{for } y > 0 \\ 1 & \text{for } y < 0 \end{cases} \end{pmatrix} \quad (66)$$

which defines the flow on $\{y = 0\}$ as $\dot{x} = 1$.

Clearly this system is not time-reversible on $y = 0$ where Φ will have infinitely branches. However suppose we take any $(x', y') \in \mathbb{R}^2$ and compute its forward time orbit $\{\phi_t(x', y') : t \in \mathbb{R}_+\}$ then calculate Φ from this orbit for each $(x, y) \in \{\phi_t(x', y') : t \in [1, \infty)\}$ calling this new quantity

$$\begin{pmatrix} p \\ q \end{pmatrix} = \Phi_{(x', y')} \begin{pmatrix} x \\ y \end{pmatrix} \quad (67)$$

Now we can differentiate the above to obtain

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \lim_{h \rightarrow \infty} \frac{\Phi_{(x', y')} \begin{pmatrix} x + \delta(x) \\ y + \delta(y) \end{pmatrix} - \Phi_{(x', y')} \begin{pmatrix} 0 \\ y \end{pmatrix}}{h} \quad (68)$$

which is fine since $(x + \delta(x), y + \delta(y)) \in \{\phi_t(x', y') : t \in [1, \infty)\}$. So we obtain the differentiable system

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 \\ \begin{cases} -1 & \text{for } q > \frac{1}{2} \\ \frac{-q^{\frac{1}{2}}}{2} & \text{for } \frac{1}{2} \geq q > 0 \\ 0 & \text{for } q = 0 \\ \frac{(-q)^{\frac{1}{2}}}{2} & \text{for } 0 > q \geq \frac{-1}{2} \\ 1 & \frac{-1}{2} > q \end{cases} \end{pmatrix} \quad (69)$$

Because of the choice of branch the switching surface is fattened by the transformation to a higher dimensional object. In this case from a line to a strip. Those points on the strip but not on the line $\{q = 0\}$ correspond to points that have joined the sliding surface less than one second ago, see fig 3,c. In the discontinuities we have looked at so far there are two non-smooth events associated with them in the transformed system, one corresponding to x crossing the discontinuity and another corresponding to $\phi_{-1}(x)$ crossing the discontinuity - the Head and the Tail of our averaged period. In the jump and switch the two events were essentially the same and therefore didn't require any extra analysis but for the slide the two are clearly very different. In this example the first non-smooth event at $q = \frac{\pm 1}{2}$ is a differentiable switch but the second at $q = 0$ is quite different with parabolic solutions converging differentially and in finite time onto the switching surface. To treat the general case of the switch we must look at these two discontinuities separately.

3.3.2 Head discontinuity

The flow is given by an ODE $\dot{x} = F(x)$ until reaching a set Σ where we instantaneously switch to an ODE $\dot{x} = H(x)$ as in (65). We will linearize everything about a point $s \in \Sigma$ to obtain a piecewise linear description of the transformed system.

Before reaching the slide the linearization of Φ is as before

$$\Phi(x) = \Phi(s) + \tilde{J}J^{-1}(x - s) \quad (70)$$

As is the flow

$$\dot{p} = s - \phi_{-1}(s) + (I - J^{-1})(x - s) \quad (71)$$

So that the transformed system flows according to the ODE $\dot{p} = \hat{F}$ whose linearization about $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (I - J^{-1})J\tilde{J}^{-1}[p - \Phi(s)] \quad (72)$$

After reaching the slide things are a little more complex. To represent points on the sliding surface we use (σ, t) co-ordinates. Here $\sigma \in \Sigma$ is the current position of the point and $t \ll 1$ is the branching index for the backwards flow, it tells us for how long the backwards orbit from σ stays on the sliding surface. See fig 5,a.

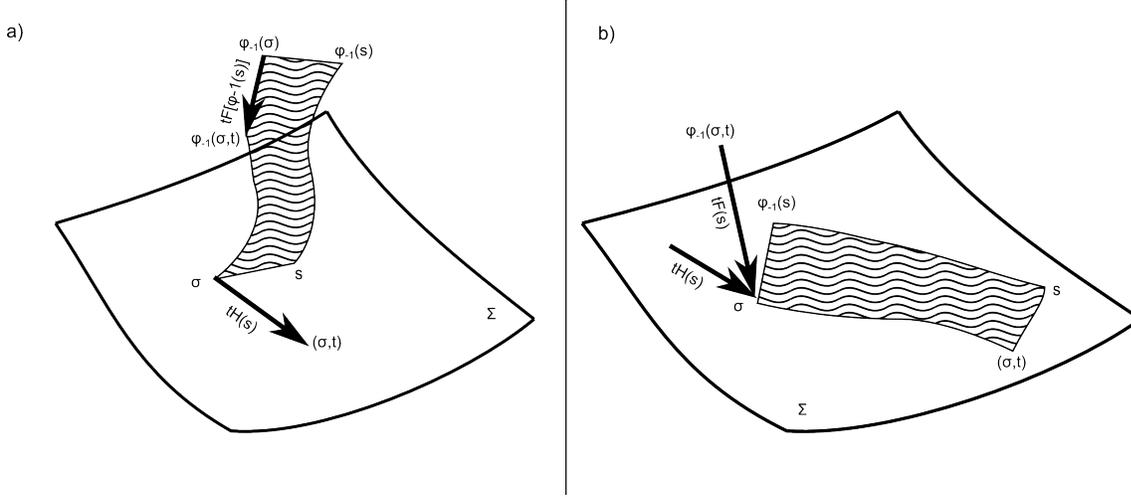


Figure 5: Configuration of components in $\Delta\Phi$; jacobian approximation (hatched) and correction terms (bold) for the slide at a) the head and b) tail discontinuities.

The linearized transformation is given by

$$\begin{aligned} \Phi(\sigma, t) &= \Phi(s) + \underbrace{\tilde{J}J^{-1}[\phi_{-t}(\sigma, t) - s]} + \int_0^t \phi_{-t}(\sigma, t) + \tau H(s) d\tau - \int_0^t \phi_{1+t}(\sigma, t) + \tau F[\phi_{-1}(s)] d\tau \\ &= \Phi(s) + \tilde{J}J^{-1}[\sigma - tH(s) - s] + t[s - \phi_{-1}(s)] \end{aligned} \quad (73)$$

where the underbraced term is the derivative of Φ applied at the point where (σ, t) first joins the sliding surface and the two integrals corrects for the short period of sliding. To invert the transformation we note that $\sigma - tH(s) - s$ is tangent to Σ at s so that if n is the corresponding normal vector then taking the dot product with $(J\tilde{J}^{-1})^T n$ gives

$$t = \frac{\langle \Phi(\sigma, t) - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle s - \phi_{-1}(s), (J\tilde{J}^{-1})^T n \rangle} \quad (74)$$

and

$$\sigma = s + J\tilde{J}^{-1}(\Phi(\sigma, t) - \Phi(s) - t[s - \phi_{-1}(s)]) + tH(s) \quad (75)$$

The flow is given by

$$\dot{p} = \sigma - \phi_{-1}(\sigma, t) = \sigma - \underbrace{(\phi_{-1}(s) + J^{-1}[\sigma - tH(s) - s])} - tF[\phi_{-1}(s)] \quad (76)$$

Where the underbraced term is the one second backwards time image of the point where (σ, t) first joins the sliding surface and the final term corrects for the fact that (σ, t) 's one second backwards time image is actually the $(1-t)$ second backwards time image of this point. So that the transformed system flows according to the ODE $\dot{p} = \tilde{G}(p)$ whose linearization about $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (J - I)\tilde{J}^{-1}(p - \Phi(s)) \quad (77)$$

$$+ \frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle s - \phi_{-1}(s), (J\tilde{J}^{-1})^T n \rangle} (I - J^{-1})[H(s) - F(s)]$$

Expressions (72) and (77) agree on the switching surface where $t = 0$ so the switch is differentiable.

