Generating the Pfaffian closure with total Pfaffian functions

Jones, Gareth and Speissegger, Patrick

2011

MIMS EPrint: 2011.53

Manchester Institute for Mathematical Sciences
School of Mathematics
The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097
Generating the Pfaffian closure with total Pfaffian functions

GARETH JONES
PATRICK SPEISSEGGER

Abstract: Given an o-minimal expansion \( R \) of the real field, we show that the structure obtained from \( R \) by iterating the operation of adding all total Pfaffian functions over \( R \) defines the same sets as the Pfaffian closure of \( R \).

2000 Mathematics Subject Classification 14P10,03C64 (primary); 58A17 (secondary)

Keywords: o-minimal structure, Pfaffian function, Rolle leaf

There are various possibilities for adding Pfaffian objects to o-minimal expansions of the real field and preserving o-minimality. One example is the Pfaffian closure of an o-minimal expansion of the real field, which was shown to be o-minimal by the second author [4]. The purpose of this note is to present a somewhat simpler construction of the Pfaffian closure. Although not as simple as the description in terms of nested leaves obtained by Lion and the second author [3], our construction has the novelty of only using total Pfaffian functions and is reminiscent of the original Pfaffian expansion of the real field constructed by Wilkie [6].

In order to state our result, we need to introduce some terminology. Suppose that \( R \) is an o-minimal expansion of the real field, and that \( U \subseteq \mathbb{R}^n \) is an \( R \)-definable open subset of \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). We say that a \( C^1 \) function \( f : U \rightarrow \mathbb{R} \) is Pfaffian over \( R \) if there exist \( R \)-definable \( C^1 \) functions \( P_i : U \times \mathbb{R} \rightarrow \mathbb{R} \), for \( i = 1, \ldots, n \) such that

\[
\frac{\partial f}{\partial x_i}(x) = P_i(x, f(x))
\]

for all \( x \in U \).

Given \( n, l \in \mathbb{N} \) such that \( l \leq n \), we let \( G^l_n \) be the Grassmannian of all linear subspaces of \( \mathbb{R}^n \) of dimension \( l \). This is an analytic manifold and is naturally definable in the real field (see [1, 3.4.2]). We also set \( G_n = \bigcup_{l=0}^n G^l_n \). Now fix an embedded \( C^1 \) submanifold \( M \) of \( \mathbb{R}^n \) and let \( l \leq n \). A \( C^1 \) map \( d : M \rightarrow G_n \) is said to be a distribution on \( M \) if \( d(x) \subseteq T_x M \) for all \( x \in M \), where \( T_x M \) is the tangent space of \( M \) at \( x \). A distribution \( d \) is an \( l \)-distribution if \( d(M) \subseteq G^l_n \). Given an \( l \)-distribution \( d \) on \( M \) and an immersed \( C^1 \) submanifold \( V \) of \( M \), we say that \( V \) is an integral manifold of \( d \) is \( T_x V = d(x) \) for all
A maximal connected integral manifold is called leaf of the distribution. Now suppose that \( d \) has codimension one. A leaf \( L \) of \( d \) is said to be a Rolle leaf of \( d \) if it is a closed embedded submanifold of \( M \) and is such that for all \( C^1 \) curves \( \gamma : [0, 1] \to M \) satisfying \( \gamma(0), \gamma(1) \in L \), we have \( \gamma'(t) \in d(\gamma(t)) \) for some \( t \in [0, 1] \). A Rolle leaf over \( \mathcal{R} \) is a Rolle leaf of an \( \mathcal{R} \)-definable codimension one distribution defined on \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). For example, a result due to Khovanskii (see [5, 1.6]) implies that if \( f : \mathbb{R}^n \to \mathbb{R} \) is Pfaffian over \( \mathbb{R} \), then the graph of \( f \) is a Rolle leaf over \( \mathbb{R} \).

We can now define the Pfaffian structures involved in our result. Given any o-minimal expansion of the real field \( \mathcal{R} \), let \( \mathcal{L}(\mathcal{R}) \) be the collection of all Rolle leaves over \( \mathcal{R} \). Now let \( \mathcal{R}_0 = \mathcal{R} \) and, for \( i \geq 0 \), let \( \mathcal{R}_{i+1} \) be the expansion of \( \mathcal{R}_i \) by all leaves in \( \mathcal{L}(\mathcal{R}_i) \). Let \( \mathcal{L} \) be the union of all the \( \mathcal{L}(\mathcal{R}_i) \) and let \( \mathcal{P}(\mathcal{R}) \) be the expansion of \( \mathcal{R} \) by all the leaves in \( \mathcal{L} \). This structure is called the Pfaffian closure of \( \mathcal{R} \). The second author showed that it is o-minimal [4].

