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MILDNESS AND THE DENSITY OF RATIONAL POINTS ON CERTAIN TRANSCENDENTAL CURVES

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Abstract. We use a result due to Rolin, Speissegger and Wilkie to show that definable sets in certain o-minimal structures admit definable parameterizations by mild maps. We then use this parameterization to prove a result on the density of rational points on curves defined by restricted Pfaffian functions.

The main result of this note is a generalization of some results of Pila ([6]) to a wider collection of curves. Before stating the result, we need some definitions. A sequence $f_1, \ldots, f_r : U \to \mathbb{R}$ of analytic functions on an open set $U \subseteq \mathbb{R}^n$ is said to be a *Pfaffian chain* of *order* r and *degree* α if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{n+j}]$ of degree at most α such that

$$df_j = \sum_{i=1}^{n} P_{i,j}(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x}))dx_i.$$

Given such a chain, we say that a function $f : U \to \mathbb{R}$ is *Pfaffian of order* r and *degree* (α, β) with chain $f_1, \ldots, f_r$, if there is a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ of degree at most β such that $f(\bar{x}) = P(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$.

Let $U \subseteq \mathbb{R}^n$ be an open set containing $[0,1]^n$. To every function $f : U \to \mathbb{R}$, we associate a new function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in [0,1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbb{R}_{\mathrm{an}}$ is the expansion of the real ordered field by all functions of the form $\hat{f}$, where $f : U \to \mathbb{R}$ is analytic, $[0,1]^n \subseteq U$ and $n \geq 1$. We let $\mathbb{R}_{\mathrm{resPfaff}}$ be the reduct of $\mathbb{R}_{\mathrm{an}}$ containing the real ordered field but in which we only add $\hat{f}$ for $f : U \to \mathbb{R}$ Pfaffian.

For $q \in \mathbb{Q}$, the *height* of q is $H(q) = \max\{|a|, b\}$, where $a, b \in \mathbb{Z}, b \geq 1, \gcd(a,b) = 1$. The height of $\bar{q} \in \mathbb{Q}^n$, again written $H(\bar{q})$, is defined as the maximum of the heights of the coordinates of $\bar{q}$. For a set $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let

$$X(\mathbb{Q}, H) = \{\bar{q} \in X \cap \mathbb{Q}^n : H(\bar{q}) \leq H\}.$$

Proposition 0.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$, and let $X = graph(f)$. Then there exist $c > 0$ and $\gamma > 0$ such that for $H \geq 3$*

$$\#X(\mathbb{Q}, H) \leq c(\log H)^{\gamma}.$$

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When f is Pfaffian, and not assumed to be definable in $\mathbb{R}_{\text{resPfaff}}$, this result is due to Pila ([6]). The extra generality here is to include functions implicitly defined by restricted Pfaffian functions.

The proof of the proposition is a modification of the proof in [5]. To this end, we need a parameterization result which, although a simple consequence of a result from [8], may be of some independent interest. We need two further definitions, the first of which is from [7].

Definitions 0.1. A smooth function $\phi : (0,1)^k \to (0,1)$ is said to be (A,C)-*mild* if

$$|D^\alpha \phi(\bar{x})| \leq \alpha!(A|\alpha|^C)^{|\alpha|}$$

for all $\alpha \in \mathbb{N}^k$ and all $\bar{x} \in (0,1)^k$. We say that a map $\Phi : (0,1)^k \to (0,1)^n$ is (A,C)-*mild* if each of its coordinate functions is (A,C)-mild.

Definitions 0.2. Fix an o-minimal structure $\tilde{\mathbb{R}}$ expanding the real field, and let $X \subseteq \mathbb{R}^n$ be definable. A *parameterization* of X is a finite set $\mathcal{S}$ of definable maps $\Phi_1, \ldots, \Phi_l : (0,1)^{\dim X} \to \mathbb{R}^n$ such that $X = \bigcup \mathrm{Im}(\Phi_i)$. A parameterization is said to be (A,C)-*mild* if each of the parameterizing maps is (A,C)-mild. We say that $\tilde{\mathbb{R}}$ admits C-*mild parameterization* if for every definable set $X \subseteq (0,1)^n$ there is an (A,C)-mild parameterization of X, for some A (depending on X).

Proposition 0.2. *Any reduct of* $\mathbb{R}_{an}$ *expanding the real ordered field admits* 0-*mild parameterization.*

We start by deriving this result from results in [8], via a more general notion of parameterization. We then prove the main result in section 2.

