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# Algorithmic Properties of $\Sigma$–definability over Positive Predicate Structures


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**Abstract.** In this paper we propose a generalisation of results in [2, 7] on semantical characterisation of $\Sigma$-definability. We prove that over every positive predicate structure a set is $\Sigma$-definable if and only if it is definable by a disjunction of a recursively enumerable set of existential formulas.


## 1 Introduction

This paper is a part of the work [2, 5, 6, 8, 7, 9, 10] on developing a logical framework for studying computability over discrete and continuous data in a common language. In order to archive this goal we represent data as a structure which could not have effective equality and employ $\Sigma$–definability theory. Our approach is based on representations of data (discrete or continuous ) by a suitable structure $\mathcal{A} = \langle A, \sigma_P, \neq \rangle$, where $A$ contains more than one element, and $\sigma_P$ is a set of basic predicates. *We assume that all predicates $Q_i \in \sigma_P$ and $\neq$ occur only positively in $\Sigma$-formulas and do not assume that the language $\sigma_P$ contains equality.* We call such structures as *positive predicate structures*. It turns out that $\Sigma$-definability without equality is rather different from $\Sigma$-definability with equality. It has been shown in [11] that there is no effective procedure which given a $\Sigma$-formula with equality defining an open set produces a $\Sigma$-formula without equality defining the same set. Therefore it is important to figure out which properties of $\Sigma$-definability hold on structures with equality likewise on structures without equality.

Some of the important properties of $\Sigma$-definability with respect to computability, i.e., existence of a universal $\Sigma$-predicate and an algorithmic characterisation of $\Sigma$-definability have been proven over structures with equality [2] and over the real numbers [7]. In this paper we show that these properties hold over every positive predicate structure. In order to do this we develop new tools and techniques to overcome difficulties arising from possible absence of equality and particular properties of the reals.

## 2 Definition and Notations

We start by introducing basic notations and definitions. In this paper we are mostly working with an arbitrary structure $\mathcal{A} = \langle A, \sigma_0 \rangle = \langle A, \sigma_P, \neq \rangle$, where $A$ contains more than one element, $\sigma_P$ is a finite set of basic predicates.

*Example 1.* 1. (The natural numbers) $\mathbb{N} = \langle \mathbb{N}, Q_1, Q_2, < \rangle$, where $Q_1$ and $Q_2$ have the following meanings:

$$\mathbb{N} \models Q_1(x) \leftrightarrow x = 0;\ \mathbb{N} \models Q_2(x, y) \leftrightarrow x = y + 1.$$

2. (The real numbers) $\mathbb{R} = \left\langle \mathbb{R}, \mathcal{M}^*_E, \mathcal{M}^*_H, \mathcal{P}^+_E, \mathcal{P}^+_H\ <, \right\rangle$ $\sigma_P = \{\mathcal{M}^*_E, \mathcal{M}^*_H, \mathcal{P}^+_E, \mathcal{P}^+_H\}$, where $\mathcal{M}^*_E, \mathcal{M}^*_H$ are interpreted as the open epigraph and the open hypograph of multiplication respectively, and $\mathcal{P}^+_E, \mathcal{P}^+_H$ are interpreted as the open epigraph and the open hypograph of addition respectively.
3. (The real-valued continuous function defined on compact intervals) $C[0,1] = \langle C[0,1], P_1, \dots, P_{10}, \neq \rangle$ where the predicates $P_1, \dots, P_{10}$ have the following meanings for every $f,\ g \in C[0,1]$:
The first group formalises relations between infimum and supremum of two functions.

$$\begin{array}{l} C[0,1] \models P_1(f,g) \leftrightarrow \sup f < \sup g; \\ C[0,1] \models P_2(f,g) \leftrightarrow \sup f < \inf g; \\ C[0,1] \models P_3(f,g) \leftrightarrow \sup f > \inf g; \\ C[0,1] \models P_4(f,g) \leftrightarrow \inf f > \inf g. \end{array}$$