3.3.3 Tail discontinuity

The flow is given by an ODE $\dot{x} = F(x)$ until reaching a set Σ close to a point s where we instantaneously switch to an ODE $\dot{x} = H(x)$ as in (65). After 1 second orbits reach the vicinity of $\phi_1(s)$. From the example we do not expect a linear description of the transformed system to adequately illustrate the dynamics so we will include some higher order terms. Also since the system $\dot{x} = H(x)$ lives on an $n - 1$ dimensional manifold we must work with some arbitrary smooth continuation of H when computing the jacobians - of course the exact nature of the continuation is unimportant - its existence just means we don't need to be too careful about what we say!

Again we will work in (σ, t) co-ordinates with $\sigma \in \Sigma$ the current position of the point only this time the branch index $t \ll 1$ tells us that the backwards time orbit of (σ, t) stays on the sliding surface for $(1 - t)$ seconds, see fig 5,b. Including some second order terms in t the transformation about $\phi_1(s)$ is given by

$$\begin{aligned} \Phi(\sigma, t) &= \Phi[\phi_1(s)] + \underbrace{\tilde{J}J^{-1}(\sigma - \phi_1(s))}_{\text{underbraced}} + \int_{-t}^0 \phi_{1-t}(\sigma) + \tau F(s) d\tau - \int_{-t}^0 \phi_{1-t}(\sigma) + \tau H(s) d\tau \\ &= \Phi[\phi_1(s)] + \tilde{J}J^{-1}(\sigma - \phi_1(s)) + \frac{t^2}{2}[H(s) - F(s)] \end{aligned} \quad (78)$$

Where the underbraced term is the derivative of Φ restricted to the backwards flow branch that stays on the sliding surface for all time and the two integrals correct for the t seconds that we are not on the sliding surface. We invert the transformation in the same way so that

$$\frac{t^2}{2} = \frac{\langle \Phi(\sigma, t) - \Phi[\phi_1(s)], (J\tilde{J}^{-1})^T n \rangle}{\langle H(s) - F(s), (J\tilde{J}^{-1})^T n \rangle} \quad (79)$$

where n is normal to Σ at $\phi_1(s)$.

$$\sigma = \phi_1(s) + J\tilde{J}^{-1}(\Phi(\sigma, t) - \Phi[\phi_1(s)]) - \frac{t^2}{2}[H(s) - F(s)] \quad (80)$$

The flow is given by

$$\dot{p} = \sigma - \phi_{-1}(\sigma, t) = \sigma - \underbrace{(s + J^{-1}[\sigma - \phi_1(s)])}_{\text{underbraced}} + t[H(s) - F(s)] \quad (81)$$

where the underbraced term is the one second backwards time image of σ restricted to the branch where we stay on the sliding surface for all time and the final term corrects for the t seconds that we are not on the sliding surface. So that the transformed system flows according to the ODE $\dot{p} = \hat{H}(p)$ whose expansion about $\Phi[\phi_1(s)]$ is given by

$$\dot{p} = a + B(p - \Phi[\phi_1(s)]) + c\sqrt{\langle p - \Phi[\phi_1(s)], d \rangle} \quad (82)$$

where

$$a + B(p - \Phi[\phi_1(s)]) = \phi_1(s) - s + (J - I)\tilde{J}^{-1}(p - \Phi[\phi_1(s)]) \quad (83)$$

$$+ \frac{\langle \Phi(\sigma, t) - \Phi[\phi_1(s)], (J\tilde{J}^{-1})^T n \rangle}{\langle H(s) - F(s), (J\tilde{J}^{-1})^T n \rangle} (J - I)\tilde{J}^{-1}[H(s) - F(s)]$$

and

$$c\sqrt{\langle p - \Phi[\phi_1(s)], d \rangle} = [H(s) - F(s)] \sqrt{\frac{\langle \Phi(\sigma, t) - \Phi[\phi_1(s)], (J\tilde{J}^{-1})^T n \rangle}{\langle H(s) - F(s), (J\tilde{J}^{-1})^T n \rangle}} \quad (84)$$

Thus as in the simple example orbits converge differentially and in finite time to the transformed sliding surface $\Phi(\Sigma) = \{\langle J\tilde{J}^{-1}(p - \Phi[\phi_1(s)]), n \rangle = 0\}$ and the distance decays like $(t - c)^2$ due to the square root in the equation for the flow.

3.4 Grazing a jump

A jump graze is defined as follows. The dynamics are governed by the ODE $\dot{x} = F(x)$ which flows into a set Σ where we instantaneously apply a map R , thereafter the dynamics are governed by a second ODE $\dot{x} = G(x)$. For $s \in \Sigma$ to be a grazing point we require $\frac{\delta d(x)}{\delta t}|_s = 0$ where $d(x)$ measures the distance from x to the switching surface Σ . Thus arbitrarily small perturbations to s result in large immediate changes to the evolution. Grazes are typically whenever we have a jump, switch or slide and are responsible for singular stability properties and novel bifurcations in non-smooth systems [2, 3, 7].

If we attempt to construct a Saltation Matrix to capture the effects of the jump at a grazing point we run into trouble. Recall that for a saltation matrix S associated with a switch we require

- $SF(s) = SG(s)$ where F and G are the value of the flows vector field before and after the switch.
- $S(a_i) = \frac{\delta R}{\delta x}|_s a_i$ where the a_i $i = 1, 2, \dots, n-1$ provide a basis for the linearization of Σ at s .

we do not know what to choose for the vector field after the jump F or G ? This doesn't really matter tho as either choice leads to disaster. Taking $H(s) = (F/G)(s)$ gives

$$S = \begin{pmatrix} H(s) & \frac{\delta R}{\delta x}|_s a_1 & \cdots & \frac{\delta R}{\delta x}|_s a_{n-1} \end{pmatrix} \begin{pmatrix} F(s) & a_1 & \cdots & a_{n-1} \end{pmatrix}^{-1} \quad (85)$$

and the second matrix is non-invertible since F must be tangential to Σ at s and therefore in the span of the a_i . We can however still obtain a piecewise linear description of the transformed system - we are just unable to make use of the saltation formulation in any of the analysis.

In the standard analysis of grazing orbits we make use of some non-linear terms which come into play as the singularity at the graze elevates them to the linear level

through a characteristic square root action. Likewise we will use the time since hitting the discontinuity as a co-ordinate which in the switching and sliding case appears in some non-linear terms. In this jumping case the smoothed system doesn't contain any square root terms but still contains a non-differentiable graze which has the required singular properties.

