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Stable topological transitivity properties of $\mathbb{R}^n$-extensions of hyperbolic transformations

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Abstract

We consider $\mathbb{R}^n$ skew-products of a class of hyperbolic dynamical systems. It was proved by Niţică and Pollicott [NP] that for an Anosov diffeomorphism ϕ of an infranilmanifold Λ there is (subject avoiding natural obstructions) an open and dense set $f : \Lambda \to \mathbb{R}^N$ for which the skew-product $\phi^f(x, s) = (\phi(x), s + f(x))$ on $\Lambda \times \mathbb{R}^N$ has a dense orbit. We prove a similar result in the context of an Axiom A hyperbolic flow on an attractor.

1 Introduction

There has been much recent interest in understanding the dynamical behaviour of partially hyperbolic dynamical systems, from both a measure-theoretic and topological viewpoint. A particularly tractable class of partially hyperbolic dynamical system is formed by constructing a group extension of a hyperbolic dynamical system, for example constructing a skew-product. By hyperbolic we essentially mean a dynamical system that satisfies Smale's Axiom A, restricted to a basic set.

We will be interested in hyperbolic flows (for example, Anosov flows on compact manifolds such as geodesic flows on compact Riemannian manifolds with negative sectional curvature, suspensions of hyperbolic diffeomorphisms, etc). Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. Let $f : \Lambda \to \mathbb{R}$ and construct the skew product flow $\phi_t^f : \Lambda \times \mathbb{R} \to \Lambda \times \mathbb{R}$ defined by

$$\phi_t^f(x, v) = \left(\phi_t(x), v + \int_0^t f(\phi_u(x))\, du \right). \tag{1}$$

More generally, we will consider an $\mathbb{R}^N$ skew-product.

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In discrete time, an $\mathbb{R}$ skew-product of a hyperbolic diffeomorphism $\phi : \Lambda \to \Lambda$ is defined by taking a function $f : \Lambda \to \mathbb{R}$ and defining $\phi^f : \Lambda \times \mathbb{R} \to \Lambda \times \mathbb{R}$ by $\phi^f(x, v) = (\phi(x), v + f(x))$. More generally, one can consider $\mathbb{R}^N$ skew-products, or skew-products where $\mathbb{R}^N$ is replaced by a connected Lie group.

A dynamical system is said to be *transitive* if there exists a point whose forward orbit under the dynamics is dense. We are interested in conditions on $f : \Lambda \to \mathbb{R}^N$ that ensure that (1) is transitive. An obvious necessary condition for (1) to be transitive is that $\int_0^t f(\phi_u(x))\, du$ takes arbitrarily large (both positive and negative) values; more generally a necessary condition for transitivity is that f satisfies the inseparability hypothesis defined in §3 below.

Several authors [N, NP, MNT] have studied skew-products with a hyperbolic base and fibre $\mathbb{R}^N$, or more generally with fibre $SO(N) \ltimes \mathbb{R}^N$. In [N], it was proved that $\mathbb{R}$ skew-products over a shift of finite type are stably transitive (in an appropriate category). In [NP], it was proved that $\mathbb{R}^N$ skew-products where the base transformation $\phi : X \to X$ is an Anosov diffeomorphism of an infranilmanifold is stably transitive in the Hölder category. The crucial fact in the analysis in [NP] is that the induced map $\phi^* : H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ does not have 1 as an eigenvalue. This is known to be the case for an Anosov diffeomorphism of an infranilmanifold and it is an important open question as to whether this is the case for all Anosov diffeomorphisms (cf [B2]) (indeed, it is an open question as to whether there are Anosov diffeomorphisms on spaces other than infranilmanifolds). We remark that there are examples [CPW] of hyperbolic attractors $\phi : X \to X$ that satisfy Smale's Axiom A but which do have 1 as an eigenvalue of the induced map ϕ^* on the first Čech cohomology group $H^1(X, \mathbb{Z})$.