The second group formalises properties of operations on $C[0,1]$.

$$\begin{array}{l} C[0,1] \models P_5(f,g,h) \leftrightarrow f(x) + g(x) < h(x) \text{ for every } x \in [0,1]; \\ C[0,1] \models P_6(f,g,h) \leftrightarrow f(x) \cdot g(x) < h(x) \text{ for every } x \in [0,1]; \\ C[0,1] \models P_7(f,g,h) \leftrightarrow f(x) + g(x) > h(x) \text{ for every } x \in [0,1]; \\ C[0,1] \models P_8(f,g,h) \leftrightarrow f(x) \cdot g(x) > h(x) \text{ for every } x \in [0,1]. \end{array}$$

The third group formalises relations between functions $f$ and the identity function $\lambda x.x$.

$$\begin{array}{l} C[0,1] \models P_9(f)\ \ \leftrightarrow f > \lambda x.x; \\ C[0,1] \models P_{10}(f) \leftrightarrow f < \lambda x.x. \end{array}$$

In order to do any kind of computation or to develop a computability theory one has to work within a structure rich enough for information to be coded and stored. For this purpose we extend the structure $A$ by the set of hereditarily finite sets $\mathrm{HF}(A)$.

The idea that the hereditarily finite sets over $A$ form a natural domain for computation is quite classical and is developed in detail in [1, 2] for the case when $\sigma_0$ contains equality.

We construct the set of hereditarily finite sets, $\mathrm{HF}(A)$, as follows:

1. $\mathrm{HF}_0(A) \rightleftharpoons A$,
2. $\mathrm{HF}_{\mathrm{n}+1}(A) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_{\mathrm{n}}(A)) \cup \mathrm{HF}_{\mathrm{n}}(A)$, where $n \in \omega$ and for every set $B$, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of $B$.

3. $\mathrm{HF}(A) \rightleftharpoons \bigcup_{n\in\omega} \mathrm{HF}_n(A)$.

We define $\mathbf{HF}(A)$ as the following model:

$$\mathbf{HF}(A) \rightleftharpoons \langle \mathrm{HF}(A), U, \sigma_0, \in \rangle \rightleftharpoons \langle \mathrm{HF}(A), \sigma \rangle \,,$$

where the binary predicate symbol $\in$ has the set-theoretic interpretation. Also we add the predicate symbol $U$ for urelements (elements from $A$).

The natural numbers $0,\ 1, \dots$ are identified with the (finite) ordinals in $\mathrm{HF}(A)$ i.e. $\emptyset,\ \{\emptyset\},\ \{\emptyset, \{\emptyset\}\}, \dots$, so in particular, $n+1 = n \cup \{n\}$ and the set $\omega$ is a subset of $\mathrm{HF}(A)$.
The atomic formulas include $U(x)$, $\neg U(x)$, $x \neq y$, $x \in s$, $x \notin s$ where $s$ ranges over sets, and also, for every $Q_i \in \sigma_P$ of the arity $n_i$, $Q_i(x_1, \dots, x_{n_i})$ which has the following interpretation:

$$\begin{aligned}&\mathbf{HF}(A) \models Q_i(x_1, \dots, x_{n_i}) \text{ if and only if}\\ &A \models Q_i(x_1, \dots, x_{n_i}) \text{ and, for every } 1 \leq j \leq n_i,\ x_j \in A.\end{aligned}$$

The set of $\Delta_0$*-formulas* is the closure of the set of atomic formulas under $\wedge, \vee$, bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$, where $(\exists x \in y)\ \Psi$ means the same as $\exists x (x \in y \wedge\ \Psi)$ and $(\forall x \in y)\ \Psi$ as $\forall x (x \in y \to \Psi)$ where $y$ ranges over sets.
The set of $\Sigma$*-formulas* is the closure of the set of $\Delta_0$-formulas under $\wedge$,$\vee$, $(\exists x \in y)$, $(\forall x \in y)$ and $\exists x$, where $y$ ranges over sets.

