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2011

MIMS EPrint: **2011.38**

ISSN 1749-9097

The Hausdorff dimension of some random invariant graphs

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3rd May, 2011

Abstract

Weierstrass' example of an everywhere continuous but nowhere differentiable function is given by $w(x) = \sum_{n=0}^{\infty} \lambda^n \cos 2\pi b^n x$ where $\lambda \in (0,1), b \geq 2$, $\lambda b > 1$. There is a well-known and widely accepted, but as yet unproven, formula for the Hausdorff dimension of the graph of w. Hunt [H] proved that this formula holds almost surely on the addition of a random phase shift. The graphs of Weierstrass-type functions appear as repellers for a certain class of dynamical system; in this note we prove formulae analogous to those for random phase shifts of $w(x)$ but in a dynamic context. Let $T : S^1 \to S^1$ be a uniformly expanding map of the circle. Let $\lambda : S^1 \to (0,1)$, $p : S^1 \to \mathbb{R}$ and define the function $w(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(T(x)) \cdots \lambda(T^{n-1}(x))p(T^n(x))$. The graph of w is a repelling invariant set for the skew-product transformation $T(x,y) = (T(x), \lambda(x)^{-1}(y-p(x)))$ on $S^1 \times \mathbb{R}$ and is continuous but typically nowhere differentiable. With the addition of a random phase shift in p, and under suitable hypotheses including a partial hyperbolicity assumption on the skew-product, we prove an almost sure formula for the Hausdorff dimension of the graph of w using a generalisation of techniques from [H] coupled with thermodynamic formalism.

§1 Introduction

The study of everywhere continuous but nowhere differentiable functions has a long history. The first, and perhaps most studied, example is the Weierstrass function

$$w(x) = \sum_{n=0}^{\infty} \lambda^n \cos 2\pi b^n x, \ \ \lambda \in (0,1), b \in \mathbb{N}. \tag{1}$$

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†A preliminary version of the results over this paper appeared in the first author's PhD thesis. Financially supported by EPSRC.

2010 *Mathematics Subject Classification:* 37C45, 37D20, 37H99.

This series converges uniformly, hence w is continuous. If $\lambda b > 1$ then w is nowhere differentiable.

More generally, consider the graph of an arbitrary function $w : [0,1] \to \mathbb{R}$:

$$\mathrm{graph}(w) = \{(x, w(x)) \mid x \in [0,1]\} \subset \mathbb{R}^2.$$

If w is differentiable then graph(w) is a 1-dimensional manifold and consequently has Hausdorff dimension 1. If w is nowhere differentiable then graph(w) is typically a fractal and the dimension of graph(w) gives an indication of how irregular w is.

Computing the box dimension of graph(w) is often straightforward. Indeed, for the Weierstrass function w one can easily check [BU] that

$$\dim_B \mathrm{graph}(w) = 2 - \frac{\log \lambda^{-1}}{\log b}. \tag{2}$$

It is widely conjectured that the Hausdorff dimension $\dim_H \mathrm{graph}(w)$ of the Weierstrass function is also given by (2).

There are examples [PU] of functions of the form (1) where $\cos 2\pi b^n x$ is replaced by the nth Rademacher function (note that this is piecewise constant but not continuous) and λ is a Pisot number for which $\dim_H \mathrm{graph}(w) < \dim_B \mathrm{graph}(w)$. More generally, given any integers $n > m > 1$, letting $\alpha = \log m / \log n$ and choosing any $s \in (1, 2-\alpha)$, one can construct [PU, M] a Hölder continuous function $w_{s,\alpha}$ of exponent α such that $\dim_B(\mathrm{graph}(w_{s,\alpha})) = 2 - \alpha$ but $\dim_H(\mathrm{graph}(w_{s,\alpha})) = s < 2 - \alpha$.

It is often the case, however, that if one introduces a random parameter into the construction of a fractal, then the conjectured value of the Hausdorff dimension for the non-random case can be proved to hold for almost every value of this parameter. Hunt proved in [H] that if $\theta = (\vartheta_n)_{n=0}^{\infty}$, where $\vartheta_n \in [0,1]$ are chosen uniformly and independently, then with

$$w_\theta(x) = \sum_{n=0}^{\infty} \lambda^n \cos 2\pi(b^n + \vartheta_n), \;\; \lambda \in (0,1), b \in \mathbb{N}, b \geq 2, \lambda b > 1 \tag{3}$$

the Hausdorff dimension of graph(w_θ) is

$$\dim_H \mathrm{graph}(w_\theta) = 2 - \frac{\log \lambda^{-1}}{\log b} \text{ a.s.}$$

Indeed, as is remarked in [H], one can replace cos in (3) with a suitably smooth periodic function p satisfying a mild condition on its critical points.

One can view graph(w) as the invariant set (indeed, a repeller) for a certain skew-product dynamical system. In order to make some of the objects below continuous, it is technically more convenient to work on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Define $T : S^1 \to S^1$ by $T(x) = bx \bmod 1$ and define

$$\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} : (x, y) \mapsto (T(x), \lambda^{-1}(y - \cos 2\pi x)).$$

Let w be defined as in (1). Then $\hat{T}(\mathrm{graph}(w)) = \mathrm{graph}(w)$. More generally, one can consider skew-products defined on $S^1 \times \mathbb{R}$ of the form $\hat{T}(x,y) = (T(x), \lambda(x)^{-1}(y - p(x))$ where $p : S^1 \to \mathbb{R}$ and $\lambda : S^1 \to (0,1)$. Define

$$w(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(T(x)) \cdots \lambda(T^{n-1}(x))p(T^n(x)). \tag{4}$$

Then $\mathrm{graph}(w)$ is a $\hat{T}$-invariant repeller for $\hat{T}$.

One can then consider the case when T is replaced by a uniformly expanding map of the circle. In the context of hyperbolic dynamics, one can often recognise the dimension of an invariant set as the solution of a certain equation, often called Bowen's equation, of the form $P(sf + g)$ where P denotes the topological pressure. For many dynamically-defined fractal sets one can also often recognise the dimension as the entropy of the underlying dynamics divided by the Lyapunov exponent, with respect to an appropriate invariant measure.

In [Be] the box dimension of the graph of a function of the form (4) is calculated to be the unique solution s to the equation $P((1-s)\log T' + \log \lambda) = 0$.

In this paper, we study the Hausdorff dimension of equations of the form (4) where the function p is modified by the addition of a random phase shift. One aim is to put the results of [H] into the context of thermodynamic formalism. Indeed, we prove:

Theorem 1.1
Suppose that T is a C^2 uniformly expanding map of the circle. Let $\lambda : S^1 \to (0,1)$ be C^1, and let $p : S^1 \to \mathbb{R}$ be, for example, a polynomial or a finite sum of trigonometric functions. Define

$$w_\theta(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(T(x)) \cdots \lambda(T^{n-1}(x))p(T^n(x) + \vartheta_n)$$

where the ϑ_n are chosen uniformly and independently from S^1. Then there exists a T-invariant probability measure μ_0 such that

$$\dim_H \mathrm{graph}(w_\theta) = 1 + \frac{h_{\mu_0}(T) + \int \log \lambda \, d\mu_0}{\int \log T' \, d\mu_0} \text{ a.s.}$$

where $h_{\mu_0}(T)$ denotes the measure-theoretic entropy of T with respect to μ_0.