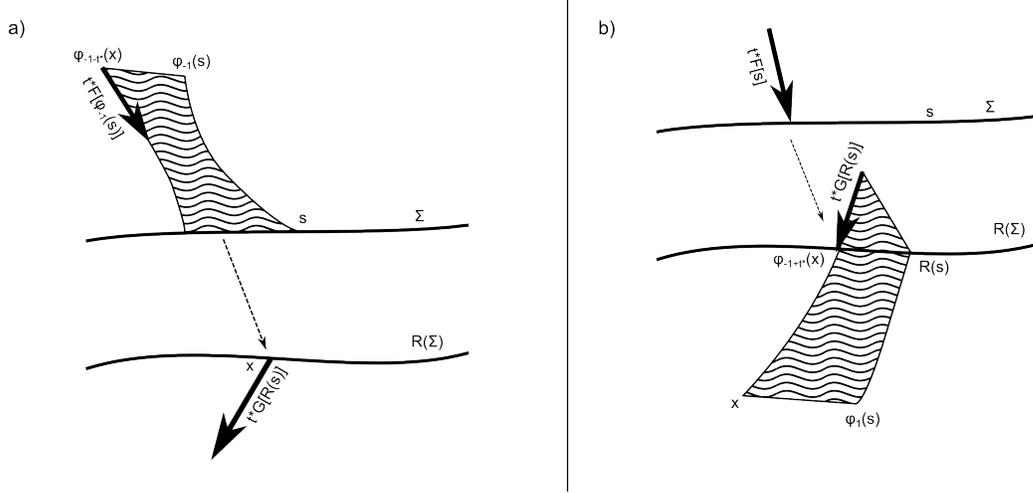


Figure 6: Configuration of components in $\Delta\Phi$; jacobian approximation (hatched) and correction terms (bold) for the jumping graze at a) the head and b) tail discontinuities.

Before the jump the analysis is exactly the same as with the non-grazing case so that the transformed system flows according to the ODE $\dot{p} = \widehat{F}(p)$ whose linearization about $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (I - J^{-1})J\tilde{J}^{-1}[p - \Phi(s)] \quad (86)$$

until we reach the set $\Phi(\Sigma)$ whose linearization about $\Phi(s)$ is given by

$$\{p : \langle J\tilde{J}^{-1}[p - \Phi(s)], n_1 \rangle = 0\} \quad (87)$$

where n_1 is normal to Σ at s . Note that $\Phi(s)$ is a grazing point of the new system since $\Phi(s) + \epsilon\dot{p}|_{\Phi(s)}$ lies in $\Phi(\Sigma)$.

$$J\tilde{J}^{-1}[\Phi(s) + \epsilon\dot{p}|_{\Phi(s)} - \Phi(s)] = J\tilde{J}^{-1}\epsilon\dot{p}|_{\Phi(s)} \quad (88)$$

$$J\tilde{J}^{-1}\epsilon[s - \phi_{-1}(s)] = J\epsilon F[\phi_{-1}(s)] = \epsilon F(s)$$

and by definition $\langle F(s), n_1 \rangle = 0$.

After the jump the linearization of Φ about $R(s)$ is given by

$$\begin{aligned} \Phi(x) = & \underbrace{\Phi(s) + \tilde{J}J^{-1}\left(\frac{\delta R}{\delta x}\Big|_s\right)^{-1}[x - t^*G[R(s)] - R(s)]}_{(89)} \\ & + \int_0^{t^*} \phi_{-t^*}(x) + \tau G[R(s)]d\tau - \int_0^{t^*} \phi_{1+t^*}(x) + \tau F[\phi_{-1}(s)]d\tau \end{aligned}$$

$$= \Phi(s) + \tilde{J}J^{-1}\left(\frac{\delta R}{\delta x}\Big|_s\right)^{-1}[x - t^*G[R(s)] - R(s)] + t^*[R(s) - \phi_{-1}(s)]$$

Where the underbraced term is the derivative of Φ at the point where x hits Σ and the integrals correct for the t^* seconds of flow after the jump, see fig 6,a, where

$$t^* = \frac{\langle x - R(s), n_2 \rangle}{\langle G[R(s)], n_2 \rangle} \quad (90)$$

with n_2 normal to $R(\Sigma)$ at $R(s)$. We invert the transformation as in 3.3.2 so that

$$t^* = \frac{\langle \Phi(x) - \Phi(s), (J\tilde{J}^{-1})^T n_1 \rangle}{\langle R(s) - \phi_1(s), (J\tilde{J}^{-1})^T n_1 \rangle} \quad (91)$$

where n_1 is normal to Σ at s .

$$x = R(s) + \frac{\delta R}{\delta x}\Big|_s J\tilde{J}^{-1}[\Phi(x) - \Phi(s) - t^*(R[s] - \phi_{-1}[s])] + t^*G[R(s)] \quad (92)$$

The flow is given by

$$\dot{p} = x - \phi_{-1}(x) = x - \underbrace{(\phi_{-1}(s) + J^{-1}\frac{\delta R}{\delta x}\Big|_s^{-1}[x - t^*G[R(s)] - R(s)] + t^*F[\phi_{-1}(s)])}_{\text{underbraced}} \quad (93)$$

Where the underbraced term is the one second backwards time image of the point where x 's orbit hits Σ and the final term corrects for the fact that x 's one second backwards time image is this point's $(1 - t^*)$ second backwards time image. So that the transformed systems flows according to the ODE $\dot{p} = \widehat{G}(p)$ whose linearization about $\Phi(s)$ is given by

$$\begin{aligned} \dot{p} &= R(s) - \phi_1(s) + (I - J^{-1}\frac{\delta R}{\delta x}\Big|_s^{-1})\frac{\delta R}{\delta x}\Big|_s \quad (94) \\ &\times \underbrace{J\tilde{J}^{-1}\left[p - \Phi(s) - \frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n_1 \rangle}{\langle R(s) - \phi_1(s), (J\tilde{J}^{-1})^T n_1 \rangle}(R[s] - \phi_{-1}[s])\right]}_{\text{underbraced}} \\ &+ \frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n_1 \rangle}{\langle R(s) - \phi_1(s), (J\tilde{J}^{-1})^T n_1 \rangle}(F[\phi_{-1}(s)] - G[R(s)]) \end{aligned}$$

where the underbraced term is tangent to Σ at s so there are no problems in applying R 's derivative or its inverse.

3.4.1 Tail discontinuity

Again the tail discontinuity requires a slightly different analysis. If we consider the backwards system as with the jump and switch we have an orbit which grazes the image of the jumping surface after going over the jump which is not qualitatively the same as the forward system. The flow is given by an ODE $\dot{x} = F(x)$ until we reach a set Σ where we instantaneously apply a map R then flow according to a second ODE $\dot{x} = G(x)$. For $s \in \Sigma$ to be a grazing point we require $\frac{\delta d(x)}{\delta t}\Big|_s = 0$ where $d(x)$

measures the distance from x to the switching surface Σ . We will linearize everything about the point $\phi_1(s)$.

