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Micro-chaos in Relay Feedback Systems with Bang-Bang Control and Digital Sampling

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Abstract: We investigate a class of linear relay feedback systems with bang-bang control and with the control input applied at discrete time instances. Using a third order system as a representative example we show that stable oscillations with so-called sliding motion, with sliding present in continuous time system, loose the sliding segment of evolution, but do not loose their stability if the open loop system is stable. We then carry on our investigations and consider a situation when stable self-sustained oscillations are generated with the unstable open loop system. In the latter case a transition from a stable limit cycle to micro-chaotic oscillations occurs. The presence of micro-chaotic oscillations is shown by considering a linearised map that maps a small neighbourhood of initial conditions back to itself. Using this map the presence of the positive Lyapunov exponent is shown. The largest Lyapunov exponent is then calculated numerically for an open set of sampling times, and it is shown that it is positive. The boundedness of the attractor is ensured for sufficiently small sampling times; with the sampling time tending to zero these switchings become faster and they turn into sliding motion. It is the presence of the underlying sliding evolution that ensures the boundedness of the chaotic attractor. Our finding implies that what may be considered as noise in systems with digital control should actually be termed as micro-chaotic behaviour. This information may be helpful in designing digital control systems where any element contributing to what appears as noise should be suppressed.

Keywords: Bang-bang control, Discontinuities, Chaotic behaviour, Dynamics, Discrete time

1. INTRODUCTION

Systems characterized by an interaction of continuous and discrete evolution abound in everyday life. Consider, for instance, a flight control: an aircraft evolves continuously in time, but it is controlled by microprocessors which operate on digital (discrete) inputs and send digital signals. Another example from outside of the field of engineering are regulatory processes in cell networks. There are continuous time processes which govern the spatial and temporal concentration of proteins inside of a single cell. However, these continuous time processes are switched on or off based on the concentration level of the protein. In fact, most processes and systems, on the macroscopic scale, combine continuous and discrete dynamics. In other words they are hybrid.

It is not surprising that investigations of hybrid systems have attracted the attention of scientists in recent years. In the context of control the problem of robust stability and controller design for hybrid systems is an active area

of research, see for instance Liberzon (2003); Sanfelice et al. (2008, 2009). The presence of combined continuous and discrete evolution leads to dynamics not observed in systems that are modelled using either: strictly continuous, or strictly discrete time framework. One of the intriguing features of hybrid systems is the possibility of a sudden onset of chaotic dynamics. Such scenarios have been observed, for instance, in the context of modelling DC/DC power converters Banerjee et al. (1998); Banerjee and Verghese (2001), and dry-friction oscillators di Bernardo et al. (2003). The onset of chaotic dynamics in these systems was shown to have been triggered by a non-trivial interaction between a system $\Omega-$limit set and the so-called switching surface. This scenario is an example of a *discontinuity induced bifurcation*. A survey of these bifurcations and tools that can be used for their analytical investigations can be found in di Bernardo et al. (2008).

In the current paper we are interested in another aspect of the dynamics of hybrid systems. Namely we consider the influence of digital sampling, or how the conversion from the analogue to digital signal may affect system dynamics. To this aim we focus on linear time invariant relay-

feedback systems with bang-bang control. We assume that the control input is not an analogue signal available in continuous time, but a discrete signal which is accessed at discrete time intervals. We disregard here any quantization noise in the signal representation. With the accuracy of current converters this effect is considered to carry much less influence on system dynamics than the digitization (we use the word digitization and discretization synonymously) of the analogue signal Vaccaro (1995); for instance, for the applications in robotics a 16-bit converter is sensitive enough for a quantization error to be negligible.

In Haller and Stépán (1996); Enikov and Stépán (1998) it has been shown that the digitization of the spatial structure by the controller can induce micro-chaotic transient dynamics. The effects of digitization on the stability of the solutions have been considered in Lee and Haddad (2002); Braslavsky et al. (2006), and in Xia and Chen (2007) the existence of different types of attractors in a simple model of a delta-modulated control system has been shown.

