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THE DENSITY OF ALGEBRAIC POINTS ON CERTAIN PFAFFIAN SURFACES

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ABSTRACT. We prove some instances of Wilkie's conjecture on the density of rational points on sets definable in the real exponential field. In particular, we prove that this conjecture is true for surfaces defined using restricted exponentiation, and that it is true for a general Pfaffian surface provided that the surface admits a certain kind of parameterization.

1. INTRODUCTION

In this paper we combine Pila's methods ([9],[10]) with a stratification result due to Gabrielov and Vorobjov ([3]) to prove some instances of Wilkie's conjecture on the density of rational points on definable sets in the expansion of the real field by the exponential function. In order to state the conjecture, we first need some definitions (which are due to Pila, see [12] for example). Suppose that X is a subset of $\mathbb{R}^n$. The *algebraic part* of X is the union of all connected semialgebraic subsets of X of positive dimension. Write X^{alg} for the algebraic part of X. The *transcendental part* of X, written X^{trans}, is $X \setminus X^{\text{alg}}$. Rather than considering only rational points, we shall in fact consider points in a fixed real number field. So, fix a number field $K \subseteq \mathbb{R}$ of degree k. Let H be the absolute multiplicative height on K (as defined in [1], for example). For $K = \mathbb{Q}$, this is given by $H(a/b) = \max\{|a|, b\}$, where $a, b \in \mathbb{Z}, \gcd(a, b) = 1, b > 0$. Extend H to K^n by taking the maximum of the heights of the coordinates. Then for X in $\mathbb{R}^n$ and T a positive real number, we set

$$\begin{aligned} X(K,T) &= \{\bar{x} \in X \cap K^n : H(\bar{x}) \leq T\}, \\ N_K(X,T) &= \#X(K,T). \end{aligned}$$

The subsets of $\mathbb{R}^n$ that we work with in this paper will be definable in some o-minimal expansion of the real field, and we shall assume that the reader is familiar with the basic theory of these structures (see van den Dries's book, [2]). Wilkie conjectures an improvement, in certain cases, of the bound in the Pila-Wilkie Theorem. One version of this theorem is as follows (see [12],[11]).

Theorem. *Suppose that $X \subseteq \mathbb{R}^n$ is definable in some o-minimal expansion of the real field. Then for all $\varepsilon > 0$ there exists $c(X, k, \varepsilon)$ such that for all $T \geq 1$,*

$$N_K(X^{trans}, T) \leq c(X, k, \varepsilon) T^{\varepsilon}.$$

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As usual, the notation $c(X, Y, Z, \ldots)$ is used to mean that c is allowed to depend only on the displayed parameters.

Now, the bound in this theorem is essentially best possible (see [12]). However, Wilkie conjectures that it can be substantially improved by restricting the structure considered. Let $\mathbb{R}_{\exp}$ be the expansion of the real ordered field by the exponential function.

Conjecture (Wilkie,Pila see [12],[10])**.** *Suppose that $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{\exp}$. Then there exist $c(X, k)$ and $\gamma(X)$ such that*

$$N_K(X^{trans}, T) \leq c(X, k)(\log T)^{\gamma(X)}$$

for all $T \geq e$.

Following Pila ([8, 9]) we will in fact work with the expansion of the real field by Pfaffian functions, rather than just the exponential function. A sequence $f_1, \ldots, f_r : U \to \mathbb{R}$ of analytic functions on an open set $U \subseteq \mathbb{R}^n$ is said to be a *Pfaffian chain* of *order* r and *degree* α if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{n+j}]$ of degree at most α such that

$$df_j = \sum_{i=1}^{n} P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x}))dx_i \qquad \text{for all } i = 1, \ldots, r \text{ and } j = 1, \ldots, n.$$

Given such a chain, we say that a function $f : U \to \mathbb{R}$ is *Pfaffian of order* r and *degree* (α, β) with chain $f_1, \ldots, f_r$, if there is a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ of degree at most β such that $f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$. Let $\mathbb{R}_{\text{Pfaff}}$ be the expansion of the real ordered field by all Pfaffian functions $f : \mathbb{R}^n \to \mathbb{R}$, for $n \geq 1$. Pila ([9]) proved the following.

Theorem. *Suppose that $a, b \in \mathbb{R} \cup \{\pm\infty\}, a < b$ and that $f : (a, b) \to \mathbb{R}$ is a transcendental Pfaffian function. Then there exist $c(f, k)$ and $\gamma(f)$ such that*

$$N_K(graph(f), T) \leq c(f, k)(\log T)^{\gamma(f)}.$$

Our first result improves this by replacing the assumption that f is Pfaffian with the assumption that f is existentially definable in $\mathbb{R}_{\text{Pfaff}}$. By Wilkie's model completeness result for $\mathbb{R}_{\exp}$ ([16]), this implies the following.

Theorem. *Wilkie's conjecture (in the form stated above) holds for one-dimensional X.*

This result has also been obtained by Lee Butler.

The constant and the exponent in this result are sufficiently uniform to enable us to prove a result for surfaces. We follow Pila's method from [10] and so we need the surface to admit a particular kind of parameterization, namely, a *mild* parameterization (see section 5 and also [5] and [14] for further information). Rather than giving the general result here, we state a corollary for a reduct of $\mathbb{R}_{\text{Pfaff}}$ in which the required parameterizations are known to exist. To every function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is an open set containing $[0, 1]^n$, we associate a new function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in [0, 1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mathbb{R}_{\text{resPfaff}}$ be the expansion of the real field by all functions $\hat{f}$, where $f : U \to \mathbb{R}$ is Pfaffian, $[0,1]^n \subseteq U$ and $n \geq 1$.