The switching surface for this second discontinuity will be given by $\Phi[\phi_1(\Sigma)]$ whose linearization at $\Phi[\phi_1(s)]$ is given by

$$\{p : \langle \tilde{J}^{-1}(p - \Phi[\phi_1(s)], n_2 \rangle = 0\} \quad (95)$$

where J maps perturbations about $R(s)$ to perturbations about $\phi_1(s)$, \tilde{J} is the integral along these perturbations evolution as before and n is normal to $R(\Sigma)$ at $R(s)$. After flowing for more than one second from the jump the transformed system flows according to the ODE $\dot{p} = \hat{G}(p)$ whose linearization about $\Phi[\phi_1(s)]$ is given by

$$\dot{p} = \phi_1(s) - R(s) + (1 - J^{-1})J\tilde{J}^{-1}(p - \Phi[\phi_1(s)]) \quad (96)$$

Before crossing the switching surface the linearization of the transformation is given by

$$\Phi(x) = \Phi[\phi_1(s)] + \underbrace{\tilde{J}J^{-1}[x - \phi_1(s)]}_{\text{underbraced}} \quad (97)$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-t^*}^0 \phi_{-1+t^*-\epsilon}(x) + \tau F(s) d\tau - \int_{-t^*}^0 \phi_{-1+t^*}(x) + \tau G[R(s)] d\tau \\ = \Phi[\phi_1(s)] + \tilde{J}J^{-1}[x - \phi_1(s)] + t^*[s - R(s)] \end{aligned} \quad (98)$$

Where the underbraced term is the derivative of Φ assuming no jump has taken place and the two integrals account for the jump that occurred $(1 - t^*)$ seconds ago, see fig 6,b, where

$$t^* = \frac{\langle J^{-1}[x - \phi_1(s)], n_2 \rangle}{\langle G[R(s)], n_2 \rangle} \quad (99)$$

We invert the transformation directly

$$x = \phi_1(s) + J\tilde{J}^{-1}[\Phi(x) - \Phi[\phi_1(s)] - t^*(s - R[s])] \quad (100)$$

and

$$t^* = \frac{\langle \tilde{J}^{-1}[\Phi(x) - \Phi[\phi_1(s)] - t^*(s - R[s])], n_2 \rangle}{\langle G[R(s)], n_2 \rangle} \quad (101)$$

so that

$$t^* = \frac{\langle \tilde{J}^{-1}(\Phi(x) - \Phi[\phi_1(s)]), n_2 \rangle}{\langle G[R(s)], n_2 \rangle (1 + \langle [s - R(s)], n_2 \rangle)} \quad (102)$$

The flow is given by

$$\dot{p} = x - \phi_{-1}(x) = x - \underbrace{\left[s + \left(\frac{\delta R}{\delta x} \Big|_s \right)^{-1} (J^{-1}[x - \phi_1(s)] + t^*G[R(s)]) \right]}_{\text{underbraced}} + t^*F(s) \quad (103)$$

Where the underbraced term is the point where x 's orbit hit Σ and the final term corrects for the t^* seconds of flow before this. So that before reaching the discontinuity the transformed system flows according to the ODE $\dot{p} = \hat{F}(p)$ whose linearization about $\Phi[\phi_1(s)]$ is given by

$$\dot{p} = \phi_1(s) - s + \left[J - \left(\frac{\delta R}{\delta x} \Big|_s \right)^{-1} \right] \tilde{J}^{-1}(p - \Phi[\phi_1(s)]) \quad (104)$$

$$+ \frac{\langle \tilde{J}^{-1}(p - \Phi[\phi_1(s)]), n_2 \rangle}{\langle G[R(s)], n_2 \rangle (1 + \langle [s - R(s)], n_2 \rangle)} (F(s) - [J - (\frac{\delta R}{\delta x}|_s)^{-1}] \tilde{J}^{-1}[s - R(s)] - (\frac{\delta R}{\delta x}|_s)^{-1} G[R(s)])$$

Note that there are no grazes associated with this non-differentiable switch. After one second

$$\langle \tilde{J}^{-1}[\phi_1(s) - R(s)], n_2 \rangle = \langle G[R(s)], n_2 \rangle \neq 0 \quad (105)$$

and before $[\phi_1(s) - s]$ bears no special relationship with \tilde{J}^{-1} so should also yield a non-zero product.

3.5 Grazing a switch

A switching graze is defined as follows. The flow is given by an ODE $\dot{x} = F(x)$ until reaching a set Σ where we instantaneously switch to a second ODE $\dot{x} = G(x)$. For $s \in \Sigma$ to be a grazing point we require $\frac{\delta d(x)}{\delta t}|_s = 0$ where $d(x)$ measures the distance from x to the switching surface Σ .

Grazing a Switch is therefore a special case of Grazing a Jump with $R = I$. Before the Switch everything is the same however we cannot use the same analysis after the switch. The linearized transformation is given by

$$\Phi(x) = \Phi(s) + \tilde{J}J^{-1}[x - t^*G(s) - s] + t^*[s - \phi_{-1}(s)] \quad (106)$$

but $s - \phi_{-1}(s) = \tilde{J}J^{-1}F(s)$ so

$$\Phi(x) = \Phi(s) + \underbrace{\tilde{J}J^{-1}[x - t^*G(s) - s + t^*F(s)]}_{\text{underbraced}} \quad (107)$$

and the underbraced term is tangent to Σ at s which is $n - 1$ dimensional. The linearized transformation is therefore not invertible. We expand Φ with some second order terms in t^*

$$\Phi(x) = \Phi(s) + \underbrace{\tilde{J}J^{-1}[x - t^*G(s) - s]}_{\text{underbraced}} \quad (108)$$

$$+ \int_0^{t^*} \phi_{-t^*}(x) + \tau G(s) d\tau - \int_0^{t^*} \phi_{1+t^*}(x) + \tau F[\phi_{-1}(s)] d\tau$$

$$= \Phi(s) + \underbrace{\tilde{J}J^{-1}[x - t^*G(s) - s] + t^*[s - \phi_{-1}(s)]}_{\text{underbraced}} + \frac{(t^*)^2}{2} [G(s) - F(\phi_{-1}[s])] \quad (109)$$

Where the underbraced term is the derivative of Φ applied at the point where x 's orbit hits Σ and the two integrals correct for the t^* seconds of flow after the switch. To invert the transformation we note that the overbraced term is tangent to $\Phi(\Sigma)$ at $\Phi(s)$ so that taking the dot product with $(J\tilde{J}^{-1})^T n$ where n is normal to Σ at s gives

$$\frac{(t^*)^2}{2} = \frac{\langle \Phi(x) - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle G(s) - F[\phi_{-1}(s)], (J\tilde{J}^{-1})^T n \rangle} \quad (110)$$

and

$$x = s + t^*G(s) + J\tilde{J}^{-1}[\Phi(x) - \Phi(s) - t^*(s - \phi_{-1}[s]) - \frac{(t^*)^2}{2}(G[s] - F[\phi_{-1}(s)])] \quad (111)$$