In this note we generalise the results of [NP] to the case of hyperbolic flows, using ideas from [W]. Recall that a subset $\{a_\alpha\}$ of an Abelian group A is linearly independent (over $\mathbb{Z}$) if $\sum n_\alpha a_\alpha = 0$ (where $n_\alpha \in \mathbb{Z}$ and only finitely many of the n_α are non-zero) implies that all the n_α are zero. The *rank* of A is the maximum cardinality of a linearly independent set. We are interested in hyperbolic flows $\phi_t : \Lambda \to \Lambda$ for which $H^1(\Lambda, \mathbb{Z})$ has finite rank. If Λ is an attractor then $H^1(\Lambda, \mathbb{Z})$ has finite rank [FP]; however there are examples of hyperbolic flows on basic sets which do not have finite rank (for example, a suspension of a Smale horseshoe [PT]). The assumption that $H^1(\Lambda, \mathbb{Z})$ has finite rank will, essentially, allow us to embed $H^1(\Lambda, \mathbb{Z})$ in a finite-dimensional vector space.

We prove the existence of an open and dense set of cocycles amongst those satisfying the inseparability hypothesis for which ϕ_t^f is transitive. Indeed, we prove

Theorem 1.1 *Let ϕ_t be a hyperbolic flow on a basic set Λ. Suppose that*

$H^1(\Lambda, \mathbb{Z})$ has finite rank. Then the set of $\mathbb{R}^N$ skew-products ϕ_t^f for which f satisfies the inseparability hypothesis contains an open dense set in the Hölder category (of fixed exponent).

We also prove the analogue of Theorem 1.1 in the case of a hyperbolic diffeomorphism on a basic set Λ for which $H^1(\Lambda, \mathbb{Z})$ has finite rank. This is a strictly weaker condition than that considered in [NP]. We remark that [MNT] prove the existence of an open dense set of functions $f : \Lambda \to \mathbb{R}^N$ satisfying the inseparability hypothesis for which the skew-product of a hyperbolic diffeomorphism $\phi : \Lambda \to \Lambda$ is transitive without any hypothesis on the cohomology of Λ. The methodology in [MNT] involves considering local perturbations of f in a neighbourhood of a certain set of homoclinic orbits for the diffeomorphism ϕ. The difference in this note is that, at least for Abelian skew-products, we can explicitly and globally describe the set of functions f that give rise to transitive skew-products in terms of the underlying homology of the set Λ (modulo natural obstructions).

2 Definitions

2.1 Hyperbolic dynamical systems

Let M be a smooth compact Riemannian manifold and let $\phi_t : M \to M$ be a C^1 flow. A ϕ_t-invariant closed set $\Lambda \subset M$ is called a *basic set* and ϕ_t restricted to Λ is called a *hyperbolic flow* if: (i) the tangent bundle $T_\Lambda M$ of M restricted to Λ can be split continuously into a Whitney sum $E^s \oplus E^u \oplus E^0$ of $D\phi_t$-invariant sub-bundles for which there exist constants $C > 0$ and $\lambda \in (0,1)$ such that for $t \geq 0$

$$\begin{aligned} \|D\phi_t(v)\| &\leq C\lambda^t\|v\|, \text{for } v \in E^s \\ \|D\phi_{-t}(v)\| &\leq C\lambda^t\|v\|, \text{for } v \in E^u; \end{aligned}$$

and E^0 is one-dimensional and tangent to the orbits of the flow; (ii) ϕ_t restricted to Λ has a dense set of periodic points, is topologically transitive, and is not a single orbit; (iii) Λ is locally maximal in the sense that there is an open neighbourhood $U \supset \Lambda$ such that $\bigcap_{t=-\infty}^{\infty} \phi_t(U) = \Lambda$.

Thus in defining a hyperbolic flow we are merely abstracting the properties of a basic set arising from the spectral decomposition of the non-wandering set of an Axiom A diffeomorphism or flow.

If $\Lambda = M$ then we call ϕ_t an Anosov flow.