*Remark 1.* We recall that all predicates $Q_i \in \sigma_P$ and $\neq$ occur only positively in $\Sigma$-formulas. Hence when $\sigma_P$ does't contain equality as a basic predicate, equality on the urelements (elements from $A$) is not representable by a $\Sigma$-formula.

*Remark 2.* Through this paper we consider also existential formulas in the language $\sigma_0$ with positive occurrences of predicate symbols from $\sigma_0$ without any further references to this restriction.

We are interested in $\Sigma$-definability of sets on $A^n$ which can be considered as generalisation of recursive enumerability. The analogy of $\Sigma$-definable and recursively enumerable sets is based on the following fact. Consider the structure $\mathbf{HF} = \langle \mathrm{HF}(\emptyset), \in \rangle$ with the hereditarily finite sets over $\emptyset$ as its universe and membership as its only relation. In $\mathbf{HF}$ the $\Sigma$-definable subsetsof $\omega$ are exactly the recursively enumerable sets [1].

The notion of $\Sigma$-definability has a natural meaning also in the structure $\mathbf{HF}(A)$.

**Definition 1.** *1. A relation $B \subseteq \mathrm{HF}(A)^n$ is $\Sigma$-definable, if there exists a $\Sigma$-formula $\Phi(\bar{a})$ such that*

$$\bar{b} \in B \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{b}).$$

*2. A function $f : \mathrm{HF}(A)^n \to \mathrm{HF}(A)^m$ is $\Sigma$-definable, if there exists a $\Sigma$-formula $\Phi(\bar{c}, \bar{d})$ such that*

$$f(\bar{a}) = \bar{b} \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{a}, \bar{b}).$$

In a similar way we introduce the notion of $\Delta_0$-definability. Let $S(\mathrm{HF}(A))$ denote the set of all sets in $\mathrm{HF}(A)$ and $S'(\mathrm{HF}(A))$ denote the set of all nonempty sets in $\mathrm{HF}(A)$.

**Lemma 1.** *1. The predicates $S(x) \rightleftharpoons$ x is a set, $\emptyset(x) \rightleftharpoons$ x is the empty set, $\neg\emptyset(x) \rightleftharpoons$ x is not the empty set and $n \in \omega$ are $\Delta_0$-definable.*
*2. The predicate $S'(x) \rightleftharpoons$ x is a nonempty set is $\Delta_0$-definable.*
*3. The following predicates are $\Delta_0$-definable: $x = y$, $x = y \cap z$, $x = y \cup z$, $x = \langle y, z\rangle$, $x = y \setminus z$ where all variables $x, y, z$ range over sets.*
*4. A function $f : \omega^n \to \omega^m$ is computable if and only if it is $\Sigma$-definable.*
*5. Let Fun(g) mean that $g : S'(\mathrm{HF}(A)) \to S'(\mathrm{HF}(A))$ is a finite function i.e.*

$$g \in S'(\mathrm{HF}(A)) \textit{ and}$$
$$g = \{\langle x, y\rangle | \textit{ for every } x \textit{ there exists a unique } y \}$$

*then the predicate Fun(g) is $\Delta_0$-definable.*
*6. If $\mathbf{HF}(A) \models$ Fun(g) then the domain of g, denoted by dom(g), is $\Delta_0$-definable.*
*7. The set $\{\gamma : \omega \to S'(\mathbf{HF}(A)) | \gamma$ is a finite function$\}$ is $\Sigma$-definable.*

*Proof.* Proofs of all properties are straightforward except (4) which can be found in [2].

For finite functions *Fun*($\gamma$) let us denote $\gamma(x) = y$ if $\langle x, y\rangle \in \gamma$.