Theorem. *Suppose that $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{resPfaff}$ and has dimension 2. Then there exist $c(X,k)$ and $\gamma(X)$ such that*

$$N_K(X^{trans}, T) \leq c(X,k)(\log T)^{\gamma(X)}$$

As Pila observes in the introduction of [10], the independence of the exponent from the number field is related to transcendence theory. At the end of the paper, we give further evidence of this by using our methods to give a new proof of a rather weak version of a theorem first proved by Waldschmidt ([15]).

2. Pfaffian functions

In this section, we recall the results on Pfaffian functions that we shall need later. We follow the presentation in [4]. First, a couple of easy lemmas.

Lemma 2.1 ([4])**.** *Suppose that f, g are Pfaffian on an open set $U \subseteq \mathbb{R}^n$ with a common chain of order r and that f and g have degree (α_1, β_1) and (α_2, β_2) respectively. Then both $f + g$ and $f \cdot g$ are Pfaffian on U, with the same chain as f and g (and so of order r) and with degree $(\max\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\})$ and $(\max\{\alpha_1, \alpha_2\}, \beta_1 + \beta_2)$ respectively.*

Lemma 2.2 ([4])**.** *Suppose that $f : U \to \mathbb{R}$, of order r and degree (α, β). Then any partial derivative $\frac{\partial f}{\partial x_i}$ of f is Pfaffian with the same chain as f and with degree $(\alpha, \alpha + \beta - 1)$.*

The foundation for the theory of Pfaffian functions is Khovanskii's theorem on the number of connected components of a Pfaffian variety ([7]). Given functions $f_1, \ldots, f_I : \mathbb{R}^n \to \mathbb{R}$, we write $V(f_1, \ldots, f_I) = \{\bar{x} \in \mathbb{R}^n : f_1(\bar{x}) = \cdots = f_I(\bar{x}) = 0\}$.

Theorem 2.3 (Khovanskii, see [4])**.** *Suppose that $f_1, \ldots, f_I : \mathbb{R}^n \to \mathbb{R}$ are Pfaffian functions, with a common chain of order r, and degree at most (α, β). Then the variety $V(f_1, \ldots, f_I)$ has at most*

$$2^{r(r-1)/2+1}\beta(\alpha + 2\beta - 1)^{n-1}((2n-1)(\alpha + \beta) - 2n + 2)^r$$

connected components.

The other result we shall need is an effective stratification theorem due to Gabrielov and Vorobjov. The sets involved are more general than varieties: an *elementary semi-Pfaffian set* is a set of the form

$$\{\bar{x} \in \mathbb{R}^n : f_1(\bar{x}) = \cdots = f_I(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_J(\bar{x}) > 0\}$$

where $f_1, \ldots, f_I, g_1, \ldots, g_J : \mathbb{R}^n \to \mathbb{R}$ are Pfaffian functions. If these functions have a common chain, order r, and degree (α, β), we say that the set has *format* (I, J, r, α, β). For a real number t, we let

$$B(t) = (\alpha + \beta + 1)^{(r+2)^{tn}}.$$

Theorem 2.4 (Gabrielov-Vorobjov, [3]). *There is an absolute constant c such that the following holds. Suppose that $X \subseteq \mathbb{R}^n$ is an elementary semi-Pfaffian set with format (I, J, r, α, β). Then there is a partition of X into at most $I^n B(c)$ smooth (not necessarily connected) elementary semi-Pfaffian sets, with all functions involved having the same chain as the functions defining X, and the format of each stratum of the partition coordinate-wise bounded by $(I^n B(c), J + 2^n, r, \alpha, B(c))$. Further, for each stratum Y of codimension m, there are, among the functions defining Y, some $h_1, \ldots, h_m$ vanishing identically along Y such that $dh_1 \wedge \cdots \wedge dh_m \neq 0$ at each point of Y.*

3. Functions implicitly defined by Pfaffian functions

In this section, we compute bounds on the number of zeros of the derivatives of an implicitly defined unary function in the structure $\mathbb{R}_{\mathrm{Pfaff}}$.

Recall ([6]) that a function $f : U \to \mathbb{R}$ definable in $\mathbb{R}_{\mathrm{Pfaff}}$, with $U \subseteq \mathbb{R}^m$ is said to be *implicitly defined* (by Pfaffian functions) if there exist $n \geq 1$, Pfaffian functions $p_1, \ldots, p_n : \mathbb{R}^{m+n} \to \mathbb{R}$ and definable smooth functions $f_1, \ldots, f_n : U \to \mathbb{R}$ such that $f_1 = f$ and

$$p_1(\bar{x}, f_1(\bar{x}), \ldots, f_n(\bar{x})) = \cdots = p_n(\bar{x}, f_1(\bar{x}), \ldots, f_n(\bar{x})) = 0,$$
$$\det\Big(\frac{\partial(p_1, \ldots, p_n)}{\partial(x_{n+1}, \ldots, x_{n+m})}\Big)(\bar{x}, f_1(\bar{x}), \ldots, f_n(\bar{x})) \neq 0,$$

for all $\bar{x} \in U$. If in this situation the functions $p_1, \ldots, p_n$ have order r, with a common chain, and degree (α, β), then we say that f has an implicit definition of *complexity* (n, r, α, β).

We now consider unary implicitly defined functions. So, suppose that $\phi_1, \ldots, \phi_n : I \to \mathbb{R}$ are smooth definable functions on some open interval $I \subseteq \mathbb{R}$ and that $p_1, \ldots, p_n : \mathbb{R}^{1+n} \to \mathbb{R}$ are Pfaffian functions such that for all $x_1 \in I$,

$$p_i(x_1, \phi(x_1)) = 0 \text{ for } i = 1, \ldots, n \tag{1}$$
$$\Delta(x_1, \phi(x_1)) \neq 0 \tag{2}$$

where $\phi = \langle \phi_1, \ldots, \phi_n \rangle, \Delta = \det J$ and $J = \frac{\partial(p_1, \ldots, p_n)}{\partial(x_2, \ldots, x_{n+1})}$. Suppose that the implicit definition has complexity (n, r, α, β).

