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Korovina, Margarita and Kudinov, Oleg

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# A Finite Language for Computable Metric Spaces *

Margarita Korovina[1] and Oleg Kudinov[2]

[1] The University of Manchester and IIS SB RAS Novosibirsk,
Margarita.Korovina@manchester.ac.uk,
[2] Sobolev Institute of Mathematics, Novosibirsk
kud@math.nsc.ru

**Abstract.** In this paper we propose a model-theoretic characterisation of computable metric spaces and computability over them in a finite language.

## 1 Introduction

The main goal of the research presented in this paper is to provide a logical framework for studying computability over computable metric spaces. We consider the following problem. Given a computable metric space is there a finite predicate language such that effective openness and computability could be characterised by appropriate formulas? In order to attack this problem we represent a computable matric space as a positive predicate structure in a finite language and employ $\Sigma$–definability theory. In this settings we prove that for every computable metric space there exists a *finite predicate language* such that the following statements hold.

1. A subset of a computable metric space is effectively open if and only if it is $\Sigma$–definable.
2. The computable functions coincide with the effectively continuous functions in the topology generated by existential formulas.
3. The existential theory of a computable metric space is computably enumerable.
4. A total real-valued function is computable if and only if the epigraph and the hypograph are $\Sigma$-definable.

## 2 Positive Predicate Structures

Our approach is based on representations of data (discrete or continuous ) by a suitable structure $\mathcal{A} = \langle A, \sigma_P \rangle$, where $A$ contains more than one element, and $\sigma_P$ is a finite set of basic predicates. *We assume that all predicates $Q_i \in \sigma_P$ occur*

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*only positively in $\Sigma$-formulas and do not require that the language $\sigma_P$ contains equality.* We call such structures as *positive predicate structures*. Examples of positive predicate structures for the real numbers, the complex numbers, and the function space will be considered later on with precise definitions of appropriate positive predicate languages. Further examples can be found in [8]. In order to do any kind of computation or to develop a computability theory one has to work within a structure rich enough for information to be coded and stored. For this purpose we extend the structure $A$ by the set of hereditarily finite sets $\mathrm{HF}(A)$.

The idea that the hereditarily finite sets over $A$ form a natural domain for computation is quite classical and is developed in detail in [1, 3] for the case when $\sigma_0$ contains equality.

We construct the set of hereditarily finite sets, $\mathrm{HF}(A)$, as follows:

1. $\mathrm{HF}_0(A) \rightleftharpoons A$,
2. $\mathrm{HF}_{\mathrm{n+1}}(A) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_{\mathrm{n}}(A)) \cup \mathrm{HF}_{\mathrm{n}}(A)$, where $n \in \omega$ and for every set $B$, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of $B$.
3. $\mathrm{HF}(A) \rightleftharpoons \bigcup_{n\in\omega} \mathrm{HF}_n(A)$.

We define $\mathbf{HF}(A)$ as the following model:

$$\mathbf{HF}(A) \rightleftharpoons \langle \mathrm{HF}(A), U, \sigma_P, \in \rangle \rightleftharpoons \langle \mathrm{HF}(A), \sigma \rangle\,,$$

where the binary predicate symbol $\in$ has the set-theoretic interpretation. Also we add the predicate symbol $U$ for urelements (elements from $A$).

The natural numbers $0,\ 1, \ldots$ are identified with the (finite) ordinals in $\mathrm{HF}(A)$ i.e. $\emptyset,\ \{\emptyset\},\ \{\emptyset, \{\emptyset\}\}, \ldots$, so in particular, $n + 1 = n \cup \{n\}$ and the set $\omega$ is a subset of $\mathrm{HF}(A)$.

The atomic formulas include $U(x)$, $x \in s$, $x \notin s$ where $s$ ranges over sets, and also, for every $Q_i \in \sigma_P$ of the arity $n_i$, $Q_i(x_1, \ldots, x_{n_i})$ which has the following interpretation:

$$\begin{aligned} &\mathbf{HF}(A) \models Q_i(x_1, \ldots, x_{n_i}) \text{ if and only if} \\ &A \models Q_i(x_1, \ldots, x_{n_i}) \text{ and, for every } 1 \le j \le n_i,\ x_j \in A. \end{aligned}$$

The set of *existential formulas* is the closure of the set of atomic formulas over $\sigma_P$ under $\wedge, \vee$ and existential quantifiers.

