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Witten-Hodge theory for manifolds with boundary and equivariant cohomology

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Abstract

We consider a compact, oriented, smooth Riemannian manifold M (with or without boundary) and we suppose G is a torus acting by isometries on M. Given X in the Lie algebra and corresponding vector field X_M on M, one defines Witten's inhomogeneous coboundary operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M} : \Omega_G^{\pm} \to \Omega_G^{\mp}$ (even/odd invariant forms on M) and its adjoint δ_{X_M}. Witten [18] showed that the resulting cohomology classes have X_M-harmonic representatives (forms in the null space of $\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2$), and the cohomology groups are isomorphic to the ordinary de Rham cohomology groups of the set $N(X_M)$ of zeros of X_M. Our principal purpose is to extend these results to manifolds with boundary. In particular, we define relative (to the boundary) and absolute versions of the X_M-cohomology and show the classes have representative X_M-harmonic fields with appropriate boundary conditions. To do this we present the relevant version of the Hodge-Morrey-Friedrichs decomposition theorem for invariant forms in terms of the operators d_{X_M} and δ_{X_M}. We also elucidate the connection between the X_M-cohomology groups and the relative and absolute equivariant cohomology, following work of Atiyah and Bott. This connection is then exploited to show that every harmonic field with appropriate boundary conditions on $N(X_M)$ has a unique X_M-harmonic field on M, with corresponding boundary conditions. Finally, we define the X_M-Poincaré duality angles between the interior subspaces of X_M-harmonic fields on M with appropriate boundary conditions, following recent work of DeTurck and Gluck.

Keywords: Hodge theory, manifolds with boundary, equivariant cohomology, Killing vector fields
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1. Introduction

Throughout we assume M to be a compact oriented smooth Riemannian manifold of dimension n, with or without boundary. For each k we denote by $\Omega^k = \Omega^k(M)$ the space of smooth differential k-forms on M. The de Rham cohomology of M is defined to be $H^k(M) = \ker \mathsf{d}_k / \operatorname{im} \mathsf{d}_{k-1}$, where d_k is the restriction of the exterior differential d to Ω^k. In other words it is the cohomology of the de Rham complex (Ω^*, d). If M has a boundary, then the relative de Rham cohomology $H^k(M, \partial M)$ is defined to be the cohomology of the subcomplex (Ω_D^*, d) where Ω_D^k is the space of *Dirichlet* k-forms—those satisfying $i^*\omega = 0$ where $i : \partial M \hookrightarrow M$ is the inclusion of the boundary.

Classical Hodge theory. Based on the Riemannian structure, there is a natural inner product on each Ω^k defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta), \tag{1.1}$$

where $\star : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator [1, 15]. One defines $\delta : \Omega^k \to \Omega^{k-1}$ by

$$\delta\omega = (-1)^{n(k+1)+1}(\star\mathsf{d}\star)\omega. \tag{1.2}$$

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If M is boundaryless, this is seen to be the formal adjoint of d relative to the inner product (1.1): $\langle \mathsf{d}\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$. The Hodge Laplacian is defined by $\Delta = (\mathsf{d}+\delta)^2 = \mathsf{d}\delta + \delta\mathsf{d}$, and a form ω is said to be *harmonic* if $\Delta\omega = 0$.

In the 1930s, Hodge [10] proved the fundamental result that (for M without boundary) each cohomology class contains a unique harmonic form. A more precise statement is that, for each k,

$$\Omega^k(M) = \mathcal{H}^k \oplus \mathsf{d}\Omega^{k-1} \oplus \delta\Omega^{k+1}. \tag{1.3}$$

The direct sums are orthogonal with respect to the inner product (1.1), and the direct sum of the first two subspaces is equal to the subspace of all closed k-forms (that is, $\ker \mathsf{d}_k$). It follows that the Hodge star operator realizes Poincaré duality at the level of harmonic forms.

Furthermore, any harmonic form $\omega \in \ker\Delta$ is both closed ($\mathsf{d}\omega = 0$) and co-closed ($\delta\omega = 0$), as

$$0 = \langle \Delta\omega, \omega\rangle = \langle \mathsf{d}\delta\omega, \omega\rangle + \langle \delta\mathsf{d}\omega, \omega\rangle = \langle \delta\omega, \delta\omega\rangle + \langle \mathsf{d}\omega, \mathsf{d}\omega\rangle = \|\delta\omega\|^2 + \|\mathsf{d}\omega\|^2. \tag{1.4}$$

For manifolds with boundary this is no longer true, and in general we write

$$\mathcal{H}^k = \mathcal{H}^k(M) = \ker \mathsf{d} \cap \ker\delta.$$

Thus for manifolds without boundary $\mathcal{H}(M) = \ker\Delta$, the space of harmonic forms.

Remark 1.1 An interesting observation which follows from the theorem of Hodge is the following. If a group G acts on M then there is an induced action on each $H^k(M)$, and if this action is trivial (for example, if G is a connected Lie group) and the action is by isometries, then each harmonic form is invariant under this action.

Witten's deformation of Hodge theory. Now suppose K is a Killing vector field on M (meaning that the Lie derivative of the metric vanishes). Witten [18] defines, for each $s \in \mathbb{R}$, an operator on differential forms by

$$\mathsf{d}_s := \mathsf{d} + s\,\iota_K\,,$$

where ι_K is interior multiplication of a form with K. This operator is no longer homogeneous in the degree of the form: if $\omega \in \Omega^k(M)$ then $\mathsf{d}_s\omega \in \Omega^{k+1} \oplus \Omega^{k-1}$. Note then that $\mathsf{d}_s : \Omega^\pm \to \Omega^\mp$, where $\Omega^\pm$ is the space of forms of even ($+$) or odd ($-$) degree. Let us write $\delta_s = \mathsf{d}_s^*$ for the formal adjoint of d_s (so given by $\delta_s = \delta + s(-1)^{n(k+1)+1}(\star\,\iota_K\star)$ on each homogeneous form of degree k). By Cartan's formula, $\mathsf{d}_s^2 = s\mathcal{L}_K$ (the Lie derivative along sK). On the space $\Omega_s^\pm = \Omega^\pm \cap \ker\mathcal{L}_K$ of invariant forms, $\mathsf{d}_s^2 = 0$ so one can define two cohomology groups $H_s^\pm := \ker \mathsf{d}_s^\pm / \operatorname{im} \mathsf{d}_s^\mp$. Witten then defines

$$\Delta_s := (\mathsf{d}_s + \delta_s)^2 : \Omega_s^\pm(M) \to \Omega_s^\pm(M),$$

(which he denotes H_s as it represents a Hamiltonian operator, but for us this would cause confusion), and he observes that using standard Hodge theory arguments, there is an isomorphism

$$\mathcal{H}_s^\pm := (\ker\Delta_s)^\pm \cong H_s^\pm(M), \tag{1.5}$$

although no details of the proof are given (the interested reader can find details in [3]). Witten also shows, among other things, that for $s \neq 0$, the dimensions of $\mathcal{H}_s^\pm$ are respectively equal to the total even and odd Betti numbers of the subset N of zeros of K, which in particular implies the finiteness of $\dim\mathcal{H}_s$. Atiyah and Bott [2] relate this result of Witten's to their localization theorem in equivariant cohomology.

It is well-known that the group of isometries of a Riemannian manifold (with or without boundary) is compact, so that a Killing vector field generates an action of a torus. In this light, and because of Remark 1.1 (and its extension to Witten's setting), Witten's analysis can be cast in the following slightly more general context.

Throughout, we let G be a torus acting by isometries on M, with Lie algebra $\mathfrak{g}$, and denote by $\Omega_G = \Omega_G(M)$ the space of smooth G-invariant forms on M. Given any $X \in \mathfrak{g}$ we denote the corresponding vector field on M by X_M, and following Witten we define $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M}$. Then d_{X_M} defines an operator

$\mathsf{d}_{X_M} : \Omega_G^{\pm} \to \Omega_G^{\mp}$, with $\mathsf{d}_{X_M}^2 = 0$. For each $X \in \mathfrak{g}$ there are therefore two corresponding cohomology groups $H^{\pm}_{X_M}(M) = \ker \mathsf{d}^{\pm}_{X_M} / \operatorname{im} \mathsf{d}^{\mp}_{X_M}$, which we call X_M-cohomology groups, and a corresponding operator we call the *Witten-Hodge-Laplacian*

$$\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2 : \Omega_G^{\pm} \to \Omega_G^{\pm}.$$

According to Witten there is an isomorphism $\mathcal{H}^{\pm}_{X_M} \cong H^{\pm}_{X_M}(M)$, where $\mathcal{H}^{\pm}_{X_M}$ is the space of X_M-harmonic forms, that is those forms annihilated by Δ_{X_M}. Of course, Witten's presentation is no less general than this, and is obtained by putting $X_M = sK$; the only difference is we are thinking of X as a variable element of $\mathfrak{g}$, while for Witten varying s only gives a 1-dimensional subspace of $\mathfrak{g}$ (although one may change K as well).

The immediate purpose of this paper is to extend Witten's results to manifolds with boundary. In order to do this, in Section 2 we outline the background to Witten's results using classical Hodge theory arguments, which in Section 3 we extend to deal with the case of manifolds with boundary. In section 4, we describe Atiyah and Bott's localization and its conclusions in the case of manifolds with boundary, and its relation to X_M-cohomology. Finally in Section 5, we extend our results to adapt ideas of DeTurck and Gluck [6] and the *Poincaré duality angles*. Section 6 provides a few conclusions.

The original motivation for this paper was to adapt to the equivariant setting some recent work of Belishev and Sharafutdinov [5] where they address the classical question, *"To what extent are the topology and geometry of a manifold determined by the Dirichlet-to-Neumann (DN) map"?* which arises in the scope of inverse problems and reconstructing a manifold from boundary measurements. They show that the DN map on the boundary of a Riemannian manifold determines the Betti numbers of the manifold. This paper provides the background necessary for the "equivariant" analogue [4] of the results of Belishev and Sharafudtinov.