The flow is given by

$$\dot{p} = x - \phi_{-1}(x) = x - \underbrace{(\phi_{-1} + J^{-1}[x - t^*G(s) - s])}_{\text{second backwards time image}} + t^*F[\phi_{-1}(s)] \quad (112)$$

where the underbraced term is the one second backwards time image of the point at which x 's orbit hits Σ and the final term corrects for the fact that this $\phi_{-1}(x)$ is this point's $(1 - t^*)$ backwards time image. So that the transformed system flows according the ODE $\dot{p} = \widehat{G}(p)$ whose expansion about $\Phi(s)$ is given by

$$\dot{p} = a + B[p - \Phi(s)] + c\sqrt{\langle p - \Phi(s), d \rangle} \quad (113)$$

where

$$\begin{aligned} a + B[p - \Phi(s)] &= s - \phi_{-1}(s) + (J - I)\tilde{J}^{-1}[p - \Phi(s)] \\ &- \frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle G(s) - F[\phi_{-1}(s)], (J\tilde{J}^{-1})^T n \rangle} (J - I)\tilde{J}^{-1}[G(s) - F(\phi_1[s])] \end{aligned} \quad (114)$$

and

$$c\sqrt{\langle p - \Phi(s), d \rangle} = [G(s) - F(s)]\sqrt{2\frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle G(s) - F[\phi_{-1}(s)], (J\tilde{J}^{-1})^T n \rangle}} \quad (115)$$

Note that (112) agrees with the flow before the switch on $t = 0$ so the graze is differentiable.

3.5.1 Tail discontinuity

The analysis follows exactly as in the case of grazing a jump only we set $R = I$. The transformed systems flow is an ODE $\dot{p} = \widehat{F}(p)$ whose linearization about $\Phi[\phi(s)]$ is given by

$$\begin{aligned} \dot{p} &= \phi_1(s) - s + (I - J^{-1})J\tilde{J}^{-1}(p - \Phi[\phi_1(s)]) \\ &+ \underbrace{\frac{\langle \tilde{J}^{-1}(p - \Phi[\phi_1(s)]), n \rangle}{\langle G(s), n \rangle}}_{\text{linearization}} [F(s) - G(s)] \end{aligned} \quad (116)$$

until reaching the set $\Phi[\phi_1(\Sigma)]$ whose linearization at $\Phi[\phi_1(s)]$ is given by

$$\{p : \langle \tilde{J}^{-1}[p - \Phi \circ \phi_1(s)], n \rangle = 0\} \quad (117)$$

Where the flow switches to the ODE $\dot{p} = \widehat{G}(p)$ whose linearization about $\Phi[\phi(s)]$ is given by

$$\dot{p} = \phi_1(s) - s + (1 - J^{-1})J\tilde{J}^{-1}(p - \Phi[\phi_1(s)]) \quad (118)$$

For a point p on the new switching surface $\tilde{J}^{-1}(p - \Phi[\phi_1(s)])$ lies on the original switching surface so the second underbraced term is zero and the expressions in (116) and (118) agree so that the switch is differentiable.

3.6 Grazing a slide

A Sliding graze is defined as follows. We have a switching surface Σ where the flow is given by an ODE $\dot{x} = F(x)$ on one side of Σ and another ODE $\dot{x} = G(x)$ on the other side of Σ . For $s \in \Sigma$ to be a sliding point we require $\langle F(s), n \rangle$ and $\langle G(s), n \rangle$ to have opposite signs where n is normal to Σ at s . As in (65) we use the Fillipov formalism to define the ODE $\dot{x} = H(x)$ on Σ . For $s \in \Sigma$ to be a grazing point we require $\frac{\delta d(x)}{\delta t}|_s = 0$ where $d(x)$ measures the distance from x to the switching surface Σ .

Therefore the set of grazing points form the boundary of the sliding region. If a point joins the sliding surface and flows along it to some grazing point s' it will then leave the sliding surface since at s' the sliding condition is violated and provided the derivative of F at s' is non-degenerate the point will then flow according to $\dot{x} = F(x)$ moving away from the switching surface. We can extend our sliding surface Σ to include all forwards time orbits from the set of grazing points and extend H the vector field on the sliding surface to this extended set by using F restricted to the new manifold, see fig 7. Within this framework we can analyze the sliding graze in more or less the same way we treated the basic slide. We will expand everything about a sliding graze point s to obtain an approximation of the transformed system.

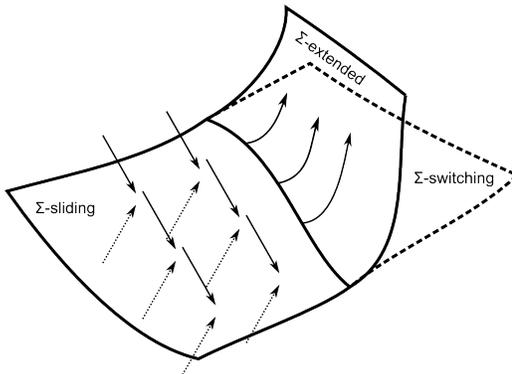


Figure 7: Extended sliding manifold Σ

Before reaching the discontinuity everything is as before so that the transformed system flows according to the ODE $\dot{p} = \tilde{F}(p)$ whose linearization about $\Phi(s)$ is given by

$$\dot{p} = s - \phi_{-1}(s) + (I - J^{-1})J\tilde{J}^{-1}[p - \Phi(s)] \quad (119)$$

After reaching the discontinuity surface we are stuck to the extended $(n - 1)$ dimensional manifold Σ . Again we use (σ, t) co-ordinates where $\sigma \in \Sigma$ is the current position of the point and $t \ll 1$ is the branching index which tells us for how long its backwards orbit stays on the sliding surface. Including second order terms in t the transformation at s is given by

$$\Phi(\sigma, t) = \Phi(s) + \underbrace{\tilde{J}J^{-1}[\phi_{-t}(\sigma, t) - s]} + \int_0^t \phi_{-t}(\sigma, t) + \tau H(s) d\tau - \int_0^t \phi_{1+t}(\sigma, t) + \tau F[\phi_{-1}(s)] d\tau \quad (120)$$

$$= \Phi(s) + \tilde{J}J^{-1}[\sigma - tH(s) - s] + t[s - \phi_{-1}(s)] + \frac{t^2}{2}(H(s) - F[\phi_{-1}(s)])$$

where the underbraced term is the derivative of Φ applied at the point where (σ, t) first joins the sliding surface and the two integrals correct for the t seconds of sliding. Note that since $F(s) = H(s)$ the terms that are linear in t cancel

$$\tilde{J}J^{-1}F(s) = s - \phi_{-1}(s) \quad (121)$$

hence the need for the second order terms. We invert the transformation as before so that

$$\frac{t^2}{2} = \frac{\langle \Phi(\sigma, t) - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle F(s) - F[\phi_{-1}(s)], (J\tilde{J}^{-1})^T n \rangle} \quad (122)$$

where n is normal to Σ at s .

$$\sigma = s + J\tilde{J}^{-1}[\Phi(\sigma, t) - \Phi(s) - \frac{t^2}{2}(F(s) - F[\phi_{-1}(s)])] \quad (123)$$