2.2 Dynamic cohomology

A function $f : \Lambda \to \mathbb{R}^N$ is Hölder-continuous of exponent $\alpha \in (0,1)$ if $|f|_\alpha := \sup_{x \neq y} \|f(x) - f(y)\|/d(x,y)^\alpha < \infty$. The vector space $C^\alpha(\Lambda, \mathbb{R}^N)$ of all such functions is a Banach space with respect to the norm $\|\cdot\|_\alpha = |\cdot|_\infty + |\cdot|_\alpha$ where $|f|_\infty = \sup_x \|f(x)\|$.

Let K denote the unit circle in $\mathbb{C}$ and define $C^\alpha(\Lambda, K)$ to be the space of Hölder-continuous, K-valued functions. This space is complete with respect to the metric inherited from $\|\cdot\|_\alpha$.

Let ϕ_t be a flow on Λ. A function $w : \Lambda \to \mathbb{C}$ is said to be *flow-differentiable* if

$$w'_\phi(x) = \lim_{t \to 0} \frac{w(\phi_t(x)) - w(x)}{t}$$

exists for each $x \in \Lambda$. We say that w is C^α flow-differentiable if $w \in C^\alpha(\Lambda, \mathbb{R})$, w'_ϕ exists and $w'_\phi \in C^\alpha(\Lambda, \mathbb{R})$.

Let $f : \Lambda \to \mathbb{R}$ be Hölder continuous. Define a *dynamic cocycle* $F_t(x) = F(t, x) : \mathbb{R} \times \Lambda \to \mathbb{R}^N$ by setting

$$F_t(x) = \int_0^t f(\phi_u x)\, du \tag{2}$$

so that

$$F_{t+s}(x) = F_t(\phi_s(x)) + F_s(x). \tag{3}$$

Any function $F_t(x)$ that satisfies (3) and for which $\lim_{t\to 0} t^{-1} F_t(x)$ exists is of the form (2).

Let $f = (f^{(1)}, \ldots, f^{(N)}) : \Lambda \to \mathbb{R}^N$ be Hölder. Then we define $F_t(x) = (F_t^{(1)}, \ldots, F_t^{(N)})$ and define the skew-product flow $\phi_t^F : \Lambda \times \mathbb{R}^N \to \Lambda \times \mathbb{R}^N$ by $\phi_t^F(x, v) = (\phi_t(x), v + F_t(x))$.

We say that a Hölder cocycle F_t on $\mathbb{R}^N$ is a *coboundary* if $F_t(x) = u(\phi_t(x)) - u(x)$ for some Hölder continuous function $u \in C^\alpha(\Lambda, \mathbb{R}^N)$. This is equivalent to requiring that $f = u'_\phi$.

Let $\mathcal{B}^\alpha(\phi, \mathbb{R}^N)$ denote the set of all coboundaries of $C^\alpha(\phi, \mathbb{R}^N)$ functions. We will often write $\mathcal{B}_\phi$ for $\mathcal{B}^\alpha(\phi, \mathbb{R}^N)$ when α, N are fixed. Let $\mathcal{H}^\alpha(\phi, \mathbb{R}^N) = C^\alpha(\Lambda, \mathbb{R}^N)/\mathcal{B}^\alpha(\phi, \mathbb{R}^N)$ denote the vector space of all dynamic cocycles.

2.3 Bruschlinsky cohomology

Typically the basic set Λ will have a very complicated topological structure and will not be a manifold. Let $N(\Lambda, K) = \{e^{2\pi i r} \mid r : \Lambda \to \mathbb{R} \text{ is continuous}\}$ denote the space of continuous circle-valued null-homotopic functions. The Bruschlinsky group is defined to be the quotient group $C(\Lambda, K)/N(\Lambda, K)$ with pointwise multiplication; it is well-known [H] that the first Čech cohomology group $H^1(\Lambda, \mathbb{Z})$ is isomorphic to the Bruschlinsky group. It is easy to see that a circle-valued continuous function may be replaced up to homotopy by a circle-valued Hölder function; hence we may write $H^1(\Lambda, \mathbb{Z}) = C^\alpha(\Lambda, K)/N^\alpha(\Lambda, K)$ where $N^\alpha(\Lambda, K)$ denotes the space of C^α null-homotopic functions. Write $[w]$ for the homotopy class of w.