Hodge theory for manifolds with boundary. In the remainder of this introduction we recall the standard extension of Hodge theory to manifolds with boundary, leading to the Hodge-Morrey-Friedrichs decompositions; details can be found in the book of Schwarz [15]. The relative de Rham cohomology and the Dirichlet forms are defined at the beginning of the introduction. One also defines $\Omega^k_N(M) = \left\{\alpha \in \Omega^k(M) \mid i^*(\star\alpha) = 0\right\}$ (Neumann boundary condition). Clearly, the Hodge star provides an isomorphism $\star : \Omega^k_D \xrightarrow{\sim} \Omega^{n-k}_N$. Furthermore, because d and i^* commute, it follows that d preserves Dirichlet boundary conditions while δ preserves Neumann boundary conditions.

As alluded to before, because of boundary terms, the null space of Δ no longer coincides with the closed and co-closed forms. Elements of $\ker\Delta$ are called *harmonic forms*, while ω satisfying $\mathsf{d}\omega = \delta\omega = 0$ are called *harmonic fields* (following Kodaira); it is clear that every harmonic field is a harmonic form, but the converse is false. In fact, the space $\mathcal{H}^k(M)$ of harmonic fields is infinite dimensional and so is much too big to represent the cohomology, and to recover the Hodge isomorphism one has to impose boundary conditions. One restricts $\mathcal{H}^k(M)$ into each of two finite dimensional subspaces, namely $\mathcal{H}^k_D(M)$ and $\mathcal{H}^k_N(M)$ with the obvious meanings (Dirichlet and Neumann harmonic k-fields, respectively). There are therefore two different candidates for harmonic representatives when the boundary is present.

The Hodge-Morrey decomposition [13] states that

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N.$$

(We make a more precise functional analytic statement below.) This decomposition is again orthogonal with respect to the inner product given above. Friedrichs [8] subsequently showed that

$$\mathcal{H}^k = \mathcal{H}^k_D \oplus \mathcal{H}^k_{\mathrm{co}}; \qquad \mathcal{H}^k = \mathcal{H}^k_N \oplus \mathcal{H}^k_{\mathrm{ex}}$$

where $\mathcal{H}^k_{\mathrm{ex}}$ are the exact harmonic fields and $\mathcal{H}^k_{\mathrm{co}}$ the coexact ones (that is, $\mathcal{H}^k_{\mathrm{ex}} = \mathcal{H}^k \cap \mathsf{d}\Omega^{k-1}$ and $\mathcal{H}^k_{\mathrm{co}} = \mathcal{H}^k \cap \delta\Omega^{k+1}$). These give the orthogonal *Hodge-Morrey-Friedrichs* [15] decompositions,

$$\begin{aligned} \Omega^k(M) &= \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_D \oplus \mathcal{H}^k_{\mathrm{co}} \\ &= \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_N \oplus \mathcal{H}^k_{\mathrm{ex}}. \end{aligned}$$

The two decompositions are related by the Hodge star operator. The consequence for cohomology is that each class in $H^k(M)$ is represented by a unique harmonic field in $\mathcal{H}^k_N(M)$, and each relative class in

$H^k(M,\partial M)$ is represented by a unique harmonic field in $\mathcal{H}^k_D(M)$. Again, the Hodge star operator acts as Poincaré duality (or rather Poincaré-Lefschetz duality) on the harmonic fields, sending Dirichlet fields to Neumann fields. And as in remark 1.1, if a group acts by isometries on $(M,\partial M)$ in a manner that is trivial on the cohomology, then the harmonic fields are invariant.

In this paper, we suppose G is a compact connected Abelian Lie group (a torus) acting by isometries on M, with Lie algebra $\mathfrak{g}$, and we let $X\in\mathfrak{g}$. If M has a boundary then the G-action necessarily restricts to an action on the boundary and X_M must therefore be tangent to the boundary. We denote by $\Omega_G=\Omega_G(M)$ the set of invariant forms on M: $\omega\in\Omega_G$ if $g^*\omega=\omega$ for all $g\in G$; in particular if ω is invariant then the Lie derivative $\mathcal{L}_{X_M}\omega=0$. Note that because the action preserves the metric and the orientation it follows that, for each $g\in G$, $\star(g^*\omega)=g^*(\star\omega)$, so if $\omega\in\Omega_G$ then $\star\omega\in\Omega_G$.

Remark on typesetting: Since the letter H plays three roles in this paper, we use three different typefaces: a script $\mathcal{H}$ for harmonic fields, a sans-serif H for Sobolev spaces and a normal (italic) H for cohomology. We hope that will prevent any confusion.

2. Witten-Hodge theory for manifolds without boundary

In this section we summarize the functional analysis behind Witten's results [18], details can be found in the first author's thesis [3]. These are needed in the next section for manifolds with boundary. We continue to use the notation from the introduction, notably the manifold M (which in this section has no boundary) and the torus G.

Fix an element $X\in\mathfrak{g}$. The associated vector field on M is X_M, and using this one defines Witten's inhomogeneous operator $\mathsf{d}_{X_M}:\Omega^{\pm}_G\to\Omega^{\mp}_G$, $\mathsf{d}_{X_M}\omega=\mathsf{d}\omega+\iota_{X_M}\omega$, and the corresponding operator (cf. eq. (1.2))

$$\delta_{X_M}=(-1)^{n(k+1)+1}\star\mathsf{d}_{X_M}\star=\delta+(-1)^{n(k+1)+1}\star\iota_{X_M}\star$$

(which is the operator adjoint to d_{X_M} by eq. (2.2) below). The resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M}:\Omega^{\pm}_G\to\Omega^{\pm}_G$ defined by $\Delta_{X_M}=(\mathsf{d}_{X_M}+\delta_{X_M})^2=\mathsf{d}_{X_M}\delta_{X_M}+\delta_{X_M}\mathsf{d}_{X_M}$. We write the space of X_M-harmonic fields

$$\mathcal{H}_{X_M}=\ker\mathsf{d}_{X_M}\cap\ker\delta_{X_M}\,,$$

which for manifolds without boundary satisfies $\mathcal{H}_{X_M}=\ker\Delta_{X_M}$. The last equality follows for the same reason as for ordinary Hodge theory, namely the argument in (1.4), with Δ replaced by Δ_{X_M} etc.

As is conventional, define $\int_M\omega=0$ if $\omega\in\Omega^k(M)$ with $k\neq n$. So, for any form $\omega\in\Omega(M)$ one has $\int_M\iota_{X_M}\omega=0$ as $\iota_{X_M}\omega$ has no term of degree n, and the following equation (2.1) follows from the ordinary Stokes' theorem. For future use, we allow M to have a boundary.

$$\int_M\mathsf{d}_{X_M}\omega=\int_{\partial M}i^*\omega. \tag{2.1}$$

For each space Ω of smooth differential forms on M, and each $s\in\mathbb{R}$, we write $\mathsf{H}^s\Omega$ for the completion of Ω under an appropriate Sobolev norm. It is not hard to prove a Green's formula in terms of d_{X_M} and δ_{X_M} which states that for $\alpha,\beta\in\mathsf{H}^1\Omega_G$,

$$\langle\mathsf{d}_{X_M}\alpha,\beta\rangle=\langle\alpha,\delta_{X_M}\beta\rangle+\int_{\partial M}i^*(\alpha\wedge\star\beta)\,, \tag{2.2}$$

Returning now to the case of a manifold without boundary, we obtain the following.

Theorem 2.1 1. *The Witten-Hodge-Laplacian Δ_{X_M} is a self-adjoint elliptic operator.*

2. *The following is an orthogonal decomposition*

$$\Omega^{\pm}_G=\mathcal{H}^{\pm}_{X_M}\oplus\mathsf{d}_{X_M}\Omega^{\mp}_G\oplus\delta_{X_M}\Omega^{\mp}_G.$$

The orthogonality is with respect to the L^2 inner product.

Part (2) is the analogue of the Hodge decomposition theorem, and is a standard consequence of the fact that Δ_{X_M} is self-adjoint. The first two summands give the X_M-closed forms.

Every elliptic operator on a compact manifold is a Fredholm operator, so has finite dimensional kernel and cokernel, and closed range. Therefore the set of X_M-harmonic (even/odd) forms $\mathcal{H}^{\pm}_{X_M} = (\ker \Delta_{X_M})^{\pm}$ is finite dimensional. One concludes with the analogue of Hodge's theorem

Proposition 2.2 *$H^{\pm}_{X_M}(M) \cong \mathcal{H}^{\pm}_{X_M}$, and in particular every X_M-cohomology class has a unique X_M-harmonic representative.*

The Hodge star operator gives a form of Poincaré duality in terms of X_M-cohomology:

$$H^{n-\pm}_{X_M}(M) \cong H^{\pm}_{X_M}(M).$$

Since Hodge star takes harmonic forms to harmonic forms, this Poincaré duality is realized at the level of harmonic forms. The full details are given in [3]. Here and elsewhere we write $n - \pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n.

Let $N(X_M)$ be the set of zeros of X_M, and $j : N(X_M) \hookrightarrow M$ the inclusion. As observed by Witten, on $N(X_M)$ one has $X_M = 0$, so that $j^*\mathsf{d}_{X_M}\omega = \mathsf{d}(j^*\omega)$, and in particular if ω is X_M-closed then its pullback to $N(X_M)$ is closed in the usual (de Rham) sense. And exact forms pull back to exact forms. Consequently, pullback defines a natural map $H^{\pm}_{X_M}(M) \to H^{\pm}(N(X_M))$, where $H^{\pm}(N(X_M))$ is the direct sum of the even/odd de Rham cohomology groups of $N(X_M)$.