For our purpose we assume that the input to the controller is delivered at discrete times. We then consider how periodic oscillations, present in the continuously sampled system, are influenced by the introduction of the digitization of the control input.

The paper is outlined as follows. In Sec. 2 the class of systems of interest is introduced. Then in Sec. 3 representative example is investigated in detail. Finally Sec. 4 concludes the paper.

2. SYSTEMS UNDER INVESTIGATION

We focus on single-input single-output linear time invariant relay feedback systems where the control variable is accessed through a negative feedback loop. Namely

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad u = -\text{sgn}y, \qquad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ are constant matrices, and $x = (x_1, x_2, x_3, \cdots, x_n)^T$ $(n \geq 1)$ is the state vector, and the "dot" symbol denotes differentiation with respect to time. The "sgn" function is defined as $\text{sgn}(y) = 1$ for $y > 0$, $\text{sgn}(y) = -1$ for $y < 0$, and $\text{sgn}(y) \in [-1, 1]$ for $y = 0$. The system trajectories of (1) evolve following $F_1 = Ax - B$ in $G_1 := \{Cx > 0\}$ and $F_2 = Ax + B$ in $G_2 := \{Cx < 0\}$.

Across the switching surface $\Sigma := \{Cx = 0\}$ system (1) may either switch between F_1 and F_2 (or *vice versa*) or it may generate so called sliding motion, that is a motion on Σ. The sliding vector field F_s that governs solutions on Σ can be obtained by means of Filippov's convex method Filippov (1988). Namely

$$F_s = \alpha F_1 + (1 - \alpha)F_2, \qquad (2)$$

where $\alpha = \dfrac{H_x F_2}{H_x(F_2 - F_1)} \in [0, 1]$, and H_x is the vector normal to Σ. The region within Σ, say $\hat{\Sigma}$, where sliding is possible can be defined by means of α. Namely $\hat{\Sigma} := \{0 \leq \alpha(x) \leq 1\}$. The boundaries of $\hat{\Sigma}$ are defined as $\partial\hat{\Sigma}^0 := \{\alpha(x) = 0\}$ and $\partial\hat{\Sigma}^1 := \{\alpha(x) = 1\}$.

We investigate the effects of digitization on the dynamics of (1) by assuming that the control variable $y = Cx$ is made available at discrete time instances with the

sampling time $\tau > 0$. The state space representation of such a modified relay feedback system is given by

$$\dot{x} = Ax + B_{i_k}, \qquad (3)$$

where

$$B_{i_{k+1}} = \begin{cases} -B & \text{if} & Cx(k\tau) > 0 \\ B & \text{if} & Cx(k\tau) < 0 \\ B_{i_k} & \text{if} & Cx(k\tau) = 0, \end{cases} \qquad (4)$$

with $B_{i_0} = -B$ (arbitrarily chosen), so that B_{i_1} is always well defined. Thus the evolution of (3) is governed by

$$\dot{x} = Ax + B_{i_k}, \qquad \text{if} \qquad (k-1)\tau \leq t < k\tau.$$

The discretization of the output variable $y = Cx$ implies that sliding motion is not possible in (3), but the discretized system exhibits the possibility of fast switchings between F_1 and F_2 (and *vice versa*).

2.1 Canonical form of the system

We assume that matrices A, B and C are given in so-called observer canonical form:

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1, \\ b_2, \\ \vdots, \\ b_{n-1}, \\ b_n \end{pmatrix}$$

and $C = (1, 0, \cdots, 0, 0)$. Using above state space representation the switching surface Σ is given by $\Sigma := \{x_1 = 0\}$. If at least one of the eigenvalues of A lies in the right half-plane then neither continuously sampled system (1) nor the discretized system (3) evolves towards the steady state of F_1 or F_2. Similarly, if the equilibrium points of F_1 and F_2 do not lie in their respective domains of definition and are bounded away from the switching surface Σ the asymptotic dynamics of (1) and (3) does not converge to the steady states of F_1 or F_2. In such cases systems (1) and (3) often exhibit self-sustained oscillations. In (1) these can be characterized by segment(s) of sliding if $b_1 > 0$. Using (2) together with the definition for α, and exploiting the fact that $\dot{x}_1 = 0$ within $\hat{\Sigma}$, we can express F_s as

