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Aubert, Anne-Marie and Baum, Paul and Plymen, Roger

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Anne-Marie Aubert, Paul Baum and Roger Plymen

Abstract

The Kazhdan-Lusztig parameters are important parameters in the representation theory of p-adic groups and affine Hecke algebras. We show that the Kazhdan-Lusztig parameters have a definite geometric structure, namely that of the extended quotient $T//W$ of a complex torus T by a finite Weyl group W. More generally, we show that the corresponding parameters, in the principal series of a reductive p-adic group with connected centre, admit such a geometric structure. This confirms, in a special case, a recent geometric conjecture in [1].

In the course of this study, we provide a unified framework for Kazhdan-Lusztig parameters on the one hand, and Springer parameters on the other hand. Our framework contains a complex parameter s, and allows us to *interpolate* between $s = 1$ and $s = \sqrt{q}$. When $s = 1$, we recover the parameters which occur in the Springer correspondence; when $s = \sqrt{q}$, we recover the Kazhdan-Lusztig parameters.

1 Introduction

The Kazhdan-Lusztig parameters are important parameters in the representation theory of p-adic groups and affine Hecke algebras. We show that the Kazhdan-Lusztig parameters have a definite geometric structure, namely that of the extended quotient $T//W$ of a complex torus T by a finite Weyl group W. More generally, we show that the corresponding parameters, in the principal series of a reductive p-adic group with connected centre, admit such a geometric structure. This confirms, in a special case, a recent geometric conjecture in [1].

In the course of this study, we provide a unified framework for Kazhdan-Lusztig parameters on the one hand, and Springer parameters on the other hand. Our framework contains a complex parameter s, and allows us to *interpolate* between $s = 1$ and $s = \sqrt{q}$. When $s = 1$, we recover the parameters which occur in the Springer correspondence; when $s = \sqrt{q}$, we recover the Kazhdan-Lusztig parameters, see §5. Here, $q = q_F$ is the cardinality of the residue field of the underlying local field F.

Let $\mathcal{G}$ denote a reductive split p-adic group with connected centre, maximal split torus $\mathcal{T}$. Let G, T denote the Langlands dual of $\mathcal{G}$, $\mathcal{T}$. Then the quotient variety T/W plays a central role. For example, we have the Satake isomorphism

$$\mathcal{H}(\mathcal{G}, \mathcal{K}) \simeq \mathcal{O}(T/W)$$

where $\mathcal{O}(T/W)$ denotes the coordinate algebra of T/W, see [18, 2.2.1], and $\mathcal{H}(\mathcal{G}, \mathcal{K})$ denotes the algebra (under convolution) of $\mathcal{K}$-bi-invariant functions of compact support on $\mathcal{G}$, where $\mathcal{K} = \mathcal{G}(\mathfrak{o}_F)$. In this article, we will show that the *extended quotient* plays a central role in the context of the Kazhdan-Lusztig parameters.

We will prove that the extended quotient $T//W$ is a model for the Kazhdan-Lusztig parameters, see §4. More generally, let

$$\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$$

be a point in the Bernstein spectrum of $\mathcal{G}$. We prove that the extended quotient $T//W^{\mathfrak{s}}$ attached to $\mathfrak{s}$ is a model of the corresponding parameters attached to $\mathfrak{s}$. This is our main result, Theorem 4.1. *The principal series of a reductive p-adic group with connected centre has a definite geometric structure. The principal series is a disjoint union: each component is the extended quotient of the dual torus T by the finite Weyl group $W^{\mathfrak{s}}$ attached to $\mathfrak{s}$.* This confirms, in a special case, a recent geometric conjecture in [1].

We also show in §4 that our bijection is compatible with base change, in the special case of the irreducible smooth representations of $\mathrm{GL}(n)$ which admit nonzero Iwahori fixed vectors.

The details of our interpolation between Springer parameters and Kazhdan-Lusztig parameters will be given in §5. Our formulation creates a projection

$$\pi_{\sqrt{q}} : T//W \to T/W$$

which provides a model of the *infinitesimal character*.

We conclude in §6 with some carefully chosen examples.