Theorem 2.3 (Witten [18]) *The pullback to $N(X_M)$ induces an isomorphism between the X_M-cohomology groups $H^{\pm}_{X_M}(M)$ and the cohomology groups $H^{\pm}(N(X_M))$.*

Witten gives a fairly explicit proof of this theorem by extending closed forms on $N(X_M)$ to X_M-closed forms on M. Atiyah and Bott [2] give a proof using their localization theorem in equivariant cohomology which we discuss, and adapt to the case of manifolds with boundary, in Section 4.

Remark 2.4 Extending remark 1.1, suppose X generates the torus $G(X)$, and G is a larger torus containing $G(X)$ and acting on M by isometries. Then the action of G preserves X_M. It follows that G acts trivially on the de Rham cohomology of $N(X_M)$, and hence on the X_M-cohomology of M, and consequently on the space of X_M-harmonic forms. In other words, $\mathcal{H}^{\pm}_{X_M} \subset \Omega^{\pm}_G$. There is therefore no loss in considering just forms invariant under the action of the larger torus in that the X_M-cohomology, or the space of X_M-harmonic forms, is independent of the choice of torus, provided it contains $G(X)$.

3. Witten-Hodge theory for manifolds with boundary

In this section we extend the results and methods of Hodge theory for manifolds with boundary to study the X_M-cohomology and the space of X_M-harmonic forms and fields for manifolds with boundary. As for ordinary (singular) cohomology, there are both absolute and relative X_M-cohomology groups. From now on our manifold will be with boundary and as before $i : \partial M \hookrightarrow M$ denotes the inclusion of the boundary, and G is a torus acting by isometries on M.

3.1. The difficulties if the boundary is present

Firstly, d_{X_M} and δ_{X_M} are no longer adjoint because the boundary terms arise when we integrate by parts, and then Δ_{X_M} will not be self-adjoint. In addition, the space of all harmonic fields is infinite dimensional and there is no reason to expect the X_M-harmonic fields $\mathcal{H}_{X_M}(M)$ to be any different. To overcome these problems, at the beginning we follow the method which is used to solve this problem in the classical case, i.e. with d and δ, by imposing certain boundary conditions on our invariant forms $\Omega_G(M)$, as described in [15]. Hence we make the following definitions.

Definition 3.1 (1) We define the following two sets of smooth invariant forms on the manifold M with boundary and with action of the torus G

$$\begin{aligned} \Omega_{G,D} &= \Omega_G \cap \Omega_D = \{\omega \in \Omega_G \mid i^*\omega = 0\} && (3.1)\\ \Omega_{G,N} &= \Omega_G \cap \Omega_N = \{\omega \in \Omega_G \mid i^*(\star\omega) = 0\} && (3.2) \end{aligned}$$

and the spaces $\mathsf{H}^s\Omega_{G,D}$ and $\mathsf{H}^s\Omega_{G,N}$ are the corresponding closures with respect to suitable Sobolev norms, for $s > \frac{1}{2}$. This can be refined to take into account the parity of the forms, so defining $\Omega^{\pm}_{G,D}$ etc. Since $\omega \in \Omega^k$ implies $\star\omega \in \Omega^{n-k}$ we write that for $\omega \in \Omega^{\pm}_G$ we have $\star\omega \in \Omega^{n-\pm}_G$.
(2) We define two subspaces of X_M-harmonic fields,

$$\begin{aligned} \mathcal{H}_{X_M,D}(M) &= \{\omega \in \mathsf{H}^1\Omega_{G,D} \mid \mathsf{d}_{X_M}\omega = 0,\ \delta_{X_M}\omega = 0\} && (3.3)\\ \mathcal{H}_{X_M,N}(M) &= \{\omega \in \mathsf{H}^1\Omega_{G,N} \mid \mathsf{d}_{X_M}\omega = 0,\ \delta_{X_M}\omega = 0\} && (3.4) \end{aligned}$$

which we call Dirichlet and Neumann X_M-harmonic fields, respectively. We will show below that these forms are smooth. Clearly, the Hodge star operator $\star$ defines an isomorphism $\mathcal{H}_{X_M,D}(M) \cong \mathcal{H}_{X_M,N}(M)$. Again, these can be refined to take the parity into account, defining $\mathcal{H}^{\pm}_{X_M,D}(M)$ etc.

As for ordinary Hodge theory, on a manifold with boundary one has to distinguish between X_M-harmonic *forms* (i.e. $\ker\Delta_{X_M}$) and X_M-harmonic *fields* (i.e. $\mathcal{H}_{X_M}(M)$) because they are not equal: one has $\mathcal{H}_{X_M}(M) \subseteq \ker\Delta_{X_M}$ but not conversely. The following proposition shows the conditions on ω to be fulfilled in order to ensure $\omega \in \ker\Delta_{X_M} \Longrightarrow \omega \in \mathcal{H}_{X_M}(M)$ when $\partial M \neq \emptyset$.

Proposition 3.2 *If $\omega \in \Omega_G(M)$ is an X_M-harmonic form (i.e. $\Delta_{X_M}\omega = 0$) and in addition any one of the following four pairs of boundary conditions is satisfied then $\omega \in \mathcal{H}_{X_M}(M)$.*

$$\begin{array}{llll} (1) & i^*\omega = 0,\ i^*(\star\omega) = 0; & (2) & i^*\omega = 0,\ i^*(\delta_{X_M}\omega) = 0;\\ (3) & i^*(\star\omega) = 0,\ i^*(\star\mathsf{d}_{X_M}\omega) = 0; & (4) & i^*(\delta_{X_M}\omega) = 0,\ i^*(\star\mathsf{d}_{X_M}\omega) = 0. \end{array}$$

PROOF: Because $\Delta_{X_M}\omega = 0$, one has $\langle \Delta_{X_M}\omega, \omega\rangle = 0$. Now applying Green's formula (2.2) to this and using any of these conditions (1)–(4) ensures ω is an X_M-harmonic field. ❒

Remark 3.3 An averaging argument shows that $\mathsf{H}^1\Omega_{G,D}$ and $\mathsf{H}^1\Omega_{G,N}$ are dense in $L^2\Omega_G$, because the corresponding statements hold for the spaces of all (not only invariant) forms.

3.2. Elliptic boundary value problem

The essential ingredients that Schwarz [15] needs to prove the classical *Hodge-Morry-Friedrichs* decomposition are Gaffney's inequality and his Theorem 2.1.5. However, these results do not appear to extend to the context of d_{X_M} and δ_{X_M}. Therefore, we use a different approach to overcome this problem, based on the ellipticity of a certain boundary value problem (BVP), namely (3.5) below. This theorem represents the keystone to extending the Hodge-Morrey and Friedrichs decomposition theorems to the present setting and thence to extending Witten's results to manifolds with boundary.

Consider the BVP

$$\begin{cases} \Delta_{X_M}\omega &= \eta \quad \text{on} \quad M\\ i^*\omega &= 0 \quad \text{on} \quad \partial M\\ i^*(\delta_{X_M}\omega) &= 0 \quad \text{on} \quad \partial M. \end{cases} \qquad (3.5)$$

Theorem 3.4

1. *The* BVP *(3.5) is elliptic in the sense of Lopatinskiĭ-Šapiro, where $\Delta_{X_M} : \Omega_G(M) \longrightarrow \Omega_G(M)$.*
2. *The* BVP *(3.5) is Fredholm of index 0.*
3. *All $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$ are smooth.*

PROOF: (1) We can see that Δ and Δ_{X_M} have the same principal symbol as $\Delta_{X_M} - \Delta$ is a first order differential operator; indeed,

$$\Delta_{X_M} = \Delta + (-1)^{n(k+1)+1}(\mathsf{d} \star \iota_{X_M} \star + \star \iota_{X_M} \star \mathsf{d} + \star \iota_{X_M} \star \iota_{X_M} + \iota_{X_M} \star \iota_{X_M} \star) + \iota_{X_M} \delta + \delta \iota_{X_M}.$$

Similarly, expanding the second boundary condition gives

$$\delta_{X_M} = \delta + (-1)^{n(k+1)+1} \star \iota_{X_M} \star$$

so δ_{X_M} and δ have the same first-order part. Hence our BVP (3.5) has the same principal symbol as the BVP

$$\begin{cases} \Delta\varepsilon &= \xi \quad \text{on} \quad M \\ i^*\varepsilon &= 0 \quad \text{on} \quad \partial M \\ i^*(\delta\varepsilon) &= 0 \quad \text{on} \quad \partial M \end{cases} \tag{3.6}$$

for $\varepsilon, \xi \in \Omega(M)$, because the principal symbol does not change when terms of lower order are added to the operator. However the BVP (3.6) is elliptic in the sense of Lopatinskiĭ-Šapiro conditions [11, 15], and thus so is (3.5).

(2) From part (1), since the BVP (3.5) is elliptic, it follows that the BVP (3.5) is a Fredholm operator and the regularity theorem holds, see for example Theorem 1.6.2 in [15] or Theorem 20.1.2 in [11]. In addition, we observe that the only differences between BVP (3.6) and our BVP (3.5) are all lower order operators and it is proved in [15] that the index of BVP (3.6) is zero but Theorem 20.1.8 in [11] asserts generally that if the difference between two BVPs are just lower order operators then they must have the same index. Hence, the index of the BVP (3.5) must be zero.