$$\dot{x} = \hat{A}x$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -b_2/b_1 & 1 & 0 & \ldots \\ 0 & -b_{n-1}/b_1 & 0 & 1 & \\ \vdots & & & \ddots & \vdots \\ 0 & -b_n/b_1 & 0 & \ldots & 0 \end{pmatrix}.$$

2.2 Stability of periodic solutions with two sliding segments

We wish to establish what are the effects of digitization on the existence and stability of simple symmetric periodic orbits with sliding. We define a simple periodic orbit using the definition similar to the one given in Johansson et al. (1999). Namely, let a limit cycle $L \subset \mathbb{R}^n$ be defined by the set of points attained by a periodic trajectory that is isolated and not an equilibrium. If there exists a trajectory in L with period T and the number $\nu > 0$ such that $Cx(t) > 0$ for $t \in (\nu, T/2)$ and $Cx(t) < 0$ for $t \in (T/2 + \nu, T)$ only, then we say that L is a simple symmetric periodic orbit with sliding. This type of

self-sustained oscillations captures a particular feature of Filippov systems which is the existence of sliding motion. Due to the symmetry of relay system (1), that is because $F_1(-x) = -F_2(x)$, typically two sliding segments built a limit cycle Tsypkin (1984).

Consider now a solution of (1) that starts on the switching surface at the boundary of the sliding region $\partial \hat{\Sigma}^1$. In such case system (1) starts its evolution by following vector field F_1. Solving the system equation gives

$$\mathbf{x_1} = \exp(\mathbf{At_{01}})\mathbf{x_0} - (\exp(\mathbf{At_{01}}) - \mathbf{I})\mathbf{A^{-1}B}, \quad (5)$$

where t_{01} is the time of evolution from the initial point $\mathbf{x_0}$ to the point $\mathbf{x_1}$ on Σ. If $\mathbf{x_1} \in \hat{\Sigma}$ then the solution follows sliding motion on $\hat{\Sigma}$, and we can assume without loss of generality that it reaches $\partial \hat{\Sigma}^0$.

Thus, the second segment of the trajectory is given by

$$\mathbf{x_2} = \exp(\mathbf{\hat{A}t_{s1}})\mathbf{x_1}, \quad (6)$$

where t_{s1} is the time required for the sliding flow to reach the boundary of the sliding region $\partial \hat{\Sigma}^0$.

The final two segments that built periodic orbits of interest are given by the solutions

$$\mathbf{x_3} = \exp(\mathbf{At_{02}})\mathbf{x_2} + (\exp(\mathbf{At_{02}}) - \mathbf{I})\mathbf{A^{-1}B}, \quad (7)$$

and

$$\mathbf{x_4} = \exp(\mathbf{\hat{A}t_{s2}})\mathbf{x_3}, \quad (8)$$

where t_{02} is the time of evolution from the boundary of the sliding region $\partial \hat{\Sigma}^0$ back to the switching surface, and t_{s2} is the time of sliding required to reach $\partial \hat{\Sigma}^1$ at $\mathbf{x_0}$ again.

Assume that a limit cycle built of segments of trajectories as described exists. To determine the stability of such a limit cycle we have to obtain the first variation of $\mathbf{x_4}$ with respect to $\mathbf{x_0}$, i.e. we wish to find $\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}}$. The stability is then given by the variation matrix which is a matrix composition of the exponential matrices $\exp(At_{01})$, $\exp(\hat{A}t_{s1})$, $\exp(At_{02})$, $\exp(\hat{A}t_{s2})$, and the saltation matrices di Bernardo et al. (2008) that we denote as DM_{ij}, with $i, j = 1, 2, s$. The saltation matrices allow to take into account switchings between flows.