Now, differentiating the above equations, we have

$$\begin{pmatrix} \phi_1'(x_1) \\ \vdots \\ \phi_n'(x_1) \end{pmatrix} = -\frac{1}{\Delta}\mathrm{Adj}J \begin{pmatrix} \frac{\partial p_1}{\partial x_1} \\ \vdots \\ \frac{\partial p_n}{\partial x_1} \end{pmatrix}, \tag{3}$$

where $\mathrm{Adj}J$ is the adjugate matrix of J and the right hand side is evaluated at the point $\langle x_1, \phi(x_1) \rangle$. Both the determinant of J and the entries of the adjugate matrix of J are polynomials of degree n in the partial derivatives of $p_1, \ldots, p_n$. Each partial $\frac{\partial p_i}{\partial x_j}$ has (by 2.2) the same chain as $p_1, \ldots, p_n$, order r, and degree $(\alpha, \alpha + \beta - 1)$. A polynomial of degree n in these partials thus has (by 2.1) the same chain as $p_1, \ldots, p_n$ (and so order r) and degree $(\alpha, n(\alpha + \beta - 1))$. So, for each

$i = 1, \ldots, n$ there is a Pfaffian function $F_i : \mathbb{R}^{1+n} \to \mathbb{R}$ with the same chain as $p_1, \ldots, p_n$ and degree $(\alpha, (n+1)(\alpha+\beta-1))$ such that

$$\phi_i'(x_1) = \frac{F_i(x_1, \phi(x_1))}{\Delta(x_1, \phi(x_1))} \tag{4}$$

for all $x_1 \in I$. We need a similar representation of the higher derivatives of the functions ϕ_i.

Proposition 3.1. *For each $i = 1, \ldots, n$ and $k \geq n$, there is a Pfaffian function $G_{i,k} : \mathbb{R}^{1+n} \to \mathbb{R}$ with the same chain as $p_1, \ldots, p_n$, of degree at most $(\alpha, (2k-1)(n+1)(\alpha+\beta-1) + (k-1)(\alpha-1))$ such that*

$$\phi_i^{(k)}(x_1) = \frac{G_{i,k}(x_1, \phi(x_1))}{\Delta^{2k-1}(x_1, \phi(x_1))}$$

for all $x_1 \in I$.

Proof. For each i, this is proved by induction on $k \geq 1$. The case $k = 1$ is (4), so suppose that $k \geq 1$ and that $G = G_{i,k} : \mathbb{R}^{1+n} \to \mathbb{R}$ is Pfaffian with the same chain as $p_1, \ldots, p_n$, has degree $(\alpha, (2k-1)(n+1)(\alpha+\beta-1) + (k-1)(\alpha-1))$ and is such that

$$\phi_i^{(k)}(x_1) = \frac{G(x_1, \phi(x_1))}{\Delta^{2k-1}(x_1, \phi(x_1))} \tag{5}$$

for all $x_1 \in I$. Differentiating (5) and using (4), we have (with the right hand sides evaluated at $\langle x_1, \phi(x_1) \rangle$):

$$\begin{aligned}
\phi_i^{(k+1)}(x_1) &= \frac{1}{\Delta^{2(2k-1)}} \Big(\Delta^{2k-1} \big(\frac{\partial G}{\partial x_1} + \sum_{j=2}^{n+1} \frac{\partial G}{\partial x_j} \cdot \frac{F_j}{\Delta} \big) \\
&\qquad -(2k-1)\Delta^{2k-2} G \big(\frac{\partial \Delta}{\partial x_1} + \sum_{j=2}^{n+1} \frac{\partial \Delta}{\partial x_j} \cdot \frac{F_j}{\Delta} \big) \Big) \\
&= \frac{1}{\Delta^{2k+1}} \Big(\Delta^2 \frac{\partial G}{\partial x_1} + \Delta \sum_{j=2}^{n+1} \frac{\partial G}{\partial x_j} F_j \\
&\qquad -(2k-1) \big(G\Delta \frac{\partial \Delta}{\partial x_1} + G \sum_{j=2}^{n+1} \frac{\partial \Delta}{\partial x_j} F_j \big) \Big).
\end{aligned}$$

This has the required form, and it follows easily from 2.1 and 2.2 that $G_{i,k+1}$ has the required degree (in fact, we can do slightly better as the degree of Δ is lower than that of F_i, but in later computations it is easier to have the bound in the form given). $\square$

Theorem 3.2. *Let $k \in \mathbb{N}, k \geq 1$. Suppose that ϕ_i is not a polynomial. Then the number of zeros of $\phi_i^{(k)}$ on I is at most*

$$2^{r(r-1)/2+1} \Big(\big(2k\alpha + 4k(n+1)(\alpha+\beta)\big)^{n+1} \big((2n+1)(k\alpha + 2k(n+1)(\alpha+\beta))\big)^r \Big).$$

Proof. Fix i and k. The zeros of $\phi_i^{(k)}$ are isolated and so by (1) and 3.1, the number of zeros of $\phi_i^{(k)}$ is bounded by the number of connected components of $V(p_1, \ldots, p_n) \cap V(G_{i,k})$. The functions $p_1, \ldots, p_n, G_{i,k}$ are Pfaffian on $\mathbb{R}^{1+n}$ of order r and degree as given above. So we can apply 2.3 and conclude that the required bound holds. □

Another application of 2.3 yields the following.

Lemma 3.3. *Suppose that $P \in \mathbb{R}[X,Y]$ has degree d and is such that $P(x_1, \phi_1(x_1))$ is not identically zero on I. Then $P(x_1, \phi_1(x_1))$ has at most*

$$2^{r(r-1)/2+1} d\beta \cdot (\alpha + 2d\beta)^n ((2n+1)(\alpha + d\beta))^r$$

zeros on I.