(3) Let $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$. If $\omega \in \mathcal{H}_{X_M,D}$ then it satisfies the BVP (3.5) with $\eta = 0$, so by the regularity properties of elliptic BVPs, the smoothness of ω follows. If on the other hand $\omega \in \mathcal{H}_{X_M,N}$ then $\star\omega \in \mathcal{H}_{X_M,D}$ which is therefore smooth and consequently $\omega = \pm \star (\star\omega)$ is smooth as well. ❒

We consider the resulting operator obtained by restricting Δ_{X_M} to the subspace of smooth invariant forms satisfying the boundary conditions

$$\overline{\Omega}_G(M) = \{\omega \in \Omega_G(M) \mid i^*\omega = 0, i^*(\delta_{X_M}\omega) = 0\} \tag{3.7}$$

Since the trace map i^* is well-defined on $\mathsf{H}^s\Omega_G$ for $s > 1/2$ it follows that it makes sense to consider $\mathsf{H}^2\overline{\Omega}_G(M)$, which is a closed subspace of $\mathsf{H}^2\Omega_G(M)$ and hence a Hilbert space. For simplicity, we rewrite our BVP (3.5) as follows: consider the restriction/extension of Δ_{X_M} to this space:

$$A = \Delta_{X_M}\big|_{\mathsf{H}^2\overline{\Omega}_G(M)} : \mathsf{H}^2\overline{\Omega}_G(M) \longrightarrow L^2\Omega_G(M).$$

and consider the BVP,

$$A\omega = \eta \tag{3.8}$$

for $\omega \in \mathsf{H}^2\overline{\Omega}_G(M)$ and $\eta \in L^2\Omega_G(M)$ instead of BVP (3.5) which are in fact compatible. In addition, from Theoremsa 3.4 we deduce that A is an elliptic and Fredholm operator and

$$\operatorname{index}(A) = \dim(\ker A) - \dim(\ker A^*) = 0 \tag{3.9}$$

where A^* is the adjoint operator of A.

From Green's formula (eq. (2.2)) we deduce the following property.

Lemma 3.5 *A is L^2-self-adjoint on $\mathsf{H}^2\overline{\Omega}_G(M)$, meaning that for all $\alpha, \beta \in \mathsf{H}^2\overline{\Omega}_G(M)$ we have*

$$\langle A\alpha, \beta\rangle = \langle \alpha, A\beta\rangle,$$

where $\langle -, -\rangle$ is the L^2-pairing.

Theorem 3.6 *The space $\mathcal{H}_{X_M,D}(M)$ is finite dimensional and*

$$L^2\Omega_G(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,D}(M)^{\perp}. \tag{3.10}$$

PROOF: We begin by showing that $\ker A = \mathcal{H}_{X_M,D}(M)$. It is clear that $\mathcal{H}_{X_M,D}(M) \subseteq \ker A$, so we need only prove that $\ker A \subseteq \mathcal{H}_{X_M,D}(M)$.

Let $\omega \in \ker A$. Then ω satisfies the BVP (3.5). Therefore, by condition (2) of Proposition 3.2, it follows that $\omega \in \mathcal{H}_{X_M,D}(M)$, as required.

Now, $\ker A = \mathcal{H}_{X_M,D}(M)$ but $\dim \ker A$ is finite, so that $\dim \mathcal{H}_{X_M,D}(M) < \infty$. This implies that $\mathcal{H}_{X_M,D}(M)$ is a closed subspace of the Hilbert space $L^2\Omega_G(M)$, hence eq. (3.10) holds. ❒

Theorem 3.7

$$\mathrm{Range}(A) = \mathcal{H}_{X_M,D}(M)^{\perp} \tag{3.11}$$

where $\perp$ denotes the orthogonal complement in $L^2\Omega_G(M)$.

PROOF: Firstly, we should observe that eq. (3.9) asserts that $\ker A \cong \ker A^*$ but Theorem 3.6 shows that $\ker A = \mathcal{H}_{X_M,D}(M)$, thus

$$\ker A^* \cong \mathcal{H}_{X_M,D}(M) \tag{3.12}$$

Since $\mathrm{Range}(A)$ is closed in $L^2\Omega_G(M)$ because A is Fredholm operator, it follows from the closed range theorem in Hilbert spaces that

$$\mathrm{Range}(A) = (\ker A^*)^{\perp} \quad \equiv \quad \mathrm{Range}(A)^{\perp} = \ker A^* \tag{3.13}$$

Hence, we just need to prove that $\ker A^* = \mathcal{H}_{X_M,D}(M)$, and to show that we need first to prove

$$\mathrm{Range}(A) \subseteq \mathcal{H}_{X_M,D}(M)^{\perp}. \tag{3.14}$$

So, if $\alpha \in \mathsf{H}^2\overline{\Omega}_G(M)$ and $\beta \in \mathcal{H}_{X_M,D}(M)$ then applying Lemma 3.5 gives

$$\langle A\alpha, \beta\rangle = 0$$

hence, eq. (3.14) holds. Moreover, equations (3.13) and (3.14) and the closedness of $\mathcal{H}_{X_M,D}(M)$ imply

$$\mathcal{H}_{X_M,D}(M) \subseteq \ker A^* \tag{3.15}$$

but eq. (3.12) and eq. (3.15) force $\ker A^* = \mathcal{H}_{X_M,D}(M)$. Hence, $\mathrm{Range}(A) = \mathcal{H}_{X_M,D}(M)^{\perp}$. ❒

Following [15], we denote the L^2-orthogonal complement of $\mathcal{H}_{X_M,D}(M)$ in the space $\mathsf{H}^2\Omega_{G,D}$ by

$$\mathcal{H}_{X_M,D}(M)^{\oplus} = \mathsf{H}^2\Omega_{G,D} \cap \mathcal{H}_{X_M,D}(M)^{\perp} \tag{3.16}$$

(although in [15] it denotes H^1-forms rather than H^2).

Proposition 3.8 *For each $\eta \in \mathcal{H}_{X_M,D}(M)^{\perp}$ there is a unique differential form $\omega \in \mathcal{H}_{X_M,D}(M)^{\oplus}$ satisfying the BVP (3.5).*

PROOF: Let $\eta \in \mathcal{H}_{X_M,D}(M)^{\perp}$. Because of Theorem (3.7) there is a differential form $\gamma \in \mathsf{H}^2\overline{\Omega}_G(M)$ such that γ satisfies the BVP (3.5). Since $\gamma \in \mathsf{H}^2\overline{\Omega}_G(M) \subseteq L^2\Omega_G(M)$ then there are unique differential forms $\alpha \in \mathcal{H}_{X_M,D}(M)$ and $\omega \in \mathcal{H}_{X_M,D}(M)^{\perp}$ such that $\gamma = \alpha + \omega$ because of eq. (3.10).

Since γ satisfies the BVP (3.5) it follows that ω satisfies the BVP (3.5) as well because $\alpha \in \mathcal{H}_{X_M,D}(M) = \ker(\Delta_{X_M}|_{\mathsf{H}^2\overline{\Omega}_G(M)})$. Since $\omega = \gamma - \alpha$, it follows that $\omega \in \mathsf{H}^2\Omega_{G,D}$, hence $\omega \in \mathcal{H}_{X_M,D}(M)^{\oplus}$ and it is unique. ❒

Remarks 3.9 (1) ω satisfying the BVP (3.5) in Proposition 3.8 can be recast to the condition

$$\langle \mathsf{d}_{X_M}\omega, \mathsf{d}_{X_M}\xi\rangle + \langle \delta_{X_M}\omega, \delta_{X_M}\xi\rangle = \langle \eta, \xi\rangle, \quad \forall \xi \in \mathsf{H}^1\Omega_{G,D} \tag{3.17}$$

(2) All the results above can be recovered but in terms of $\mathcal{H}_{X_M,N}(M)$ because the Hodge star operator defines an isomorphism $L^2\Omega_G \cong L^2\Omega_G$ which restricts to $\mathcal{H}_{X_M,D}(M) \cong \mathcal{H}_{X_M,N}(M)$.

3.3. Decomposition theorems

The results above provide the basic ingredients needed to extend the Hodge-Morrey and Freidrichs decompositions arising for Hodge theory on manifolds with boundary, to the present setting with d_{X_M} and δ_{X_M}. Depending on these results, the proofs in this subsection rely heavily on the analogues of the corresponding statements for the usual Laplacian Δ on a manifold with boundary, as described in the book of Schwarz [15]. Therefore, we omit the proofs here while full details are given in the first author's thesis [3].

Definition 3.10 Define the following two sets of invariant exact and coexact forms on M,

$$\begin{aligned}\mathcal{E}_{X_M}(M) &= \{\mathsf{d}_{X_M}\alpha \mid \alpha \in \mathsf{H}^1\Omega_{G,D}\} \subseteq L^2\Omega_G(M),\\ \mathcal{C}_{X_M}(M) &= \{\delta_{X_M}\beta \mid \beta \in \mathsf{H}^1\Omega_{G,N}\} \subseteq L^2\Omega_G(M).\end{aligned}$$

Clearly, $\mathcal{E}_{X_M}(M) \perp \mathcal{C}_{X_M}(M)$ because of eq. (2.2). We denote by $L^2\mathcal{H}_{X_M}(M) = \overline{\mathcal{H}_{X_M}(M)}$ the L^2-closure of the space $\mathcal{H}_{X_M}(M)$.

Proposition 3.11 (Algebraic decomposition and L^2-closedness) *(a) Each $\omega \in L^2\Omega_G(M)$ can be split uniquely into*

$$\omega = \mathsf{d}_{X_M}\alpha_\omega + \delta_{X_M}\beta_\omega + \kappa_\omega$$

where $\mathsf{d}_{X_M}\alpha_\omega \in \mathcal{E}_{X_M}(M)$, $\delta_{X_M}\beta_\omega \in \mathcal{C}_{X_M}(M)$ and $\kappa_\omega \in (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$.

(b) The spaces $\mathcal{E}_{X_M}(M)$ and $\mathcal{C}_{X_M}(M)$ are closed subspaces of $L^2\Omega_G(M)$.

(c) Consequently there is the following orthogonal decomposition

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$$

Now we can present the main theorems for this section; all orthogonality is with respect to the L^2 inner product.