1. $\mathcal{C}$-PARAMETERIZATION

In this section we observe that the results in [8] imply a parameterization result. So, we work in the setting of [8], and fix, for every compact box $B \subseteq \mathbb{R}^n$ and every $n \in \mathbb{N}$, an $\mathbb{R}$-algebra $\mathcal{C}_B$ of functions $f : B \to \mathbb{R}$ such that the following hold.

($\mathcal{C}_1$) Each of the projection functions $\langle x_1, \ldots, x_n \rangle \mapsto x_i$, restricted to B, is in $\mathcal{C}_B$, and for every function $f \in \mathcal{C}_B$ the restriction of f to the interior of B is smooth.

($\mathcal{C}_2$) If $B' \subseteq \mathbb{R}^m$ is a compact box and $g_1, \ldots, g_n \in \mathcal{C}_{B'}$ are such that $g(B') \subseteq B$, where $g = \langle g_1, \ldots, g_n \rangle$, then for every $f \in \mathcal{C}_B$, the composition $f \circ g$ is in $\mathcal{C}_{B'}$.

($\mathcal{C}_3$) For every compact box $B' \subseteq B$ and function $f \in \mathcal{C}_B$, the restriction of f to B' is in $\mathcal{C}_{B'}$. For every $f \in \mathcal{C}_B$ there is a compact box $B' \subseteq \mathbb{R}^n$, the interior of which contains B, and a function $g \in \mathcal{C}_{B'}$ such that $g|_B = f$.

($\mathcal{C}_4$) For every $f \in \mathcal{C}_B$ and $i = 1, \ldots, n$, the partial derivative $\frac{\partial f}{\partial x_i}$ is in $\mathcal{C}_B$.

Note that the partial derivatives in ($\mathcal{C}_4$) exists by ($\mathcal{C}_1$) and ($\mathcal{C}_3$). Since we shall not need the precise statements of the remaining assumptions, we only state rough versions of them. The full details can be found in [8].

($\mathcal{C}_5$) For each $n \geq 1$ and each box $B \in \mathbb{R}^n$ containing the origin, the collection of germs at the origin of functions in $\mathcal{C}_B$ forms a quasianalytic class.

($\mathcal{C}_6$) This collection of germs is closed under extraction of implicit functions.

($\mathcal{C}_7$) This collection of germs is closed under monomial division.

The example which will interest us is as follows. Suppose that $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of the real field. For each compact box, let $\mathcal{C}_B$ be the collection of definable smooth functions $f : B \to \mathbb{R}$. By well known properties of o-minimal structures ([2],[4]) these algebras satisfy the above requirements. In particular, if $\tilde{\mathbb{R}}$ is a reduct of $\mathbb{R}_{\mathrm{an}}$, then each function f in $\mathcal{C}_B$ is the restriction to B of an analytic function defined in a neighborhood of B, and hence there exist positive constants A and K such that

$$|D^\alpha f(x)| \leq \alpha! K A^{|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$.

We now recall some further definitions from [8]. Given a polyradius $\bar{r} = \langle r_1, \ldots, r_n \rangle \in (0, \infty)^n$ we let $I_{\bar{r}} = \prod(-r_i, r_i)$ and let $\bar{I}_{\bar{r}}$ be the topological closure of $I_{\bar{r}}$. Write $\mathcal{C}_{n,\bar{r}}$ for $\mathcal{C}_{\bar{I}_{\bar{r}}}$.

Definition 1.1. A set $A \subseteq \mathbb{R}^n$ is called a *basic $\mathcal{C}$-set* if there are $\bar{r} \in (0, \infty)^n$ and $f, g_1, \ldots, g_k \in \mathcal{C}_{n,\bar{r}}$ such that

$$A = \{\bar{x} \in I_{\bar{r}} : f(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_k(\bar{x}) > 0\}.$$

A finite union of basic $\mathcal{C}$-sets is called a *$\mathcal{C}$-set.* A set $A \subseteq \mathbb{R}^n$ is called *$\mathcal{C}$-semianalytic* if for every $\bar{a} \in \mathbb{R}^n$ there is an $\bar{r} \in (0, \infty)^n$ such that

$$(A - \bar{a}) \cap I_{\bar{r}}$$

is a $\mathcal{C}$-set. If A is also a manifold, we call A a $\mathcal{C}$-semianalytic manifold.