Proof. By (1) the number of zeros of $P(x_1, \phi_1(x_1))$ on I is at most the number of connected components of $V(\tilde{P}) \cap V(p_1, \ldots, p_n)$, where $\tilde{P}(x_1, x_2, \ldots, x_{n+1}) = P(x_1, x_2)$. Now apply 2.3. □

Finally, all of the above also applies to the inverses of the functions ϕ_i (where they exist), by the following.

Lemma 3.4. *Suppose that $J \subseteq I$ is an interval on which ϕ_i' does not vanish. Then ϕ_i^{-1} (exists and) has an implicit definition on $\phi_i(J)$ with the same complexity as the definition of ϕ_i.*

Proof. Use the same functions $p_1, \ldots, p_n$ but let x_{i+1} be the independent variable. The required Jacobian does not vanish since ϕ_i' does not vanish. □

4. Wilkie's conjecture for existentially definable Pfaffian curves

We now use the bounds from the previous section to prove Wilkie's conjecture for curves which can be existentially defined using Pfaffian functions. We follow Pila's proof for curves which are the graphs of Pfaffian functions of one variable (see [11]). Accordingly, we need to introduce another height function.

Definition 4.1. Suppose that α is an algebraic number. The *denominator* of α is the smallest positive integer $\text{den}(\alpha)$ such that $\text{den}(\alpha) \cdot \alpha$ is an algebraic integer. We let

$$H^{\text{size}}(\alpha) = \max\{\text{den}(\alpha), |\alpha_1|, \ldots, |\alpha_k|\}$$

where $\alpha_1, \ldots, \alpha_k$ are the conjugates of α.

If α is algebraic of degree k then $H^{\text{size}}(\alpha) \leq H(\alpha)^k$ (see [11, Section 6]) and so, in order to prove the theorems in the introduction, we may work with H^{size} rather than H. So, for $X \subseteq \mathbb{R}^n$ and $K \subseteq \mathbb{R}$ a number field, we let

$$X^{\text{size}}(K,T) = \{\bar{x} \in X(K) : H^{\text{size}}(\bar{x}) \leq T\}$$

and

$$N_K^{\text{size}}(X,T) = \#X^{\text{size}}(K,T).$$

Now fix a number field $K \subseteq \mathbb{R}$ of degree k. The following result is due to Pila ([11, 6.8]).

Proposition 4.2. *Let $d \geq 1, D = (d+1)(d+2)/2, T \geq e$ and let $I \subseteq \mathbb{R}$ be an interval of length L, where $\frac{1}{T^{4k}} \leq L \leq 2T$. Suppose that $\phi : I \to \mathbb{R}$ is a C^D function with $|\phi'| \leq 1$ and with the derivatives $\phi^{(j)}$ either identically zero on I or non-vanishing on the interior of I, for $j = 1, \dots, D$. Let X be the graph of ϕ. Then $X^{size}(K,T)$ is contained in the union of at most*

$$12(4k+3) \cdot 6^k D \cdot T^{\frac{4(4k+1)}{3(d+3)}} \cdot \log T$$

intersections of X with real algebraic curves of degree at most d.

Using this result together with the bounds in the previous section, we can prove Wilkie's conjecture for implicitly defined functions of one variable. The proof is the same as the proof of [11, 6.2] but we include the details as, in order to use this result in the case of surfaces, we need to be rather careful with the constant and the exponent.

Proposition 4.3. *Suppose that $I \subseteq \mathbb{R}$ is an open interval in $\mathbb{R}$ and that $\phi : I \to \mathbb{R}$ is a transcendental function and is implicitly definable in $\mathbb{R}_{Pfaff}$, with complexity (n, r, α, β). Then there are explicit constants $c_1(k), c_2(n, r, \alpha, \beta)$ such that for $T \geq e$*

$$N_K^{size}(X,T) \leq c_1 \cdot c_2 \cdot (\log T)^{3n+3r+8}.$$

Furthermore, for fixed r, n and α, c_2 is a polynomial in β of degree $2(n+r+1)$.

Proof. For now, fix a natural number $d \geq 1$ and let $D = (d+1)(d+2)/2$. Divide I into intervals on which either $|\phi'| \leq 1$ or $|\phi'| > 1$. Then further divide these subintervals into intervals on the interior of which either ϕ has non-vanishing derivatives up to order D or the inverse ψ of ϕ has non-vanishing derivatives up to order D. The total number of intervals needed is at most

$$\#V(\phi' + 1) + \#V(\phi' - 1) + \sum_{i=1}^{D} \#V(\phi^{(i)}) + \sum_{i=1}^{D} \#V(\psi^{(i)}) + 1.$$

By 3.2 and 3.4 this number is bounded by

$$2^{r(r-1)/2+2}\Big((3\alpha + 4(n+1)(\alpha+\beta))^{n+1}(2(2n+1)\alpha + 2(2n+1)(n+1)(\alpha+\beta))^r +$$
$$\sum_{i=1}^{D} \left((2i\alpha + 4i(n+1)(\alpha+\beta))^{n+1}((2n+1)i\alpha + 2i(n+1)(\alpha+\beta))^r\right)\Big) + 1$$

which is less than

$$2^{r(r-1)/2+2}\Big((3\alpha + 4(n+1)(\alpha+\beta))^{n+1}(2(2n+1)\alpha + 2(2n+1)(n+1)(\alpha+\beta))^r +$$
$$D\big((2D\alpha + 2D(n+1)(\alpha+\beta))^{n+1}((2n+1)D\alpha + 2D(n+1)(\alpha+\beta))^r\big)\Big)$$

which is bounded by

$$c_3(n, r, \alpha, \beta) D^{n+r+2}$$

where, for fixed n, r and α the constant c_3 is a polynomial in β of degree $n+r+1$.