Theorem 3.12 (X_M-Hodge-Morrey decomposition theorem) *The following is an orthogonal direct sum decomposition:*

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus L^2\mathcal{H}_{X_M}(M)$$

Theorem 3.13 (X_M-Friedrichs decomposition theorem) *The space $\mathcal{H}_{X_M}(M) \subseteq \mathsf{H}^1\Omega_G(M)$ of X_M- harmonic fields can respectively be decomposed as orthogonal direct sums into*

$$\begin{aligned}\mathcal{H}_{X_M}(M) &= \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,\mathrm{co}}(M)\\ \mathcal{H}_{X_M}(M) &= \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,\mathrm{ex}}(M),\end{aligned}$$

where the right hand terms are the X_M-coexact and exact harmonic forms respectively:

$$\begin{aligned}\mathcal{H}_{X_M,\mathrm{co}}(M) &= \{\eta \in \mathcal{H}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\}\\ \mathcal{H}_{X_M,\mathrm{ex}}(M) &= \{\xi \in \mathcal{H}_{X_M}(M) \mid \xi = \mathsf{d}_{X_M}\sigma\}\end{aligned}$$

For $L^2\mathcal{H}_{X_M}(M)$ these decompositions are valid accordingly.

Combining Theorems 3.12 and 3.13 gives the following.

Corollary 3.14 (The X_M-Hodge-Morrey-Friedrichs decompositions) *The space $L^2\Omega_G(M)$ can be decomposed into L^2-orthogonal direct sums as follows:*

$$\begin{aligned}L^2\Omega_G(M) &= \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus \mathcal{H}_{X_M,D}(M) \oplus L^2\mathcal{H}_{X_M,\mathrm{co}}(M)\\ L^2\Omega_G(M) &= \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus \mathcal{H}_{X_M,N}(M) \oplus L^2\mathcal{H}_{X_M,\mathrm{ex}}(M)\end{aligned}$$

Remark 3.15 All the results above can be refined in terms of $\pm$-spaces, for instance,

$$\mathcal{H}^{\pm}_{X_M,D}(M) \cong \mathcal{H}^{n-\pm}_{X_M,N}(M), \quad L^2\Omega^{\pm}_G(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,D}(M) \oplus L^2\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)$$

...etc.

3.4. Relative and absolute X_M-cohomology

Using d_{X_M} and δ_{X_M} we can form a number of $\mathbb{Z}_2$-graded complexes. A $\mathbb{Z}_2$-graded complex is a pair of Abelian groups $C^\pm$ with homomorphisms between them:

$$C^+ \underset{\mathsf{d}_-}{\overset{\mathsf{d}_+}{\rightleftarrows}} C^-$$

satisfying $\mathsf{d}_+ \circ \mathsf{d}_- = 0 = \mathsf{d}_- \circ \mathsf{d}_+$. The two (co)homology groups of such a complex are defined in the obvious way: $H^\pm = \ker \mathsf{d}_\pm / \operatorname{im} \mathsf{d}_\mp$. The complexes we have in mind are,

$$\begin{array}{cc} (\Omega_G^\pm, \mathsf{d}_{X_M}) & (\Omega_G^\pm, \delta_{X_M}) \\ (\Omega_{G,D}^\pm, \mathsf{d}_{X_M}) & (\Omega_{G,N}^\pm, \delta_{X_M}). \end{array}$$

The two on the lower line are subcomplexes of the corresponding upper ones, because i^* commutes with d_{X_M}. By analogy with the de Rham groups, we denote

$$H_{X_M}^\pm(M) := H^\pm(\Omega_G, \mathsf{d}_{X_M}) \quad \text{and} \quad H_{X_M}^\pm(M, \partial M) := H^\pm(\Omega_{G,D}, \mathsf{d}_{X_M}).$$

The decomposition theorems above lead to the following result.

Theorem 3.16 (X_M-Hodge Isomorphism) *Let $X \in \mathfrak{g}$. There are the following isomorphisms of vector spaces:*

(a) $H_{X_M}^\pm(M, \partial M) \cong \mathcal{H}_{X_M,D}^\pm(M) \cong H^\pm(\Omega_G^\pm, \delta_{X_M})$;

(b) $H_{X_M}^\pm(M) \cong \mathcal{H}_{X_M,N}^\pm(M) \cong H^\pm(\Omega_{G,N}^\pm, \delta_{X_M})$;

(c) (X_M-Poincaré-Lefschetz duality): The Hodge star operator $\star$ on $\Omega_G(M)$ induces an isomorphism

$$H_{X_M}^\pm(M) \cong H_{X_M}^{n-\pm}(M, \partial M).$$

PROOF: The proofs use the decomposition theorems above. For the first isomorphism in (a), Theorem 3.12 (the X_M-Hodge-Morrey decomposition theorem) implies a unique splitting of any $\gamma \in \Omega_{G,D}^\pm$ into,

$$\gamma = \mathsf{d}_{X_M}\alpha_\gamma + \delta_{X_M}\beta_\gamma + \kappa_\gamma$$

where $\mathsf{d}_{X_M}\alpha_\gamma \in \mathcal{E}_{X_M}^\pm(M)$, $\delta_{X_M}\beta_\gamma \in \mathcal{C}_{X_M}^\pm(M)$ and $\kappa_\gamma \in L^2\mathcal{H}_{X_M}^\pm(M)$. If $\mathsf{d}_{X_M}\gamma = 0$ then $\delta_{X_M}\beta_\gamma = 0$, but $i^*\gamma = 0$ implies $i^*(\kappa_\gamma) = 0$ so that $\kappa_\gamma \in \mathcal{H}_{X_M,D}^\pm(M)$. Thus,

$$\gamma \in \ker \mathsf{d}_{X_M}|_{\Omega_{G,D}} \Longleftrightarrow \gamma = \mathsf{d}_{X_M}\alpha_\gamma + \kappa_\gamma.$$

This establishes the isomorphism $H_{X_M}^\pm(M, \partial M) \cong \mathcal{H}_{X_M,D}^\pm(M)$.

For the second isomorphism in (a), the second X_M-Hodge-Morrey-Friedrichs decomposition of Corollary 3.14 implies as well a unique splitting of any $\gamma \in \Omega_G^\pm(M)$ into,

$$\gamma = \mathsf{d}_{X_M}\xi_\gamma + \delta_{X_M}\eta_\gamma + \delta_{X_M}\zeta_\gamma + \lambda_\gamma$$

where $\mathsf{d}_{X_M}\xi_\gamma \in \mathcal{E}_{X_M}^\pm(M)$, $\delta_{X_M}\eta_\gamma \in \mathcal{C}_{X_M}^\pm(M)$, $\delta_{X_M}\zeta_\gamma \in L^2\mathcal{H}_{X_M,\mathrm{co}}^\pm(M)$ and $\lambda_\gamma \in \mathcal{H}_{X_M,D}^\pm(M)$.

If $\delta_{X_M}\gamma = 0$, then $\mathsf{d}_{X_M}\xi_\gamma = 0$, and hence

$$\gamma \in \ker \delta_{X_M} \Longleftrightarrow \gamma = \delta_{X_M}(\eta_\gamma + \zeta_\gamma) + \lambda_\gamma.$$

This establishes the isomorphism $\mathcal{H}_{X_M,D}^\pm(M) \cong H_{X_M}^\pm(\Omega_G^\pm, \delta_{X_M})$.

Part (b) is proved similarly, and part (c) follows from (a) and (b) and the fact that the Hodge star operator defines an isomorphism $\mathcal{H}_{X_M,D}^\pm(M) \cong \mathcal{H}_{X_M,N}^{n-\pm}(M)$. ❐

The theorem of Hodge is often quoted as saying that every (de Rham) cohomology class on a compact Riemannian manifold without boundary contains a unique harmonic form. The corresponding statement for X_M-cohomology on a manifold with boundary is,

Corollary 3.17 *Each absolute X_M-cohomology class contains a unique Neumann X_M-harmonic field, and each relative X_M-cohomology class contains a unique Dirichlet X_M-harmonic field.*

4. Relation with equivariant cohomology

When the manifold in question has no boundary, Atiyah and Bott [2] discuss the relationship between equivariant cohomology and X_M-cohomology by using their localization theorem. In this section we will relate our relative and absolute X_M-cohomology with the relative and absolute equivariant cohomology $H_G^{\pm}(M,\partial M)$ and $H_G^{\pm}(M)$; the arguments are no different to the ones in [2]. First we recall briefly the basic definitions of equivariant cohomology, and the relevant localization theorem, and then state the conclusions for the relative and absolute X_M-cohomology.

If a torus G acts on a manifold M (with or without boundary), the Cartan model for the equivariant cohomology is defined as follows. Let $\{X_1,\dots,X_\ell\}$ be a basis of $\mathfrak{g}$ and $\{u_1,\dots,u_\ell\}$ the corresponding coordinates. The *Cartan complex* consists of polynomial[1] maps from $\mathfrak{g}$ to the space of invariant differential forms, so is equal to $\Omega_G^*(M)\otimes R$ where $R=\mathbb{R}[u_1,\dots,u_\ell]$, with differential

$$\mathsf{d}_{\mathrm{eq}}(\omega)=\mathsf{d}\omega+\sum_{j=1}^{\ell}u_j\,\iota_{X_j}\omega.$$

The equivariant cohomology $H_G^*(M)$ is the cohomology of this complex. The relative equivariant cohomology $H_G^*(M,\partial M)$ (if M has non-empty boundary) is formed by taking the subcomplex with forms that vanish on the boundary $i^*\omega=0$, with the same differential.

The cohomology groups are graded by giving the u_i weight 2 and a k-form weight k, so the differential d_{eq} is of degree 1. Furthermore, as the cochain groups are R-modules, and d_{eq} is a homomorphism of R-modules, it follows that the equivariant cohomology is an R-module. The localization theorem of Atiyah and Bott [2] gives information on the module structure (there it is only stated for absolute cohomology, but it is equally true in the relative setting, with the same proof; see also Appendix C of [9]).

First we define the following subset of $\mathfrak{g}$,

$$Z:=\bigcup_{\widehat{K}\subsetneq G}\mathfrak{k}$$

where the union is over proper isotropy subgroups $\widehat{K}$ (and $\mathfrak{k}$ its Lie algebra) of the action on M. If M is compact, then Z is a finite union of proper subspaces of $\mathfrak{g}$. Let $F=\mathrm{Fix}(G,M)=\{x\in M\mid G\cdot x=x\}$ be the set of fixed points in M. It follows from the local structure of group actions that F is a submanifold of M, with boundary $\partial F=F\cap\partial M$.

