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Jones, G. O. and Miller, D. J. and Thomas, M. E. M.

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MILDNESS AND THE DENSITY OF RATIONAL POINTS ON CERTAIN TRANSCENDENTAL CURVES

G. O. JONES, D. J. MILLER, AND M. E. M. THOMAS

Abstract. We use a result due to Rolin, Speissegger and Wilkie to show that definable sets in certain o-minimal structures admit definable parameterizations by mild maps. We then use this parameterization to prove a result on the density of rational points on curves defined by restricted Pfaffian functions.

The main result of this note is a generalization of some results of Pila ([6]) to a wider collection of curves. Before stating the result, we need some definitions. A sequence \( f_1, \ldots, f_r : U \to \mathbb{R} \) of analytic functions on an open set \( U \subseteq \mathbb{R}^n \) is said to be a Pfaffian chain of order \( r \) and degree \( \alpha \) if there are polynomials \( P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{n+j}] \) of degree at most \( \alpha \) such that

\[
df_j = \sum_{i=1}^{n} P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x})) \, dx_i.
\]

Given such a chain, we say that a function \( f : U \to \mathbb{R} \) is Pfaffian of order \( r \) and degree \((\alpha, \beta)\) with chain \( f_1, \ldots, f_r \), if there is a polynomial \( P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r] \) of degree at most \( \beta \) such that

\[
f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x})).
\]

Let \( U \subseteq \mathbb{R}^n \) be an open set containing \([0,1]^n\). To every function \( f : U \to \mathbb{R} \), we associate a new function \( \hat{f} : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
\hat{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in [0,1]^n, \\ 0 & \text{otherwise}. \end{cases}
\]

Recall that \( \mathbb{R}_{an} \) is the expansion of the real ordered field by all functions of the form \( \hat{f} \), where \( f : U \to \mathbb{R} \) is analytic, \([0,1]^n \subseteq U \) and \( n \geq 1 \). We let \( \mathbb{R}_{resPfaff} \) be the reduct of \( \mathbb{R}_{an} \) containing the real ordered field but in which we only add \( \hat{f} \) for \( f : U \to \mathbb{R} \) Pfaffian.

For \( q \in \mathbb{Q} \), the height of \( q \) is \( H(q) = \max\{|a|, b| \), where \( a, b \in \mathbb{Z}, b \geq 1, \gcd(a, b) = 1 \). The height of \( \bar{q} \in \mathbb{Q}^n \), again written \( H(\bar{q}) \), is defined as the maximum of the heights of the coordinates of \( \bar{q} \). For a set \( X \subseteq \mathbb{R}^n \) and \( H \geq 1 \), we let

\[
X(\mathbb{Q}, H) = \{ \bar{q} \in X \cap \mathbb{Q}^n : H(\bar{q}) \leq H \}.
\]

Proposition 0.1. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a transcendental analytic function definable in \( \mathbb{R}_{resPfaff} \), and let \( X = \text{graph}(f) \). Then there exist \( c > 0 \) and \( \gamma > 0 \) such that for \( H \geq 3 \)

\[
\#X(\mathbb{Q}, H) \leq c(\log H)^\gamma.
\]

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When $f$ is Pfaffian, and not assumed to be definable in $\mathbb{R}_{\text{realPfaff}}$, this result is due to Pila ([6]). The extra generality here is to include functions implicitly defined by restricted Pfaffian functions.

The proof of the proposition is a modification of the proof in [5]. To this end, we need a parameterization result which, although a simple consequence of a result from [8], may be of some independent interest. We need two further definitions, the first of which is from [7].

**Definitions 0.1.** A smooth function $\phi : (0, 1)^k \to (0, 1)$ is said to be $(A, C)$-mild if

$$|D^\alpha \phi(\bar{x})| \leq \alpha! (|A|^C)^{\alpha}$$

for all $\alpha \in \mathbb{N}^k$ and all $\bar{x} \in (0, 1)^k$. We say that a map $\Phi : (0, 1)^k \to (0, 1)^n$ is $(A, C)$-mild if each of its coordinate functions is $(A, C)$-mild.

**Definitions 0.2.** Fix an o-minimal structure $\tilde{\mathbb{R}}$ expanding the real field, and let $X \subseteq \mathbb{R}^n$ be definable. A parameterization of $X$ is a finite set $S$ of definable maps $\Phi_1, \ldots, \Phi_l : (0, 1)^{\dim X} \to \mathbb{R}^n$ such that $X = \bigcup \text{Im}(\Phi_i)$. A parameterization is said to be $(A, C)$-mild if each of the parameterizing maps is $(A, C)$-mild. We say that $\tilde{\mathbb{R}}$ admits $C$-mild parameterization if for every definable set $X \subseteq (0, 1)^n$ there is an $(A, C)$-mild parameterization of $X$, for some $A$ (depending on $X$).