## 3 Gandy's Theorem and Inductive Definitions

Let us recall Gandy's Theorem for $\mathbf{HF}(A)$ which will be essentially used in all proofs of the main results. Let $\Phi(a_1, \dots, a_n, P)$ be a $\Sigma$-formula, where $P$ occurs positively in $\Phi$ and the arity of $P$ is equal to $n$. We think of $\Phi$ as defining an *effective operator* $\Gamma : \mathcal{P}(\mathrm{HF}(A)^n) \to \mathcal{P}(\mathrm{HF}(A)^n)$ given by

$$\Gamma(Q) = \{\bar{a} | (\mathbf{HF}(A), Q) \models \Phi(\bar{a}, P)\}.$$

Since the predicate symbol $P$ occurs only positively we have that the corresponding operator $\Gamma$ is monotone, i.e., for all sets $B$ and $C$, from $B \subseteq C$ follows $\Gamma(B) \subseteq \Gamma(C)$, and continuous with respect to Scott topology on $\mathcal{P}(\mathrm{HF}(A)^n)$. By monotonicity, the operator $\Gamma$ has a least (w.r.t. inclusion) fixed point which can be described as follows. We start from the empty set and apply operator $\Gamma$ until we reach the fixed point:

$$\Gamma^0 = \emptyset, \quad \Gamma^{n+1} = \Gamma(\Gamma^n), \quad \Gamma^\gamma = \cup_{n<\gamma}\Gamma^n,$$

where $\gamma$ is a limit ordinal.

One can easily check that the sets $\Gamma^n$ form an increasing chain of sets: $\Gamma^0 \subseteq \Gamma^1 \subseteq \dots$. By set-theoretical reasons, there exists the least ordinal $\gamma$ such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$. This $\Gamma^\gamma$ is the least fixed point of the given operator $\Gamma$.

**Theorem 1 (Gandy's Theorem for HF$(A)$).**
*Let $\Gamma : \mathcal{P}(\mathrm{HF}(A)^n) \to \mathcal{P}(\mathrm{HF}(A)^n)$ be an effective operator. Then the least fixed-point of $\Gamma$ is $\Sigma$-definable and the least ordinal such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$ is less or equal to $\omega$.*

*Proof.* See [8].

**Definition 2.** *A relation $B \subset A^n$ is called $\Sigma$-inductive if it is the least-fixed point of an effective operator.*

**Corollary 1.** *Every $\Sigma$-inductive relation is $\Sigma$-definable.*

## 4 Universal $\Sigma$-predicate

In order to obtain a result on the existence of a universal $\Sigma$-predicate we first prove $\Sigma$-definability of the predicate $TR^\forall$ introduced below.

We fix a standard effective Gödel numbering of formulas of the language $\sigma$ by finite ordinals which are elements of HF$(\emptyset)$. Let $\lceil \Phi \rceil$ denote the codes of a formula $\Phi$. It is worth noting that the type of an expression is effectively recognisable by its code. We also can obtain effectively from the codes of expressions the codes of their subexpressions and vice versa. Since equality is $\Delta_0$-definable in HF$(\emptyset)$, we can use the well-known characterisation which states that all effective procedures over ordinals are $\Sigma$-definable. Thus, for example, the following predicates

$$\begin{aligned} Code_{elem_0}(n,j) &\rightleftharpoons n = \lceil U(x_j) \rceil, \\ Code_{elem_i}(n,j_1,\dots,j_{n_i}) &\rightleftharpoons n = \lceil Q_i(x_{j_1},\dots,x_{j_{n_i}}) \rceil, \\ Code_{\wedge}(n,i,j) &\rightleftharpoons n = \lceil \Phi \wedge \Psi \rceil \wedge i = \lceil \Phi \rceil \wedge j = \lceil \Psi \rceil \end{aligned}$$

are $\Sigma$-definable. Hence, in $\Sigma$-formulas we can use such predicates.

