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Abstract

We consider a switched linear dynamical system described by

$$\begin{array}{rcl} \delta x(t) & = & A_\sigma x(t) + B_\sigma u(t), \qquad x(t_0) = x_0, \\ y(t) & = & C_\sigma x(t), \end{array} \tag{1}$$

where $x(t)$ is the state, $u(t)$ is the controlled input, $y(t)$ is the measured output, σ is the piecewise constant signal taking values from an index set $M = \{1, \ldots, l\}$, and A_k, B_k and C_k, $k \in M$ are matrices of appropriate dimensions. The switched system is a multi-model which is a special case of hybrid systems [3, 4].

This talk is about model reduction of switched systems which has received relatively little attention in the Numerical Linear Algebra community. We will present several new Gramian-based methods. These Gramians are matrix energy functions and they are, in theory, solutions of certain complicated Lyapunov equations. Here we propose to solve a set of simpler Lyapunov equations and to use linear combinations of these solutions to obtain the Gramians. We propose also a balanced truncation-like method with these two Gramians. We will also present another new algorithm based on Lyapunov stability analysis. We will show how to solve the underlying set of Linear Matrix Inequalities for two common solutions. These two solutions are used to come up with a balanced truncation-like method. With this approach we will preserve the stability for the reduced model. We will suppose implicitly that each subsystem is stable.

The switching controlled by σ is introducing a form of uncertainty and the composite system comprises a certain number of subsystems where, at every time step, there is a certain probability that a particular subsystem will be switched on. Any operation on the composite system involves many numerical difficulties especially for stability or reachability analysis and formal verification [3, 4]. The use of numerical model reduction techniques has the potential to make feasible the computational investigation of a class of currently intractable systems. Gramian-based methods are one major class of the model reduction techniques [1, 5]. But it is a challenge, since almost all reduction methods cannot be directly applied to switched systems [2].

In general, a switching signal may depend on the time, its own past value, the state/output, and/or possibly an external signal as well

$$\sigma(t+) = \sigma(t, \sigma(t), x(t)/y(t), z(t)) \quad \forall t$$

where $z(t)$ is an external signal produced by other devices, $\sigma(t+)$ is the next switching signal. The switching is also making every operation on the system more complex as the number of involved subsystems is large. For example, a simple check of the stability, which for a single subsystem is equivalent to the eigenvalue problem of the dynamic matrix, is now a multiple eigenvalue problem of different matrix pencils. For instance, we have to check the eigenvalues of all linear combinations of each triplet (A_i, A_j, A_k) and (A_i, A_j^{-1}, A_k) for all $i \neq j \neq k \in M$.

In the computer science community, which has for some time considered these systems, the switching signal has been classified according to different laws on time-driven or event-driven dynamics. In view of the applications we have in mind and for the purpose of using similar ideas that were developed for classic dynamical systems we will distinguish two cases for the switching signal: time-transmission switching and time independent switching.

For the time-transmission switching, the switching path is known a priori (i.e., we know a priori at what moment t_k the system is switching to which subsystem i_k). Given a switching sequence $\{x_0, (t_0, i_0), (t_1, i_1), \ldots, (t_l, i_l)\}$, the state is given at any time $t \in [t_k, t_{k+1})$ by

$$x(t) = \phi(t, t_0, x_0, u, \sigma) = \Phi(t, t_0, \sigma, x_0)x_0 + \sum_{j=1}^{k} \Phi(t, t_j, \sigma, x_0) \int_{t_{j-1}}^{t_j} e^{A_{i_{j-1}}(t_j - \tau)} B_{i_{j-1}} u(\tau) d\tau$$

where the transition matrix

$$\Phi(t_1, t_2, \sigma, x_0) = \psi(t_1, \sigma, x_0)(\psi(t_2, \sigma, x_0))^{-1} \text{ and } \quad \psi(t, \sigma, x_0) := e^{A_{i_k}(t - t_k)} \prod_{j=k}^{1} e^{A_{i_{j-1}}(t_j - t_{j-1})}.$$

The key idea is to use the sub-Gramians, which are the Gramians (energy matrix functions) of each subsystem and solution of the Lyaponuv equations

$$A_j \mathcal{P}_j + \mathcal{P}_j A_j^T + B_j B_j^T = 0, \quad A_j^T \mathcal{Q}_j + \mathcal{Q}_j A_j + C_j^T C_j = 0.$$

The Gramians for the switched system are $\mathcal{P} = \sum_{j=1}^{l} \alpha_j \mathcal{P}_j$, $\mathcal{Q} = \sum_{j=1}^{l} \beta_j \mathcal{Q}_j$, where α_j and β_j define how much each subsystem is involved in the system. These parameters can be chosen following different strategies: for example $\alpha_j = \beta_j = \frac{t_j - t_{j-1}}{t_l - t_0}$ or $= 1/l$ simply, if we know that each subsystem is visited at least once. Other strategies will be presented.

For the time-independent switching, the switching is based on the notion of guards. Once a guard is hit the switching occurs. To be able to reduce the system without destroying its nature (switching) we have to use properties of the system like the stability. The stability of the switched system is equivalent to the existence of two common solutions to the following systems of LMIs

$$\begin{aligned} \mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{N \times N}, \quad \mathcal{P} \succ 0, \quad A_\sigma \mathcal{P} + \mathcal{P} A_\sigma^T + B_\sigma B_\sigma^T \prec 0, \\ \mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{N \times N}, \quad \mathcal{Q} \succ 0, \quad A_\sigma^T \mathcal{Q} + \mathcal{Q} A_\sigma + C_\sigma^T C_\sigma \prec 0. \end{aligned}$$

Until now, no efficient algorithm exists for solving these LMIs in the general case. Even for some special simple cases, solving numerically these LMIs is still a challenge once both N and l are large. We propose a new algorithm to compute efficiently $\mathcal{P}$ and $\mathcal{Q}$.

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