2.4 Dynamic versus Bruschlinsky cohomology

Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. For each $w \in C(\Lambda, K)$ it is clear that $w\phi_t$ is homotopic to w. Hence for each $t \in \mathbb{R}$ there exists a continuous function $G_t : \Lambda \to \mathbb{R}$ such that

$$\frac{w(\phi_t(x))}{w(x)} = \exp 2\pi i G_t(x).$$

If w is flow-differentiable then $g(x) = \lim_{t\to 0} G_t(x)/t$ exists and

$$\frac{w(\phi_t(x))}{w(x)} = \exp 2\pi i \int_0^t g(\phi_u(x)). \tag{4}$$

If w is C^α flow-differentiable then $g \in C^\alpha(\Lambda, \mathbb{R})$.

Suppose that $w_1, w_2 \in C^\alpha(\Lambda, K)$ are flow-differentiable and homotopic. Define g_1, g_2 by

$$\frac{w_1(\phi_t(x))}{w_1(x)} = \exp 2\pi i \int_0^t g_1(\phi_u(x))\, du, \quad \frac{w_2(\phi_t(x))}{w_2(x)} = \exp 2\pi i \int_0^t g_2(\phi_u(x))\, du. \tag{5}$$

If $w_1/w_2 = \exp 2\pi i u$ then it is easy to see that $g_1 = g_2 + u'_\phi$. That is, homotopic circle-valued functions give rise to cohomologous real-valued functions. Thus we have a well-defined map

$$\iota : H^1(\Lambda, \mathbb{Z}) \to \mathcal{H}^\alpha(\phi, \mathbb{R}).$$

If w_1, w_2, g_1, g_2 satisfy (5) then

$$\frac{w_1(\phi_t(x))w_2(\phi_t(x))}{w_1(x)w_2(x)} = \exp 2\pi i \int_0^t (g_1(\phi_u(x)) + g_2(\phi_u(x))\, du.$$

Hence $\iota([w_1w_2]) = \iota([w_1]) + \iota([w_2])$ so that ι is a homomorphism.

If $\iota([w]) = 0$ then there exists $u \in C^\alpha(\Lambda, \mathbb{R})$ such that $w(\phi_t(x))/w(x) = \exp 2\pi i(u(\phi_t(x)) - u(x))$. Hence $w \exp(-2\pi i u)$ is a ϕ_t-invariant continuous function, and therefore constant by the transitivity of ϕ_t. Hence w is null-homotopic and it follows that ι is injective.

For each $[w] \in H^1(\Lambda, \mathbb{Z})$, choose $w \in C^\alpha(\Lambda, K)$ in that homotopy class and construct g as in (4) so that $\iota([w]) = g + \mathcal{B}_\phi$. Let $\mathcal{Z}_\mathbb{R}$ denote the $\mathbb{R}$-linear span of the gs. Although $\mathcal{Z}_\mathbb{R}$ is not uniquely determined (due to the choice of w up to a null-homotopic function in the homotopy class $[w]$), it is well-defined up to the addition of coboundaries, i.e. one can write

$$\iota(H^1(\Lambda, \mathbb{Z})) \otimes \mathbb{R} = (\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi)/\mathcal{B}_\phi. \tag{6}$$

We shall need the following result.

Proposition 2.1 *Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. Suppose that $H^1(\Lambda, \mathbb{Z})$ has finite rank. Then:*

(i) $\mathcal{B}_\phi$ is a closed subspace of $C^\alpha(\Lambda, \mathbb{R})$;

(ii) $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$ is a closed subspace of $C^\alpha(\Lambda, \mathbb{R})$;

(iii) $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$ has infinite codimension in $C^\alpha(\Lambda, \mathbb{R})$.

First recall the well-known Livšic periodic data criteria for functions to be coboundaries.