**Proposition 1.** *For every $\mathcal{A}$ of cardinality $> 1$ there exists a $\Sigma$-definable set $TR^\forall \subseteq \omega \times [\omega \to S'(\mathrm{HF}(A))]$ with the following properties.*

1. *If $n$ is the Gödel number of a $\Sigma$-formula $\Phi$ and $\gamma : \omega \to S'(\mathrm{HF}(A))$ is a finite function defined by an assignment function $f : FV(\Phi) \to \mathrm{HF}(A)$ as $\gamma(i) = \{f(x_i)\}$ for all $i : x_i \in dom(f)$ then $\langle n, \gamma \rangle \in TR^\forall$.*
2. *If $\langle n, \gamma \rangle \in TR^\forall$ then $n$ is the Gödel number of a $\Sigma$-formula $\Phi$ and $\gamma : \omega \to S'(\mathrm{HF}(A))$ is a finite function such that, for every assignment function $f : FV(\Phi) \to \mathrm{HF}(A)$ with the property $f(x_i) \in \gamma(i)$, $\mathbf{HF}(A) \models \Phi[f]$.*

*Proof.* The predicate $TR^\forall$ is the least fixed point of the operator defined by the following formula:

$$\begin{aligned} \Psi(n,\gamma,P) \rightleftharpoons \text{Gödel}(n) \wedge \text{Correct}(n,\gamma) \wedge (&\Psi_{elem}(n,\gamma) \vee \Psi_\wedge(n,\gamma,P) \vee \\ &\Psi_\vee(n,\gamma,P) \vee \Psi_{\exists\in}(n,\gamma,P) \vee \Psi_{\forall\in}(n,\gamma,P) \vee \Psi_\exists), \end{aligned}$$

where $n$, $\gamma$ are free variables and $P$ is a new predicate symbol. The formula $\Psi(n,\gamma,P)$ represents the inductive definition of the predicate $TR^\forall$ where the immediate subformulas have the following meaning. The first two formulas recognise the properties of $n$ and $\gamma$. The formula Gödel$(n)$ represents that $n$ is the Gödel number of a $\Sigma$-formula $\Phi$; the formula Correct$(n,\gamma)$ represents than $\gamma$ is a finite function from $\omega$ to $S'(\mathrm{HF}(A))$ such that $i \in dom(\gamma)$ if and only if $x_i \in FV(\Phi)$. The formula $\Psi_{elem}(n,\gamma)$ defines the basis of the inductive definition and captures the cases when $n$ is the Gödel number of an atomic formula. The remaining formulas represent inductive steps for conjunctions, disjunctions, bounded quantifiers, and existential quantifiers. By Lemma 1, the formulas Gödel$(n)$ and Correct$(n,\gamma)$ are equivalent to $\Sigma$-formulas. We illustrate constructions of the rest of the formulas. The basis of the inductive definition is given by the formula

$$\Psi_{elem}(n,\gamma) \rightleftharpoons \Psi_U(n,\gamma) \vee \Psi_{\neg U}(n,\gamma) \vee \Psi_{\in}(n,\gamma) \vee \Psi_{\notin}(n,\gamma),$$

where the subformulas can be done in the following way.

$$\Psi_U(n,\gamma) \rightleftharpoons \exists i \left(n = \ulcorner U(x) \urcorner \wedge i = \ulcorner x \urcorner \wedge \forall z \in \gamma(i) U(z)\right);$$

$$\Psi_{\neg U}(n,\gamma) \rightleftharpoons \exists i \left(n = \ulcorner \neg U(x) \urcorner \wedge i = \ulcorner x \urcorner \wedge \forall z \in \gamma(i) \neg U(z)\right);$$

$$\begin{aligned}\Psi_{\in}(n,\gamma) \rightleftharpoons \exists i \exists j \exists a (n = \ulcorner \neg x \in y \urcorner \wedge i = \ulcorner x \urcorner \wedge j = \ulcorner y \urcorner \wedge \\ S'(a) \wedge \gamma(j) = \{a\} \wedge \gamma(i) \subseteq A);\end{aligned}$$

$$\begin{aligned}\Psi_{\notin}(n,\gamma) \rightleftharpoons \exists i \exists j (n = \ulcorner \neg x \in y \urcorner \wedge i = \ulcorner x \urcorner \wedge j = \ulcorner y \urcorner \wedge \exists a (S'(a) \wedge \\ \gamma(j) = \{a\} \wedge \gamma(i) \cap A = \emptyset) \vee \forall z \in \gamma(j) U(z) \vee \forall z \in \gamma(j) \emptyset(z)).\end{aligned}$$