Since the crossed product algebra $\mathcal{O}(T) \rtimes W$ is isomorphic to

$$\mathbb{C}[X(T)] \rtimes W \simeq \mathbb{C}[X(T) \rtimes W],$$

we obtain a bijection

$$\mathrm{Prim}\,\mathbb{C}[X(T) \rtimes W] \to T/\!/W$$

where Prim denotes primitive ideals. By composing this bijection with the bijection μ in Theorem 4.1, we finally get a bijection

$$\mathrm{Prim}\,\mathbb{C}[X(T) \rtimes W] \to \mathfrak{P}(G)$$

where $\mathfrak{P}(G)$ denotes the Kazhdan-Lusztig parameters. Let $\mathcal{I}$ be a standard Iwahori subgroup in $\mathcal{G}$ and let $\mathcal{H}(\mathcal{G}, \mathcal{I})$ denote the corresponding Iwahori-Hecke algebra, *i.e.,* the algebra (for the convolution product) of compactly supported $\mathcal{I}$-biinvariant functions on $\mathcal{G}$. The algebra is isomorphic to

$$\mathcal{H}(X(T) \rtimes W, q)$$

the Hecke algebra of the extended affine Weyl group $X(T) \rtimes W$, with parameter q. The simple modules of $\mathcal{H}(\mathcal{G}, \mathcal{I})$ are parametrized by $\mathfrak{P}(G)$ [7].

Hence $\mathfrak{P}(G)$ provides a parametrization of the simple modules of both the Iwahori-Hecke algebra $\mathcal{H}(X(T) \rtimes W, q)$ and of the group algebra of $X(T) \rtimes W$ (that is, the algebra $\mathcal{H}(X(T) \rtimes W, 1)$).

Note that the existence of a bijection between these sets of simple modules was already proved by Lusztig (see for instance [9, p. 81, assertion (a)]). Lusztig's construction needs to pass through the asymptotic Hecke algebra J, while we have replaced the use of J by the use of the extended quotient $T/\!/W$ (which is much simpler to construct).

2 Extended quotients

Let $\mathcal{O}(T)$ denote the coordinate algebra of the complex torus T. In noncommutative geometry, one of the elementary, yet fundamental, concepts is that of *noncommutative quotient* [8, Example 2.5.3]. The *noncommutative quotient* of T by W is the crossed product algebra

$$\mathcal{O}(T) \rtimes W.$$

This is a noncommutative unital $\mathbb{C}$-algebra. We need to filter this idea through periodic cyclic homology. We have an isomorphism

$$\mathrm{HP}_*(\mathcal{O}(T) \rtimes W) \simeq H^*(T/\!/W; \mathrm{C})$$

where HP_* denotes periodic cyclic homology, H^* denotes cohomology, and $T/\!/W$ is the extended quotient of T by W, see [3]. We recall the definition of the extended quotient $T/\!/W$.

Definition 2.1. *Let*

$$\widetilde{T} = \{(t, w) \in T \times W : w \cdot t = t\}.$$

The extended quotient is the quotient

$$T//W := \widetilde{T}/W$$

where W acts via $\alpha(t, w) = (\alpha \cdot t, \alpha w \alpha^{-1})$ with $\alpha \in W$.

Let $W(t)$ denote the isotropy subgroup of t. Let $\mathrm{conj}(W(t))$ denote the set of conjugacy classes in $W(t)$, and let $[w]$ denote the conjugacy class of w in $W(t)$. The map

$$\{(t, w) : t \in T, w \in W(t)\} \to \{(t, c) : t \in T, c \in \mathrm{conj}(W(t))\}$$

$$(t, w) \mapsto (t, [w])$$

induces a canonical bijection

$$\{(t, w) : t \in T, w \in W(t)\}/W \to \{(t, c) : t \in T, c \in \mathrm{conj}(W(t))\}/W$$

where W acts via $\alpha(t, c) = (\alpha \cdot t, [\alpha x \alpha^{-1}])$ with $x \in c$.

Let $\mathrm{Irr}(W(t))$ denote the set of equivalence classes of irreducible representations of $W(t)$. A choice of bijection between $\mathrm{conj}(W(t))$ and $\mathrm{Irr}(W(t))$ then creates a bijection

$$T//W \simeq \{(t, \tau) : t \in T, \tau \in \mathrm{Irr}(W(t))\}/W$$

where W acts via $\alpha(t, \tau) = (\alpha \cdot t, \alpha_*(\tau))$. Here, $\alpha_*(\tau)$ is the push-forward of τ to an irreducible representation of $W(\alpha \cdot t)$.