Lemma 2.2 ([L]) *Let ϕ_t be a hyperbolic flow.*

(i) Suppose $f \in C^\alpha(\Lambda, \mathbb{R})$ is such that $f^T(x) = 0$ whenever $\phi_T(x) = x$. Then there exists $u \in C^\alpha(\Lambda, \mathbb{R})$ such that $f = u'_\phi$.

(ii) Suppose $f \in C^\alpha(\Lambda, \mathbb{R})$ is such that $\exp 2\pi i \int_0^T f(\phi_u(x))\, du = 1$ whenever $\phi_T(x) = x$. Then there exists $w \in C^\alpha(\Lambda, K)$ such that

$$\frac{w(\phi_t(x))}{w(x)} = \exp 2\pi i \int_0^t f(\phi_u(x))\, du.$$

Proof of Proposition 2.1. Let $h_n = (u_n)'_\phi \in \mathcal{B}_\phi$ be such that $h_n \to h$ in $C^\alpha(\Lambda, \mathbb{R})$. Suppose that $\phi_T(x) = x$. Then $\int_0^T h_n(\phi_u(x))\, du = 0$, hence $\int_0^T h(\phi_u(x))\, du = 0$. It follows from Lemma 2.2(i) that $h \in \mathcal{B}_\phi$.

To prove (ii) it is sufficient to prove that $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$ is finite dimensional. Let β denote the rank of $H^1(\Lambda, \mathbb{Z})$ and choose a maximal set of integrally independent elements $[w_j] \in H^1(\Lambda, \mathbb{Z})$, $1 \le j \le \beta$. For each j, let $w_j \in C^\alpha(\Lambda, K)$ be in the homotopy class $[w]$ and define g_j by (4). Then any element of $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$ is a real linear combination of the $g_j + \mathcal{B}_\phi$, so that $\dim \mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi \le \beta$.

To prove (iii) it is sufficient to prove that $\mathcal{B}_\phi$ has infinite codimension in $C^\alpha(\Lambda, \mathbb{R})$. First note that if μ is a ϕ-invariant measure then $\int u'_\phi\, d\mu = 0$. Hence any invariant measure defines a linear functional on $C^\alpha(\Lambda, \mathbb{R})/\mathcal{B}_\phi$. If $C^\alpha(\Lambda, \mathbb{R})/\mathcal{B}_\phi$ were finite dimensional, then its dual space would be finite dimensional. This is a contradiction as invariant measures supported on periodic orbits generate an infinite-dimensional subspace. ❏

3 Criteria for topological transitivity

Throughout, ϕ_t will be a hyperbolic flow on a basic set Λ. Let $f : \Lambda \to \mathbb{R}^N$ be a Hölder function. Let $\mathcal{L}$ denote the set of weights of periodic orbits of ϕ_t:

$$\mathcal{L} = \{F_T(x) \mid \phi_T(x) = x\} \subset \mathbb{R}^N.$$

Let $V = \{\sum m_i p_i \mid m_i \in \mathbb{Z}^+, p_i \in \mathcal{L}\}$ denote the semigroup of $\mathbb{R}^N$ generated by $\mathcal{L}$.

Definition. We say that f satisfies the *inseparability hypothesis* if f is not cohomologous to a function that takes values in a half-space bounded by a hyperplane in $\mathbb{R}^N$ passing through the origin.

Remark. By the positive Livšic theorem for hyperbolic flows [PS], it follows that f satisfies the inseparability hypothesis if and only if, for all $v \in \mathbb{R}^N \setminus \{0\}$, $\langle V, v\rangle$ takes both positive and negative values.

Definition. We say that V is *co-lattice valued* if there exists $v \in \mathbb{R}^N \setminus \{0\}$ and $a \in \mathbb{R}$ such that $\langle V, v\rangle \subset a\mathbb{Z}$.