The set of *$\Delta_0$-formulas* is the closure of the set of atomic formulas under $\wedge, \vee$, bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$, where $(\exists x \in y)\ \Psi$ means the same as $\exists x (x \in y \wedge\ \Psi)$ and $(\forall x \in y)\ \Psi$ as $\forall x (x \in y \to \Psi)$ where $y$ ranges over sets.

The set of *$\Sigma$-formulas* is the closure of the set of $\Delta_0$-formulas under $\wedge$,$\vee$, $(\exists x \in y)$, $(\forall x \in y)$ and $\exists x$, where $y$ ranges over sets.

*Remark 1.* It is worth noting that all predicates $Q_i \in \sigma_P$ occur only positively in all formulas above. Hence if equality is not a basic predicate then in $\Sigma$-formulas we don't allow equality on the urelements (elements from $A$).

We are interested in $\Sigma$-definability of sets on $A^n$ which can be considered as a generalisation of recursive enumerability. The analogy of $\Sigma$-definable and recursively enumerable sets is based on the following fact. Consider the structure $\mathbf{HF} = \langle \mathrm{HF}(\emptyset), \in \rangle$ with the hereditarily finite sets over $\emptyset$ as its universe and membership as its only relation. In **HF** the $\Sigma$-definable subsets of $\omega$ are exactly the recursively enumerable sets [1].

The notion of $\Sigma$-definability has a natural meaning also in the structure $\mathbf{HF}(A)$.

**Definition 1.** *1. A relation $B \subseteq \mathrm{HF}(A)^n$ is $\Sigma$-definable, if there exists a $\Sigma$-formula $\Phi(\bar{a})$ such that*

$$\bar{b} \in B \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{b}).$$

*2. A function $f : \mathrm{HF}(A)^n \to \mathrm{HF}(A)^m$ is $\Sigma$-definable, if there exists a $\Sigma$-formula $\Phi(\bar{c}, \bar{d})$ such that*

$$f(\bar{a}) = \bar{b} \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{a}, \bar{b}).$$

In similar way we introduce the notion of $\Delta_0$-definability.

**Theorem 1.** *A set $B \subseteq A^n$ is $\Sigma$-definable if and only if there exists an effective sequence of existential formulas in the language $\sigma_P$, $\{\varphi_s(\bar{x})\}_{s\in\omega}$, such that*

$$(x_1, \ldots, x_n) \in B \leftrightarrow \mathcal{A} \models \bigvee_{s\in\omega} \varphi_s(x_1, \ldots, x_n).$$

*Proof.* See [6].

Now we discuss links between structures and topological spaces. Let $\mathcal{A}$ be a positive predicate structure. *The topology* $\tau_\Sigma^\mathcal{A}$ is formed by a base which is the set of subsets definable by existential formulas in the language $\sigma_P$. The following proposition shows that the topology $\tau_\Sigma^\mathcal{A}$ is natural with respect to $\Sigma$-definability.

**Theorem 2.** *Every subset of $A$ is effectively open in the topology $\tau_\Sigma^\mathcal{A}$ if and only if it is $\Sigma$-definable.*

*Proof.* The claim follows from Theorem 1.

## 3 A Finite Language for Computable Metric Spaces

For the definition of computable metric space we refer to [14, 15]. Let $\mathcal{M} = (M, \nu, \mathbf{B}, d)$ be a computable metric space, where $\mathbf{B} = \{b_r\}_{r\in\omega} \subseteq M$ is countable and dense in $M$, $\nu : \omega \to \mathbf{B}$ is a numbering, and $d : M \times M \to \mathbb{R}$ is a distance function computable on $(\mathbf{B}, \nu)$.

### 3.1 Case Studies

Now we consider examples of structures for the classical computable metric spaces.