Similarly, we let \( \mathcal{L}'(\mathcal{R}) \) be the collection of all functions \( f : \mathbb{R}^n \to \mathbb{R} \), for all \( n \in \mathbb{N} \) that are Pfaffian over \( \mathcal{R} \). We define \( \mathcal{R}'_i \) and then \( \mathcal{P}'(\mathcal{R}) \) by mimicking the previous paragraph. The structure \( \mathcal{P}'(\mathcal{R}) \) is a reduct of \( \mathcal{P}(\mathcal{R}) \) (by the example above) and it is the purpose of this note to show that they are in fact the same from the point of view of definability.

**Theorem 1** A set \( X \subseteq \mathbb{R}^n \) is definable in \( \mathcal{P}(\mathcal{R}) \) if and only if it is definable in \( \mathcal{P}'(\mathcal{R}) \).

If \( \mathcal{R} \) admits analytic cell decomposition, then so too does \( \mathcal{P}'(\mathcal{R}) \) (see [5]) and it follows that in this case, the reduct of \( \mathcal{P}'(\mathcal{R}) \) in which only analytic functions are added also defines the same sets as \( \mathcal{P}(\mathcal{R}) \).

Given the definition of \( \mathcal{P}'(\mathcal{R}) \), in order to prove the theorem it suffices to show that if \( L \) is a Rolle leaf over \( \mathcal{P}(\mathcal{R}) \) then \( L \) is definable in \( \mathcal{P}(\mathcal{R}) \). For the proof of this, we assume that the reader is familiar with both o-minimality (as presented in [2]) and the theory of Pfaffian sets (as in [5] for example). From now on, we use the word definable to mean \( \mathcal{P}'(\mathcal{R}) \)-definable. In particular, cell means \( \mathcal{P}'(\mathcal{R}) \)-definable cell. First, an easy observation.

**Lemma 2** Suppose that \( C \subseteq \mathbb{R}^n \) is an open \( C^2 \) cell and that \( f : C \to \mathbb{R} \) is Pfaffian over \( \mathcal{P}'(\mathcal{R}) \). Then \( f \) is definable.

The proof, using a definable diffeomorphism between \( C \) and \( \mathbb{R}^n \), is left to the reader. Now suppose that \( C \subseteq \mathbb{R}^n \) is a bounded open \( C^2 \) cell, and that \( \alpha, \beta, \gamma, \delta : C \to \mathbb{R} \) are definable bounded \( C^2 \) functions such that

\[
\gamma(x) < \alpha(x) < \beta(x) < \delta(x)
\]
The following proposition suffices to prove the theorem.

**Proposition 4** Let \( L \subseteq \mathbb{R}^n \) be a Rolle leaf over \( \mathcal{P}'(\mathbb{R}) \). Then \( L \) is definable in \( \mathcal{P}'(\mathbb{R}) \).

**Proof** The proof is by induction on \( n \). The \( n = 1 \) case is trivial, so we assume that \( n > 1 \) and that the proposition is true for Rolle leaves over \( \mathcal{P}'(\mathbb{R}) \) contained in \( \mathbb{R}^m \) with \( m < n \). Thus if \( C \subseteq \mathbb{R}^n \) is a \( C^2 \) cell of dimension less than \( n \) and \( V \subseteq C \) is a Rolle leaf of a definable codimension one distribution on \( C \), then \( V \) is definable.

Suppose that \( L \subseteq \mathbb{R}^n \) is a Rolle leaf over \( \mathcal{P}'(\mathbb{R}) \). Then \( L \) is a closed embedded proper submanifold of \( \mathbb{R}^n \), and so there are \( p \in \mathbb{R}^n \setminus L \) and \( r > 0 \) such that \( B(p, 2r) \cap L = \emptyset \),
where $B(a, \varepsilon)$ is the open ball around $a$ of radius $\varepsilon$. Perhaps after translating and stretching, we may assume that $p = 0$ and that $r = 1$. Let $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the semialgebraic diffeomorphism $\phi(x) = \frac{x}{|x|^2}$. Then $\phi(L)$ is contained in $B(0, 1/2)$ and $\text{cl} (\phi(L)) \subseteq \phi(L) \cup \{0\}$. So, after replacing $L$ by $\phi(L)$, we may assume that $L$ is a Rolle leaf of a definable integrable $(n - 1)$-distribution $d$ on $B'(0, 1) := B(0, 1) \setminus \{0\}$, that $L \subseteq B(0, 1/2)$ and that $\text{cl} L \subseteq L \cup \{0\}$.