Exactly one of the following holds:

(i) the closure $\bar{V} = \mathbb{R}^N$,

(ii) $\bar{V}$ is co-latticed valued and there exists $v \in \mathbb{R}^N \setminus \{0\}$ such that $\langle V, v\rangle \subset \mathbb{R}$ is not contained on one side of zero,

(iii) $\bar{V}$ is co-latticed valued and there exists $v \in \mathbb{R}^N \setminus \{0\}$ such that $\langle V, v\rangle \subset \mathbb{R}$ is contained on one side of zero.

Lemma 3.1 *Suppose that $\mathcal{L}$ satisfies the inseparability hypothesis and is not co-latticed valued. Then $\bar{V} = \mathbb{R}^N$.*

Proof. In the above trichotomy, (iii) cannot happen by the inseparability hypothesis. As (ii) cannot hold, it follows that $\bar{V} = \mathbb{R}^N$. ❑

We need the following result:

Lemma 3.2 ([PS]) *Suppose that $f \in C^\alpha(\Lambda, \mathbb{R})$ generates a real-valued cocycle $F_t(x) = \int_0^t f(\phi_u(x))\, du$ such that $F_T(x) \geq 0$ whenever $\phi_T(x) = x$. Then F is bounded below: $\inf_{t\in\mathbb{R}, x\in X} F_t(x) > -\infty$.*

The following two results links transitivity to our hypotheses.

Proposition 3.3 *The following are equivalent:*

(i) $\mathcal{L}$ satisfies the inseparability hypothesis and is not co-lattice valued;

(ii) the skew-product flow ϕ_t^F is transitive.

Proof. If (i) holds then, by Lemma 3.1, $\bar{V} = \mathbb{R}^N$. The proof that $\bar{V} = \mathbb{R}^N$ in the case of discrete time is, by now, well-known [NP], and use properties of orbit-shadowing for hyperbolic diffeomorphisms. The modifications required to prove the result for hyperbolic flows are standard, but notationally cumbersome. For the benefit of the reader we briefly sketch the main ideas; the complete details can be found in [M]. Choose a countable dense set X of points in Λ and construct a sequence $x_1, x_2, x_3, \ldots$ of points where $x_k \in D$ and each point in D occurs infinitely often in this sequence. Choose a dense set $v_k \in \mathbb{R}^N$. Let $\varepsilon_k \searrow 0$. The aim is to construct a point $x \in \Lambda$ and a sequence of times t_k such that, for each k, $d(\phi_{t_j}(x), x_k) < \varepsilon_k$ and $\|F_{t_k}(x) - v_k\| < \varepsilon_k$ (here d denotes the metric on Λ). It then follows that the orbit of $(x, 0) \in \Lambda \times \mathbb{R}^N$ under ϕ_t^f is dense.

To construct x we use the shadowing property of hyperbolic flows: given collection of orbit segments, one can construct a single orbit that shadows, at various times, these orbit segments to a pre-assigned degree of accuracy. Specifically: given periodic orbits $q_1, \ldots, q_\ell$, with corresponding periods $\tau(q_1), \ldots, \tau(q_\ell)$, and points z_1, z_2, we use a shadowing argument (cf [B1]) to construct a point y whose orbit shadows: (i) firstly the orbit of z_1 for some prescribed amount of time, then (ii) the periodic orbits $q_1, \ldots, q_\ell$, then (iii) the orbit of z_2 for some prescribed period of time (the amount of time it takes the orbit of y to travel from z_1 to q_1, from q_1 to q_2, etc, depends only on the degree of accuracy required, and not on the orbits).

The point x is constructed inductively as follows. Suppose we have constructed x so that we know its orbit for time $0 < t < t_{k-1}$. Further, suppose that we have chosen periodic points $q_1, \ldots, q_\ell$ of periods $\tau(q_1), \ldots, \tau(q_\ell)$ such that the semi-group of $\mathbb{R}^N$ generated by $F_{\tau(q_1)}, \ldots, F_{\tau(q_\ell)}$ is ε_{k+1}-dense. We assume that x has been constructed so that for time $t_{k-1} < t < t'_k$, the orbit of x shadows the periodic orbits $q_1, \ldots, q_\ell$ in turn.