The flow is given by

$$\begin{aligned} \dot{p} &= \sigma - \phi_{-1}(\sigma, t) = \sigma - \underbrace{(\phi_{-1}(s) + J^{-1}[\sigma - tF(s) - s])}_{\text{underbraced}} + tF[\phi_{-1}(s)] \quad (124) \\ &= s - \phi_{-1} + (I - J^{-1})(\sigma - s) \end{aligned}$$

Where underbraced term is the one second backwards time image of the point where (σ, t) first joins the sliding surface and the final term corrects for the fact that (σ, t) 's one second backwards time image is actually the $(1-t)$ second backwards time image of this point. The transformed system flows according to the ODE $\dot{p} = \hat{G}(p)$ whose linearization about $\Phi(s)$ is given by

$$\begin{aligned} \dot{p} &= s - \phi_{-1}(s) + (J - I)\tilde{J}^{-1}[p - \Phi(s)] \quad (125) \\ &+ \frac{\langle p - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle F(s) - F[\phi_{-1}(s)], (J\tilde{J}^{-1})^T n \rangle} (J - I)\tilde{J}^{-1}[F[\phi_{-1}(s)] - F(s)] \end{aligned}$$

Expressions (119) and (124) agree on the switching surface $t = 0$ so the switch is differentiable.

3.6.1 Tail discontinuity

The flow is given by an ODE $\dot{x} = F(x)$ until reaching a set Σ where we instantaneously switch to an ODE $\dot{x} = H(x)$ as in (33). Solutions then flow for 1 second along the extended sliding surface Σ until reaching the vicinity of $\phi_1(s)$, where s is a grazing point. Again we use (σ, t) co-ordinates where $\sigma \in \Sigma$ is the current position of the point on the extended sliding surface and $t \ll 1$ is the branch index which tells us that its backwards time orbit stays on the extended sliding surface for $(1-t)$ seconds. If we attempt to expand Φ about $\phi_1(s)$ as in the basic slide tail-discontinuity we run into some difficulties, in particular since $H(s) = F(s)$ the second order terms cancel. We therefore include third order terms in t

$$\Phi(\sigma, t) = \Phi[\phi_1(s)] + \underbrace{\tilde{J}J^{-1}[\sigma - \phi_1(s)]}_{\text{underbraced}} + \int_0^t \phi_{-1+\tau}(x, t) d\tau - \int_0^t \phi_{-1+\tau}(x, 0) d\tau \quad (126)$$

Where the underbraced term is the derivative of Φ applied to σ restricted to the backwards flow branch where we stay on Σ for all time and the two integrals correct for the t seconds that we are not on the extended sliding surface.

In order to expand the first integral to third order in t we approximate the flow $\dot{x} = F(x)$ in the vicinity of s by

$$\dot{x} = F(s) + \underbrace{\frac{\delta F}{\delta x}}|_s(x - s) \quad (127)$$

Where we will denote the underbraced matrix A . This equations solution to second order in t is given by

$$x(t) = x(0) + t[A(x(0) - s) + F(s)] + \frac{t^2}{2}[A^2(x(0) - s) + AF(s)] \quad (128)$$

and for $x(0) = \phi_{-1+t}(\sigma) = [s + J^{-1}(\sigma - \phi_1(s)) + tF(s)]$

$$\int_{-t}^0 x(t)dt = t[s + J^{-1}(\sigma - \phi_1(s)) + tF(s)] - \frac{t^2}{2}[A[J^{-1}(\sigma - \phi_1(s)) + tF(s)] + F(s)] \quad (129)$$

$$+ \frac{t^3}{6}[A^2[J^{-1}(\sigma - \phi_1(s)) + tF(s)] + AF(s)]$$

Likewise we approximate $\dot{x} = H(x)$ by

$$\dot{x} = H(s) + \underbrace{\frac{\delta H}{\delta x}}|_s(x - s) \quad (130)$$

for the second integral. To obtain

$$\Phi(\sigma, t) = \Phi(\phi_1(s)) + \tilde{J}J^{-1}[\sigma - \phi_1(s)] + \frac{t^3}{3}\left(\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s\right)F(s) \quad (131)$$

which is invertible in the usual way

$$\frac{t^3}{3} = \frac{\langle \Phi(\sigma, t) - \Phi(s), (J\tilde{J}^{-1})^T n \rangle}{\langle (\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s)F(s), (J\tilde{J}^{-1})^T n \rangle} \quad (132)$$

where n is normal to Σ at $\phi_1(s)$.

$$\sigma = \phi_1(s) + J\tilde{J}^{-1}[\Phi(\sigma, t) - \Phi[\phi_1(s)] - \frac{t^3}{3}\left(\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s\right)F(s)] \quad (133)$$

The flow is given by

$$\dot{p} = \sigma - \phi_{-1}(\sigma, t) = \sigma - \underbrace{[s + J^{-1}[\sigma - \phi_1(s)]]} + \frac{t^2}{2}\left(\frac{\delta F}{\delta x}|_s - \frac{\delta H}{\delta x}|_s\right)F(s) \quad (134)$$

where the underbraced term is the one second backwards time image of x restricted to the branch where we stay on the sliding surface for all time and the final term corrects

for the t seconds of flow off the sliding surface. So that the transomed system flows according to the ODE $\dot{p} = \widehat{H}(p)$ whose expansion about $\Phi[\phi(s)]$ is given by

$$\dot{p} = a + B(p - \Phi[\phi_1(s)]) + c \sqrt[3]{\langle p - \Phi[\phi_1(s)], d \rangle} \quad (135)$$

where

$$\begin{aligned} a + B(p - \Phi[\phi_1(s)]) &= \phi_1(s) - s + (J - I)\widetilde{J}(p - \Phi[\phi_1(s)]) \\ &- \frac{\langle \Phi(\sigma, t) - \Phi(s), (J\widetilde{J}^{-1})^T n \rangle}{\langle (\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s)F(s), (J\widetilde{J}^{-1})^T n \rangle} (J - I)\widetilde{J}(\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s)F(s) \end{aligned} \quad (136)$$

and

$$c \sqrt[3]{\langle p - \Phi[\phi_1(s)], d \rangle} = \frac{1}{2}(\frac{\delta F}{\delta x}|_s - \frac{\delta H}{\delta x}|_s)F(s) \sqrt[3]{3 \frac{\langle \Phi(\sigma, t) - \Phi(s), (J\widetilde{J}^{-1})^T n \rangle}{\langle (\frac{\delta H}{\delta x}|_s - \frac{\delta F}{\delta x}|_s)F(s), (J\widetilde{J}^{-1})^T n \rangle}} \quad (137)$$

So that as in the basic sliding case orbits converge differentially and in finite time to the image of the extended sliding surface $\Phi(\Sigma, 0)$ except that in this grazing case the distance decays like $(t - c)^3$ due to the power of $\frac{2}{3}$ in the flow.

4 Computer example

The Moving Average Filter can easily be applied to time series data from a computer simulation of a non-smooth system. Since this data is equivalent to unsmoothed output from the smoothed system we can use it to study the systems obtained by applying the Moving Average Transformation to non-smooth systems.