Fix $T \geq e$ and intersect I with the interval $[-T, T]$. Then each of the subintervals found above has length $\leq 2T$. Suppose that J is one of these subintervals, and has length $> \frac{1}{T^2}$. If J is an interval on which $|\phi'| \leq 1$ then let X_J be the graph of

ϕ restricted to J and otherwise let X_J be the graph of ψ restricted to J. By 4.2, X_J^{size} is contained in a most

$$12(4k+3)\cdot 6^k \cdot T^{\frac{4(4k+1)}{3(d+3)}} D \log T$$

intersections of X_J with algebraic curves of degree at most d. The number of points in each of these intersections is at most

$$2^{r(r-1)/2+1} d \cdot \beta(\alpha+2d\beta)^n \big((2n+1)(\alpha+d\beta)\big)^r,$$

by 3.3. Write this bound as $c_4(n,r,\alpha,\beta)d^{n+r+1}$, where c_4 is, for fixed n,r,α, a polynomial in β of degree $n+r+1$.

Any subinterval J of length $< \frac{1}{T^2}$ contains at most one point of height T, so combining the estimates above we have

$$\begin{aligned} N_K^{\text{size}}(X,T) &\le c_3(n,r,\alpha,\beta)D^{n+r+2} 12(4k+3)\cdot 6^k \cdot \\ &\qquad T^{4(4k+3)/3(d+3)} \cdot D \cdot \log T \cdot c_4(n,r,\alpha,\beta)d^{n+r+1} \\ &\le c_5(n,r,\alpha,\beta)d^{3n+3r+7}\cdot(4k+3)6^k \cdot T^{4(4k+3)/3(d+3)} \cdot \log T. \end{aligned}$$

For fixed n,r,α we can express c_5 as a polynomial in β of degree $2(n+r+1)$.

Taking $d = [\log T]$, where $[\cdot]$ is the integer part, the expression $T^{4(4k+3)/3(d+3)}$ is bounded by some $c_1(k)$ and we have

$$N_K^{\text{size}}(X,T) \le c_1(k)c_2(n,r,\alpha,\beta)(\log T)^{3n+3r+8},$$

where the constants have the required properties. □

Now suppose that $\phi : I \to \mathbb{R}$ is existentially definable in $\mathbb{R}_{\text{Pfaff}}$. Then by Wilkie's method (see for example [6]) there are intervals $I_1,\ldots,I_m \subseteq I$ which cover I up to a finite set, such that on each interval I_i the function ϕ is implicitly defined. We can then apply the proposition above to each restriction in turn. This proves the following.

Theorem 4.4. *Suppose that $I \subseteq \mathbb{R}$ is an interval and that $\phi : I \to \mathbb{R}$ is existentially definable in $\mathbb{R}_{Pfaff}$ and transcendental. Then there exist c,γ, depending only on ϕ, such that*

$$N_K^{size}(X,T) \le c_1 c(\log T)^\gamma.$$

In particular, the result applies to any unary function definable in a model complete reduct of $\mathbb{R}_{\text{Pfaff}}$.

Corollary 4.5. *Suppose that $\tilde{\mathbb{R}} = \langle \bar{\mathbb{R}}, f_1,\ldots,f_r\rangle$ is a model complete expansion of the real field by a Pfaffian chain on $\mathbb{R}^n$. Suppose that $I \subseteq \mathbb{R}$ is an interval and that $\phi : I \to \mathbb{R}$ is a transcendental function definable in $\tilde{\mathbb{R}}$. Then there exist c,γ, depending only on ϕ, such that*

$$N_K^{size}(X,T) \le c_1 c(\log T)^\gamma.$$

Combining this Corollary with Wilkie's model completeness result for the real exponential field ([16]), we see that Wilkie's conjecture holds for one dimensional sets.

5. Surfaces

This section contains the main results of the paper. As mentioned in the introduction, we need a certain kind of parameterization. Before introducing this, we first recall some standard notation. Given a function $\phi : U \to \mathbb{R}$ on some open set $U \subseteq \mathbb{R}^n$, and a mutiindex $\alpha \in \mathbb{N}^n$, we let $|\alpha| = \sum_{i=1}^n \alpha_i, \alpha! = \prod_{i=1}^n \alpha_i!$ and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We need some definitions and results due to Pila ([10]) (for a discussion of the relationship between mildness and the similar notion of Gevrey see the introduction of [5]).

Definition 5.1. Let $A > 0, C \geq 0$. A function $\phi : (0,1)^l \to (0,1)$ is said to be (A,C)-*mild* if it is smooth and for all $\alpha \in \mathbb{N}^l$ and $\bar{x} \in (0,1)^l$,

$$|D^\alpha \phi(\bar{x})| \leq \alpha!(A|\alpha|^C)^{|\alpha|}.$$

We call a map $\theta : (0,1)^l \to (0,1)^n$ (A,C)-*mild* if each of its coordinates functions is (A,C)-mild. Finally, we say that a set $X \subseteq (0,1)^n$, definable in some o-minimal structure, admits a (J,A,C)-*mild parameterization* if there exist (A,C)-mild maps $\theta_1, \dots, \theta_J : (0,1)^{\dim X} \to (0,1)^n$ such that

$$X = \bigcup_{i=1}^J \theta_i\big((0,1)^{\dim X}\big).$$

The following result links mild parameterizations and rational points.

Theorem 5.2 (Pila, [10])**.** *Suppose that $X \subseteq (0,1)^3$ has dimension 2 and admits a (J,A,C)-mild parameterization. Let k be a positive integer and $K \subseteq \mathbb{R}$ a number field of degree k over $\mathbb{Q}$. Then $Y^{size}(K,T)$ is contained in the union of at most*

$$Jc_6^r A^{c_7} (\log T)^{c_8 C}$$

intersections of Y with algebraic surfaces of degree $[(\log T)^2]$, for some absolute constants c_6, c_7, c_8.