Now we construct the formulas for the inductive steps. For conjunctions and disjunctions:

$$\Psi_{\wedge}(n,\gamma,P) \rightleftharpoons \exists m \exists k \left(n = \ulcorner \Phi \wedge \Psi \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge k = \ulcorner \Psi \urcorner \wedge P(m,\gamma) \wedge P(k,\gamma)\right);$$

$$\Psi_{\vee}(n,\gamma,P) \rightleftharpoons \exists m \exists k \left(n = \ulcorner \Phi \wedge \Psi \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge k = \ulcorner \Psi \urcorner \wedge (P(m,\gamma) \vee P(k,\gamma))\right);$$

For bounded quantifiers:

$$\begin{aligned}\Psi_{\exists\in}(n,\gamma,P) \rightleftharpoons\ & \exists i \exists j \exists a \exists v \exists \gamma^* \exists m \, (n = \ulcorner \exists x \in y \Phi \urcorner \wedge i = \ulcorner x \urcorner \wedge j = \ulcorner y \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge \\ & S'(A) \wedge \gamma(j) = \{A\} \wedge \gamma \cup \{\langle i, v \rangle\} = \gamma^* \wedge i \notin dom(\gamma) \wedge \\ & P(m,\gamma^*) \wedge v \subseteq a) \,)) \, ;\end{aligned}$$

$$\begin{aligned}\Psi_{\forall\in}(n,\gamma,P) \rightleftharpoons\ & \exists i \exists j \exists a \exists \gamma^* \exists m \, (n = \ulcorner \forall x \in y \Phi \urcorner \wedge i = \ulcorner x \urcorner \wedge j = \ulcorner y \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge \\ & \forall z \in \gamma(j) U(z) \vee \forall z \in \gamma(i) \emptyset(z) \vee (S'(a) \wedge \gamma(j) = \{a\} \wedge \\ & P(m,\gamma^*) \wedge i \in dom(\gamma^*) \wedge \gamma \subseteq \gamma^* \wedge \gamma^*(i) \subseteq a \,)) \, .\end{aligned}$$

The formula $\Psi_{\exists}(n,\gamma,P)$ can be given as follows.

$$\Psi_{\exists}(n,\gamma,P) \rightleftharpoons \exists i \exists m \exists v \exists w\,(n=\ulcorner \exists x\Phi \urcorner \wedge i=\ulcorner x \urcorner \wedge m=\ulcorner \Phi \urcorner \wedge i \notin dom(\gamma) \wedge \\ S'(v) \wedge w=\gamma \cup \{\langle i,v\rangle\} \wedge P(m,w))\,.$$

From Gandy's theorem (c.f. Section 2.2) it follows that the least fixed point $TR^{\forall}$ of the effective operator defined by $\Psi$ is $\Sigma$-definable.

**Theorem 2.** *For every $n \in \omega$ there exists a $\Sigma$-formula $Univ_{n+1}(m,x_0,\dots,x_n)$ such that for any $\Sigma$-formula $\Phi(x_0,\dots,x_n)$*

$$\mathbf{HF}(A) \models \Phi(r_0,\dots,r_n) \leftrightarrow Univ_{n+1}(\ulcorner \Phi \urcorner, r_0,\dots,r_n).$$

*Proof.* It is easy to see that the following formula defines a universal $\Sigma$-predicate for the $\Sigma$-formulas of arity $n+1$.