Consider $F_{t'_k}(x) - v_{n_k}$. One can find $m_1, \ldots, m_\ell \in \mathbb{Z}^+$ such that $\|F_{t_y}(y) - v_{n_k} - \sum_{j=0}^{\ell} m_j F_{\tau(q_j)}(q_j)\| = o(\varepsilon_k)$. One now constructs a new point x' that shadows x for time $0 < t < t_{k-1}$, then shadows q_1 with multiplicity m_1, q_2 with multiplicity m_2, etc. One then checks that with, $t_k = t'_k + \sum_{j=1}^{\ell-1}(m_j - 1)$ (i.e. t'_k plus the extra amount of time taken to travel the q_j with multiplicity), one has $\|F_{t_k}(x) - v_{n_k}\| = O(\varepsilon_k)$.

We prove (ii) implies (i). Suppose $\mathcal{L}$ does not satisfy the inseparability hypothesis. Then there exists $0 \neq v \in \mathbb{R}^N$ such that $\langle \mathcal{L}, v \rangle \subset \mathbb{R}^+$. Let $f_v = \langle f, v \rangle : \Lambda \to \mathbb{R}$. Then Lemma 3.2 implies that F_v is bounded below and so $\phi_t^{F_v}$, hence ϕ_t^F, cannot be transitive.

Finally, suppose that ϕ_t^F is transitive with dense orbit $(x_0, v_0) \in \Lambda \times \mathbb{R}^N$ but $\langle \mathcal{L}, v \rangle \subset a\mathbb{Z}$ for some $0 \neq v \in \mathbb{R}^N$ and $a > 0$. Choose $a' \in (0, a)$. Let $\varepsilon > 0$. Then there exists $T > 0$ such that $d(\phi_T^F(x_0, v_0), (x_0, v_0 + a'v)) < \varepsilon$. By the Anosov Closing Lemma [KH, p.548] (cf [B1]) there exists a periodic point p close to x_0 such that $\phi_{T'}(p) = p$ for some T' close to T. Moreover, the estimate $\|F_{T'}(p) - F_T(x_0)\| \leq C\varepsilon$ holds, for some constant $C > 0$. One can then estimate $|\langle F_{T'}(p), v \rangle - a'| \leq C'\varepsilon$, for some constant $C' > 0$ depending only on F and ϕ_t. As ε is arbitrary, it follows that $a' \in \mathcal{L}$, a contradiction.

❑

Proposition 3.4 *Let $f = (f^{(1)}, \ldots, f^{(N)}) \in C^\alpha(\Lambda, \mathbb{R}^N)$. Suppose that f is linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$. Then $\mathcal{L}$ is not co-lattice valued.*

Proof. Suppose that $\mathcal{L}$ is co-lattice valued. Then there exists $0 \neq v \in \mathbb{R}^N$ and $a \in \mathbb{R}$ such that $\langle \mathcal{L}, v\rangle \subset a\mathbb{Z}$, that is $\langle v, F_T(x)\rangle \in a\mathbb{Z}$ whenever $\phi_T(x) = x$. Hence

$$\exp 2\pi i \left\langle \frac{v}{a}, F_T(x) \right\rangle = 1$$

whenever $\phi_T(x) = x$. By Lemma 2.2(ii), it follows that $\exp 2\pi i\langle v/a, F_t\rangle$ is a circle-valued coboundary, i.e. there exists $w \in C^\alpha(\Lambda, K)$ such that

$$\exp 2\pi i \left\langle \frac{v}{a}, F_t(x) \right\rangle = \frac{w(\phi_t(x))}{w(x)}.$$

Equivalently, $\iota[w] = \langle v/a, f\rangle$. Hence, $\langle v/a, f\rangle \in \mathcal{Z}_\mathbb{Z} + \mathcal{B}_\phi$ so that f is linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$. ❑

Theorem 3.5 *Suppose that $H^1(\Lambda, \mathbb{Z})$ has finite rank. Let $f = (f^{(1)}, \ldots, f^{(N)}) \in C^\alpha(\Lambda, \mathbb{R}^N)$. Suppose that f is linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$. Then the following are equivalent:*

(i) ϕ_t^F is transitive,

(ii) ϕ_t^F is stably transitive,

(iii) ϕ_t^F has orbits that are unbounded in all directions,

(iv) $\mathcal{L}$ satisfies the inseparability hypothesis.