The precise statement and hypotheses are given below in Theorem 2.4. In particular, μ_0 can be identified to be the equilibrium state of a certain Hölder continuous potential. We also make precise the conditions assumed on p.

In §2 we give the necessary background and state two results which give an upper and lower bound, respectively, on $\dim_H(\text{graph}(w_\theta))$. In §3 we prove the upper bound. In §4, we prove the lower bound; the key estimate in §4 is a generalisation of the method used in [H].

§2 Preliminaries and statement of results

§2.1 Expanding circle maps

Let $T : S^1 \to S^1$ be a $C^{1+\varepsilon}$ map, i.e. T is continuous, continuously differentiable and the derivative is Hölder continuous. By replacing T with T^2 if necessary, there is no loss in assuming that T is orientation-preserving. There exists a partition of S^1 into intervals $I_j = [a_{j-1}, a_j]$, $1 \leq j \leq N$, (where the intervals are taken mod 1 if necessary) such that the restriction $T : I_j \to S^1$ is a homeomorphism on I_j and a diffeomorphism on the interior of I_j. We assume that T is uniformly expanding, in the sense that there exists $\beta > 1$ such that $T'(x) \geq \beta$ for all $x \in S^1$.

Let $T_j : S^1 \to I_j$, $1 \leq j \leq N$, denote the inverse branches of T. Let $x_0, x_1, \ldots, x_{n-1} \in \{1, \ldots, N\}$. Define

$$[x_0, x_1, \ldots, x_{n-1}] = T_{x_0} T_{x_1} \cdots T_{x_{n-1}}(S^1)$$

and note that this is an interval. We call such sets cylinders of rank n. Let $\mathcal{C}_n$ denote the set of all cylinders of rank n and note that, for each $n \geq 1$, $\mathcal{C}_n$ is a partition of S^1.

§2.2 Pressure

Let $g : S^1 \to \mathbb{R}$ be Hölder continuous. Define $|g|_\alpha = \sup_{x \neq y} |g(x) - g(y)|/d(x,y)^\alpha$. There are many equivalent ways of defining the pressure $P(g)$ of g, and here we briefly review those that we will need in what follows.

§2.2.1 Pressure via cylinders

Let $g : S^1 \to \mathbb{R}$ be Hölder continuous. Define the pressure of g to be

$$P(g) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{[x_0, x_1, \ldots, x_{n-1}] \in \mathcal{C}_n} \exp \sum_{j=0}^{n-1} g(T^j(x)) \tag{5}$$

where the x in the summand is taken to be any point in $[x_0, x_1, \ldots, x_{n-1}]$. It follows immediately from the definition and the following lemma that $P(g)$ is independent of the choice of points $x \in [x_0, x_1, \ldots, x_{n-1}]$.

Lemma 2.1
Let $g : S^1 \to \mathbb{R}$ be Hölder continuous of exponent α. If x, y are in the same cylinder of rank n then

$$\left| \sum_{j=0}^{n-1} g(T^j(x)) - g(T^j(y)) \right| \leq |g|_\alpha \frac{1}{1 - \beta^{-\alpha}}.$$

Proof. Recall that $T_j : S^1 \to I_j$ denote the inverse branches of T. As $[x_0, x_1, \ldots, x_{n-1}] = T_{x_0} \circ T_{x_1} \circ \cdots \circ T_{x_{n-1}}(S^1)$ it follows that $\mathrm{diam}[x_0, x_1, \ldots, x_{n-1}] \leq 1/\beta^n$. A straightforward calculation shows that if $x, y \in [x_0, x_1, \ldots, x_{n-1}]$ then

$$\left| \sum_{j=0}^{n-1} g(T^j(x)) - g(T^j(y)) \right| \leq |g|_\alpha \sum_{j=0}^{n-1} d(T^j(x), T^j(y))^\alpha \leq |g|_\alpha \frac{1}{1 - \beta^{-\alpha}}.$$

❑

§2.2.2 Pressure via spanning sets

We can also define pressure via spanning sets [W, for example]. Let $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$ denote the metric on S^1 inherited from $[0, 1]$ with 0 identified with 1. Define a family of metrics d_n by $d_n(x, y) = \max\{d(T^j(x), T^j(y)) \mid 0 \leq j \leq n - 1\}$. For $\delta > 0$ and $x \in S^1$ define the (n, δ)-*Bowen ball* $B_{n,\delta}(x) = \{y \in S^1 \mid d_n(x, y) < \delta\}$. A subset $F \subset S^1$ is said to be (n, δ)-*spanning* if $\bigcup_{x \in F} B_{n,\delta}(x) = S^1$.

Let $g : S^1 \to \mathbb{R}$ be continuous. Define

$$P_n(g, \delta) = \inf \sum_{x \in F} \exp \sum_{j=0}^{n-1} g(T^j(x)) \tag{6}$$

where the infimum is taken over all (n, δ)-spanning sets.

Define

$$P(g, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(g, \delta).$$

Then one can show that if $\delta_1 < \delta_2$ then $P(g, \delta_1) \geq P(g, \delta_2)$ so that $P(g) = \lim_{\delta \to 0} P(g, \delta)$ exists.

Recall that a dynamical system T is said to be *expansive* with expansivity constant $\delta_0 > 0$ if $d(T^n(x), T^n(y)) \leq \delta_0$ for all $n \geq 0$ implies $x = y$. If T is a uniformly expanding map of S^1 then T is expansive with expansivity constant $1/\beta$. One can show [W, Theorem 9.6] that if δ_0 is an expansivity constant then $P(g) = P(g, \delta)$ whenever $\delta < \delta_0/4$.

§2.2.3 Pressure via the variational principle

We can also define $P(g)$ via the variational principle [W]. Let $g : S^1 \to \mathbb{R}$ be continuous. Then

$$P(g) = \sup \left\{ h_\mu(T) + \int g \, d\mu \right\} \tag{7}$$

where $h_\mu(T)$ denotes the entropy of T and the supremum is taken over all T-invariant probability measures μ.

For each Hölder continuous $g : S^1 \to \mathbb{R}$ there exists a unique T-invariant probability measure μ_g which attains the supremum in (7), i.e. $P(g) = h_{\mu_g}(T) + \int g \, d\mu_g$. We call μ_g the *equilibrium state* with potential g.

Equilibrium states have the following Gibbs property. Let $g : S^1 \to \mathbb{R}$ be Hölder continuous and let $\delta > 0$. Then there exists a constant $C(g, \delta) > 1$ such that for any $x \in S^1$ and any Bowen ball $B_{n,\delta}(x)$ we have

$$\frac{1}{C(g,\delta)} \leq \frac{\mu_g(B_{n,\delta}(x))}{\exp \sum_{j=0}^{n-1} g(T^j(x)) - nP(g)} \leq C(g, \delta). \tag{8}$$

§2.2.4 Properties of pressure

Let f, g be Hölder continuous. It is clear from any of the definitions of pressure that if $f \leq g$ then $P(f) \leq P(g)$. In particular if $f \geq 0$ and g is any function then $s \mapsto P(sf + g)$ and $s \mapsto P(-sf + g)$ are increasing and decreasing functions of s, respectively. It is also well-known that the dependence of $P(f)$ is continuous [W, for example] (indeed, analytic) on f, so that $s \mapsto P(sf + g), P(-sf + g)$ are continuous functions of s.