In order be able to use this theorem, we shall need to study the intersection of definable surfaces with algebraic surfaces. So, suppose that $f : U \to \mathbb{R}$ is an implicitly defined function on an open analytic cell $U \subseteq \mathbb{R}^2$, and that the definition has complexity (n,r,α,β). That is, there exist $n \geq 1$, Pfaffian functions $p_1, \dots, p_n : \mathbb{R}^{2+n} \to \mathbb{R}$ and functions $f_1, \dots, f_n : U \to \mathbb{R}$ such that $f = f_1$ and

$$\begin{aligned} p_i(x_1, x_2, f_1(x_1,x_2), \dots, f_n(x_1,x_2)) &= 0 \text{ for } i = 1, \dots, n, \\ \det\Big(\frac{\partial(p_1,\dots,p_n)}{\partial(x_3,\dots,x_{n+2})}\Big)(x_1, x_2, f_1(x_1,x_2), \dots, f_n(x_1,x_2)) &\neq 0, \end{aligned}$$

and the functions $p_1, \dots, p_n$ have a common chain, order r and degree (α,β). We also assume that $\det\Big(\frac{\partial(p_1,\dots,p_n)}{\partial(x_3,\dots,x_{n+2})}\Big)$ is non-vanishing at all points of $V(p_1,\dots,p_n)$. This can be arranged by adding a further equation

$$\det\Big(\frac{\partial(p_1,\dots,p_n)}{\partial(x_3,\dots,x_{n+2})}\Big)(x_1,\dots,x_{n+2}) \cdot x_{n+3} - 1 = 0$$

and so increasing n by 1. We fix $k \in \mathbb{N}$ and suppose that $K \subseteq \mathbb{R}$ is a number field of degree k. Let $X = \operatorname{graph}(f)$.

Proposition 5.3. *There exist positive integers $\gamma(n,r), N(n,r)$ and a polynomial $Q \in \mathbb{R}[X]$ of degree N with coefficients depending only on n, r, α, β such that for all $d \in \mathbb{N}$ if $P : \mathbb{R}^3 \to \mathbb{R}$ is a polynomial of degree d then*

$$N_K^{size}((X \cap V(P))^{trans}, T) \leq c_1(k)Q(d)(\log T)^\gamma$$

Proof. Fix a polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of degree d. First suppose that $\dim X \cap V(P) = 2$. Then there is some open set $U_0 \subseteq U$ such that $P(x_1, x_2, f(x_1, x_2)) = 0$ for all $\langle x_1, x_2 \rangle \in U_0$. This equation extends to all of U so that in this case f is algebraic, and the transcendental part of X (and so of $X \cap V(P)$) is empty.

So we can suppose that the dimension of $X \cap V(P)$ is at most one. Let $\tilde{P} : \mathbb{R}^{2+n} \to \mathbb{R}$ be defined by $\tilde{P}(\bar{x}) = P(x_1, x_2, x_3)$, so $\tilde{P}$ is a polynomial of degree d and can be considered as a Pfaffian function of order 0 and degree $(1, d)$. We apply 2.4 to find a stratification of $V(p_1, \ldots, p_n) \cap V(\tilde{P})$, with at most

$$(n+1)^{n+2} \cdot B \tag{6}$$

strata, and with each stratum an elementary Pfaffian set defined by Pfaffian functions with the same chain as $p_1, \ldots, p_n$ and with degree (α, B). Recall that

$$B = (\alpha + \max\{\beta, d\} + 1)^{(r+2)^{c_9(n+2)}},$$

for an absolute constant c_9. Note that B is a polynomial in d, of degree $(r+2)^{c_9(n+2)}$ with coefficients depending only on n, r, α and β.

We count the points on each stratum, so fix a stratum Y. Since the projection map π from $V(p_1, \ldots, p_n)$ to X is finite-to-one, the dimension of Y is at most one. If $\dim Y = 0$ then Y is contained in a finite Pfaffian variety defined by functions of order r and degree (α, B), and $\#\pi(Y) \leq \#Y$. By 2.3, this latter will certainly be bounded by the bounds below. Suppose that $\dim Y = 1$. Then, as Y is a stratum in a weak elementary stratification, there are Pfaffian functions $h_1, \ldots, h_{n+1} : \mathbb{R}^{n+2} \to \mathbb{R}$ with the same chain as $p_1, \ldots, p_n$ and with degree (α, B) such that these functions vanish identically along Y and such that $dh_1 \wedge \cdots \wedge dh_{n+1} \neq 0$ at every point of Y. In particular, Y is a (not necessarily connected) embedded submanifold of $\mathbb{R}^{n+2}$. Let

$$Y_1 = \Big\{\bar{x} \in Y : \det\Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_2, \ldots, x_{n+2})}\Big)(\bar{x}) = 0\Big\},$$

$$Y_2 = \Big\{\bar{x} \in Y : \det\Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_1, x_3, \ldots, x_{n+2})}\Big)(\bar{x}) = 0\Big\}.$$

We look separately at $\pi(Y \setminus Y_1), \pi(Y \setminus Y_2)$ and $\pi(Y_1 \cap Y_2)$. The method for $\pi(Y \setminus Y_1)$ and $\pi(Y \setminus Y_2)$ is the same, so we only give the details for $\pi(Y \setminus Y_1)$. Each connected component of $Y \setminus Y_1$ is the graph of some analytic map from I to $\mathbb{R}^{n+1}$, for some open interval $I \subseteq \mathbb{R}$. Rather than estimating the number of components of $Y \setminus Y_1$ we in fact work with a slightly larger set. So, let

$$Y' = \Big\{\bar{x} \in \mathbb{R}^{n+2} : h_1(\bar{x}) = \cdots = h_{n+1}(\bar{x}) = 0, \det\Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_2, \ldots, x_{n+2})}\Big)(\bar{x}) \neq 0\Big\}.$$