Theorem 4.1 (Atiyah-Bott [2, Theorem 3.5]) *The inclusion $j:F\hookrightarrow M$ induces homomorphisms of R-modules*

$$H_G^*(M)\xrightarrow{j^*}H_G^*(F)$$

$$H_G^*(M,\partial M)\xrightarrow{j^*}H_G^*(F,\partial F)$$

whose kernel and cokernel have support in Z.

In particular, this means that if $f\in I(Z)$ (the ideal in R of polynomials vanishing on Z) then the localizations[2] $H_G^*(M)_f$ and $H_G^*(F)_f$ are isomorphic R_f-modules. Notice that the act of localization destroys the integer grading of the cohomology, but since the u_i have weight 2, it preserves the parity of the grading, so that the separate even and odd parts are maintained: $H_G^{\pm}(M)_f\cong H_G^{\pm}(F)_f$. The same reasoning applies to the cohomology relative to the boundary, so $H_G^{\pm}(M,\partial M)_f\cong H_G^{\pm}(F,\partial F)_f$

[1] we use real valued polynomials, though complex valued ones works just as well, and all tensor products are thus over $\mathbb{R}$, unless stated otherwise

[2] The localized ring R_f consists of elements of R divided by a power of f and if K is an R-module, its localization is $K_f:=K\otimes_R R_f$; they correspond to restricting to the open set where f is non-zero. See the notes by Libine [12] for a good discussion of localization in this context.

Since the action on F is trivial, it is immediate from the definition that there is an isomorphism of R-modules, $H^*_G(F) \cong H^*(F) \otimes R$ so that the localization theorem shows j^* induces an isomorphism of R_f-modules,

$$H^{\pm}_G(M)_f \xrightarrow{j^*} H^{\pm}(F) \otimes R_f. \tag{4.1}$$

It follows that $H^{\pm}_G(M)_f$ is a free R_f module whenever $f \in I(Z)$. Of course, analogous statements hold for the relative versions. Since localization does not alter the rank of a module (it just annihilates torsion elements), we have that

$$\operatorname{rank} H^{\pm}_G(M) = \dim H^{\pm}(F), \qquad \operatorname{rank} H^{\pm}_G(M, \partial M) = \dim H^{\pm}(F, \partial F).$$

For $X \in \mathfrak{g}$, define $N(X_M) = \{x \in M \mid X_M(x) = 0\}$, the set of zeros of the vector field X_M. Since X generates a torus action, $N(X_M)$ is a manifold with boundary $\partial N(X_M) = N(X_M) \cap \partial M$. Clearly $N(X_M) \supset F$, and $N(X_M) = F$ if and only if $X \notin Z$.

Theorem 4.2 *Let $X = \sum_j s_j X_j \in \mathfrak{g}$. If the set of zeros of the corresponding vector field X_M is equal to the fixed point set for the G-action (i.e. $N(X_M) = F$) then*

$$H^{\pm}_{X_M}(M, \partial M) \cong H^{\pm}_G(M, \partial M)/\mathfrak{m}_X H^{\pm}_G(M, \partial M), \tag{4.2}$$

and

$$H^{\pm}_{X_M}(M) \cong H^{\pm}_G(M)/\mathfrak{m}_X H^{\pm}_G(M) \tag{4.3}$$

where $\mathfrak{m}_X = \langle u_1 - s_1, \dots, u_l - s_l \rangle$ is the ideal of polynomials vanishing at X.

PROOF: Our assumption $N(X_M) = F$ is equivalent to $X \in \mathfrak{g} \setminus Z$. Therefore there is a polynomial $f \in I(Z)$ such that $f(X) \neq 0$. In addition, we can use f and replace the ring R by R_f and then localize $H^{\pm}_G(M)$ and $H^{\pm}_G(M, \partial M)$ to make $H^{\pm}_G(M)_f$ and $H^{\pm}_G(M, \partial M)_f$ which are free R_f-modules.

We now apply the lemma stated below, in which the left-hand side is obtained by putting $u_i = s_i$ before taking cohomology, so results in $H^{\pm}_{X_M}(M)$ (or similar for the relative case), while the right-hand side is the right-hand side of (4.2) and (4.3), so proving the theorem. ❒

Lemma 4.3 (Atiyah-Bott [2, Lemma 5.6]) *Let (C^*, d) be a cochain complex of free R-modules and assume that, for some polynomial f, $H(C^*, d)_f$ is a free module over the localized ring R_f. Then, if $s \in \mathbb{R}^l$ with $f(s) \neq 0$,*

$$H^{\pm}(C^*_s, d_s) \cong H^{\pm}(C^*, d) \bmod \mathfrak{m}_s$$

where $\mathfrak{m}_s$ is the (maximal) ideal $\langle u_1 - s_1, \dots, u_l - s_l \rangle$ at X in $\mathbb{R}[\mathfrak{g}]$.

Corollary 4.4 *Let $X \in \mathfrak{g}$ and $j_X : N(X_M) \hookrightarrow M$, then j_X^* induces the following isomorphisms*

1- $H^{\pm}_{X_M}(M) \cong H^{\pm}(N(X_M))$,

2- $H^{\pm}_{X_M}(M, \partial M) \cong H^{\pm}(N(X_M), \partial N(X_M))$.

PROOF: First suppose $X \notin Z$. Then the isomorphisms above follow by reducing equation (4.1) modulo $\mathfrak{m}_X$ and applying Theorem 4.2.

If on the other hand, $X \in Z$, then let G' be the corresponding isotropy subgroup, so that $N(X_M) = F' := \mathrm{Fix}(G', M)$ (it is clear that $G' \supset G(X)$, the subgroup of G generated by X). The considerations above show that $H^{\pm}_{X_M, G'}(M, \partial M) \cong H^{\pm}(F', \partial F')$ and $H^{\pm}_{X_M, G'}(M) \cong H^{\pm}(F')$, where $H^{\pm}_{X_M, G'}(M)$ and $H^{\pm}_{X_M, G'}(M, \partial M)$ are defined using G'-invariant forms, and $\mathfrak{m}_{G', X}$ is the maximal ideal at X in the ring $\mathbb{R}[\mathfrak{g}']$. Moreover, all classes in $H^{\pm}_{X_M, G'}(M)$ and $H^{\pm}_{X_M, G'}(M, \partial M)$ have representatives which are G-invariant, not only G'-invariant (either by an averaging argument, or by using the unique X_M-harmonic representatives). So, this gives $H^{\pm}_{X_M, G}(M) \cong H^{\pm}_{X_M, G'}(M)$ and $H^{\pm}_{X_M, G}(M, \partial M) \cong H^{\pm}_{X_M, G'}(M, \partial M)$, $\forall X \in \mathfrak{g}' \subset \mathfrak{g}$ as desired. ❒

Remark 4.5 If M is a compact manifold with boundary then $H^k(M) \cong H_k(M)$ and $H^k(M,\partial M) \cong H_k(M,\partial M)$, where $H_k(M)$ and $H_k(M,\partial M)$ are the absolute and relative singular homology with real coefficients. We observe that this fact together with corollary 4.4 give us the following isomorphisms

$$H^{\pm}_{X_M}(M) \cong H_{\pm}(N(X_M)) \quad \text{and} \quad H^{\pm}_{X_M}(M,\partial M) \cong H_{\pm}(N(X_M),\partial N(X_M)),$$

where $H_+(N(X_M)) = \oplus_i H_{2i}(N(X_M))$ and $H_-(N(X_M),\partial N(X_M)) = \oplus_i H_{2i+1}(N(X_M),\partial N(X_M))$, by using the map

$$[\omega]_{X_M}(\{c\}) = \int_c j^*\omega, \tag{4.4}$$

where ω is X_M-closed $\pm$-form representing the absolute (or relative) X_M-cohomology class $[\omega]_{X_M}$ on M and c is a $\pm$-cycle representing the absolute (or relative) singular homology class $\{c\}$ on $N(X_M)$. In this light, eq. (2.1), corollary 4.4 and the bijection (4.4) prove the following statement:

An X_M-closed form ω is X_M-exact iff all the periods of $j^*\omega$ over all $\pm$-cycles of $N(X_M)$ vanish.

5. Interior and boundary subspaces

In this section we visit some recent work of DeTurck and Gluck [6] on harmonic fields and cohomology (see also [16, 17] for details), and adapt it to X_M-harmonic fields.

5.1. Interior and boundary subspaces after DeTurck and Gluck

Given the usual manifold M with boundary, there is a long exact sequence in cohomology associated to the pair $(M,\partial M)$ and one can use this to define two subspaces of $H^k(M)$ and $H^k(M,\partial M)$ as follows:

- the *interior* subspace $IH^k(M)$ of $H^k(M)$ is the kernel of $i^* : H^k(M) \to H^k(\partial M)$
- the *boundary* subspace $BH^k(M,\partial M)$ of $H^k(M,\partial M)$ is the image of $\mathsf{d} : H^{k-1}(\partial M) \to H^k(M,\partial M)$

Note that if M has no boundary, then $IH^k = H^k$ and $BH^k = 0$, as should be expected from their names.