1. (The real numbers) $\mathbb{R} = \langle \mathbb{R}, \mathcal{M}^*_E, \mathcal{M}^*_H, \mathcal{P}^+_E, \mathcal{P}^+_H, < \rangle$, where $\mathcal{M}^*_E, \mathcal{M}^*_H$ are interpreted as the open epigraph and the open hypograph of multiplication respectively, and $\mathcal{P}^+_E, \mathcal{P}^+_H$ are interpreted as the open epigraph and the open hypograph of addition respectively.
2. (The complex numbers) $\mathbb{C} = \langle \mathbb{C}, , P_1, \dots, P_{12} \rangle$, where the predicates $P_1, \dots, P_{12}$ have the following meanings for every $x,\ y,\ z \in \mathbb{C}$.
   The first group formalises relations between $Re$ and $Im$ of two complex numbers.
   $$\mathbb{C} \models P_1(x,y) \leftrightarrow Re\, x < Re\, y; \mathbb{C} \models P_2(x,y) \leftrightarrow Im\, x < Im\, y;$$
   $$\mathbb{C} \models P_3(x,y) \leftrightarrow Re\, x < Im\, y; \mathbb{C} \models P_4(x,y) \leftrightarrow Im\, x < Re\, y.$$
   The second group formalises properties of operations.
   $$\begin{aligned}
   \mathbb{C} &\models P_5(x,y,z) \leftrightarrow Re\, x + Re\, y < Re\, z;\\
   \mathbb{C} &\models P_6(x,y,z) \leftrightarrow Re\, x + Re\, y > Re\, z;\\
   \mathbb{C} &\models P_7(x,y,z) \leftrightarrow Re\, x \cdot Re\, y < Re\, z;\\
   \mathbb{C} &\models P_8(x,y,z) \leftrightarrow Re\, x \cdot Re\, y > Re\, z;\\
   \mathbb{C} &\models P_9(x,y,z) \leftrightarrow Im\, x + Im\, y < Im\, z;\\
   \mathbb{C} &\models P_{10}(x,y,z) \leftrightarrow Im\, x + Im\, y > Im\, z;\\
   \mathbb{C} &\models P_{11}(x,y,z) \leftrightarrow Im\, x \cdot Im\, y < Im\, z;\\
   \mathbb{C} &\models P_{12}(x,y,z) \leftrightarrow Im\, x \cdot Im\, y > Im\, z;
   \end{aligned}$$
3. (The function space) $C[0,1] = \langle C[0,1], P_1, \dots, P_{10} \rangle$ where the predicates $P_1, \dots, P_{10}$ have the following meanings for every $f,\ g \in C[0,1]$:
   The first group formalises relations between infimum and supremum of two functions.
   $$\begin{aligned}
   C[0,1] &\models P_1(f,g) \leftrightarrow \sup f < \sup g;\\
   C[0,1] &\models P_2(f,g) \leftrightarrow \sup f < \inf g;\\
   C[0,1] &\models P_3(f,g) \leftrightarrow \sup f > \inf g;\\
   C[0,1] &\models P_4(f,g) \leftrightarrow \inf f > \inf g.
   \end{aligned}$$
   The second group formalises properties of operations on $C[0,1]$.
   $$\begin{aligned}
   C[0,1] &\models P_5(f,g,h) \leftrightarrow f(x) + g(x) < h(x);\ \text{for every } x \in [0,1];\\
   C[0,1] &\models P_6(f,g,h) \leftrightarrow f(x) \cdot g(x) < h(x) \text{ for every } x \in [0,1];\\
   C[0,1] &\models P_7(f,g,h) \leftrightarrow f(x) + g(x) > h(x) \text{ for every } x \in [0,1];\\
   C[0,1] &\models P_8(f,g,h) \leftrightarrow f(x) \cdot g(x) > h(x) \text{ for every } x \in [0,1].
   \end{aligned}$$
   The third group formalises relations between functions $f$ and the identity function $\lambda x.x$.
   $$\begin{aligned}
   C[0,1] &\models P_9(f) \leftrightarrow f > \lambda x.x;\\
   C[0,1] &\models P_{10}(f) \leftrightarrow f < \lambda x.x.
   \end{aligned}$$

**Proposition 1.** *1. For the structure* $\mathbb{R} = \langle \mathbb{R}, \mathcal{M}_E^*, \mathcal{M}_H^*, \mathcal{P}_E^+, \mathcal{P}_H^+ <\rangle$ *the topology* $\tau_\Sigma^{\mathbb{R}}$ *coincides with the real line topology.*

2. *For the structure* $\mathbb{C} = \langle \mathbb{C}, , P_1, \ldots, P_{12} \rangle$ *the topology* $\tau_\Sigma^{\mathbb{C}}$ *coincides with the plane topology.*
3. *For the structure* $C[0,1] = (C[0,1], P_1, \ldots, P_{10})$*, the topology* $\tau_\Sigma^{C[0,1]}$ *coincides with the topology* $\tau_{||\ ||}$ *induced by the supremum norm.*