Clearly $Y \setminus Y_1 \subseteq Y'$ and again, each connected component of Y' is the graph of some analytic map defined on an open interval. Let Z be a connected component of

$Y \setminus Y_1$. Note that $\pi(Z)$ is contained in the algebraic part of $X \cap V(P)$ if and only if the first two coordinate functions of the map corresponding to Z are algebraic. By analytic continuation, this holds if and only if the first two coordinate functions of the map corresponding to Z' are algebraic, where Z' is the connected component of Y' containing Z. So we may work with the connected components of Y' rather than the connected components of $Y \setminus Y_1$. Since

$$\begin{aligned} Y' \quad = \quad & \Big\{ \bar{x} \in \mathbb{R}^{n+2} : \exists z (h_1(\bar{x}) = \cdots = h_{n+1}(\bar{x}) \\ & \qquad = \det \Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_2, \ldots, x_{n+2})} \Big)(\bar{x}) z - 1 = 0) \Big\}, \end{aligned}$$

Y' is the projection of a Pfaffian variety in $\mathbb{R}^{n+3}$ defined by functions with a common chain of order r and degree bounded by $(\alpha, (n+1)(\alpha+B))$. Therefore Khovanskii's theorem (2.3) implies that Y' has at most

$$2^{r(r-1)/2+1}(\alpha + 2(n+1)(\alpha + B))^{n+3}((2n+5)(\alpha + (n+1)(\alpha + B))^r \tag{7}$$

connected components. Write this quantity as $c_{10}(n, r, \alpha, B)$ and note that it is a polynomial in B of degree $n + r + 3$ with coefficients depending only on n, r, α and β.

Fix a component Z' of Y'. Then $\pi(Z')$ has the form $\{\langle x_1, \phi(x_1), \psi(x_1)\rangle : x_1 \in I\}$ for some open interval $I \subseteq \mathbb{R}$ and analytic functions $\phi, \psi : I \to \mathbb{R}$. If $\pi(Z') \cap X$ is not contained in the algebraic part of X then at least one of the functions ϕ, ψ is transcendental, say ϕ. Now, ϕ is implicitly defined by the functions $h_1, \ldots, h_{n+1}$. These functions have the same chain as $p_1, \ldots, p_n$ and have degree (α, B). So the implicit definition has complexity $(n + 1, r, \alpha, B)$, and we can apply 4.3 and obtain

$$N_K^{\text{size}}(\text{graph}(\phi), T) \leq c_1(k)c_2(n + 1, r, \alpha, B)(\log T)^\gamma$$

where $\gamma = 3(n + r) + 11$ depends only on n and r, and c_2 is polynomial in B with degree $2(n+r+2)$ and coefficients depending only on n, r, α and β. So we certainly have

$$N_K^{\text{size}}(\pi(Z'), T) \leq c_1 c_2 (\log T)^\gamma.$$

Applying this bound to each connected component of Y', and using (7), we have

$$N_K^{\text{size}}(\pi(Y')^{trans}, T) \leq c_{10} c_1 c_2 (\log T)^\gamma \tag{8}$$

and so

$$N_K^{\text{size}}(\pi(Y \setminus Y_1)^{trans}, T) \leq c_{10} c_1 c_2 (\log T)^\gamma. \tag{9}$$

Repeating the above with Y_2 in place of Y_1 we have

$$N_K^{\text{size}}((\pi(Y \setminus Y_1) \cup \pi(Y \setminus Y_2))^{trans}, T) \leq 2c_{10} c_1 c_2 (\log T)^\gamma. \tag{10}$$

To finish bounding the points coming from $\pi(Y)$, we need to consider $\pi(Y_1 \cap Y_2)$. By Sard's theorem and o-minimality, the set $Y_1 \cap Y_2$ is finite. The number of points in $Y_1 \cap Y_2$ is bounded by the number of connected components of

$$\begin{aligned} & \Big\{ \bar{x} \in \mathbb{R}^{n+2} : h_1(\bar{x}) = \cdots = h_{n+1}(\bar{x}) = \\ & \qquad \det \Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_2, \ldots, x_{n+2})} \Big)(\bar{x}) = \det \Big(\frac{\partial(h_1, \ldots, h_{n+1})}{\partial(x_1, x_3, \ldots, x_{n+2})} \Big)(\bar{x}) = 0 \Big\}. \end{aligned} \tag{11}$$

This is a variety defined by Pfaffian functions with a common chain, order r, and degree at most $(\alpha, (n+1)(\alpha+B))$ and so, by 2.3, has at most $c_{11}(n, r, \alpha, B)$ connected components, where c_{11} is polynomial in B of degree $n+r+2$ with coefficients depending only on n, r, α, β. Combining this bound with those above, we have

$$N_K^{\text{size}}(\pi(Y), T) \leq 3 \cdot c_{11}c_{10}c_1c_2(\log T)^\gamma.$$

Finally, we apply this method to each of the $(n+1)^{(n+2)}B$ strata of our stratification to obtain

$$N_K^{\text{size}}((X \cap V(P))^{trans}, T) \leq 3(n+1)^{(n+2)}Bc_{11}c_{10}c_1c_2(\log T)^\gamma.$$

As observed above, c_2, c_{10} and c_{11} are polynomials in B of degrees $2(n+r+2), n+r+3$ and $n+r+2$, respectively, and all have coefficients which depend only on n, r, α and β. Now, B itself is a polynomial in d, of degree

$$(r+2)^{c_9(n+2)},$$

again with coefficients depending only on n, r, α and β. It follows that

$$3(n+1)^{(n+2)}Bc_{11}c_{10}c_2$$

is a polynomial in d, with degree N depending only on n and r and coefficients depending only on n, r, α and β. So we are done, taking Q to be this polynomial. $\square$

We know have everything in place to prove our main result.