$$\mathrm{Univ}_{n+1}(m,x_0,\dots x_n) \rightleftharpoons \exists y_0 \dots \exists y_n \exists \gamma (S'(y_0) \wedge \dots \wedge S'(y_n) \wedge \\ \gamma=\{\langle 0,y_0\rangle,\dots,\langle n,y_n\rangle\} \wedge TR^{\forall}(m,\gamma) \wedge \bigwedge_{0\le i\le n} x_i \in y_i)$$

## 5 Semantic characterisation of $\Sigma$-definability

In this section we prove that a relation over $A$ is $\Sigma$-definable if and only if it is definable by a disjunction of a recursively enumerable set of existential formulas in the language $\sigma_0$.

**Definition 3.** *Let a set of distinct variables $X=\{x_i | i \in \omega\}$ and an injective function (assignment) $f : X \to A$ be given. For $z \in \mathrm{HF}(X)$, define $sp(z)$ and $[z]_f$ as follows:*

*(i) if $z$ is a variable, then $sp(z)=\{z\}$ and $[z]_f=f(z)$;*
*(ii) if $z$ is the set $\{z_1,\dots,z_k\}$ then $sp(z)=\bigcup_{i\le k} sp(z_i)$ and $[z]_f=\{[z_1]_f,\dots,[z_k]_f\}$.*

**Definition 4.** *We say that $z \in \mathrm{HF}(X)$ structurally represents $y \in \mathrm{HF}(A)$ if $[z]_f=y$ for an assignment $f : X \to A$.*

**Proposition 2.** *Suppose $\varphi(y_1,\dots,y_s)$ is a $\Delta_0$-formula and $z_1,\dots,z_s$ structurally represent $y_1,\dots,y_s$ with the same assignment $f : X \to A$. Then we can effectively construct a quantifier-free formula $\psi$ such that $FV(\psi) \subseteq sp(\{z_1,\dots,z_s\})$ and*

$$\mathcal{A} \models \psi[f] \Leftrightarrow \mathbf{HF}(A) \models \varphi([z_1]_f,\dots,[z_s]_f).$$

*The choice of $\psi$ depends on the tuple $\bar{z}=(z_1,\dots,z_s)$ and $\psi$, and does not depend on $f$.*

*Proof.* In order to simplify the proof, without loss of generality, we assume that all assignments are injective and every formula has subformulas distinguishing free variables. Using induction on the structure of a $\Delta_0$-formula $\varphi$, we show how to obtain a required formula $\psi$. Let $\top$ denotes a logical truth which can be represented by the formula $\exists x \exists y\, x \neq y$ and $\perp$ denotes a logical false which can be represented by the formula $\exists x\, x \neq x$.
*Atomic case.*
1. If $\varphi(\bar{y}) \rightleftharpoons P(\bar{y})$ and $P \in \sigma_P$ or $\varphi(y_i, y_j) \rightleftharpoons y_i \neq y_j$, and $\bar{x}$ structurally represent $\bar{y}$, then $\psi(\bar{x}) \rightleftharpoons \varphi(\bar{x})$.
2. Suppose $\varphi(y_1, y_2) \rightleftharpoons y_1 \in y_2$ and $z_1,\ z_2$ structurally represent $y_1,\ y_2$. If $z_1 \in z_2$ then $\psi \rightleftharpoons \top$ else $\psi \rightleftharpoons \perp$. The subcase $\varphi(y_1, y_2) \rightleftharpoons y_1 \notin y_2$ can be considered by analogy.
*Disjunction* and *Conjunction.* If $\varphi \rightleftharpoons \varphi_1 \tau \varphi_2$, where $\tau$ is $\vee$ or $\wedge$, and $\psi_1,\ \psi_2$ are already constructed for $\varphi_1,\ \varphi_2$ then $\psi \rightleftharpoons \psi_1 \tau \psi_2$.
*Bounded quantifier cases.*
Suppose $\varphi(\bar{y}) \rightleftharpoons (\exists v \in y_j)\nu(v, \bar{y})$ and $z_j$ structurally represents $y_j$. If $z_j \in X$, then the formula $\varphi$ is false, so $\psi \rightleftharpoons \perp$. Suppose $z_j = \{z_j^1, \dots, z_j^k\}$. By inductive assumption, for every $\Delta_0$-formula $\nu(z_j^i, \bar{y})$, where $1 \leq i \leq k$, there exists a required $\psi_i$. Put $\psi \rightleftharpoons \bigvee_{1\leq i\leq k} \psi_i$. For the subcase $\varphi(y_1, y_2) \rightleftharpoons (\forall v \in y_j)\nu(v, \bar{y})$, we put $\psi \rightleftharpoons \bigwedge_{1\leq i \leq k} \psi_i$.