*Proof.* The first two claims are trivial. Let us prove the last statement. $\subseteq$). It is easy to see that $\{\bar{x} | \mathbf{HF}(C[0,1]) \models P_i(\bar{x})\} \in \tau_{||\ ||}$ for every $1 \le i \le 10$. Since $\left(C[0,1], d_{||\ ||}\right)^m$ is a metric space, a projection of an open set is again open. So, $\{\bar{x} | \mathbf{HF}(C[0,1]) \models Q(\bar{x}),\ Q \text{ is a } \exists\text{-formula}\} \in \tau_{||\ ||}$. By induction, $\tau_\Sigma^{C[0,1]} \subseteq \tau_{||\ ||}$.
$\supseteq$). First, recall that a base of the topology $\tau_{||\ ||}$ is the following:

$$\tau_{||\ ||}^* = \{\{f | \, ||f - p_i|| < \epsilon\} | p \text{ is a polynomial with rational coefficients, } \epsilon \in \mathbb{Q}\}.$$

Since the set $\{p | \, p$ is a polynomial with rational coefficients$\}$ is dense in $C[0,1]$, it is sufficient to show that $f > p$ and $f < p$ are $\exists$-definable. This claim follows from the following equivalences:

$$\begin{array}{ll} f > 0 & \leftrightarrow f + f > f; \\ f > 1 & \leftrightarrow \exists g \, (f \cdot g > g \wedge g > 0)\,; \\ f < 0 & \leftrightarrow f + f < f; \\ f < 1 & \leftrightarrow \exists g \, (f < 0 \vee g > 0 \wedge f \cdot g < g)\,; \\ f > x^2 & \leftrightarrow \exists g \, (g > \lambda x.x \wedge f > g \cdot g)\,; \\ f < x^2 & \leftrightarrow \exists g \, (g < \lambda x.x \wedge f < g \cdot g)\,; \\ f > \frac{x}{n} & \leftrightarrow \exists g \, (g > \lambda x.x \wedge (f + \cdots + f) > g)\,; \\ f < \frac{x}{n} & \leftrightarrow \exists g \, (g < \lambda x.x \wedge (f + \cdots + f) < g)\,; \ldots \end{array}$$

So, the set $\{f | \, ||f - p|| < \epsilon,\, p$ is a polynomial with rational coefficients, $\epsilon \in \mathbb{Q}\}$ is $\exists$-definable for every considered $p$ and $\epsilon$. Therefore $\tau_\Sigma^{\mathcal{C}} \supseteq \tau_{||\ ||}$.

### 3.2 The General Case

Let $\mathbb{M} = (M, \mathbf{B}, d)$ be a computable metric space. We define the corresponding positive predicate structure $\mathcal{M} = (M, \sigma_P) = (M, R_0, S, D_1, D_2, D_3)$, where the predicates have the following meanings

$$\begin{gathered} D_1(x, y, u, v) \rightleftharpoons d(x, y) < d(u, v), \\ D_2(y, z, v) \rightleftharpoons d(y, z) - d(y, v) < 1, \\ D_3(y, z, t, w, s) \rightleftharpoons 2\,(d(y, z) - d(y, v)) < d(t, w) - d(t, s), \\ R_n(x, y, z, v) \rightleftharpoons 2d(x, b_n) < d(y, z) - d(y, z) \wedge (\forall i < n)\, 2d(x, b_i) > d(y, z) - d(y, z), \\ S(x, y, z, v, a, b, c, d) \leftrightarrow \bigvee_{n \in \omega} (R_n(x, y, z, v) \wedge R_{n+1}(a, b, c, d)). \end{gathered}$$

**Theorem 3.** *Let* $\mathbb{M}^n$ *be a computable metric space and* $\mathcal{M}$ *be the corresponding positive predicate structure defined above. Then the topology* $\tau_\Sigma^{\mathcal{M}}$ *coincides with the topology* $\tau_d$ *induced by the metric in an effective way that means that the lists of effectively open sets coincide and one can compute corresponding indices from each other.*

*Proof.* $\subseteq$). By definition the predicates $S$, $D_1$, $D_2$, $D_3$, and $R_0$ define sets which are open in the product topology. Since $\mathbb{M}$ is a metric space, a projection of an open set is again open. So, by induction, every $\exists$-definable subset of $M$ belongs to $\tau_d$.
$\supseteq$). It is sufficient to show that the balls $B(b_r, a)$, where $b_r \in \mathbf{B}$ and $a \in \mathbb{Q}^+$, are uniformly $\Sigma$-definable.

First we show by induction on $n$, that $R_n$ are uniformly $\Sigma$-definable.
$n = 0$. By the definition $R_0 \in \sigma_P$.
$n \to n+1$. Since by definition $R_n \cap R_m = \emptyset$ for $n \neq m$, we can $\Sigma$-define $R_{n+1}$ as follows.

$$R_{n+1}(a, b, c, d) \rightleftharpoons \exists x \exists y \exists z \exists v \left(R_n(x, y, z, v) \wedge S(x, y, z, v, a, b, c, d)\right).$$

The next step is to define predicates $A_s^m$ with the following properties: for every $m \in \omega$ and $s \in \omega$, the set $\{x | \mathbf{HF}(M) \models A_s^m(x)\}$ is a subset of the ball $B(b_s, \frac{1}{2^m})$, and for all $x \in M$ and $m \in \omega$ there exists $s \in \omega$ such that $\mathbf{HF}(M) \models A_s^m(x)$.