4.1 Friction oscillator

Systems with static and dynamic friction can exhibit sliding behavior, there is a (so far as possible) humorous mismatch in terminology here since mathematical sliding in the sense of section 3.3 actually corresponds to an object being stuck to a surface by static friction whilst sliding along experiencing dynamic friction has no mathematical sliding associated with it at all! In our model we consider a mass on a linear spring subjected to a periodic force resting on a rough moving belt with a piecewise linear friction force

$$Fric(y) = \begin{cases} ay + b & y > 0 \\ -ay - b & y < 0 \end{cases} \quad (138)$$

where y is the velocity of the mass relative to the surface of the belt. The force acting on the block is given by

$$F(x, \dot{x}, t) = kx - d\dot{x} + l \sin(\omega t) + Fric(\dot{x} - u) \quad (139)$$

The state space of this dynamical system is three dimensional (position, velocity, forcing-phase). We will embed the phase variable as an interval $[0, 2\pi]$ rather than on the circle as it enables us to embed the the whole state space in \mathbb{R}^3 and the jump from 2π to 0 gives us some discontinuities in the flow which we can resolve with the transformation.

4.1.1 Procedure

For a given set of parameter values we simulate the oscillator on a computer and record a time series in the three dimensional state variable. The recording is just a long sequence $(x_i, \dot{x}_i, \tau_i)_{i=1}^N$ where we use a time-step of $0.05s$. The smoothing filter is just a summed average applied to the now discrete time data.

$$\Phi(x, \dot{x}, \tau)_i = \frac{1}{40} \sum_{j=i-39}^i (x_j, \dot{x}_j, \tau_j) \quad (140)$$

So that we are averaging over a period of $2s$. The accuracy of our routine is actually slightly better than this as we can approximate the average between time steps during the integration routine which improves the resolution of the sum. To obtain a differentiable time series we simply apply the same filter a second time to the data.

4.1.2 Data

We use the following parameters $k = 1.2$, $d = 0.0$, $w = 1.02$, $l = 1.9$, $u = 3.4$, $a = 0.04$ and $b = 0.1$ whose dynamics exhibit a grazing orbit that gives rise to chaotic dynamics on what appears to be a fully 2 dimensional attractor. We plot the time series for the three systems, discontinuous, continuous/non-differentiable and differentiable, see fig 8,1.

4.1.3 Results

Both applications of the filter give us useful insight into the original systems dynamics. When we transform the system from a discontinuous system to a continuous system we change the topology of the attractor. As outlined in the introduction we claim that the topology of the continuous system gives a better characterization of the dynamics and that is the case here. Homologically the discontinuous system is equivalent to two points whereas the continuous system is a figure of eight, the two possible loops in the attractor provide the topological mechanism for chaos, see fig 8,b.

On the second application of the filter we arrive at a differentiable system which can be thought of as an ODE with continuous RHS. We can therefore locally approximate this ODE from our data and use this to compute lyupanov exponents along the flow. The clear positive exponent shows that we have a chaotic system, see fig 8,c.

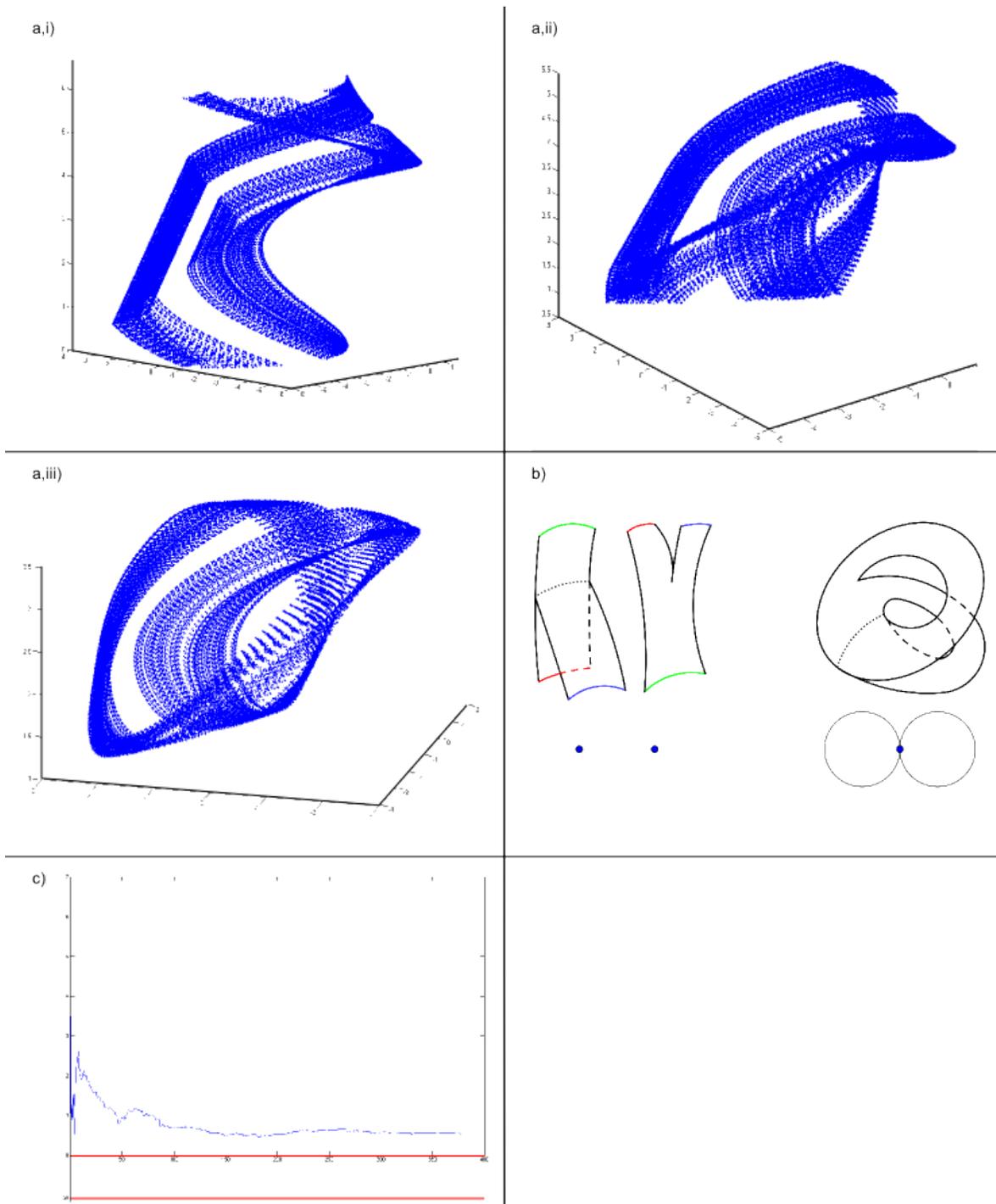


Figure 8: a) Time series recorded from i) Original discontinuous system, ii) Once transformed continuous system, iii) Twice transformed differentiable system, b) Convergence of Lyapunov exponent calculation, c) Topology of invariant set for discontinuous and transformed continuous system.