Namely we have

$$DM_{ij} = \left(I + \frac{(F_j - F_i)N}{NF_i} \right) \quad (9)$$

where F_i is the vector field which generates the flow that in forward time reaches the switching point, F_j is the vector field which in backward time reaches the switching point, I is the identity matrix, and N is a vector normal to some surface on which switching between vector fields occurs. In our case N is either H_x (the vector normal to Σ), or α_x (the vector normal to the surface that defines the boundaries $\partial \hat{\Sigma}^0$ or $\partial \hat{\Sigma}^1$ – the boundaries of sliding region $\hat{\Sigma}$). In this notation DM_{1s} is the saltation matrix which captures the effect of the switching between vector field F_1 and sliding vector field F_s around point $\mathbf{x_1}$. The subscripts signify between which vector fields there switchings occur. For a limit cycle as described we have $\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}} = DM_{s1} \exp(\hat{A}t_{s2}) DM_{2s} \exp(At_{02})$ $DM_{s2} \exp(\hat{A}t_{s1}) DM_{1s} \exp(At_{01})$, with DM_{s1} and DM_{s2} being the identity matrix since on $\partial \hat{\Sigma}^1$ the vector field $F_1 = F_s$, and on $\partial \hat{\Sigma}^0$ it is the vector field $F_2 = F_s$.
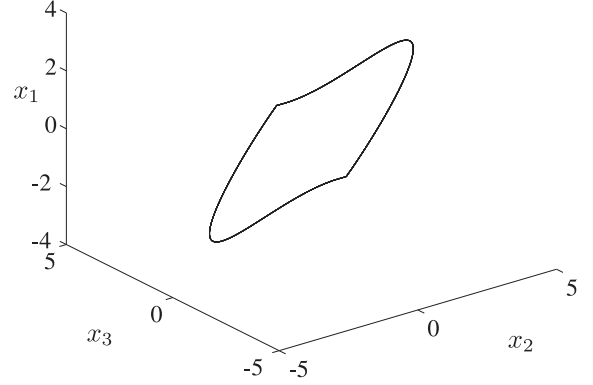

Fig. 1. Periodic orbit of system (10) with a segment of sliding when no digital sampling is applied to the system. Parameters are set to $\zeta = 0.1$, $\omega = 1$, $k = 1$, $\sigma = -1$ and $\rho = 0.4$.
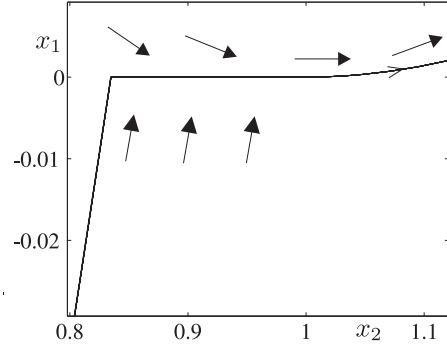

Fig. 2. Zoom into the region of sliding of periodic orbit from Fig. 1.

3. REPRESENTATIVE EXAMPLE

In the remaining part of the paper we focus our attention on a third order relay feedback system given by

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad u = -\text{sgn}y, \quad (10)$$

where

$$A = \begin{pmatrix} -(2\zeta\omega + \lambda) & 1 & 0 \\ -(2\zeta\omega\lambda + \omega^2) & 0 & 1 \\ -\lambda\omega^2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} k \\ 2k\sigma\rho \\ k\rho^2 \end{pmatrix},$$

$x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$, $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. The switching surface is given by a zero-level set $\{y = Cx = x_1 = 0\}$, vector fields $F_1 = Ax - B$ and $F_2 = Ax + B$. Assuming that the system is controlled digitally with a zero-order hold the output $y(t)$ is given at discrete sampling times $n\tau$ where τ is the length of the sampling interval and n is a positive integer. Note that in the current framework switchings between vector fields F_1 and F_2 occur at $n\tau$ time instances when $y(n\tau)$ changes its sign.