This leads us to

Definition 2.2. *The extended quotient of the second kind is*

$$(T//W)_2 := \{(t, \tau) : t \in T, \tau \in \mathrm{Irr}(W(t))\}/W$$

We then have a non-canonical bijection

$$T//W \simeq (T//W)_2.$$

Let T^w denote the fixed set $\{t \in T : w \cdot t = t\}$, and let $Z(w)$ denote the centralizer of w in W. We have

$$T//W = \bigsqcup T^w/Z(w) \tag{1}$$

where one w is chosen in each conjugacy class in W. Therefore $T/\!/W$ is a complex affine algebraic variety. The number of irreducible components in $T/\!/W$ is bounded below by $|\mathrm{conj}(W)|$.

The Jacobson topology on the primitive ideal spectrum of $\mathcal{O}(T) \rtimes W$ induces a topology on $(T/\!/W)_2$ such that the identity map

$$T/\!/W \to (T/\!/W)_2$$

is continuous. From the point of view of noncommutative geometry [8], the extended quotient of the second kind is a *noncommutative complex affine algebraic variety.*

The transformation groupoid $T \rtimes W$ is naturally an étale groupoid, see [8, p. 45]. Its groupoid algebra $\mathbb{C}[T \rtimes W]$ is the crossed product algebra

$$\mathcal{O}(T) \rtimes W.$$

In the groupoid $T \rtimes W$, we have

$$\mathrm{source}(t, w) = t, \quad \mathrm{target}(t, w) = w \cdot t$$

so that the set

$$\{(t, w) \in T \times W : w \cdot t = t\}$$

comprises all the arrows which are *loops*.

The decomposition of the groupoid $T \rtimes W$ into transitive groupoids leads naturally to Eqn. (1). The groupoid $T \rtimes W$ seems to be a bridge between $T/\!/W$ and $(T/\!/W)_2$.

In the context of algebraic geometry, the extended quotient is known as the inertia stack [13], in which case the notation is

$$I(T) := \widetilde{T}, \qquad [I(T)/W] := T/\!/W.$$

3 The parameters for the principal series

Let $\mathcal{W}_F$ denote the Weil group of F, let I_F be the inertia subgroup of $\mathcal{W}_F$. Let $\mathrm{Frob} \subset \mathcal{W}_F$ denote a geometric Frobenius (a generator of $\mathcal{W}_F/I_F \simeq \mathbb{Z}$). We have $\mathcal{W}_F/I_F =< \mathrm{Frob} >$. We will think of this as a multiplicative group, with identity element 1.

Let $\mathfrak{P}(G)$ denote the set of conjugacy classes in G of pairs (Φ, ρ) such that Φ is a morphism

$$\Phi\colon \mathcal{W}_F/I_F \times \mathrm{SL}(2, \mathbb{C}) \to G$$

which is *admissible*, *i.e.,* $\Phi(1,-)$ is a morphism of complex algebraic groups, $\Phi(\mathrm{Frob},1)$ is a semisimple element in G, and ρ is defined in the following way.

We will adopt the formulation of Reeder [16]. Choose a Borel subgroup B_2 in $\mathrm{SL}(2,\mathbb{C})$ and let $S_\Phi = \Phi(\mathcal{W}_F \times B_2)$, a solvable subgroup of G. Let $\mathbf{B}^\Phi$ denote the variety of Borel subgroups of G containing S_Φ. Let G_Φ be the centralizer in G of the image of Φ. Then G_Φ acts naturally on $\mathbf{B}^\Phi$, and hence on the singular homology $H_*(\mathbf{B}^\Phi,\mathbb{C})$. Then ρ is an irreducible representation of G_Φ which appears in the action of G_Φ on $H_*(\mathbf{B}^\Phi,\mathbb{C})$.

A Reeder parameter (Φ,ρ) determines a Kazhdan-Lusztig parameter (σ,u,ρ) in the following way. Let

$$u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

and set

$$u = \Phi(1,u_0), \quad \sigma = \Phi(\mathrm{Frob}, T_{\sqrt{q}})$$

where q is the cardinality of the residue field k_F. Then the triple (σ,u,ρ) is a Kazhdan-Lusztig parameter. Since Φ is a homomorphism and

$$T_{\sqrt{q}}\, u_0\, T_{\sqrt{q}}^{-1} = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = u_0^q$$

it follows that

$$\sigma u \sigma^{-1} = u^q.$$

It is worth noting that the set $\mathfrak{P}(G)$ is q-independent.