4.1.4 Remarks on averaging period

In the theory laid out earlier in the paper the length of the averaging period really didn't matter. If we set

$$\Phi_\alpha(x) = \int_{-\alpha}^0 \phi_\tau(x) d\tau \quad (141)$$

then the Φ_α transformed systems will all be topologically equivalent and admit the same sort of local descriptions at the images and tail images of discontinuities. In the practical world of numerics this is obviously not the case. There are two strategic factors in play. Firstly if we make α to large then we loose a lot of information when we take the average as Φ_α will be too contracting, mix this with some natural round up error and the transformed system will begin to look like a smooth blob with noise and little resemblance to the original system. On the other hand if α is too small then smoothed discontinuities will still have very large changes in second derivatives which will lead to more numerical errors.

A theoretical basis for choosing α is still a work in progress but for non-smooth oscillators like this one an averaging period roughly half of a typical orbit period seems to work well as a rule of thumb. Moving averages of Markov chains with exponentially decaying memory of the form

$$\Phi(x) = \int_{-\infty}^0 \phi_\tau(x) e^{\beta\tau} d\tau \quad (142)$$

are of some interest in approximating the behavior of coupled systems with switching network architecture. Here the entropy of the Markov chain provides a natural time scale when considering the choice of β [12], perhaps something like that can be said here.

4.1.5 Remarks on alternative methods

Of course the results outlined above could be obtained without making use of the Moving Average Transformation, traditional methods are however quite difficult to apply in a systematic way. In order to identify the topology of the continuous system it would be necessary to identify points before and after the time reset jumps as being connected then add these connections to something like the rips complex of the series.

To compute the lyupanov exponent from the data as the integral of the derivative along the flow it would be necessary to divide the points in the series up into 5 categories

1. Points that are not sliding and not about to start siding or time reset.
2. Points that are not sliding and are about to start sliding
3. Points that are not sliding and are about to time reset
4. Points that are sliding and are not about to time reset
5. Points that are sliding that are about to time reset

The vector field/saltation matrix can then be adequately approximated at a point in category i say, by a local approximation using other close by points in that same category. This wouldn't be too difficult from a computer simulation like this where it would be easy to categorize points as such but would be extremely difficult when dealing with experimental data where it would be impossible to verify the categorization. Of course it is possible to compute Lyapunov exponents from data by constructing a return map which placed away from the discontinuity would not be too problematic, however this is very far from systematic doesn't give us an approximation of the vector field.

5 Conclusion

We have shown that the Moving Average Transformation can be used to transform discontinuous or non-differentiable systems into dynamically equivalent differentiable systems. That stability and orbit structure are dynamical equivalence invariants follows from the fact that away from the discontinuities we have a standard smooth change of variables and on the discontinuities the theory in section 3 shows how in a quite natural way this information is integrated into the smoothed system. We have also demonstrated that the transformation can be applied numerically especially well to time-series data where we can replace the state space transformation with a simple finite impulse response (FIR) filter on the data. There are several questions left open all of which I hope to address in future work

5.1 Numerical implementation

There are many questions here. Firstly as discussed in 4.1.4 how should we choose the length of the averaging period? Whilst the time-series based method is very easy to implement it is not necessarily the best approach, for example we are only able to collect reasonably dense data from attracting objects. How should one best go about numerically implementing a method like that used in section 2? Then there are all the usual questions associated with numerics such as cost, the effect of noise (FIR filters like the moving average filter is routinely used to remove high frequency noise in signal processing) and rounding error which could be very important here especially if we want to use a longer averaging period where the more contracting transformation can lose sensitivity.

In section 4 we were able to compute the Lyapunov spectrum of the Friction Oscillator from a time-series using our new technique. Do the benefits of this approach which include systematic implementation, use of the whole vector field and implementation of well tested/researched standard smooth methods outweigh the potential problems such as the introduction of systematic numerical error through the transformation and increased dimension through use of delay maps?

5.2 More applications

This paper has focused on computing the stability of non-smooth systems by smoothing them then applying standard techniques that really rely on differentiability which

would not have been applicable before the smoothing. Of course stability calculations are not the only standard techniques that rely on smoothness. What other standard techniques can be applied to the smoothed systems to get results that would have been impossible or more difficult to obtain from the original non-smooth system? Numerical continuation for non-smooth systems is very difficult because of the non-differentiability [4]. Can we use our smoothing technique to do this in a simpler more systematic way?

5.3 Generic Smoothed non-smooth systems

Non-smooth systems can generically exhibit bifurcation structures which would be impossible or of high codimension in the space of smooth systems. Since dynamically equivalent systems have the same dynamics and therefore bifurcate in the same way the Moving Average Transformation provides us with a family of smooth systems (the smoothed non-smooth systems) which behave in an atypical way. How can we characterize these systems? What does a generic smoothed system look like and what insight into non-smooth dynamics does this answer give us, if any? we conjecture that a generic smoothed system can be expressed as a generic smooth flow on special type of branching manifold that splits up and folds back into itself along non-transverse intersections with grazes and slides respectively responsible for the two types of irregularity.

Acknowledgements

I would like to thank my supervisor Professor David Broomhead as well as Professor Paul Glendinning for their useful discussions, support and guidance.

References

- [1] Alart P, Maisonneuve O, Rockafellar R.T. (2006). Nonsmooth Mechanics and Analysis. *Springer*
- [2] Bernardo M, Budd C, Champneys A.R, Kowalczyk P. (2008). Piecewise-smooth Dynamical Systems. *Springer*.
- [3] Kunze M, Küpper, T. (2001). Nonsmooth dynamical systems: An overview. *Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems*. pages 431-452.
- [4] Arango I, Taborda J. A. (2007). Continuation of nonsmooth bifurcations in filippov systems using singular point tracking. *International Journal of Applied Mathematics and Informatics*. Issue 1, volume 1, pages 36-49.
- [5] Llibre J, Da Silva P.R, Teixeira M.A. (2007). Regularization of Discontinuous Vector Fields on \mathbb{R}^3 via Singular Perturbation. *Journal of Dynamics and Differential Equations*. Volume 19, number 2, pages 309-331.

- [6] Gang T, Frank L. (2001). Adaptive Control of Nonsmooth Dynamic Systems. *Springer*.
- [7] Di Bernardo M, Hogan S.J. (2010) Discontinuity-induced bifurcations of piecewise smooth dynamical systems. *Phil. Trans. R. Soc. A*. Volume 368, number 1930, pages 4915-4935.
- [8] Kunze M, Küpper T. (1996). Qualitative bifurcation analysis of a non-smooth friction-oscillator model. *ZAMP*. Volume 48, pages 87-101.
- [9] Huke J. (1993) Embedding nonlinear dynamical systems: A guide to Takens' theorem. *Technical Report DRA, Malvern*.
- [10] Filippov A.F. (1960). Differential equations with discontinuous right-hand side. *Mat. Sb. (N.S.)* Volume 51, number 1, Pages 99-128.
- [11] Darbyshire G. and Broomhead D.S. (1996). The calculation of Liapunov exponents from time series data. *Physica D*. Volume 89, pages 287-305.
- [12] Skufca J.D, Bollt E.M. (2004). Communication and Synchronization in Disconnected Networks with Dynamic Topology Moving Neighborhood Networks. *Mathematical Biosciences and Engineering*. Volume 1.