The dynamics of (10) with continuous time bang-bang control has been extensively studied in di Bernardo et al. (2001); Kowalczyk and di Bernardo (2001) where stable self-sustained oscillations as well as chaotic dynamics have been reported.
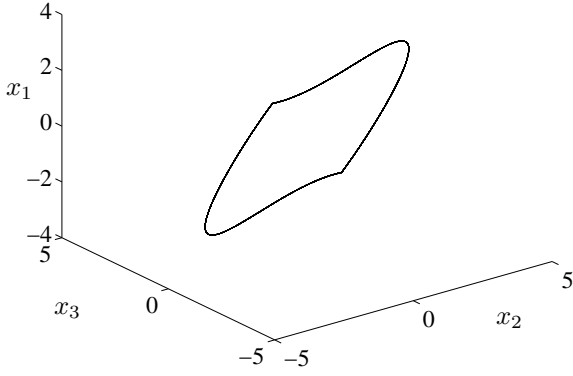
Fig. 3. Periodic orbit of system (10) with digital sampling $\tau = 0.005$. Parameters are set to $\zeta = 0.1$, $\omega = 1$, $k = 1$, $\sigma = -1$ and $\rho = 0.4$.

3.1 Stable self-sustained oscillations

The poles of the transfer function of the open loop system are given by the eigenvalues of the matrix A, and are given by $\mu_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega$, $\mu_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega$ and $\mu_3 = -\lambda$.

System (10) can exhibit stable self-sustained oscillations in the case when the open loop system is stable (all the poles of the transfer function lie in the left half-plane of the complex plane), but also in the case when the open loop system is unstable (at least one of the poles of the transfer function lies in the right half-plane of the complex plane). We can calculate the stability of such orbits analytically using explicit solutions (5)–(8) together with the expressions for the saltation matrices (9).

We start our investigations by considering a stable limit cycle with sliding existing for $\zeta = 0.1$, $\omega = 1$, $k = 1$, $\sigma = -1$, and $\rho = 0.4$. As depicted in Fig. 1 for these parameter values the system exhibits symmetric limit cycles with a segment of sliding motion. Sliding can be clearly seen in Fig. 2. To calculate the stability of the limit cycle we compute the eigenvalues of the matrix composition

$$\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}} = Dm_{s1} \times \exp(\hat{A}t_{s1}) \times DM_{2s} \times \exp(At_{02})$$
$$\times DM_{s2} \times \exp(\hat{A}t_{s1}) \times DM_{1s} \times \exp(At_{01}) \qquad (11)$$

where DM_{1s}, DM_{s2}, DM_{2s} and DM_{s1} are the saltation matrices calculated at points $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ and $\mathbf{x_4} = \mathbf{x_0}$ respectively. In the case depicted in Fig. 1 $\mathbf{x_0} = [0, 1, 2.71]^T$, time $t_{01} = 3.89$, $\mathbf{x_1} = [0, -0.83, -2.72]^T$, $t_{s1} = 0.048$, $\mathbf{x_2} = [0, -1, -2.71]^T$, $t_{02} = 3.89$, $\mathbf{x_3} =$, $[0, 0.83, 2.72]^T$, $t_{s2} = 0.048$, and the matrices

$$A = \begin{pmatrix} -1.2 & 1 & 0 \\ -1.2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.8 & 1 \\ 0 & 0.16 & 0 \end{pmatrix}.$$

Using these numerical values we find $\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}}$ and we determine the stability using the multipliers of $\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}}$. The two non-trivial multipliers of $\frac{\partial \mathbf{x_4}}{\partial \mathbf{x_0}}$ are $\lambda_1 = 0$ and $\lambda_2 = 0.2623$, and hence the limit cycle is stable. The multiplier 0 is present due to the existence of sliding.
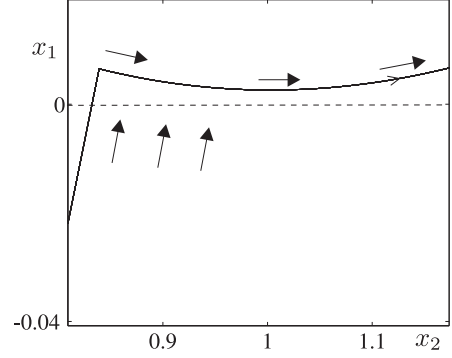