Put

$$\begin{aligned}
A_s^1(x) &\rightleftharpoons \exists y \exists z \exists v (R_s(x, y, z, v) \wedge d(y, z) - d(y, v) < 1);\\
A_s^{m+1}(x) &\rightleftharpoons \exists y \exists z \exists v \exists s_1 \ldots \exists s_m \exists w_1 \ldots \exists w_m \exists t_1 \ldots \exists t_m (R_s(x, y, z, v)) \wedge \\
&\quad 2(d(y, z) - d(y, v)) < d(t_1, w_1) - d(t_1, s_1) \wedge \\
&\quad d(t_m, w_m) - d(t_m, s_m) < 1 \wedge \\
&\quad \bigwedge_{1 \le i < m-1} 2(d(t_i, w_i) - d(t_i, s_i)) < d(t_{i+1}, w_{i+1}) - d(t_{i+1}, s_{i+1})).
\end{aligned}$$

By definition the first property holds. We prove the second one. Let $x \in M$ and $m \in \omega$. We find the first $s \in \omega$ such that $2d(x, b_s) < d(y_m, z_m) < \frac{1}{2^m}$, where $y_m$, $z_m$, $v_m$ are inductively constructed as follows.

$$\begin{aligned}
&0 < d(y_0, z_0) - d(y_0, v_0) < 1;\\
&0 < d(y_1, z_1) - d(y_1, v_1) < \frac{d(y_0, z_0) - d(y_0, v_0)}{2}.
\end{aligned}$$

In order to avoid the case $d(x, b_i) = d(y_m, z_m)$ for $i < s$, we choose $v \in \mathbf{B}$ such that for all $i < s$ we have $2d(x, b_i) > d(y_m, z_m) - d(y_m, v)$ and $2d(x, b_s) < d(y_m, z_m) - d(y_m, v)$. Then $\mathbf{HF}(M) \models R_s(x, y_m, z_m, v)$ and $d(y_m, z_m) < \frac{1}{2^m}$. So $x \in A_s^m$. Now we are ready to prove that the balls $B(b_r, a)$ are $\Sigma$-definable. For this we show the following equivalence.

$$d(x, b_r) < a \leftrightarrow \mathbf{HF}(M) \models \exists s \exists m \left(d(b_r, b_s) < a - \frac{1}{2^m} \wedge x \in A_s^m\right).$$

$\leftarrow$). If $x \in A_s^m$ then as we have shown above $d(x, b_s) < \frac{1}{m}$. So $d(b_r, x) < d(b_r, b_s) + d(x, b_s) < a$.
$\rightarrow$). Since $a - d(x, b_r) > 0$, we can find $N \in \omega$ such that $a - d(x, b_r) > \frac{1}{2^N}$. We already proved that for $x$ and $N$ there exists $s$ such that $x \in A_s^{N+1}$. So $d(x, b_s) < \frac{1}{2^{N+1}}$. Finally, $d(b_r, b_s) < d(b_r, x) + d(x, b_s) < a - \frac{1}{2^{N+1}}$.

**Corollary 1.** *Every subset of a computable metric space is effectively open if and only if it is $\Sigma$-definable.*

*Proof.* The claim follows from Theorem 2 and Theorem 3.

**Corollary 2.** *Every function over a computable metric space is computable if and only if it is effectively continuous in $\tau_{\Sigma}^{M}$ topology.*

*Proof.* The claim follows from Theorem 3 and [14, 16].

**Corollary 3.** *If $\mathbb{M}$ is a computable metric space then $Th_{\exists}(M)$ is computably enumerable.*

*Proof.* The claim follows from Theorem 3 and [7].

**Corollary 4.** *A total function $F : M \to \mathbb{R}$ is computable if and only if the epigraph and the hypograph are $\Sigma$-definable.*

*Proof.* The claim follows from Theorem 3 and [10, 7].

## 4 Conclusion

A finite language is preferable for many applications where effective representation of continuous data are required. The main challenge of this work is to keep the language finite yet powerful to express computability. The obtained results show that many effectively enumerable $T_2$-spaces admit effective structurisations.

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