We now move on to the rest of the principal series. We recall that $\mathcal{G}$ denotes a reductive split p-adic group *with connected centre*, maximal split torus $\mathcal{T}$, and G, T denote the Langlands dual of $\mathcal{G}$, $\mathcal{T}$. We assume in addition that the residual characteristic of F is not a torsion prime for G.

Let $\mathfrak{Q}(G)$ denote the set of conjugacy classes in G of pairs (Φ,ρ) such that Φ is a continuous morphism

$$\Phi\colon \mathcal{W}_F \times \mathrm{SL}(2,\mathbb{C}) \to G$$

which is rational on $\mathrm{SL}(2,\mathbb{C})$ and such that $\Phi(\mathcal{W}_F)$ consists of semisimple element in G, and ρ is defined in the following way.

Choose a Borel subgroup B_2 in $\mathrm{SL}(2,\mathbb{C})$ and let $S_\Phi = \Phi(\mathcal{W}_F \times B_2)$. Let $\mathbf{B}^\Phi$ denote the variety of Borel subgroups of G containing S_Φ. The variety $\mathbf{B}^\Phi$ is non-empty if and only if Φ factors through the topological abelianization

$\mathcal{W}_F^{\text{ab}} := \mathcal{W}_F/\overline{[\mathcal{W}_F, \mathcal{W}_F]}$ of $\mathcal{W}_F$ (see [16, § 4.2]). We will assume that $\mathbf{B}^\Phi$ is non-empty, and we will still denote by Φ the homomorphim

$$\Phi \colon \mathcal{W}_F^{\text{ab}} \times \text{SL}(2, \mathbb{C}) \to G.$$

Let I_F^{ab} denote the image of I_F in $\mathcal{W}_F^{\text{ab}}$. The choice of Frobenius Frob determines a splitting

$$\mathcal{W}_F^{\text{ab}} = I_F^{\text{ab}} \times \langle \text{Frob} \rangle. \tag{2}$$

Let G_Φ be the centralizer in G of the image of Φ. Then G_Φ acts naturally on $\mathbf{B}^\Phi$, and hence on the singular homology of $H_*(\mathbf{B}^\Phi, \mathbb{C})$. Then ρ is an irreducible representation of G_Φ which appears in the action of G_Φ on $H_*(\mathbf{B}^\Phi, \mathbb{C})$.

Let χ be a smooth quasicharacter of $\mathcal{T}$ and let $\mathfrak{s} = [\mathcal{T}, \chi]_{\mathcal{G}}$ be the point in the Bernstein spectrum $\mathfrak{B}(\mathcal{G})$ determined by χ. Let

$$W^{\mathfrak{s}} = \{w \in W : w \cdot \mathfrak{s} = \mathfrak{s}\}. \tag{3}$$

Let X denote the rational co-character group of $\mathcal{T}$, identified with the rational character group of T. Let $\mathcal{T}_0$ be the maximal compact subgroup of $\mathcal{T}$. By choosing a uniformizer in F, we obtain a splitting

$$\mathcal{T} = \mathcal{T}_0 \times X,$$

according to which

$$\chi = \lambda \otimes t,$$

where λ is a character of $\mathcal{T}_0$, and $t \in T$. Let $r_F \colon \mathcal{W}_F^{\text{ab}} \to F^\times$ denote the reciprocity isomorphism of abelian class field theory, and let

$$\widehat{\lambda} \colon I_F^{\text{ab}} \to T \tag{4}$$

be the unique homomorphism satisfying

$$\eta \circ \widehat{\lambda} = \lambda \circ \eta \circ r_F, \quad \text{for all } \eta \in X, \tag{5}$$

where η is viewed as a character of T on the left side and as a co-character of $\mathcal{T}$ on the right side of (5).

Let H denote the centralizer in G of the image of $\widehat{\lambda}$:

$$H = G_{\widehat{\lambda}}. \tag{6}$$

The assumption that G has simply-connected derived group implies that the group H is connected (see [17, p. 396]). Note that H itself does not have simply-connected derived group in general (for instance, if G is the exceptional group of type G_2, and σ is the tensor square of a ramified quadratic character of $F^\times$ then $H = \mathrm{SO}(4, \mathbb{C})$).