Fig. 4. Zoom into the neighbourhood of the switching surface of the periodic orbit from Figure 3 with digital sampling $\tau = 0.005$.
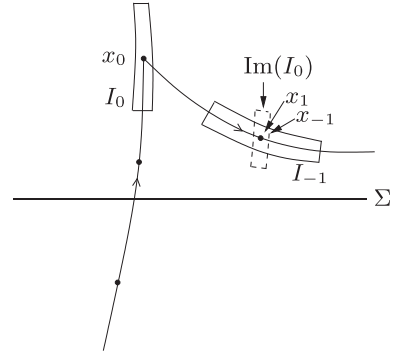

Fig. 5. Schematic sketch of the locking of switchings in digitally sampled system (10).

Keeping the same parameters we introduce sampling time τ to the output variable y thus obtaining discretized system (3). In Fig. 3 we depict a periodic orbit existing in the system for $\tau = 0.005$. Digitization destroys the segment of sliding cf. Fig. 2 and 4. We must note, however, that the introduction of sampling implies the extension of phase space dimensions to four; the phase of switching across the switching manifold is the fourth state variable, call it ψ, where $\psi \in [0, \tau]$.

To calculate the stability of the limit cycle it is convenient to consider a fixed time map from a switching point back to itself. In G_1 the switching between ϕ_2 and ϕ_1 occurs at $[0.0065, 0.8390, 2.7174]$, and in G_2 at $-[0.0065, 0.8390, 2.7174]$. The time of evolution between these two points is $t = 3.9329$. We can calculate the stability of the periodic orbit by considering the variational equation only. That is we do not need to apply the saltation matrices. The Floquet multipliers are $\lambda_1 = 0.01257 + 0.45522i$, $\lambda_2 = 0.01257 - 0.45522i$ and $\lambda_3 = 0.00038$, and they all lie within the unit circle of the complex plane. Note that in the case of continuous time sampling the number of non-trivial Floquet multipliers is 1, but in the discrete case we have 3 non-trivial Floquet multipliers. The reason behind the extension of phase space lies in the fact that digital sampling: (a) introduces additional state variable related to the phase of crossing of the flow with respect to the switching surface, and (b) sampling destroys sliding motion (the existence of sliding implies the loss of system dimension).
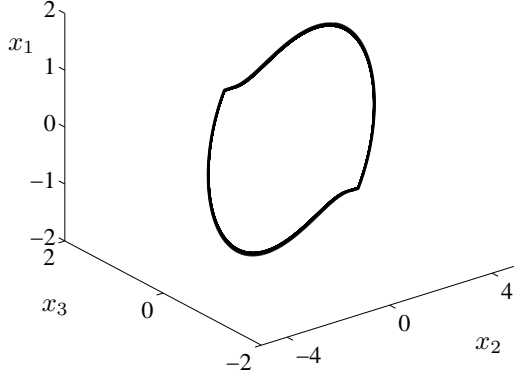
Fig. 6. Chaotic attractor of system (10) with digital sampling $\tau = 0.005$. Parameters are set to $\zeta = -0.07$, $\omega = 1$, $k = 1$, $\sigma = -1$, and $\rho = 0.05$.

We should also note that in the current case the phase of switching is locked. The mechanism behind the locking of switching points is schematically explained in Fig. 5. Consider a rectangle of initial conditions at time τ after the switching between ϕ_2 to ϕ_1 takes place (the rectangle labelled by I_{-1} in Fig. 5, where x_{-1} denotes the middle point in the box with respect to the phase of different points). This box is then mapped by ϕ_1 and ϕ_2 onto I_0 (assuming the size of I_{-1} is such that it is mapped across Σ within time τ which we can assume without loss of generality). Note that the length of I_0 in the direction normal to Σ is roughly the same as the length of I_{-1} along Σ. However the width of I_0 is very small due to the contractions induced by flows ϕ_2 and ϕ_1, but it is the width of I_0 which will differentiate between trajectories characterised by different phases (x_0 is the image of x_{-1} under ϕ_2 and ϕ_1). After time τ the rectangle I_0 is mapped to $Im(I_0)$ which, along Σ, lies within I_{-1}. For $t \to \infty$ the subsequent images of I_{-1} tend to 0, and hence the phase of switchings is locked.