At the level of cohomology there is no 'natural' definition for the boundary part of the absolute cohomology nor the interior part of the relative cohomology. However, DeTurck and Gluck [6] use the metric and harmonic representatives to provide these. Firstly the subspaces defined above are realized as

$$\begin{aligned} \mathcal{IH}^k_N &= \{\omega \in \mathcal{H}^k_N(M) \mid i^*\omega = \mathsf{d}\theta, \text{ for some } \theta \in \Omega^{k-1}(\partial M)\} \\ \mathcal{BH}^k_D &= \mathcal{H}^k_D(M) \cap \mathcal{H}^k_{\text{ex}} \end{aligned}$$

respectively (these are denoted $\mathcal{E}_\partial\mathcal{H}^k_N(M)$ and $\mathcal{EH}^k_D(M)$ respectively in [6, 16, 17]). They then use the Hodge star operator to define the other spaces:

$$\begin{aligned} \mathcal{BH}^k_N &= \mathcal{H}^k_N(M) \cap \mathcal{H}^k_{\text{co}} \\ \mathcal{IH}^k_D &= \{\omega \in \mathcal{H}^k_D(M) : i^*\star\omega = \mathsf{d}\kappa, \text{ for some } \kappa \in \Omega^{n-k-1}(\partial M)\} \end{aligned}$$

(denoted $c\mathcal{EH}^k_N(M)$ and $c\mathcal{E}_\partial\mathcal{H}^k_D(M)$ in [6, 16, 17]). The first theorem of DeTurck and Gluck on this subject is

Theorem 5.1 (DeTurck and Gluck [6]) *Both $\mathcal{H}^k_D$ and $\mathcal{H}^k_N$ have orthogonal decompositions,*

$$\begin{aligned} \mathcal{H}^k_N(M) &= \mathcal{IH}^k_N \oplus \mathcal{BH}^k_N \\ \mathcal{H}^k_D(M) &= \mathcal{BH}^k_D \oplus \mathcal{IH}^k_D. \end{aligned}$$

Furthermore, the two boundary subspaces are mutually orthogonal inside $L^2\Omega$.

However the interior subspaces are not orthogonal, and they prove

Theorem 5.2 (DeTurck-Gluck [6]) *The principal angles between the interior subspaces $\mathcal{IH}^k_N$ and $\mathcal{IH}^k_D$ are all acute.*

Part of the motivation for considering these principal angles, called *Poincaré duality angles*, is that they should measure in some sense how far the Riemannian manifold M is from being closed. That these angles are non-zero follows from the fact that $\mathcal{H}_N^k \cap \mathcal{H}_D^k = 0$, see [15]. Another consequence of this, pointed out by DeTurck and Gluck is that the Hodge-Morrey-Freidrichs decomposition can be refined to a 5-term decomposition,

$$\Omega^k(M) = \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus (\mathcal{H}_D^k + \mathcal{H}_N^k) \oplus \mathcal{H}_{\mathrm{ex,co}}^k, \tag{5.1}$$

where $\mathcal{H}_{\mathrm{ex,co}}^k = \mathcal{H}_{\mathrm{ex}}^k \cap \mathcal{H}_{\mathrm{co}}^k$ and the symbol $+$ indicates a direct sum whereas $\oplus$ indicates an orthogonal direct sum.

In his thesis [16], Shonkwiler measures these Poincaré duality angles in interesting examples of manifolds with boundary derived from complex projective spaces and Grassmannians and shows that in these examples the angles do indeed tend to zero as the boundary shrinks to zero, see alternatively [17].

5.2. *Extension to X_M-cohomology*

It seems reasonable to think that we can extend further to the style of DeTurck-Gluck, and break down the Neumann and Dirichlet X_M-harmonic fields into interior and boundary subspaces. If so, does the natural extension of corollary 4.4 hold? The answer is affirmative and contained in the proof of theorem 5.6.

Answering this question will indeed give more concrete understanding of these isomorphisms and consequently will give a precise extension to Witten's results when $\partial M \neq \emptyset$ (see Section 6).

Refinement of the X_M-Hodge-Morrey-Friedrichs decomposition. In [4], we prove that

$$\mathcal{H}_{X_M,N}^{\pm}(M) \cap \mathcal{H}_{X_M,D}^{\pm}(M) = \{0\},$$

which implies that the sum $\mathcal{H}_{X_M,N}^{\pm}(M) + \mathcal{H}_{X_M,D}^{\pm}(M)$ is a direct sum, and by using Green's formula (2.2), one finds that the orthogonal complement of $\mathcal{H}_{X_M,N}^{\pm}(M) + \mathcal{H}_{X_M,D}^{\pm}(M)$ inside $\mathcal{H}_{X_M}^{\pm}(M)$ is $\mathcal{H}_{X_M,\mathrm{ex,co}}^{\pm}(M) = \mathcal{H}_{X_M,\mathrm{ex}}^{\pm}(M) \cap \mathcal{H}_{X_M,\mathrm{co}}^{\pm}(M)$. Therefore, we can refine the X_M-Friedrichs decomposition (theorem 3.13) into

$$\mathcal{H}_{X_M}^{\pm}(M) = (\mathcal{H}_{X_M,N}^{\pm}(M) + \mathcal{H}_{X_M,D}^{\pm}(M)) \oplus \mathcal{H}_{X_M,\mathrm{ex,co}}^{\pm}(M).$$

Consequently, following DeTurck and Gluck's decomposition (5.1), we can refine the X_M-Hodge-Morrey-Friedrichs decompositions (Corollary 3.14) into the following five terms decomposition:

$$\Omega_G^{\pm}(M) = \mathcal{E}_{X_M}^{\pm}(M) \oplus \mathcal{C}_{X_M}^{\pm}(M) \oplus (\mathcal{H}_{X_M,N}^{\pm}(M) + \mathcal{H}_{X_M,D}^{\pm}(M)) \oplus \mathcal{H}_{X_M,\mathrm{ex,co}}^{\pm}(M). \tag{5.2}$$

Here as usual, $\oplus$ is an orthogonal direct sum, while $+$ is just a direct sum.

Interior and boundary portions of X_M-cohomology. Following the ordinary case described above, we can define interior and boundary portions of the X_M-cohomology and X_M-harmonic fields by

$$\begin{aligned} IH_{X_M}^{\pm}(M) &= \ker[i^* : H_{X_M}^{\pm}(M) \to H_{X_M}^{\pm}(\partial M)] \\ BH_{X_M}^{\pm}(M,\partial M) &= \mathrm{im}[\mathrm{d}_{X_M} : H_{X_M}^{\mp}(\partial M) \to H_{X_M}^{\pm}(M,\partial M)]. \end{aligned} \tag{5.3}$$

Here d_{X_M} is the standard construction: given a closed form λ on ∂M, let $\tilde{\lambda}$ be an extension to M. Then $\mathrm{d}_{X_M}\tilde{\lambda}$ defines a well-defined element of $H_{X_M}(M,\partial M)$. These spaces are realized through corollary 3.17 as

$$\begin{aligned} \mathcal{IH}_{X_M,N}^{\pm} &= \{\omega \in \mathcal{H}_{X_M,N}^{\pm}(M) \mid i^*\omega = \mathrm{d}_{X_M}\theta,\ \text{for some } \theta \in \Omega^{\mp}(\partial M)\} \\ \mathcal{BH}_{X_M,D}^{\pm} &= \mathcal{H}_{X_M,D}^{\pm}(M) \cap \mathcal{H}_{X_M,\mathrm{ex}}^{\pm} \end{aligned}$$

respectively. Now use the Hodge star operator to define the other spaces:

$$\begin{aligned} \mathcal{IH}_{X_M,D}^{\pm} &= \{\omega \in \mathcal{H}_{X_M,D}^{\pm}(M) : i^* \star \omega = \mathrm{d}_{X_M}\kappa,\ \text{for some } \kappa \in \Omega^{n-\mp}(\partial M)\} \\ \mathcal{BH}_{X_M,N}^{\pm} &= \mathcal{H}_{X_M,N}^{\pm}(M) \cap \mathcal{H}_{X_M,\mathrm{co}}^{\pm}. \end{aligned}$$

Note that Hodge star maps boundary to boundary and interior to interior; it follows that, for example $\mathcal{BH}_{X_M,N}^{\pm} \cong \mathcal{BH}_{X_M,D}^{n-\pm}$.

Theorem 5.3 *The boundary subspace $\mathcal{BH}^{\pm}_{X_M,N}(M)$ is the largest subspace of $\mathcal{H}^{\pm}_{X_M,N}(M)$ orthogonal to all of $\mathcal{H}^{\pm}_{X_M,D}(M)$ while the boundary subspace $\mathcal{BH}^{\pm}_{X_M,D}(M)$ is the largest subspace of $\mathcal{H}^{\pm}_{X_M,D}(M)$ orthogonal to all of $\mathcal{H}^{\pm}_{X_M,N}(M)$.*

PROOF: The orthogonality follows immediately from Green's formula (2.2) while the rest of the proof follow immediately from the X_M-Friedrichs decomposition theorem (theorem 3.13) (restricted to smooth invariant forms). ❒

The main goal of this subsection is to prove the following theorem and to answer the question above.

Theorem 5.4 *Analogous to theorem 5.1, we have the orthogonal decompositions*

$$\begin{aligned}
\mathcal{H}^{\pm}_{X_M,N}(M) &= \mathcal{IH}^{\pm}_{X_M,N} \oplus \mathcal{BH}^{\pm}_{X_M,N} \\
\mathcal{H}^{k}_{X_M,D}(M) &= \mathcal{BH}^{\pm}_{X_M D} \oplus \mathcal{IH}^{\pm}_{X_M,D}.
\end{aligned}$$

Remark 5.5 The proof by DeTurck and Gluck of the analogous result uses the duality between de Rham cohomology and singular homology. However, we do not have such a result on M (though perhaps a proof using the equivariant homology described in [14] would be possible), so we give a direct proof involving only the cohomology—the same argument can be used to prove DeTurck and Gluck's original theorem (replacing $\pm$ by k everywhere). An alternative argument can be given using the localization to the fixed point set (corollary 4.4)—details of which can be found in [3].