Given $m \leq n$ and an injective $\lambda : \{1, \ldots, m\} \to \{1, \ldots, n\}$, we write $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ for the projection $\bar{x} \mapsto \langle x_{\lambda(1)}, \ldots, x_{\lambda(m)} \rangle$.

Definition 1.2. Let $\bar{r} \in (0, \infty)^n$. A set $M \subseteq I_{\bar{r}}$ is said to be *$\mathcal{C}$-trivial* if one of the following holds:

(i) $M = \{\bar{x} \in I_{\bar{r}} : x_1 \square_1 0, \ldots, x_n \square_n 0\}$, where $\square_i \in \{<, =, >\}$ for each i;
(ii) there exist a permutation λ of $\{1, \ldots, n\}$, a $\mathcal{C}$-trivial $N \subseteq I_{\bar{s}}$ and a $g \in \mathcal{C}_{n-1,\bar{s}}$, where $\bar{s} = \langle r_{\lambda(1)}, \ldots, r_{\lambda(n-1)} \rangle$, such that $g(I_{\bar{s}}) \subseteq (-r_{\lambda(n)}, r_{\lambda(n)})$ and $\pi_\lambda(M) = \mathrm{graph}(g|_N)$.

Note that $\mathcal{C}$-trivial sets are necessarily manifolds; we shall refer to them as $\mathcal{C}$-trivial manifolds. A $\mathcal{C}$-seminanalytic manifold $m \subseteq \mathbb{R}^n$ is called *trivial* if there exist $\bar{a} \in \mathbb{R}^n$ and a $\mathcal{C}$-trivial manifold $N \subseteq \mathbb{R}^n$ such that $M = N + \bar{a}$.

We need the following results which are due to Rolin, Speissegger and Wilkie.

Theorem 1.3. *([8, 4.7]) Suppose that $A \subseteq \mathbb{R}^n$ is a bounded $\mathcal{C}$-semianalytic set and that $k \leq n$. Then there are trivial $\mathcal{C}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{n_i}$ for some $n_i \geq n, i = 1, \ldots J$, such that*

$$\pi_k(A) = \pi_k(N_1) \cup \cdots \cup \pi_k(N_J)$$

where $\pi_k|_{N_i}$ is an immersion, for each i. (Here, π_k is projection onto the first k coordinates.)

Let $\mathbb{R}_{\mathcal{C}}$ be the expansion of $\bar{\mathbb{R}}$ by all functions $\hat{f}$, for $f \in \mathcal{C}_{n,\bar{r}}, n \in \mathbb{N}, \bar{r} \in (0, \infty)^n$, where $\hat{f}$ is as defined in the introduction.

Theorem 1.4. *([8, 5.2]) The structure $\mathbb{R}_{\mathcal{C}}$ is o-minimal, model complete and polynomially bounded.*

We now use these results to prove a parameterization result. We work in the structure $\mathbb{R}_{\mathcal{C}}$.

Definitions 1.1. Let $X \subseteq \mathbb{R}^n$ be definable. A *$\mathcal{C}$-parameterization* of X is a finite set $\mathcal{S}$ of maps $\phi_1, \ldots, \phi_l \in \mathcal{C}_{[0,1]^{\dim X}}$ such that $\{\phi_i|_{(0,1)^{\dim X}} : i = 1, \ldots, l\}$ is a parameterization of X.

Lemma 1.5. *Suppose that $M \subseteq \mathbb{R}^n$ is a $\mathcal{C}$-trivial manifold. Then there is a $\mathcal{C}$-parameterization $\mathcal{S}$ of M with $\#\mathcal{S} = 1$.*

Proof. This follows from the definitions by induction on n. □

Proposition 1.6. *Suppose that $X \subseteq \mathbb{R}^n$ is a bounded definable set. Then X has a $\mathcal{C}$-parameterization.*

Proof. By model completeness, there is an $m \geq 0$ and a quantifier-free definable set $A \subseteq \mathbb{R}^{n+m}$ such that $X = \pi(A)$. Using the fact that $\mathbb{R}_{\mathcal{C}}$ is an expansion of the real field, we may assume that A is bounded and that A is $\mathcal{C}$-semianalytic. By Theorem 1.3

$$X = \pi(N_1) \cup \cdots \cup \pi(N_k)$$

for some $\mathcal{C}$-trivial manifolds $N_1, \ldots, N_k$, were each $\pi|_{N_i}$ is a an immersion. Thus $\dim(X) = \max\{\dim(N_1), \ldots, \dim(N_k)\}$. A $\mathcal{C}$-parameterization of X can be constructed by composing the functions in the $\mathcal{C}$-parameterizations of each of the N_i with the projections π, and then trivially extending any of these functions to $(0,1)^{\dim X}$ if their domain is $(0,1)^{\dim N_i}$ with $\dim N_i < \dim(X)$. □

Note that Proposition 0.2 follows immediately, by applying the above to the given reduct of $\mathbb{R}_{\text{an}}$.