Proof. Note that the hypothesis on f ensures that $\mathcal{L}$ is not co-lattice valued. Recall from Proposition 2.1 that $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\phi$ is a closed subspace; hence the hypothesis on f is an open condition.

(i) implies (iii) is clear. That (ii) is equivalent to (iv) follows from Propositions 3.3 and 3.4, noting that the inseparability hypothesis is an open condition on f.

To see that (iii) implies (iv) note that if the inseparability hypothesis is not satisfied then there exists $0 \neq v \in \mathbb{R}^N$ such that $\langle f, v\rangle$ takes only non-negative or non-positive values on periodic orbits. By Lemma 3.2, ϕ_t^F cannot have orbits that are unbounded in the v-direction.

That (iv) implies (i) follows from Proposition 3.3. ❑

Corollary 3.6 *Suppose that $H^1(\Lambda, \mathbb{Z})$ has finite rank. Then, amongst the cocycles in $C^\alpha(\Lambda, \mathbb{R}^N)$ that satisfy the inseparability hypothesis, there exists an open dense set of transitive cocycles.*

Proof. By Proposition 2.1, $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\phi$ is a closed, infinite-codimension subspace of $C^\alpha(\Lambda, \mathbb{R}^N)$. Hence the set of $f \in C^\alpha(\Lambda, \mathbb{R}^N)$ that are linearly independent mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\phi$ is open and dense. ❑

4 Discrete time

One can prove analogous results to Theorem 3.5 and Corollary 3.6 for hyperbolic flows using the methodology of [FP] to construct the analogue of the inclusion $\iota : H^1(\Lambda, \mathbb{Z}) \to \mathcal{H}^1(\phi, \mathbb{R})$ as follows. Let $\phi : \Lambda \to \Lambda$ be a hyperbolic diffeomorphism on a basic set Λ. The map $\phi : \Lambda \to \Lambda$ induces an automorphism $\phi^* : H^1(\Lambda, \mathbb{Z}) \to H^1(\Lambda, \mathbb{Z})$ on the Bruschlinsky group defined by $\phi^*[w] = [w\phi]$. Let $[w] \in \ker(\phi^* - \mathrm{id})$. Then there exists a continuous function $g : \Lambda \to \mathbb{R}$ such that

$$\frac{w(\phi(x))}{w(x)} = \exp 2\pi i g(x). \tag{7}$$

One then defines $\iota : \ker(\phi^* - \mathrm{id}) \to C^\alpha(\Lambda, \mathbb{R})$ by $\iota([w]) = g + \mathcal{B}_\phi$, where $\mathcal{B}_\phi = \{u\phi - u \mid u \in C^\alpha(\Lambda, \mathbb{R})\}$ denotes the set of coboundaries. The above arguments then go through, with the hypothesis that $H^1(\Lambda, \mathbb{R})$ has finite rank replaced by the hypothesis that $\ker(\phi^* - \mathrm{id})$ has finite rank. One then finds a subspace $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\phi$ of $C^\alpha(\Lambda, \mathbb{R}^N)$ defined by the analogue (6) such that the corresponding versions of Theorem 3.5 and Corollary 3.6 hold (with $\ker(\phi^* - \mathrm{id})$ replacing $H^1(\Lambda, \mathbb{Z})$).

Remark. In the case of a hyperbolic diffeomorphism we only require $\ker(\phi^* - \mathrm{id})$ to have finite rank. (Note that 1 is not an eigenvalue of ϕ^* precisely when $\ker(\phi^* - \mathrm{id})$ is trivial.)

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