**Theorem 3.** *A set $B \subseteq A^n$ is $\Sigma$-definable if and only if there exists an effective sequence of existential formulas in the language $\sigma_0$, $\{\varphi_s(\bar{x})\}_{s\in\omega}$, such that*

$$(x_1, \dots, x_n) \in B \leftrightarrow \mathcal{A} \models \bigvee_{s\in\omega} \varphi_s(x_1, \dots, x_n).$$

*Proof.* $\rightarrow$) Without loss of generality suppose $B$ is $\Sigma$-definable by the formula $\exists y \psi(y, \bar{x})$. It worth noting that for every $y \in \mathrm{HF}(A)$ there exists $z \in \mathrm{HF}(X)$ which structurally represents $y$, and we can effectively enumerate $\mathrm{HF}(X)$. Using Proposition 2 we effectively construct the set of formulas $\psi_j(\bar{x}_j, \bar{x})$ such that

$$\mathbf{HF}(A) \models \exists y \psi(y, \bar{x}) \leftrightarrow A \models \bigvee_{j\in\omega} \exists \bar{x}_j \psi_j(\bar{x}_j, \bar{x}).$$

$\leftarrow$) Let $B \subset A^n$ be definable by $\bigvee_{s\in\omega} \varphi_s(x_1, \dots, x_n)$. By Theorem 2, there exists a universal $\Sigma$-predicate $Univ_n(m, \bar{x})$ for $\Sigma$ formulas with variables from $\{x_1, \dots, x_n\}$. Let the computable function $f : \omega \to \omega$ enumerate the Gödel numbers of the formulas $\varphi_i$, $i \in \omega$. It is easy to see that the following formula is required.

$$\Phi(\bar{x}) \rightleftharpoons \exists i Univ_n(f(i), \bar{x}).$$

It is worth noting that both of the directions of this characterisation are important. The right-to-left direction revels an algorithmic property of $\Sigma$-definability, i.e., gives us an effective procedure which generates existential formulas approximating $\Sigma$-relations. The converse direction provides tools for descriptions of the results of effective infinite approximating processes by finite formulas. Now we consider a structure $\mathcal{A}$ with the topology $\tau_{\Sigma}^{\mathcal{A}}$ formed by a base which is the set of subsets definable by existential formulas in the language $\sigma_0$.

**Theorem 4.** *Every subset of $A$ is effectively open in the topology $\tau_{\Sigma}^{\mathcal{A}}$ if and only if it is $\Sigma$-definable.*

*Proof.* The claim follows from Theorem 3.

## 6 Future work

In this paper we have shown that over every positive predicate structure $\Sigma$-definability has algorithmic properties, i.e., a set is $\Sigma$-definable if and only if it is definable by a disjunction of a recursively enumerable set of existential formulas. In [5] we proved the Uniformity principle for $\Sigma$-definability over the real numbers. We employed the Uniformity principle to show that quantifiers bounded by computable compact sets, rational numbers, polynomials, and computable functions as well can be used in $\Sigma$-formulas without enlarging the class of $\Sigma$-definable sets. It will be interesting to find requirements on a positive predicate structure under which the Uniformity principle holds.

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