PROOF: The orthogonality of the right hand sides follows from Green's formula (2.2). It follows that

$$\mathcal{IH}^{\pm}_{X_M,N} \oplus \mathcal{BH}^{\pm}_{X_M,N} \subset \mathcal{H}^{\pm}_{X_M,N}(M) \quad \text{and} \quad \mathcal{BH}^{\pm}_{X_M D} \oplus \mathcal{IH}^{\pm}_{X_M,D} \subset \mathcal{H}^{k}_{X_M,D}(M). \tag{5.4}$$

Now consider the long exact sequence in X_M-cohomology derived from the inclusion $i : \partial M \hookrightarrow M$,

$$\cdots \xrightarrow{i^*} H^{\mp}_{X_M}(\partial M) \xrightarrow{\mathrm{d}_{X_M}} H^{\pm}_{X_M}(M,\partial M) \xrightarrow{\rho^*} H^{\pm}_{X_M}(M) \xrightarrow{i^*} H^{\pm}_{X_M}(\partial M) \xrightarrow{\mathrm{d}_{X_M}} H^{\mp}_{X_M}(M,\partial M) \longrightarrow \cdots$$

It follows from the exactness that

$$IH^{\pm}_{X_M}(M) = \operatorname{im}\rho^*, \quad \text{and} \quad BH^{\pm}_{X_M}(M,\partial M) = \ker\rho^*.$$

Thus, $H^{\pm}_{X_M}(M,\partial M) \cong BH^{\pm}_{X_M}(M,\partial M) + IH^{\pm}_{X_M}(M)$, (direct sum) or equivalently

$$\mathcal{H}^{\pm}_{X_M,D} \cong \mathcal{BH}^{\pm}_{X_M,D} + \mathcal{IH}^{\pm}_{X_M,N}. \tag{5.5}$$

It follows from equations (5.4) and (5.5) that $\dim(\mathcal{IH}^{\pm}_{X_M,D}) \leq \dim(\mathcal{IH}^{\pm}_{X_M,N})$. However, the Hodge star operator identifies $\mathcal{IH}^{\pm}_{X_M,N}$ with $\mathcal{IH}^{n-\pm}_{X_M,D}$ which implies that the inequality in dimensions is in fact an equality, and the result follows. ❒

Theorem 5.6 *Let $F' = N(X_M)$. We have isomorphisms,*

$$\begin{aligned}
&\mathcal{IH}^{\pm}_{X_M,N}(M) \cong \mathcal{IH}^{\pm}_{N}(F'), \quad \mathcal{BH}^{\pm}_{X_M,D}(M) \cong \mathcal{BH}^{\pm}_{D}(F'), \\
&\mathcal{IH}^{\pm}_{X_M,D}(M) \cong \mathcal{IH}^{\pm}_{D}(F'), \quad \mathcal{BH}^{\pm}_{X_M,N}(M) \cong \mathcal{BH}^{\pm}_{N}(F').
\end{aligned}$$

PROOF: We prove the first two; the other two follow by applying the Hodge star operator (on M and on F'). Denote by j_X the inclusion of the pair, $j_X : (F',\partial F') \hookrightarrow (M,\partial M)$. Then j_X induces a chain map between the long exact sequences of X_M cohomology on M and de Rham cohomology on F', which by corollary 4.4 is an isomorphism.

Since the interior part of the absolute cohomology and the boundary part of the relative cohomology are defined from these long exact sequences, it follows that j_X induces isomorphisms

$$IH_{X_M}(M)^{\pm} \cong IH^{\pm}(F'), \quad \text{and} \quad BH^{\pm}_{X_M}(M,\partial M) \cong BH^{\pm}(F',\partial F').$$

It then follows from the X_M-Hodge theorem 3.16 that there are isomorphisms $\mathcal{IH}_{X_M,N}(M) \cong \mathcal{IH}^{\pm}_N(F')$ and $\mathcal{BH}^{\pm}_{X_M,D}(M) \cong \mathcal{BH}^{\pm}_D(F')$. ❒

The analogue of Gluck and DeTurck's theorem for the Poincaré duality angles (theorem 5.2) also holds. The X_M-Poincaré duality angles are defined in the obvious way, as the principal angles between $\mathcal{IH}^{\pm}_{X_M,D}$ and $\mathcal{IH}^{\pm}_{X_M,N}$.

Proposition 5.7 *The X_M-Poincaré duality angles are all acute.*

PROOF: These angles can be neither 0 nor $\pi/2$, firstly because $\mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M) = \{0\}$ (shown in [4]), and secondly because of theorem 5.3. Hence they must all be acute. ❒

The results above and in [4] would allow us to extend most of the results of [16] to the context of X_M-cohomology and X_M-Poincaré duality angles but we leave this for future work.

6. Conclusions

In previous sections, we began with the action of a torus G; here we state results for a given Killing vector field K on a compact Riemannian manifold M (with or without boundary), more in keeping with Witten's original work [18]. Recall that the group $\mathrm{Isom}(M)$ of isometries of M is a compact Lie group, and the smallest closed subgroup $G(K)$ containing K in its Lie algebra is Abelian, so a torus. Furthermore, the submanifold $N(K)$ of zeros of K coincides with $\mathrm{Fix}(G(K),M)$.

The equivariant cohomology constructions of Section 4 give us the proof of the following result, which extends the theorem of Witten (our Theorem 2.3) to manifolds with boundary.

Theorem 6.1 *Let K be a Killing vector field on the compact Riemannian manifold M (with or without boundary), and let $N(K)$ be the submanifold of zeros of K. Then pullback to N induces isomorphisms*

$$H^{\pm}_K(M) \cong H^{\pm}(N(K)), \quad \textit{and} \quad H^{\pm}_K(M,\partial M) \cong H^{\pm}(N(K),\partial N(K)).$$

PROOF: Apply Corollary 4.4 to the equivariant cohomology for the action of the torus $G(K)$. ❒

Furthermore, using the Hodge star operator, the Poincaré-Lefschetz duality of Theorem 3.16(c) corresponds under the isomorphisms in the theorem above, to Poincaré-Lefschetz duality on the fixed point space.

Translating this theorem into the language of harmonic fields, shows

$$\mathcal{H}^{\pm}_{K,N}(M) \cong \mathcal{H}^{\pm}_N(N(K)) \quad \text{and} \quad \mathcal{H}^{\pm}_{K,D}(M) \cong \mathcal{H}^{\pm}_D(N(K)). \tag{6.1}$$

where $\mathcal{H}^{\pm}_N(N(K))$ and $\mathcal{H}^{\pm}_D(N(K))$ are the ordinary Neumann and Dirichlet harmonic fields on $N(K)$ respectively. The fact that theorem 6.1 and eq. (6.1) can be refined to the style of theorem 5.6 which gives a more precise meaning for these isomorphisms.

Corollary 6.2 *Given any harmonic field on $N(K)$ with either Dirichlet or Neumann boundary conditions, there is a unique K-harmonic field on M with the corresponding boundary conditions whose restriction on $N(K)$ is cohomologous to the given field.*

Note that if $\partial N(K) = \emptyset$ then the boundary condition on $N(K)$ is non-existent, and so every harmonic form (= field) on $N(K)$ has corresponding to it both a unique Dirichlet and a unique Neumann K-harmonic field on M. Moreover, since in this case there is no boundary part of the cohomology of $N(K)$, it follows from theorem 5.6 that $\mathcal{BH}_{X_M,N} = \mathcal{BH}_{X_M,D} = 0$.

In other words, it means that all the de Rham cohomology of $N(K)$ must come only from the interior portion, i.e. $H^{\pm}(N(K)) = H^{\pm}(N(K),\partial N(K))$, which shows that every interior de Rham cohomology class has corresponding to it both a unique relative and a unique absolute K-cohomology class on M.

As an application, we have the fact that theorem 6.1 and corollary 6.2 can be used to extend the other results of Witten in [18] and we hope that this extension will be useful in quantum field theory and other mathematical and physical applications when $\partial M \neq \emptyset$.

Euler characteristics. As is well known, given a complex of $\mathbb{R}[s]$ (or $\mathbb{C}[s]$) modules whose cohomology is finitely generated, the Euler characteristic of the complex is independent of s. This remains true for a $\mathbb{Z}_2$-graded complex, for the same reasons (briefly, the cohomology is the direct sum of a torsion module and a free module, and the torsion cancels in the Euler characteristic).

Applying this to the complexes for X_M-cohomology, with $X_M = sK$, it follows that $\chi(M) = \chi(N)$ and $\chi(M,\partial M) = \chi(N,\partial N)$ (where $N = N(K)$), and furthermore applying the same arguments to the manifold ∂M, one has $\chi(\partial M) = \chi(\partial N)$, i.e.

$$\chi(M) = \chi(\partial M) + \chi(M,\partial M) = \chi(\partial N) + \chi(N,\partial N) = \chi(N).$$

Other Applications:. We have shown that the Witten-Hodge theory can shed light to give additional equivariant geometric and topological insight. In addition, the fact that we can use the new decompositions of $L^2\Omega^{\pm}_G(M)$ given in theorem 3.12 and corollary 3.14 and also the relation between the X_M-cohomology and X_M-harmonic fields (theorem 3.16) as powerful tools (under topological aspects) in the theory of differential equations on $L^2\Omega^{\pm}_G(M)$ to obtain the solubility of various BVPs. In particular, we can extend most of the results of chapter three of [15] on $L^2\Omega^{\pm}_G(M)$ to the context of the operators d_{X_M}, δ_{X_M} and Δ_{X_M}. Moreover, the classical Hodge theory plays a fundamental role in incompressible hydrodynamics and it has applications to many other area of mathematical physics and engineering [1]. So, following these, we hope that the Witten-Hodge theory will be using as tools in these applications as well.

Geometric question:. Finally, we proved that $\mathcal{IH}^{\pm}_{X_M,N}(M) \cong \mathcal{IH}^{\pm}_N(N(X_M))$ and $\mathcal{IH}^{\pm}_{X_M,D}(M) \cong \mathcal{IH}^{\pm}_D(N(X_M))$ and that the principal angles between the corresponding interior subspaces are all acute. Hence, it would be interesting to answer the following

How do the X_M-Poincaré duality angles between the interior subspaces $\mathcal{IH}^{\pm}_{X_M,N}(M)$ and $\mathcal{IH}^{\pm}_{X_M,D}(M)$ depend on X, and how do they compare to the Poincaré duality angles between the interior subspaces $\mathcal{IH}^{\pm}_N(N(X_M))$ and $\mathcal{IH}^{\pm}_D(N(X_M))$.

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