Let $\mathfrak{Q}(G)_{\widehat{\lambda}}$ be the subset of $\mathfrak{Q}(G)$ consisting of the G-conjugacy classes of all the pairs (Φ, ρ) such that Φ factors through $\mathcal{W}_F^{\mathrm{ab}}$ and

$$\Phi|_{I_F^{\mathrm{ab}}} = \widehat{\lambda}.$$

The group $W^{\mathfrak{s}}$ defined in (3) is a Weyl group: it is the Weyl group of H (indeed, in the decomposition of [17, Lemma 8.1 (i)] the group C_χ is trivial as proven on [17, p. 396]):

$$W^{\mathfrak{s}} = W_H.$$

4 Main result

Theorem 4.1. *There is a canonical bijection of the extended quotient of the second kind $(T//W^{\mathfrak{s}})_2$ onto the set $\mathfrak{Q}(G)_{\widehat{\lambda}}$ of conjugacy classes of Reeder parameters attached to the point $\mathfrak{s}$ in the Bernstein spectrum of $\mathcal{G}$. It follows that there is a bijection*

$$\mu^{\mathfrak{s}} : T//W^{\mathfrak{s}} \simeq \mathfrak{Q}(G)_{\widehat{\lambda}}$$

so that the extended quotient $T//W^{\mathfrak{s}}$ is a model for the Reeder parameters attached to the point $\mathfrak{s}$.

The proof of this theorem requires a series of Lemmas. We recall that

$$W^{\mathfrak{s}} = W_H.$$

The plan of our proof is to begin with an element in the extended quotient of the second kind $(T//W_H)_2$. Lemmas 4.2 and 4.3 allow us to infer that $W_H(t)$ is a semidirect product $W_{\mathfrak{G}(t)} \rtimes A_H(t)$. We now combine the Springer correspondence for $W_{\mathfrak{G}(t)}$ with Clifford theory for semidirect products (Clifford theory is a noncommutative version of the Mackey machine). This creates 4 parameters (t, x, ϱ, ψ). With this data, and the character λ determined by the point $\mathfrak{s}$, we construct a Reeder parameter (Φ, ρ) such that $\Phi(\mathrm{Frob}, 1) = t$, $\Phi(1, u_0) = \exp x$ and the restriction of ρ contains ϱ.

Lemma 4.2. *Let M be a reductive algebraic group. Let M^0 denote the connected component of the identity in M. Let T be a maximal torus of M^0 and let B be a Borel subgroup of M^0 containing T. Let*

$$W_{M^0}(T) := \mathrm{N}_{M^0}(T)/T$$

denote the Weyl group of M^0 with respect to T. We set

$$W_M(T) := \mathrm{N}_M(T)/T.$$

(1) *The group $W_M(T)$ has the semidirect product decomposition:*

$$W_M(T) = W_{M^0}(T) \rtimes (\mathrm{N}_M(T,B)/T),$$

where $\mathrm{N}_M(T,B)$ denotes the normalizer in M of the pair (T,B).

(2) *We have*

$$\mathrm{N}_M(T,B)/T \simeq M/M^0 = \pi_0(M).$$

Proof. The group $W_{M^0}(T)$ is a normal subgroup of $W_M(T)$. Indeed, let $n \in \mathrm{N}_{M^0}(T)$ and let $n' \in \mathrm{N}_M(T)$, then $n'nn'^{-1}$ belongs to M^0 (since the latter is normal in M) and normalizes T, that is, $n'nn'^{-1} \in \mathrm{N}_{M^0}(T)$. On the other hand, $n'(nT)n'^{-1} = n'nn'^{-1}(n'Tn'^{-1}) = n'nn'^{-1}T$.

Let $w \in W_M(T)$. Then wBw^{-1} is a Borel subgroup of M^0 (since, by definition, the Borel subgroups of an algebraic group are the maximal closed connected solvable subgroups). Moreover, wBw^{-1} contains T. In a connected reductive algebraic group, the intersection of two Borel subgroups always contains a maximal torus and the two Borel subgroups are conjugate by a element of the normalizer of that torus. Hence B and wBw^{-1} are conjugate by an element w_1 of $W_{M^0}(T)$. It follows that $w_1^{-1}w$ normalises B. Hence

$$w_1^{-1}w \in W_M(T) \cap \mathrm{N}_M(B) = \mathrm{N}_M(T,B)/T,$$

that is,

$$W_M(T) = W_{M^0}(T) \cdot (\mathrm{N}_M(T,B)/T).$$

Finally, we have

$$W_{M^0}(T) \cap (\mathrm{N}_M(T,B)/T) = \mathrm{N}_{M^0}(T,B)/T = \{1\},$$

since $\mathrm{N}_{M^0}(B) = B$ and $B \cap \mathrm{N}_{M^0}(T) = T$. This proves (1).