2. Curves

We now prove Proposition 0.1. In fact, we prove a result about the number of points in a fixed number field $k \subseteq \mathbb{R}$ of degree l. We use the absolute multiplicative height H on k, which agrees with the height on $\mathbb{Q}$ given in the introduction (for the definition of H, see [1]). For $X \subseteq \mathbb{R}^n$ and $H \geq 1$, we let $X(k, H) = X \cap \{\bar{a} \in k^n : H(\bar{a}) \leq H\}$. The following is a special case of [7, Corollary 3.3].

Proposition 2.1. *Suppose that $X \subseteq (0,1)^2$ has dimension 1 and that $\mathcal{S}$ is an $(A, 0)$-mild parameterization of X. Then there is an absolute constant c_0 such that $X(k, H)$ is contained in a union of at most*

$$\#\mathcal{S} \cdot c_0^l \cdot A^{2(1+o(1))}$$

intersections of X with algebraic curves of degree $\lfloor l \cdot \log H \rfloor$. Here the $1 + o(1)$ is taken as $H \to \infty$ with absolute implied constant, and $\lfloor \cdot \rfloor$ denotes integer part.

Given a function $F : \mathbb{R}^m \to \mathbb{R}$, we let $V(F) = \{\bar{x} \in \mathbb{R}^m : F(\bar{x}) = 0\}$.

Lemma 2.2. *Suppose that $f:(0,1)\to(0,1)$ is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$. Suppose further that graph$(f)=\pi(V(F))$ where $F:\mathbb{R}^{2+n}\to\mathbb{R}$ is a Pfaffian function of order r and degree (α,β), where π is the projection on to the first two coordinates. If $P:\mathbb{R}^2\to\mathbb{R}$ is a polynomial of degree d then*

$$\#(graph(f)\cap V(P))\leq 2^{r(r+1)/2+1}(n+2)^r(\alpha+2d')^{n+r+2} \tag{1}$$

where $d'=\max\{d,\beta\}$.

Proof. Let $\tilde{P}:\mathbb{R}^{2+n}\to\mathbb{R}$ be given by $\tilde{P}(x,y,\bar{z})=P(x,y)$. Then graph$(f)\cap V(P)=\pi(V(F)\cap V(\tilde{P}))$. The number of points in graph$(f)\cap V(P)$ is thus bounded by the number of connected components of $V(f)\cap V(\tilde{P})$ (there are only finitely many points as we have assumed that f is transcendental). By Kovanskii's theorem (as presented in [3, 3.3]) there are at most

$$2^{r(r-1)/2+1}d'(\alpha+2d'-1)^{n+1}((2(n+2)-1)(\alpha+d')-2n-2)^r$$

such components, and clearly this is less than the right hand side of (1). □

Proposition 2.3. *Suppose that $f:(0,1)\to(0,1)$ is a transcendental analytic function definable in $\mathbb{R}_{resPfaff}$ and let $X=graph(f)$. Then there are $c,\gamma>0$ such that (for $H\geq e$)*

$$\#X(k,H)\leq c(\log H)^\gamma.$$

Proof. By model completeness of $\mathbb{R}_{\mathrm{resPfaff}}$ (see [9]) we may suppose that $X=\pi(V(F))$ for some Pfaffian function $F:\mathbb{R}^{2+n}\to\mathbb{R}$ and some $n\geq 0$. Suppose that F has order r and degree (α,β). By Proposition 0.2 we can take an $(A,0)$-mild parameterization $\mathcal{S}$ of X, for some A. Combining Proposition 2.1 with Lemma 2.2 (with $d=\lfloor l\log H\rfloor$), we have

$$\begin{aligned}\#X(k,H) &\leq \#\mathcal{S}\cdot c_0^l\cdot A^{2(1+o(1))}2^{r(r+1)/2+1}(n+2)^r(\alpha+2\max\{\beta,d\})^{n+r+2}\\ &\leq c(\log H)^\gamma\end{aligned}$$

where $\gamma=n+r+2$. □

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