3.2 Micro-chaos due to digitization

Considering the same parameter values we now decrease ζ and ρ to be able to find a stable limit cycle with unstable open loop in system (10). A limit cycle with the unstable open loop was found for $\zeta = -0.07$, $\omega = 1$, $k = 1$, $\sigma = -1$ and $\rho = 0.05$. The non-trivial Floquet multipliers corresponding to this orbit lie within the unit circle ($\lambda_1 = 0$ and $\lambda_2 = 0.7919$) hence the limit cycle is stable. The multipliers are calculated by means of (11). It is the effect of the saltation matrices (9) that ensures the stability. Let us now consider the effect of introducing digitization to the control variable y. We observe apparent thickening of the attractor. In Fig. 6 we depict the corresponding attractor for $\tau = 0.005$.

It turns out that the observed attractor is chaotic.

The boundedness can be seen as follows. Consider some region of phase space about the point on $\partial\hat{\Sigma}^1$ where the limit cycle with continuous time sampling lifts off the switching plane. In Glendinning and Kowalczyk (2010) it has been shown that for sufficiently small τ sliding motion is a limit of the "zig-zag" motion present in the system due to switchings.

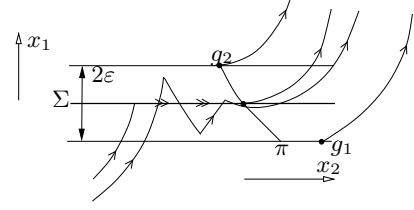
Fig. 7. Schematic representation of the bounding region along the limit cycle for sufficiently small sampling time τ.