**Proposition 0.2.** Any reduct of $\mathbb{R}_{\text{an}}$ expanding the real ordered field admits 0-mild parameterization.

We start by deriving this result from results in [8], via a more general notion of parameterization. We then prove the main result in section 2.

1. **$C$-Parameterization**

In this section we observe that the results in [8] imply a parameterization result. So, we work in the setting of [8], and fix, for every compact box $B \subseteq \mathbb{R}^n$ and every $n \in \mathbb{N}$, an $\mathbb{R}$-algebra $C_B$ of functions $f : B \to \mathbb{R}$ such that the following hold.

- **(C1)** Each of the projection functions $\langle x_1, \ldots, x_n \rangle \mapsto x_i$, restricted to $B$, is in $C_B$, and for every function $f \in C_B$ the restriction of $f$ to the interior of $B$ is smooth.
- **(C2)** If $B' \subseteq \mathbb{R}^n$ is a compact box and $g_1, \ldots, g_n \in C_{B'}$ are such that $g(B') \subseteq B$, where $g = \langle g_1, \ldots, g_n \rangle$, then for every $f \in C_B$, the composition $f \circ g$ is in $C_{B'}$.
- **(C3)** For every compact box $B' \subseteq B$ and function $f \in C_B$, the restriction of $f$ to $B'$ is in $C_{B'}$. For every $f \in C_B$ there is a compact box $B' \subseteq \mathbb{R}^n$, the interior of which contains $B$, and a function $g \in C_{B'}$ such that $g|_B = f$.
- **(C4)** For every $f \in C_B$ and $i = 1, \ldots, n$, the partial derivative $\frac{df}{dx_i}$ is in $C_B$.

Note that the partial derivatives in (C4) exists by (C1) and (C3). Since we shall not need the precise statements of the remaining assumptions, we only state rough versions of them. The full details can be found in [8].

- **(C5)** For each $n \geq 1$ and each box $B \in \mathbb{R}^n$ containing the origin, the collection of germs at the origin of functions in $C_B$ forms a quasianalytic class.
- **(C6)** This collection of germs is closed under extraction of implicit functions.
This collection of germs is closed under monomial division.

The example which will interest us is as follows. Suppose that \( \mathbb{R} \) is a polynomially bounded o-minimal expansion of the real field. For each compact box, let \( C_B \) be the collection of definable smooth functions \( f : B \to \mathbb{R} \). By well known properties of o-minimal structures ([2],[4]) these algebras satisfy the above requirements. In particular, if \( \mathbb{R} \) is a reduct of \( \mathbb{R}_{an} \), then each function \( f \) in \( C_B \) is the restriction to \( B \) of an analytic function defined in a neighborhood of \( B \), and hence there exist positive constants \( A \) and \( K \) such that

\[
|D^n f(x)| \leq \alpha! KA^{[\alpha]}
\]

for all \( \alpha \in \mathbb{N}^n \).

We now recall some further definitions from [8].

Given a polyradius \( \bar{r} = (r_1, \ldots, r_n) \in (0, \infty)^n \) we let \( I_{\bar{r}} = \prod (-r_i, r_i) \) and let \( I_{\bar{r}} \) be the topological closure of \( I_{\bar{r}} \). Write \( C_{n, \bar{r}} \) for \( C_{I_{\bar{r}}} \).

**Definition 1.1.** A set \( A \subseteq \mathbb{R}^n \) is called a **basic \( C \)-set** if there are \( \bar{r} \in (0, \infty)^n \) and \( f, g_1, \ldots, g_k \in C_{n, \bar{r}} \) such that

\[
A = \{ \bar{x} \in I_{\bar{r}} : f(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_k(\bar{x}) > 0 \}.
\]

A finite union of basic \( C \)-sets is called a \( C \)-set. A set \( A \subseteq \mathbb{R}^n \) is called **\( C \)-semianalytic** if for every \( \bar{a} \in \mathbb{R}^n \) there is an \( \bar{r} \in (0, \infty)^n \) such that

\[
(A - \bar{a}) \cap I_{\bar{r}}
\]

is a \( C \)-set. If \( A \) is also a manifold, we call \( A \) a \( C \)-semianalytic manifold.

Given \( m \leq n \) and an injective \( \lambda : \{1, \ldots, m\} \to \{1, \ldots, n\} \), we write \( \pi_{\lambda} : \mathbb{R}^n \to \mathbb{R}^m \) for the projection \( \bar{x} \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(m)}) \).