It is also clear from (7) that if c is a constant then $P(g + c) = P(g) + c$ and that, if $c \geq 1$, then $P(cg) \leq cP(g)$.

§2.3 Hausdorff dimension

Let $E \subset \mathbb{R}^n$ and $s \geq 0$. The s-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \sum_j \operatorname{diam}(I_j)^s$$

where the infimum is taken over all countable open covers I_j such that $E \subset \bigcup_j I_j$ and the diameter $\operatorname{diam} I_j \leq \delta$. The Hausdorff dimension of E is defined by

$$\dim_H E = \inf\{s \mid \mathcal{H}^s(E) = 0\} = \sup\{s \mid \mathcal{H}^s(E) = \infty\}.$$

One can also characterise Hausdorff dimension in terms of energy integrals. Let μ be a probability measure supported on E. For $s \geq 0$ define the s-energy of μ to be

$$I_s(\mu) = \int\int \frac{d\mu(x) \, d\mu(y)}{\|x - y\|^s}$$

and define the correlation dimension of μ to be $\sup\{s \mid I_s(\mu) < \infty\}$. Then the Hausdorff dimension $\dim_H(E)$ is the supremum of the correlation dimensions over all probability measures supported on E.

§2.4 Random dynamical systems

Let $\lambda : S^1 \to (0,1)$ be Hölder continuous. Let $p : S^1 \to \mathbb{R}$ be continuous. Let $T : S^1 \to S^1$ be a uniformly expanding map of S^1. Define

$$\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} : (x,y) \mapsto (T(x), \lambda(x)^{-1}(y - p(x))).$$

We introduce a random phase-shift as follows. Equip S^1 with Lebesgue measure. Let $\Omega = \{(\vartheta_j)_{j=0}^{\infty} \mid \vartheta_j \in S^1\}$. Equip Ω with the measure given by the Cartesian product of Lebesgue measure; we will denote this measure by $d\theta$.

Let $\tau : \Omega \to \Omega$ be the left shift map, so that if $\theta = (\vartheta_j)_{j=0}^{\infty}$ then $(\tau(\theta))_j = \vartheta_{j+1}$. Define the random dynamical system $\tilde{T}$ by

$$\tilde{T} : \Sigma \times \mathbb{R} \times \Omega \to \Sigma \times \mathbb{R} \times \Omega : (x, y, \theta) = (T(x), \lambda(x)^{-1}(y - p(x + \vartheta_0), \tau(\theta)))$$

and consider the projection onto the (x, y) co-ordinates

$$\hat{T}_\theta(x,y) = (T(x), \lambda(x)^{-1}(y - p(x + \vartheta_0))).$$

Define $\lambda^n(x) = \lambda(x)\lambda(T(x)) \cdots \lambda(T^{n-1}(x))$, $\lambda^0(x) = 1$. Then for each $\theta = (\vartheta_j)_{j=0}^{\infty} \in \Omega$, the function

$$w_\theta(x) = \sum_{n=0}^{\infty} \lambda^n(x) p(T^n(x) + \vartheta_n) \tag{9}$$

is a continuous function (as the sum converges uniformly), and graph(w_θ) is $\hat{T}_\theta$-invariant. To see this, simply observe that

$$\begin{aligned}
\hat{T}_\theta(x, w_\theta(x)) &= \left(T(x), \lambda^{-1}(x) \left(\sum_{n=0}^{\infty} \lambda^n(x) p(T^n(x) + \vartheta_n) - p(x + \vartheta_0) \right) \right) \\
&= \left(T(x), \sum_{n=1}^{\infty} \lambda(T(x)) \cdots \lambda(T^{n-1}(x)) p(T^n(x) + \vartheta_n) \right) \\
&= (T(x), w_{\tau(\theta)}(T(x))).
\end{aligned}$$

§2.5 Statement of results

We assume that $\hat{T}$ is *partially hyperbolic*, i.e. there exists $\rho > 1$ such that

$$1 < \rho \le \inf_{x \in S^1} \lambda(x) \inf_{x \in S^1} T'(x). \tag{10}$$

That is $\hat{T}$ is partially hyperbolic if the maximum rate of exponential expansion in the $\mathbb{R}$-direction is strictly less than the maximum rate of exponential expansion in the S^1-direction. In the case where λ is constant and $T(x) = bx \bmod 1$, this reduces to $\lambda b > 1$.

There is an obvious obstruction to the regularity of w: if there exists a smooth solution u to the cohomological equation $p(x) = \lambda(x)u(T(x)) - u(x)$ then $w = u$ and the graph of w is as smooth as u. Generically this does not happen [HNW].

The following gives an upper bound for $\dim_H(\mathrm{graph}(w_\theta))$; note that in this case the bound holds for all $\theta \in \Omega$.

Proposition 2.2 (Upper bound)
Suppose that $T : S^1 \to S^1$ is a $C^{1+\varepsilon}$ uniformly expanding map of the circle. Let $\lambda : S^1 \to (0,1)$ be C^1. Let $p : S^1 \to \mathbb{R}$ be C^1. Suppose that the partial hyperbolicity assumption (10) holds. Then there exists a unique $s > 0$ such that

$$P((1-s)\log T' + \log \lambda) = 0. \tag{11}$$

Moreover $\dim_H(\mathrm{graph}(w_\theta)) \le s$ for every $\theta \in \Omega$.

For the lower bound we need some additional smoothness assumptions on p. Recall that $p(\vartheta)$ has a critical point of order k if $p'(\vartheta) = p''(\vartheta) = \cdots = p^{(k-1)}(\vartheta) = 0$ but $p^{(k)}(\vartheta) \neq 0$. We say that a smooth function p satisfies the *critical point hypothesis* if there exists $r > 0$ such that for all $a \in (0,1)$ and $c \in \mathbb{R}$, the critical points of $p(a+\vartheta) - cp(\vartheta)$ have orders strictly less than r. This assumption is satisfied by any polynomial, any finite sum of trigonometric functions, and we would expect it to hold generically for smooth functions on S^1.

Proposition 2.3 (Lower bound)
Suppose that $T : S^1 \to S^1$ is a $C^{1+\varepsilon}$ uniformly expanding map of the circle. Let $\lambda : S^1 \to (0,1)$ be such that $\log \lambda$ is Hölder continuous. Let $p : S^1 \to \mathbb{R}$ satisfy the critical point hypothesis. Suppose that the partial hyperbolicity assumption (10) holds.

Let $g : S^1 \to \mathbb{R}$ be Hölder continuous. Then there exists a unique solution $s_g > 0$ to

$$P((s_g - 1)\log T' + 2(g - P(g)) - \log \lambda) = 0. \tag{12}$$

Moreover $s_g \le \dim_H(\mathrm{graph}(w_\theta))$ for almost every $\theta \in \Omega$.

We can now state the main result.

Theorem 2.4
Suppose that $T : S^1 \to S^1$ is a $C^{1+\varepsilon}$ uniformly expanding map of the circle. Let $\lambda : S^1 \to (0,1)$ be C^1. Let $p : S^1 \to \mathbb{R}$ satisfy the critical point hypothesis. Suppose that the partial hyperbolicity assumption (10) holds.