In Fig. 7 we depict a segment of sliding limit cycle in the projection onto (x_1, x_2) coordinate plane as well as a trajectory existing in some neighbourhood of the limit cycle for some sufficiently small sampling time τ. We can make all admissible "zig-zag" trajectories bounded in the x_1, x_2, x_3 coordinates along the sliding part of the cycle within a parallelepiped of some width, say 2ε, by appropriately decreasing the sampling time τ. For the boundedness we have to ensure that all the trajectories can be made to leave the parallelepiped arbitrarily close to the point, say $\mathbf{x_g}$, at which the sliding cycle leaves the switching surface. Let $\pi := \{x \in \mathbb{R}^3 : H_x F_1 = 0\}$ be the zero velocity surface containing $\mathbf{x_g}$. We wish to establish that any trajectory is ejected from the parallelepiped along the x_2 coordinate in a way that can be made arbitrarily small by decreasing τ. Let n_1, n_2 be the maximum velocity of vector fields F_1 and F_2 along the x_2 direction for all pints (x_1, x_2, x_3) within the parallelepiped in the region of interest. Let m_1 be the minimum velocity of the vector field F_2 along the x_1 direction for all points (x_1, x_2, x_3) within the parallelepiped in the region of interest. The maximum value of x_2 that a "zig-zag" trajectory leaving the switching surface can attain is $g_1 = x_2 = x_{2g} + n_1\tau + n_2\lceil\frac{\varepsilon}{m_1\tau}\rceil\tau$, where x_{2g} is the component of $\mathbf{x_g}$ along the x_2 coordinate. Clearly g_1 can be made arbitrarily close to x_g by decreasing τ. This ensures that for sufficiently small τ there exists a bounded region in the neighbourhood of the region of phase space where sliding cycle exists in continuously sampled system. For a detailed argument on boundedness proved for the planar case we refer to Glendinning and Kowalczyk (2010). We now establish the existence of a positive Lyapunov exponent. The average behaviour of state vectors from some region, say D, containing an attractor, back to itself, can be established by considering a mapping from the tangent space of D back to itself. The logarithms of the eigenvalues of the linear map so obtained correspond to the sought Lyapunov exponents. We can compute these eigenvalues by computing the multipliers of $\exp(At_i)$ where t_i is the average time needed to reach a region in G_1 where switching between ϕ_2 and ϕ_1 takes place. Note that t_i will include the times of evolution following both flows ϕ_1 and ϕ_2. As we stated before, the eigenvalues of the characteristic equation of the matrix A are $\mu_{1,2} = -\zeta\omega \pm \sqrt{\zeta^2 - 1}$, $\mu_3 = -\lambda$. Therefore for ζ negative two of these eigenvalues are positive. This already suffices to show the existence of a positive Lyapunov exponent along the attractor since in the sampled system there are no saltation matrices that could induce contraction. For the sake of completeness we calculate the eigenvalues of the first variation along the flow that map points from some
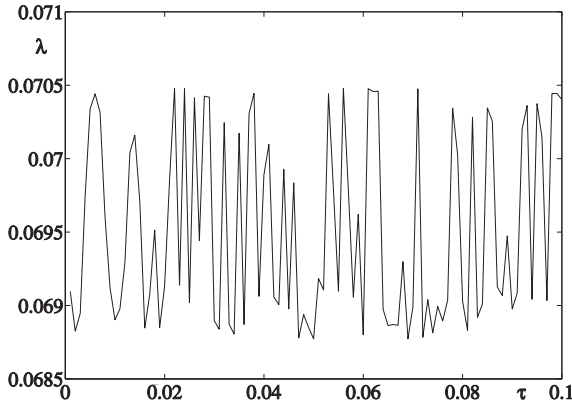
Fig. 8. The dominant lyapunov exponent indicating the presence of chaotic dynamics in the digitally sampled relay system. Parameters are set to $\zeta = -0.07$, $\omega = 1$, $k = 1$, $\sigma = -1$ and $\rho = 0.05$.

neighbourhood of $\mathbf{x_g}$ back to itself – these are the averaged multipliers of the linear map. We find the numerical values of these multipliers to be $\lambda_1 = -0.1801 + 1.7385i$, $\lambda_2 = -0.1801 - 1.7385i$ and $\lambda_3 = 0.00034$. We note that the first two of the eigenvalues lie outside of the unit circle. The logarithms of their absolute values are positive, and hence the Lyapunov exponents along the flow are positive.

Our numerics suggests that the chaotic attractor depicted in Fig 6 exists for an open set of sampling times τ. In Fig. 8 we depict the largest Lyapunov exponent versus the sampling time τ for $\tau \in (0.001, 0.1)$.

4. CONCLUSIONS

In the paper we study a class of single-input single-output relay feedback systems with discretized control. In particular, we are investigating the effects of discretization on periodic oscillations present in continuously sampled system. Using a third order relay feedback system as a representative example, in the case when the open loop system is stable, discretization implies the destruction of a segment of sliding and an effective increase of state space dimensions by two. In the case when the open loop system is unstable digitization leads to an onset of micro-chaotic oscillations. We compute the largest Lyapunov exponent of the chaotic attractor for an open set of sampling times and show it to be positive. The dynamics of the continuously sampled system is used to explain the mechanism behind the onset of micro-chaos. The mechanism which we unraveled in our system is quite general and we believe that any digitally controlled switched systems with the state space dimension $n \geq 2$ can exhibit the chaotic dynamics as described in our paper. Moreover, what in these systems is currently attributed to noise may in fact result from the digitization and be of a deterministic nature. This, in turn, may be useful in applications as it may, for example, inform a designer of a digital control system that a seemingly noisy output is produced due to digitization. This information can then be used to reduce noise using a different control strategy, which could be important if the noise levels happen to be of critical importance.

Further work is aimed at verifying if liner, planar relay feedback systems can produce chaotic dynamics as described in our paper.

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