**Definition 1.2.** Let \( \bar{r} \in (0, \infty)^n \). A set \( M \subseteq I_{\bar{r}} \) is said to be **\( C \)-trivial** if one of the following holds:

(i) \( M = \{ \bar{x} \in I_{\bar{r}} : x_i \square_1 0, \ldots, x_n \square_0 0 \} \), where \( \square_i \in \{<,=,>\} \) for each \( i \);

(ii) there exist a permutation \( \lambda \) of \( \{1, \ldots, n\} \), a \( C \)-trivial \( N \subseteq I_{\bar{r}} \) and a \( g \in C_{n-1, s} \), where \( \bar{s} = (r_{\lambda(1)}, \ldots, r_{\lambda(n-1)}) \), such that \( g(I_{\bar{s}}) \subseteq (-r_{\lambda(n)}, r_{\lambda(n)}) \) and \( \pi_{\lambda}(M) = \text{graph}(g\vert_N) \).

Note that \( C \)-trivial sets are necessarily manifolds; we shall refer to them as \( C \)-trivial manifolds. A \( C \)-semianalytic manifold \( m \subseteq \mathbb{R}^n \) is called **trivial** if there exist \( \bar{a} \in \mathbb{R}^n \) and a \( C \)-trivial manifold \( N \subseteq \mathbb{R}^n \) such that \( M = N + \bar{a} \).

We need the following results which are due to Rolin, Speissegger and Wilkie.

**Theorem 1.3.** ([8, 4.7]) Suppose that \( A \subseteq \mathbb{R}^n \) is a bounded \( C \)-semianalytic set and that \( k \leq n \). Then there are trivial \( C \)-semianalytic manifolds \( N_i \subseteq \mathbb{R}^n \) for some \( n_i \geq n, i = 1, \ldots, J \), such that

\[
\pi_k(A) = \pi_k(N_1) \cup \cdots \cup \pi_k(N_J)
\]

where \( \pi_k\vert_{N_i} \) is an immersion, for each \( i \). (Here, \( \pi_k \) is projection onto the first \( k \) coordinates.)

Let \( \mathbb{R}_C \) be the expansion of \( \mathbb{R} \) by all functions \( \hat{f} \), for \( f \in C_{n, \bar{r}}, n \in \mathbb{N}, \bar{r} \in (0, \infty)^n \), where \( \hat{f} \) is as defined in the introduction.
Theorem 1.4. ([8, 5.2]) The structure \( \mathbb{R}_C \) is o-minimal, model complete and polynomially bounded.

We now use these results to prove a parameterization result. We work in the structure \( \mathbb{R}_C \).

Definitions 1.1. Let \( X \subseteq \mathbb{R}^n \) be definable. A \( C \)-parameterization of \( X \) is a finite set \( S \) of maps \( \phi_1, \ldots, \phi_l \in C_{[0,1]} \) such that \( \{ \phi_i|_{(0,1)^{\dim X}} : i = 1, \ldots, l \} \) is a parameterization of \( X \).

Lemma 1.5. Suppose that \( M \subseteq \mathbb{R}^n \) is a \( C \)-trivial manifold. Then there is a \( C \)-parameterization \( S \) of \( M \) with \( \#S = 1 \).

Proof. This follows from the definitions by induction on \( n \). \( \square \)

Proposition 1.6. Suppose that \( X \subseteq \mathbb{R}^n \) is a bounded definable set. Then \( X \) has a \( C \)-parameterization.

Proof. By model completeness, there is an \( m \geq 0 \) and a quantifier-free definable set \( A \subseteq \mathbb{R}^{n+m} \) such that \( X = \pi(A) \). Using the fact that \( \mathbb{R}_C \) is an expansion of the real field, we may assume that \( A \) is bounded and that \( A \) is \( C \)-semianalytic. By Theorem 1.3

\[
X = \pi(N_1) \cup \cdots \cup \pi(N_k)
\]

for some \( C \)-trivial manifolds \( N_1, \ldots, N_k \), were each \( \pi|_{N_i} \) is a an immersion. Thus \( \dim(X) = \max\{\dim(N_1), \ldots, \dim(N_k)\} \). A \( C \)-parameterization of \( X \) can be constructed by composing the functions in the \( C \)-parameterizations of each of the \( N_i \) with the projections \( \pi \), and then trivially extending any of these functions to \( (0,1)^{\dim X} \) if their domain is \( (0,1)^{\dim N_i} \) with \( \dim N_i < \dim(X) \). \( \square \)

Note that Proposition 0.2 follows immediately, by applying the above to the given reduct of \( \mathbb{R}_{an} \).