Let s_0 be the unique solution to (11) and let μ_0 be the unique equilibrium state of the potential $g_0 = (1-s_0)\log T' + \log\lambda$. Then for almost every $\theta \in \Omega$

$$\dim_H(\mathrm{graph}(w_\theta)) = 1 + \frac{h_{\mu_0}(T) + \int \log\lambda \, d\mu_0}{\int \log T' \, d\mu_0}. \tag{13}$$

Remark. For an invariant probability measure μ and continuous function $g : S^1 \to \mathbb{R}$, the measure-theoretic pressure of g with respect to μ is defined to be $P_\mu(g) = h_\mu(T) + \int g \, d\mu$ and can be regarded as a generalisation of entropy. Thus we can regard the right-hand side of (13) as the sum of the dimension of $\mathrm{graph}(w_\theta)$ in the S^1-direction and the dimension of $\mathrm{graph}(w_\theta)$ in the $\mathbb{R}$-direction, where the latter has the form of (a generalisation of) entropy divided by the Lyapunov exponent of T with respect to μ_0.

Remark. If $T(x) = bx \bmod 1$ where $b \geq 2$ is an integer, then μ_0 is the equilibrium state of $\log\lambda$. In this case, (13) takes the form

$$\dim_H(\mathrm{graph}(w_\theta)) = 1 + \frac{P(\log\lambda)}{\log b} \quad \text{a.e..}$$

If in addition $\lambda(x) = \lambda$ is constant then μ_0 is Lebesgue measure, $h_{\mu_0}(T) = \log b$ and we rederive the result in [H].

Proof of Theorem 2.4. Let s_0, g_0 and μ_0 be as in the statement of the theorem. Note that $P(g_0) = 0$. By the variational principle, we have that

$$h_{\mu_0}(T) + (1-s_0)\int \log T' \, d\mu_0 + \int \log\lambda \, d\mu_0 = 0.$$

It follows that

$$s_0 = 1 + \frac{h_{\mu_0}(T) + \int \log\lambda \, d\mu_0}{\int \log T' \, d\mu_0}$$

and from Proposition 2.2 that $\dim_H(\mathrm{graph}(w_\theta)) \leq s_0$ for almost every $\theta \in \Omega$.

Let s_1 be the unique solution to (12) with potential g_0. It follows from Proposition 2.3 that $s_1 \leq \dim_H(\mathrm{graph}(w_\theta))$ for almost every $\theta \in \Omega$. It remains to show that $s_1 = s_0$. First note that from (12) we have that

$$P((s_1 - 1)\log T' - \log\lambda + 2(1-s_0)\log T' + 2\log\lambda) = 0,$$

that is

$$P((1 - 2s_0 + s_1)\log T' + \log\lambda) = 0.$$

As s_0 is the unique solution to $P((1-s_0)\log T' + \log\lambda) = 0$, it follows that $1 - s_0 = 1 - 2s_0 + s_1$, i.e. $s_0 = s_1$, and the result follows. ❑

§3 Proof of Proposition 2.2

The upper bound continues to be true without the addition of the random phase shift θ. Indeed, [Be] considers invariant graphs that include those of the form $w(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(T(x)) \cdots \lambda(T^{n-1}(x))p(T^n(x))$ and proves that the box dimension of graph(w) is given by (11). The following proposition in a non-random context is proved in [Be]. For a function $\gamma : S^1 \to \mathbb{R}$ and an interval $I \subset [0,1]$, the *height of* γ *over* I, denoted by $\text{height}_I(\gamma)$, is defined to be

$$\text{height}_I(\gamma) = \sup_{s,t\in I} |\gamma(s) - \gamma(t)|.$$

Proposition 3.1
Let $[x_0, x_1, \ldots, x_{n-1}] \in \mathcal{C}_n$ *be a cylinder of rank* n*. Then there exists* $C > 0$ *(independent of* x, n*) such that*

$$\text{height}_{[x_0,x_1,\ldots,x_{n-1}]}(w_\theta) \le C\lambda^n(x).$$

Proof. Let T_i be the inverse branches of T. For each $\vartheta \in S^1$ define

$$\hat{T}_{i,\vartheta}(x, y, \theta) = (T_i(x), \lambda(T_i(x))y + p(T_i(x) + \vartheta), \vartheta\theta)$$

where if $\theta = (\vartheta_0, \vartheta_1, \ldots)$ then $\vartheta\theta$ denotes the sequence $(\vartheta, \vartheta_0, \vartheta_1, \ldots)$. Then $\hat{T}\hat{T}_{i,\vartheta}(x, y, \theta) = (x, y, \theta)$ so that $\hat{T}_{i,\vartheta}$ are the inverse branches of $\hat{T}$.

Consider the action of $\hat{T}_{i,\vartheta}$ on the first two coordinates. This has derivative

$$\begin{pmatrix} T_i'(x) & 0 \\ S_{i,\vartheta}(x,y) & \lambda(T_i(x)) \end{pmatrix}$$

where $S_{i,\vartheta}(x, y) = \lambda'(T_i(x))T_i'(x)y + p'(T_i(x) + \vartheta)T_i'(x)$.

First note that

$$\begin{aligned}
&(\hat{T}_{x_0,\vartheta_0} \circ \hat{T}_{x_1,\vartheta_1})'(x, y) \\
&\quad= \hat{T}'_{x_0,\vartheta_0}(\hat{T}_{x_1,\vartheta_1}(x, y))\hat{T}'_{x_1,\vartheta_1}(x, y) \\
&\quad= \begin{pmatrix} T'_{x_0}(T_{x_1}(x)) & 0 \\ S_{x_0,\vartheta_0}(\hat{T}_{x_1,\vartheta_1}(x, y)) & \lambda(T_{x_0}T_{x_1}(x)) \end{pmatrix} \begin{pmatrix} T'_{x_1}(x) & 0 \\ S_{x_1,\vartheta_1}(x, y) & \lambda(T_{x_1}(x)) \end{pmatrix} \\
&\quad= \begin{pmatrix} T'_{x_0}(T_{x_1}(x))T'_{x_1}(x) & 0 \\ S_{x_0,\vartheta_0}(\hat{T}_{x_1}(x, y))T'_{x_1}(x) + \lambda(T_{x_0}T_{x_1}(x))S_{x_1,\vartheta_1}(x, y) & \lambda(T_{x_0}T_{x_1}(x))\lambda(T_{x_1}(x)) \end{pmatrix}.
\end{aligned}$$

Induction then allows us to write the derivative of $(\hat{T}_{x_0,\vartheta_0}\hat{T}_{x_1,\vartheta_1} \cdots \hat{T}_{x_{n-1},\vartheta_{n-1}})(x, y)$ in the form

$$\begin{pmatrix} \prod_{j=0}^{n-1} T'_{x_j}(T_{x_{j+1}} \cdots T_{x_{n-1}}(x)) & 0 \\ (*) & \prod_{j=0}^{n-1} \lambda(T_{x_j} \cdots T_{x_{n-1}}(x)) \end{pmatrix}$$

where

$$(*) \quad = \quad \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \lambda(T_{x_j} \cdots T_{x_{n-1}}(x)) S_{x_k,\vartheta_k}(\hat{T}_{x_{k+1},\vartheta_{k+1}} \cdots \hat{T}_{x_{n-1},\vartheta_{n-1}}(x,y)) \times \prod_{j=k+1}^{n-1} T'_{x_j}(T_{x_{j+1}} \cdots T_{x_{n-1}}(x))$$

and products such as $\prod_{n}^{n-1}$, etc, are interpreted as being empty.