Let $\Pi_{n-1}$ be the projection onto the first $n - 1$ coordinates. For each coordinate permutation $\sigma$ on $\mathbb{R}^n$, the set $B_\sigma = \{ x \in B'(0, 1) : \Pi_{n-1}|_{\sigma^*(d(\sigma^{-1}(x)))} \text{ has rank } n - 1 \}$ is open and together these sets cover $B'(0, 1)$. So it suffices to show that $L \cap B_\sigma$ is definable for each $\sigma$. Fix $\sigma$, which we may assume to be the identity. Let $\mathcal{C}$ be a $C^2$ cell decomposition of $B'(0, 1)$ compatible with $B_{\text{id}}$, $B'(0, 1/2)$ and $d$. We show that $C \cap L$ is definable for each cell $C \subseteq \mathcal{C}$ such that $C \subseteq B_{\text{id}}$. If $C \in \mathcal{C}$ is not open then $L \cap C$ is definable, by Khovanskii theory and the inductive hypothesis. So, suppose that $C \subseteq \mathcal{C}$ is open and that $C \subseteq B_{\text{id}}$. Let $N$ be a component of $L \cap C$. Since $N$ is a Rolle leaf of $d|_{\mathcal{C}}$ and $C$ is a cell, $N$ is the graph of a function $f : \Pi_{n-1}(N) \to \mathbb{R}$. Let $\alpha, \beta : \Pi_{n-1}(C) \to \mathbb{R}$ be the functions such that $\text{graph}_\alpha$ and $\text{graph}_\beta$ are the two cells in $\mathcal{C}$ forming the ‘bottom’ and ‘top’ of $C$. Then the graph of $\alpha$ is compatible with $d$ and so it is either tangent to $d$ or transverse to $d$. Since $\text{graph}_\alpha$ is connected, if it is tangent to $d$, then either $\text{graph}_\alpha \subseteq L$ or $L \cap \text{graph}_\alpha = \emptyset$. If the graph of $\alpha$ is transverse to $d$ then by Khovanskii theory and the inductive hypothesis, $L \cap \text{graph}_\alpha$ is definable. By Lemma 3, $\text{fr} N \cap \text{graph}_\alpha$ is a clopen subset of $L \cap \text{graph}_\alpha$ and so $\text{fr} N \cap \text{graph}_\alpha$ is also definable. This all also holds with the graph of $\beta$ in place of the graph of $\alpha$. Since $N$ is bounded and the graph of a continuous function, $x \in \text{fr} \Pi_{n-1}(N)$ if and only if there is a $y$ such that $(x, y) \in \text{fr} N$. So the set $\text{fr} \Pi_{n-1}(N) \cap \Pi_{n-1}(C)$ is definable. Let $\mathcal{D}$ be a cell decomposition of $\Pi_{n-1}(C)$ compatible with $\text{fr} \Pi_{n-1}(N) \cap \Pi_{n-1}(C)$. Then for each $D \in \mathcal{D}$ we either have $D \subseteq \Pi_{n-1}(N)$ or $D \cap \Pi_{n-1}(N) = \emptyset$. For each non-open cell $D \in \mathcal{D}$ such that $D \subseteq \Pi_{n-1}(N)$, let $E_D = (\alpha|_{D}, \beta|_{D})_D$. Take a cell decomposition of $E_D$ compatible with $d$. Let $E'$ be a cell in this decomposition such that $\text{graph}_f|_D \cap E'$ is non-empty. Then by Khovanskii theory, $\text{graph}_f|_D \cap E'$ is either a finite union of Rolle leaves of the pullback of $d$ to $E'$ and so definable by the inductive hypothesis, or is equal to $E'$ (in the case that $E'$ is tangent to $d$). So the graph of $f|_D$ is definable. Finally, for each open cell $D \in \mathcal{D}$ such that $D \subseteq \Pi_{n-1}(N)$, the restriction of $f$ to $D$ is Pfaffian over $\mathcal{P}(\mathcal{R})$ and so is definable by Lemma 2. So $N$ is definable, as required.

\[\Box\]
Acknowledgements

The first author is supported by EPSRC, and the second author is supported by NSERC.

References


School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK.
Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada.
gareth.jones-3@manchester.ac.uk, speisseg@math.mcmaster.ca

Received: aa bb 20YY Revised: cc dd 20ZZ