Theorem 5.4. *Suppose that $\tilde{\mathbb{R}} = \langle \bar{\mathbb{R}}, f_1, \ldots, f_r \rangle$ is a model complete expansion of the real field by a Pfaffian chain on $\mathbb{R}^n$. If $X \subseteq (0,1)^3$ is definable in $\tilde{\mathbb{R}}$, has dimension 2 and admits a mild parameterization, then there are $c(X)$ and $\gamma(X)$ such that*

$$N_K^{size}(X^{trans}, T) \leq c_1c(\log T)^\gamma.$$

Proof. The structure $\tilde{\mathbb{R}}$ has analytic cell decomposition (in fact, model completeness is unnecessary for this, see [13]) so we can take a cell decomposition $C_1, \ldots, C_m$ of X. By model completeness, each of these cells is existentially definable and so we may assume, perhaps after decomposing further, and perhaps permuting coordinates (which has no effect on algebraic points or the transcendental part), that each of the two dimensional cells is the graph of an implicitly defined function defined on an open cell in $\mathbb{R}^2$. Suppose that the cells are numbered so that $C_1, \ldots, C_l$ are two-dimensional and $C_{l+1}, \ldots, C_m$ are at most one-dimensional. By 4.5 there exist c', γ' such that

$$N_K^{\text{size}}((C_{l+1} \cup \cdots \cup C_m)^{trans}, T) \leq c'(\log T)^{\gamma'}$$

Now, fix n, α and β such that the implicit definitions of the functions giving the cells $C_1, \ldots, C_l$ have complexity bounded by (n, r, α, β). Let γ, N and Q be as in the previous proposition. Now, suppose that X admits a (J, A, C)-mild parameterization, for some $J, A > 0$ and $C \geq 0$. Fix $T > e$. By 5.2, the set $X^{\text{size}}(K, T)$ is contained in the union of at most

$$M(T) = Jc_6^rA^{c_7}(\log T)^{c_8C}$$

intersections of X with algebraic surfaces of degree at most $[(\log T)^2]$. Let $P_1, \dots, P_{M(T)}$ be polynomials of degree at most $[(\log T)^2]$ defining these surfaces. By the previous proposition applied to each cell, we have

$$N_K^{\text{size}}(((C_1 \cup \cdots \cup C_l) \cap V(P_i))^{trans}, T) \leq lc_1 Q([(\log T)^2])(\log T)^\gamma$$

for each $i = 1, \dots, M(T)$. So we have

$$N_K^{\text{size}}(X^{trans}, T) \leq M(T) \cdot lc_1 Q([(\log T)^2])(\log T)^\gamma + c'(\log T)^{\gamma'}$$

which has the required form. □

In particular, this shows that if a surface definable in $\mathbb{R}_{\exp}$ admits a mild parameterization then Wilkie's conjecture holds for that surface. In the reduct $\mathbb{R}_{\text{resPfaff}}$ of $\mathbb{R}_{\text{Pfaff}}$ defined in the introduction, we know that mild parameterizations exist and so we obtain a result for arbitrary definable surfaces.

Corollary 5.5. *Suppose that $X \subseteq \mathbb{R}^3$ is definable in $\mathbb{R}_{resPfaff}$ and has dimension 2. Then there exist $c(X)$ and $\gamma(X)$ such that*

$$N_K^{size}(X^{trans}, T) \leq c_1 c(\log T)^\gamma.$$

Proof. We can assume that $X \subseteq (0,1)^3$. Since X is subanalytic, we can use uniformization to find a mild parameterization of X. In fact we can also take the parameterizing maps to be definable in $\mathbb{R}_{\text{resPfaff}}$ (see [5] for the details of this). Now, X is definable in some expansion of $\bar{\mathbb{R}}$ by a restricted Pfaffian chain, and such expansions are model complete ([16]). So we can apply the theorem to finish. □

We conclude with an example of how to obtain a transcendence result using our theorem. As the result is (a rather weak case of something) known, we shall be brief. Given real α and β, consider the surface $X = \{\langle x, y, z\rangle \in (0,\infty)^3 : z = x^\alpha y^\beta\}$. We assume that $1, \alpha$ and β are linearly independent over $\mathbb{Q}$. It is not hard to see that this implies that X^{alg} is empty. Now, Pila ([8]) has shown that the function $t \mapsto \exp(-1/t)$ is mild and it follows that the map

$$\begin{array}{ccc} (0,1)^2 & \to & (0,1)^3 \\ \langle u, v\rangle & \mapsto & \langle \exp(-1/\alpha u), \exp(-1/\beta v), \exp(-1/u) \cdot \exp(-1/v)\rangle \end{array}$$

is mild. The image of this map is an open subset X' of X. Clearly X' is definable in $\mathbb{R}_{\exp}$ and so we can apply 5.4 to obtain $c(X')$ and $\gamma(X') \in \mathbb{N}$ such that

$$\#X'(K, T) \leq c_1(k)c(\log T)^\gamma$$

for any number field $K \subseteq \mathbb{R}$ of degree k. Now, using the group structure on X' and that fact that γ is independent of k, we obtain the following.

Proposition 5.6. *Suppose that $x_1, \dots, x_\gamma, y_1, \dots, y_\gamma$ are two multiplicatively independent tuples of positive real numbers. Then at least one of the numbers $x_i^\alpha y_i^\beta$ is transcendental.*

As mentioned above, this result it not new. In fact, much more precise results (in an arbitrary number of variables) have been obtained by Waldschmidt ([15]).

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