Let $J \subset I$ be a subinterval and let $\gamma(t) = (\gamma_H(t), \gamma_V(t))$ be a differentiable curve in $J \times \mathbb{R}$ (we use H, V to denote the 'horizontal' (along I) and 'vertical' (along $\mathbb{R}$) directions, respectively, and write π_H, π_V to denote the corresponding projections). Choose points $x^+, x^- \in [x_0, x_1, \ldots, x_{n-1}]$ such that

$$\mathrm{height}_{[x_0,x_1,\ldots,x_{n-1}]}(w_\theta) = w_\theta(x^+) - w_\theta(x^-).$$

Let γ_0 denote the straight-line segment joining $(x^+, w_\theta(x^+))$ to $(x^-, w_\theta(x^-))$ and let $\gamma = \hat{T}\gamma_0$. The vertical height of w_θ over $[x_0, \ldots, x_{n-1}]$ is then bounded by

$$\begin{aligned}
&\mathrm{height}_{[x_0,x_1,\ldots,x_{n-1}]}(\hat{T}_{x_0,\vartheta_0}\hat{T}_{x_1,\vartheta_1} \cdots \hat{T}_{x_{n-1},\vartheta_{n-1}}\gamma) \\
&\leq \int \left| \pi_V(\hat{T}_{x_0,\vartheta_0}\hat{T}_{x_1,\vartheta_1} \cdots \hat{T}_{x_{n-1},\vartheta_{n-1}}\gamma)'(t) \right| dt \\
&\leq \int \left| \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \lambda(T_{x_j} \cdots T_{x_{n-1}}\gamma_H(t)) S_{x_k,\vartheta_k}(\hat{T}_{x_{k+1},\vartheta_{k+1}} \cdots \hat{T}_{x_{n-1},\vartheta_{n-1}}(\gamma(t))) \right. \\
&\qquad \left. \times \prod_{j=k+1}^{n-1} T'_{x_j}(T_{x_{j+1}} \cdots T_{x_{n-1}}(\gamma_H(t))) \right| |\gamma'_H(t)|\, dt \\
&\qquad + \int \left| \prod_{j=0}^{n-1} \lambda(T_{x_j} \cdots T_{x_{n-1}}(\gamma_H(t))) \right| |\gamma'_V(t)|\, dt. \qquad (14)
\end{aligned}$$

By the partial hyperbolicity assumption (10), we have that

$$\sup_{x,i} T'_i(x) \leq \rho^{-1}\lambda(x)$$

where $0 < \rho^{-1} < 1$.

Note that $|w_\theta(x)| \leq |p|_\infty/(1 - |\lambda|_\infty)$. Let $A = \sup_{i,\vartheta} \sup_{x,y} |S_{i,\vartheta}(x)| < \infty$ where the supremum over x, y is taken over $x \in I$, $|y| \leq |p|_\infty/(1 - |\lambda|_\infty)$. Then we can bound the first integral in (14) by

$$A \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \lambda(T_{x_j} \cdots T_{x_{n-1}}(\gamma_H(t))) \prod_{j=k+1}^{n-1} \lambda(T_{x_{j+1}} \cdots T_{x_{n-1}}(\gamma_H(t)))\rho^{j-n}|\gamma'_H(t)|.$$

By Lemma 2.1 we can bound this by

$$\prod_{j=0}^{n-1} \lambda(T^j(x)) \times C \sum_{k=0}^{n-1} \rho^{-k} \leq C' \prod_{j=0}^{n-1} \lambda(T^j(x))$$

for some constants $C, C' > 0$. Hence

$$\begin{aligned}
&\text{height}_{[x_0,x_1\ldots,x_{n-1}]}(\hat{T}_{x_0,\vartheta_0}\hat{T}_{x_1,\vartheta_1}\cdots\hat{T}_{x_{n-1},\vartheta_{n-1}}\gamma) \\
&= \ w_\theta(x^+) - w_\theta(x^-) \\
&\leq \ \left(C' \int |\gamma_H'(t)| + |\gamma_V'(t)|\, dt\right) \prod_{j=0}^{n-1} \lambda(T^j(x)) \\
&\leq \ (C'|\gamma|_H + |\gamma|_V) \prod_{j=0}^{n-1} \lambda(T^j(x)) \\
&\leq \ (C' + |w_\theta|_\infty) \prod_{j=0}^{n-1} \lambda(T^j(x))
\end{aligned}$$

and the result follows. ❑

Proof of Proposition 2.2. First note that $(1-s)\log T' + \log\lambda \leq -s\log\beta + \|\log T' + \log\lambda\|_\infty$. Hence

$$P((1-s)\log T' + \log\lambda) \leq -s\log\beta + \|\log T' + \log\lambda\|_\infty + h_{\text{top}}(T)$$

where $h_{\text{top}}(T) = P(0)$ is the topological entropy of T. As $\log\beta > 0$, it follows that $P((1-s)\log T' + \log\lambda) \to -\infty$ as $s \to \infty$. By partial hyperbolicity, $\log T' + \log\lambda \geq \log\rho > 0$ so that when $s = 0$, $P((1-s)\log T' + \log\lambda) \geq P(\log\rho) > 0$. As the pressure depends continuously on s, it follows that there is a unique value $s > 0$ that solves (11).

Let s_0 be the unique solution to $P((1-s_0)\log T' + \log\lambda)$. Let $s > s_0$. Then $P((1-s)\log T' + \log\lambda) = -\eta < 0$. By the partial hyperbolicity hypothesis, $\log\lambda + \log T' > 0$, hence $P(-s\log T') \leq -\eta < 0$. Hence there exists N such that if $n \geq N$ then

$$\sum_{[x_0,x_1,\ldots,x_{n-1}]\in\mathcal{C}_n} \prod_{j=0}^{n-1} T'(T^j(x))^{1-s} \prod_{j=0}^{n-1} \lambda(T^j(x)) < e^{\frac{-n\eta}{2}}$$

and

$$\sum_{[x_0,x_1,\ldots,x_{n-1}]\in\mathcal{C}_n} \prod_{j=0}^{n-1} T'(T^j(x))^{-s} < e^{\frac{-n\eta}{2}}$$

Let $\delta > 0$. Then there exists N such that if $n \geq N$ then $\text{diam}[x_0, x_1, \ldots, x_{n-1}] < \delta$ for all cylinders of rank n. By the Mean Value Theorem, for each $[x_0, x_1, \ldots, x_{n-1}] \in$

$\mathcal{C}_n$, choose $x \in [x_0, x_1, \ldots, x_{n-1}]$ such that

$$\operatorname{diam}[x_0, x_1, \ldots, x_{n-1}] = \prod_{j=0}^{n-1} T'(T^j(x))^{-1}.$$