2. Curves

We now prove Proposition 0.1. In fact, we prove a result about the number of points in a fixed number field \( k \subseteq \mathbb{R} \) of degree \( l \). We use the absolute multiplicative height \( H \) on \( k \), which agrees with the height on \( \mathbb{Q} \) given in the introduction (for the definition of \( H \), see [1]). For \( X \subseteq \mathbb{R}^n \) and \( H \geq 1 \), we let \( X(k,H) = X \cap \{ \bar{a} \in k^n : H(\bar{a}) \leq H \} \). The following is a special case of [7, Corollary 3.3].

Proposition 2.1. Suppose that \( X \subseteq (0,1)^2 \) has dimension 1 and that \( S \) is an \( (A,0) \)-mild parameterization of \( X \). Then there is an absolute constant \( c_0 \) such that \( X(k,H) \) is contained in a union of at most

\[
\#S \cdot c_0 \cdot A^{2(1+\alpha(1))}
\]

intersections of \( X \) with algebraic curves of degree \( [l \cdot \log H] \). Here the \( 1 + \alpha(1) \) is taken as \( H \to \infty \) with absolute implied constant, and \( [\cdot] \) denotes integer part.

Given a function \( F : \mathbb{R}^m \to \mathbb{R} \), we let \( V(F) = \{ \bar{x} \in \mathbb{R}^m : F(\bar{x}) = 0 \} \).
Lemma 2.2. Suppose that \( f : (0, 1) \to (0, 1) \) is a transcendental analytic function definable in \( \mathbb{R}_{\text{resPfaff}} \). Suppose further that \( \text{graph}(f) = \pi(V(F)) \) where \( F : \mathbb{R}^{2+n} \to \mathbb{R} \) is a Pfaffian function of order \( r \) and degree \((\alpha, \beta)\), where \( \pi \) is the projection on to the first two coordinates. If \( P : \mathbb{R}^2 \to \mathbb{R} \) is a polynomial of degree \( d \) then

\[
\#(\text{graph}(f) \cap V(P)) \leq 2^{r(r+1)/2 + 1} (n + 2)^r (\alpha + 2d')^{n+r+2}
\]

where \( d' = \max\{d, \beta\} \).

Proof. Let \( \tilde{P} : \mathbb{R}^{2+n} \to \mathbb{R} \) be given by \( \tilde{P}(x, y, z) = P(x, y) \). Then \( \text{graph}(f) \cap V(P) = \pi(V(F) \cap V(\tilde{P})) \). The number of points in \( \text{graph}(f) \cap V(P) \) is thus bounded by the number of connected components of \( V(f) \cap V(\tilde{P}) \) (there are only finitely many points as we have assumed that \( f \) is transcendental). By Kovanskii’s theorem (as presented in \([3, 3.3]\)) there are at most

\[
2^{r(r-1)/2 + 1} d'(\alpha + 2d' - 1)^{n+1} ((2(n + 2) - 1)(\alpha + d') - 2n - 2)^r
\]

such components, and clearly this is less than the right hand side of (1). \( \square \)

Proposition 2.3. Suppose that \( f : (0, 1) \to (0, 1) \) is a transcendental analytic function definable in \( \mathbb{R}_{\text{resPfaff}} \) and let \( X = \text{graph}(f) \). Then there are \( c, \gamma > 0 \) such that (for \( H \geq c \))

\[
\#X(k, H) \leq c(\log H)^\gamma.
\]

Proof. By model completeness of \( \mathbb{R}_{\text{resPfaff}} \) (see \([9]\)) we may suppose that \( X = \pi(V(F)) \) for some Pfaffian function \( F : \mathbb{R}^{2+n} \to \mathbb{R} \) and some \( n \geq 0 \). Suppose that \( F \) has order \( r \) and degree \((\alpha, \beta)\). By Proposition 0.2 we can take an \((A, 0)\)-mild parameterization \( S \) of \( X \), for some \( A \). Combining Proposition 2.1 with Lemma 2.2 (with \( d = \lfloor l \log H \rfloor \)), we have

\[
\#X(k, H) \leq \#S \cdot c_0^r \cdot A^{2(1+o(1))} 2^{r(r+1)/2 + 1} (n + 2)^r (\alpha + 2 \max\{\beta, d\})^{n+r+2} \leq c(\log H)^\gamma
\]

where \( \gamma = n + r + 2 \). \( \square \)

References


E-mail address: gareth.jones-3@manchester.ac.uk

E-mail address: dmille10@emporia.edu

E-mail address: margaret.thomas-2@manchester.ac.uk