Consider the graph of w_θ over the cylinder $[x_0, x_1, \ldots, x_{n-1}]$. This has height at most $C\lambda^n(x)$. Hence at most

$$\frac{C\lambda^n(x)}{\operatorname{diam}[x_0, x_1, \ldots, x_{n-1}]} + 1 = C\left(\prod_{j=0}^{n-1} T'(T^j(x)) \prod_{j=0}^{n-1} \lambda(T^j(x))\right) + 1$$

sets of diameter at most $\operatorname{diam}[x_0, x_1, \ldots, x_{n-1}]$ are needed to cover the graph of w_θ over $[x_0, x_1, \ldots, x_{n-1}]$. Taking all such sets over all cylinders of rank n gives an open cover $\mathcal{U}_n$ of $\operatorname{graph}(w_\theta)$ of diameter at most δ. Hence

$$\begin{aligned}
&\mathcal{H}^s_\delta(\operatorname{graph}(w_\theta)) \\
&\quad\le \sum_{U \in \mathcal{U}_n} (\operatorname{diam} U)^s \\
&\quad\le \sum_{[x_0, x_1, \ldots, x_{n-1}] \in \mathcal{C}_n} \left(C \prod_{j=0}^{n-1} T'(T^j(x)) \prod_{j=0}^{n-1} \lambda(T^j(x)) + 1\right) (\operatorname{diam} U)^s \\
&\quad\le C \sum_{[x_0, x_1, \ldots, x_{n-1}] \in \mathcal{C}_n} \prod_{j=0}^{n-1} T'(T^j(x))^{1-s} \prod_{j=0}^{n-1} \lambda(T^j(x)) \\
&\qquad + \sum_{[x_0, x_1, \ldots, x_{n-1}] \in \mathcal{C}_n} \prod_{j=0}^{n-1} T'(T^j(x))^{-s} \\
&\quad\le (C+1) e^{\frac{-n\eta}{2}}.
\end{aligned}$$

Letting $n \to \infty$ we have that $\mathcal{H}^s_\delta(\operatorname{graph}(w_\theta)) = 0$. Letting $\delta \to 0$, we have that $\mathcal{H}^s(\operatorname{graph}(w_\theta)) = 0$. Hence $\dim_H \operatorname{graph}(w_\theta) \le s$. As $s > s_0$ is arbitrary, the result follows. ❑

§4 Proof of Proposition 2.3

We first need the following bounded distortion estimate on Bowen balls.

Lemma 4.1
Let $f : S^1 \to \mathbb{R}$ be Hölder continuous of exponent α. Let δ be less than the injectivity radius of T. Then there exists a constant $C > 0$ such that for all balls $B_{n,\delta}(z)$ and all $x, y \in B_{n,\delta}(z)$ we have

$$\left| \sum_{j=0}^{n-1} f(T^j(x)) - f(T^j(y)) \right| \le C |f|_\alpha \delta^\alpha.$$

Proof. To see this note that if x, y, δ are as in the statement of the lemma, then $d(x,y) \le \sup_{x \in S^1} (T^{-1})'(x) d(x,y) \le \beta^{-1} d(x,y)$. Inductively we obtain that

$$\begin{aligned} \left| \sum_{j=0}^{n-1} f(T^j(x)) - f(T^j(y)) \right| &\le |f|_\alpha \sum_{j=0}^{n-1} \frac{\delta^\alpha}{\beta^{j\alpha}} \\ &\le \frac{|f|_\alpha}{1-\beta^{-\alpha}} \delta^\alpha. \end{aligned}$$

❏

Remark. As λ is C^1, $\log \lambda$ is Hölder continuous. It follows immediately from Lemma 4.1 that, if $\delta > 0$ is less than the injectivity radius of T, then there exists $C_\lambda > 0$ such that for all balls $B_{n,\delta}(z)$ and all $x, y \in B_{n,\delta}(z)$ we have

$$\frac{1}{C_\lambda} \le \frac{\lambda^n(x)}{\lambda^n(y)} \le C_\lambda. \tag{15}$$

Proof of Proposition 2.3. First note that

$$(s-1)\log T' + 2(g - P(g)) - \log \lambda \ge s \log \beta + 2(g - P(g)) - \| \log T' + \log \lambda \|_\infty.$$

Hence $P((s-1)\log T' + 2(g-P(g)) - \log\lambda) \ge s\log\beta + P(2(g-P(g))) - \|\log T' + \log\lambda\|_\infty$ so that $P((s-1)\log T' + 2(g-P(g)) - \log\lambda) \to \infty$ as $s \to \infty$. Note that $-\log T' + 2(g-P(g)) - \log\lambda \le -\log\rho + 2(g - P(g))$. Hence when $s = 0$,

$$\begin{aligned} &P((s-1)\log T' + 2(g-P(g)) - \log\lambda) \qquad (16) \\ &= P(-\log T' + 2(g-P(g)) - \log\lambda) \\ &\le -\log\rho + P(2g) - 2P(g). \qquad (17) \end{aligned}$$

As $P(2g) - 2P(g) < 0$, we see that $P((s-1)\log T' + 2(g-P(g)) - \log\lambda) < 0$ when $s = 0$. By the continuity of pressure, there exists a unique value $s_g > 0$ solving (12).

Let $g : S^1 \to \mathbb{R}$ be Hölder continuous and let μ_g be the associated equilibrium state. Let $\theta = (\vartheta_j)_{j=0}^\infty \in \Omega$. Define a measure $\hat{\mu}_g$ on $S^1 \times \mathbb{R}$ supported on graph(w_θ) by $\hat{\mu}_g(E) = \mu_g\{x \in S^1 \mid (x, w_\theta(x)) \in E\}$. We want to show that if $s < s_g$ where s_g is determined by (12) then

$$I_s(\hat{\mu}_g) = \iint_{S^1 \times S^1} \frac{d\mu_g(x)\, d\mu_g(y)}{((x-y)^2 + (w_\theta(x) - w_\theta(y))^2)^{s/2}} < \infty$$

for $d\theta$-almost every $\theta \in \Omega$. To do this, it is sufficient to assume that $s > 1$ and prove that

$$E_s = \int_\Omega I_s(\hat{\mu}_g)\, d\theta < \infty.$$

By Fubini's theorem we can write

$$E_s = \iint_{S^1\times S^1} \int_\Omega \frac{d\theta\, d\mu_g(x)\, d\mu_g(y)}{((x-y)^2 + (w_\theta(x) - w_\theta(y))^2)^{s/2}}.$$

Let $r > 0$ be determined by the critical point hypothesis. Choose $\delta < 1/4\beta$ and shrink δ further, if necessary, so that $\|T'\|_\infty^r \delta$ is less than the injectivity radius of T. Note that δ is an expansivity constant for T. Let

$$X_n^r = \{(x,y) \in S^1 \times S^1 \mid d_n(x,y) < \delta, \delta \le d(T^n(x), T^n(y))\}.$$

Then clearly $\bigcup_{n=0}^\infty X_n^r \subset \{(x,y) \in S^1 \times S^1 \mid d(x,y) < \delta\} = \Delta_\delta$, a neighbourhood of the diagonal in $S^1 \times S^1$.

Note that if $(x,y) \in (S^1 \times S^1) \setminus \Delta_\delta$ then $|x-y| \ge \delta$. Hence

$$\begin{aligned} &\iint_{(S^1\times S^1)\setminus\Delta_\delta} \int_\Omega \frac{d\theta\, d\mu_g(x)\, d\mu_g(y)}{((x-y)^2 + (w_\theta(x) - w_\theta(y))^2)^{s/2}} \\ \le\ & \iint_{(S^1\times S^1)\setminus\Delta_\delta} \int_\Omega \frac{d\theta\, d\mu_g(x)\, d\mu_g(y)}{|x-y|^s} \\ \le\ & \frac{1}{\delta^s}. \end{aligned}$$

Hence

$$E_s = \frac{1}{\delta^s} + \iint_{\Delta_\delta} \int_\Omega \frac{d\theta\, d\mu_g(x)\, d\mu_g(y)}{((x-y)^2 + (w_\theta(x) - w_\theta(y))^2)^{s/2}} \tag{18}$$

and it remains to show that the second term in (18) is finite.

Fix $x, y \in X_n^r$. Let $z_{x,y}(\theta) = w_\theta(x) - w_\theta(y)$ and let $h_{x,y}$ denote the density of $z_{x,y}$. Then the second term in (18) can be written as

$$E_s(\delta) = \sum_{n=0}^\infty \iint_{X_n^r} \int_{-\infty}^\infty \frac{h_{x,y}(z_{x,y})\, dz_{x,y}\, d\mu_g(x)\, d\mu_g(y)}{\left((x-y)^2 + z_{x,y}^2\right)^{s/2}}.$$

Let $z_{x,y} = |x-y|u_{x,y}$ so that $dz_{x,y} = |x-y| du_{x,y}$. Then

$$E_s(\delta) = \sum_{n=0}^\infty \iint_{X_n^r} \int_{-\infty}^\infty \frac{|x-y|^{1-s} h_{x,y}(|x-y|u_{x,y})\, du_{x,y}\, d\mu_g(x)\, d\mu_g(y)}{(1 + u_{x,y}^2)^{s/2}}.$$

Let

$$K(s) = \int_{-\infty}^\infty \frac{du}{(1+u^2)^{s/2}}$$

and note that $K(s) < \infty$ if $s > 1$. Then

$$E_s(\delta) \le K(s) \sum_{n=0}^\infty \iint_{X_n^r} |x-y|^{1-s} \sup_{u\in\mathbb{R}} h_{x,y}(u)\, d\mu_g(x)\, d\mu_g(y).$$

Let F_n be an (n,δ)-spanning set which achieves the infimum in (6) for the potential $(1-s)\log T' + \log\lambda$. For $z \in F_n$ let

$$X_n^r(z) = X_n^r \cap \left(B_{n,2\delta}(z) \times B_{n,2\delta}(z)\right).$$

Let $(x,y) \in X_n^r$. As F_n is (n,δ)-spanning, there exists $z \in F_n$ such that $d_n(x,z) < \delta$. Hence $d_n(y,z) \leq d(y,x) + d(x,z) \leq 2\delta$. Clearly $d_n(x,z) < 2\delta$, so it follows that $(x,y) \in X_n^r(z)$. Hence

$$X_n^r = \bigcup_{z \in F_n} X_n^r(z).$$

Hence

$$E_s(\delta) \leq K(s) \sum_{n=0}^{\infty} \sum_{z \in F_n} \int\!\!\int_{X_n^r(z)} |x-y|^{1-s} \sup_{u \in \mathbb{R}} h_{x,y}(u)\, d\mu_g(x)\, d\mu_g(y).$$

We first bound $h_{x,y}$.

Lemma 4.2
Let $x, y \in B_{n,2\delta}(z)$. Then $\sup_{u\in\mathbb{R}} h_{x,y}(u) < C_h \lambda^n(z)^{-1}$ where $C_h > 0$ is a constant independent of x, y, n.

Proof. Write

$$\begin{aligned} z_{x,y}(\theta) &= w_\theta(x) - w_\theta(y) \\ &= \sum_{k=0}^{\infty} \lambda^k(x) p(T^k(x) + \vartheta_k) - \lambda^k(y) p(T^k(y) + \vartheta_k) \\ &= \sum_{k=0}^{\infty} z_{k,x,y}(\vartheta_k). \end{aligned}$$

Let $h_{k,x,y}$ denote the density of $z_{k,x,y}$. As the ϑ_ks are independent, the density of $z_{x,y}$ is the convolution of the densities of the $z_{k,x,y}$. Hence

$$h_{x,y} = \mathop{\Huge *}_{k=0}^{\infty} h_{k,x,y}. \tag{19}$$

Now a bound on the convolution of the $h_{k,x,y}$ for finitely many values of n will automatically be a bound on the infinite convolution in (19). Hence

$$h_{x,y} \leq \mathop{\Huge *}_{j=0}^{r-1} h_{n+j,x,y}$$

where n is chosen so that x, y are in the same Bowen ball $B_{n,2\delta}(z)$. By Hölder's inequality we can bound

$$\mathop{\Huge *}_{j=0}^{r-1} h_{n+j,x,y} \leq \|h_{n,x,y} * \cdots * h_{n+r-2,x,y}\|_r \|h_{n+r-1,x,y}\|_{\frac{r}{r-1}}.$$

Repeated applications of Young's inequality then implies that

$$\|h_{n,x,y} * \cdots * h_{n+r-2,x,y}\|_r \leq \|h_{n,x,y}\|_{\frac{r}{r-1}} \cdots \|h_{n+r-2,x,y}\|_{\frac{r}{r-1}}.$$

Hence

$$h_{x,y} \leq \prod_{j=0}^{r-1} \|h_{n+j,x,y}\|_{\frac{r}{r-1}}. \tag{20}$$

Define $z'_{i,x,y}(\vartheta_i)$ by

$$\begin{aligned} z_{i,x,y}(\vartheta_i) &= \lambda^i(x)\left(p(T^i(x)+\vartheta_i) - \frac{\lambda^i(y)}{\lambda^i(x)}p(T^i(y)+\vartheta_i)\right) \\ &= \lambda^i(x) z'_{i,x,y}(\vartheta_i). \end{aligned}$$

Let $h'_{i,x,y}$ denote the density of $z'_{i,x,y}(\vartheta_i)$. Then

$$h_{i,x,y}(u) = \frac{1}{\lambda^i(x)} h'_{i,x,y}\left(\frac{u}{\lambda^i(x)}\right). \tag{21}$$

We will prove that

$$\|h'_{n+j,x,y}\|_{\frac{r}{r-1}} \leq M \tag{22}$$

for $j = 0, \ldots, r-1$, for some M independent of x, y. It then follows from (21) that

$$\|h_{n+j,x,y}\|_{\frac{r}{r-1}} \leq M\lambda^{n+j}(x)^{-1/r} \leq \|\lambda\|_\infty^{-j/r} M\lambda^n(x)^{-1/r}. \tag{23}$$

Hence from (20)

$$h_{x,y} \leq M^r \|\lambda\|_\infty^{(r-1)/2} \lambda^n(x)^{-1} \leq C_h \lambda^n(z)^{-1}.$$

where the last equality follows from the remark following Lemma 4.1.

It remains to prove (22). Write $\vartheta'_j = T^j(y) + \vartheta_j$ so that

$$z'_{n+j,x,y}(\vartheta_{n+j}) = p(T^{n+j}(x) - T^{n+j}(y) + \vartheta'_{n+j}) - \frac{\lambda^{n+j}(y)}{\lambda^{n+j}(x)} p(\vartheta'_{n+j}).$$

Now $x, y \in X_n^r(z)$. Hence $d(T^j(x), T^j(y)) \leq \delta$ for $0 \leq j \leq n-1$ and $\delta \leq d(T^n(x), T^n(y))$. Recalling that $\beta \leq \inf_{x \in S^1} T'(x)$ it follows that $\beta^{j-1}\delta \leq d(T^{n+j}(x), T^{n+j}(y)) < \|T'\|_\infty^j \delta$ for $j = 0, 1, \ldots, r-1$. In particular, there exists $\kappa > 0$ such that

$$\kappa \leq |T^{n+j}(x) - T^{n+j}(y)| < 1 - \kappa$$

for $j = 0, 1, \ldots, r-1$.

Let $\Lambda = \sup_{x \in S^1} \lambda(x) / \inf_{x \in S^1} \lambda(x)$. By (15) there exists $C_\lambda > 0$ such that

$$\frac{1}{C_\lambda \Lambda^r} \leq \frac{\lambda^{n+j}(x)}{\lambda^{n+j}(y)} \leq C_\lambda \Lambda^r$$

for $j = 0, 1, \ldots, r-1$.

Suppose that $q : S^1 \to \mathbb{R}$ has a critical point of order k at $x_0 \in S^1$ and $q^{(k)}(x_0) = b$. Then the density of q in a neighbourhood of $q(x_0)$ behaves like $C(b)t^{(1-k)/k}$ where the constant $C(b)$ is of the order $O(b^{-1/k})$. Hence if all the critical points of q have order less than r then the density h_q of q is such that $\|h_q\|_{\frac{r}{r-1}} < \infty$. Suppose we have a family $q_j : S^1 \to \mathbb{R}$, $j \in J$, which have critical points of order less than r. If the values of $q_j^k(x_0) = b$ where x_0 is a critical point of order k for q_j, as j ranges over J, $k < r$, are uniformly bounded away from 0 then there exists $M > 0$ such that $\|h_{q_j}\|_{\frac{r}{r-1}} \leq M$ for all $j \in J$.

By the critical point hypothesis, the critical points of

$$z'_{n+j,x,y}(\vartheta_{n+j}) = p(T^{n+j}(x) - T^{n+j}(y) + \vartheta'_{n+j}) - \frac{\lambda^{n+j}(y)}{\lambda^{n+j}(x)} p(\vartheta'_{n+j})$$

have orders less than r, and the corresponding kth derivatives ($1 \leq k < r$) take values bounded away from zero as $|T^{n+j}(x) - T^{n+j}(y)| \in [\kappa, 1-\kappa]$, $\lambda^n(x)/\lambda^n(y) \in [(C_\lambda \Lambda^r)^{-1}, C_\lambda \Lambda^r]$, a compact set. Hence there exists $M > 0$ such that $\|h'_{n+j,x,y}\|_{\frac{r}{r-1}} \leq M$ for $j = 0, \ldots, r-1$ and all $x, y \in S^1$. ❏

To complete the estimate on the bound of E_s we need the following result.

Lemma 4.3
Let $x, y \in B_{n,2\delta}(z)$. Then $|x-y|^{1-s} \leq C_T (T^n)'(z)^{s-1}$ where $C_T > 0$ is a constant independent of x, y, n.

Proof. This follows immediately from Lemma 4.1 and the fact that $s > 1$. ❏

From Lemmas 4.2 and 4.3 it follows that

$$E_s(\delta) \leq K(s) C_h C_T \sum_{n=0}^{\infty} \sum_{z \in F_n} \iint_{X_n^r(z)} (T^n)'(z)^{s-1} \lambda^n(z)^{-1}\, d\mu_g(x)\, d\mu_g(y)$$

As the integrand is constant on each ball $B_{n,2\delta}(z)$, each x and y in the integrand are in the same ball $B_{n,\delta}(z)$, and $X_n^r(z) \subset B_{n,2\delta}(z) \times B_{n,2\delta}(z)$, we can bound $E_s(\delta)$ by

$$\begin{aligned} & K(s) C_h C_T \sum_{n=0}^{\infty} \sum_{z \in F_n} (T^n)'(z)^{s-1} \lambda^n(z)^{-1} \iint_{B_{n,2\delta}(z) \times B_{n,2\delta}(z)} d\mu_g(x)\, d\mu_g(y) \\ \leq\ & K(s) C_h C_T \sum_{n=0}^{\infty} \sum_{z \in F_n} (T^n)'(z)^{s-1} \lambda^n(z)^{-1} \mu_g(B_{n,2\delta}(z)) \mu_g(B_{n,2\delta}(z)). \end{aligned}$$

Hence, by (8), $E_s(\delta)$ is bounded above by

$$K(s)C_hC_TC(g,2\delta)^2\sum_{n=0}^{\infty}\sum_{z\in F_n}\exp\sum_{j=0}^{n-1}\big((s-1)\log T'(T^j(z))-\log\lambda(T^j(z))$$
$$+2(g(T^j(z))-P(g))\big).$$

As $1<s<s(g)$, where $s(g)$ is the the unique solution to (12), we have that $P((s-1)\log T'-\log\lambda+2(g-P(g)))=-\eta<0$. Then there exists $C>0$ such that for all $n\geq 0$ we have

$$\sum_{z\in F_n}\exp\sum_{j=0}^{n-1}(s-1)\log T'(T^j(z))-\log\lambda(T^j(z))+2(g(T^j(z))-P(g))$$
$$\leq\ C\exp^{\frac{-n\eta}{2}}.$$

In particular, if $1<s<s(g)$ then

$$E_s(\delta)\leq CK(s)C_hC_TC(g,2\delta)^2\sum_{n=0}^{\infty}\exp^{\frac{-n\eta}{2}}<\infty.$$

By choosing s arbitrarily close to $s(g)$, the result follows. ❏

References

[Be] T. Bedford, *The box dimension of self-affine graphs and repellers*, Nonlinearity **2** (1989), 53–71.

[BU] A.S. Besicovitch and H.D. Ursell, *Sets of fractional dimensions, V: On dimensional numbers of some continuous curves*, J. London Math. Soc. **12** (1937), 18–25.

[HNW] D. Hadjiloucas, M. J. Nicol, and C. P. Walkden, *Regularity of invariant graphs over hyperbolic systems*, Ergod. Th. & Dyn. Syst. **22** (2002), 469–482.

[H] B. Hunt, *The Hausdorff dimension of graphs of Weierstrass functions*, Proc. Amer. Math. Soc. **126** (1998), 791–800.

[M] C. McMullen, *The Hausdorff dimension of general Sierpiński carpets*, Nagoya Math. J. **96** (1984), 19.

[PU] F. Przytycki and M. Urbański, *On the Hausdorff dimension of some fractal sets*, Studia Math. **93**(1989), 155186.

[W] P. Walters, *An introduction to ergodic theory*, Springer, Berlin, 1982.

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