We will now prove (2). We consider the following map:

$$(*) \qquad \mathrm{N}_M(T,B)/T \to M/M^0 \qquad mT \mapsto mM^0.$$

It is injective. Indeed, let $m, m' \in \mathrm{N}_M(T,B)$ such that $mM^0 = m'M^0$. Then $m^{-1}m' \in M^0 \cap \mathrm{N}_M(T,B) = \mathrm{N}_{M^0}(T,B) = T$ (as we have seen above). Hence $mT = m'T$.

On the other hand, let m be an element in M. Then $m^{-1}Bm$ is a Borel subgroup of M^0, hence there exists $m_1 \in M^0$ such that $m^{-1}Bm = m_1^{-1}Bm_1$.

It follows that $m_1m^{-1} \in \mathrm{N}_M(B)$. Also $m_1m^{-1}Tmm_1^{-1}$ is a torus of M^0 which is contained in $m_1m^{-1}Bmm_1^{-1} = B$. Hence T and $m_1m^{-1}Tmm_1^{-1}$ are conjugate in B: there is $b \in B$ such that $m_1m^{-1}Tmm_1^{-1} = b^{-1}Tb$. Then $n := bm_1m^{-1} \in \mathrm{N}_M(T, B)$. It gives $m = n^{-1}bm_1$. Since $bm_1 \in M^0$, we obtain $mM^0 = n^{-1}M^0$. Hence the map $(*)$ is surjective. □

In order to approach the notation in [4, p.471], we let $\mathfrak{G}(t)$ denote the identity component of the centralizer $C_H(t)$:

$$\mathfrak{G}(t) := C_H^0(t).$$

Let $W_{\mathfrak{G}(t)}$ denote the Weyl group of $\mathfrak{G}(t)$.

Lemma 4.3. *Let $t \in T$. The isotropy subgroup $W_H(t)$ is the group of $\mathrm{N}_{C_H(t)}(T)/T$, and we have*

$$W_H(t) = W_{\mathfrak{G}(t)} \rtimes A_H(t) \quad \textit{with } A_H(t) := \pi_0(C_H(t)).$$

In the case when H has simply-connected derived group, the group $C_H(t)$ is connected and $W_H(t)$ is then the Weyl group of $C_H(t) = \mathfrak{G}(t)$.

Proof. Let $t \in T$. Note that

$$\begin{aligned} W_H(t) &= \{w \in W_H : w \cdot t = t\} \\ &= \{w \in W_H : wtw^{-1} = t\} \\ &= \{w \in W_H : wt = tw\} \\ &= W \cap C_H(t). \end{aligned}$$

Note that H and $C_H(t)$ have a common maximal torus T. Now

$$\begin{aligned} W_H \cap C_H(t) &= \mathrm{N}_H(T)/T \cap C_H(t) \\ &= \mathrm{N}_{C_H(t)}(T)/T \\ &= W_{C_H(t)}(T). \end{aligned}$$

The result follows by applying Lemma 4.2 with $M = C_H(t)$.

If H has simply-connected derived group, then the centralizer $C_H(t)$ is connected by Steinberg's theorem [4, §8.8.7]. □

Let τ be an irreducible representation of $W_{\mathfrak{G}(t)}$. Now we apply the Springer correspondence to τ. Note: the Springer correspondence that we are considering here coincides with that constructed by Springer for a reductive group over a field of positive characteristic and is obtained from the correspondence constructed by Lusztig by tensoring the latter by the sign representation of $W_{\mathfrak{G}(t)}$ (see [6]).

Let $\mathfrak{c}(t)$ denote the Lie algebra of $\mathfrak{G}(t)$, for $x \in \mathfrak{c}(t)$, let $Z_{\mathfrak{G}(t)}(x)$ denote the centralizer of x in $\mathfrak{G}(t)$, via the adjoint representation of $\mathfrak{G}(t)$ on $\mathfrak{c}(t)$, and let

$$A_x = \pi_0(Z_{\mathfrak{G}(t)}(x)) \tag{7}$$

Let $\mathbf{B}_x$ denote the variety of Borel subalgebras of $\mathfrak{c}(t)$ that contain x.

All the irreducible components of $\mathbf{B}_x$ have the same dimension $d(x)$ over $\mathbb{R}$, see [4, Corollary 3.3.24]. The finite group A_x acts on the set of irreducible components of $\mathbf{B}_x$ [4, p. 161].

Definition 4.4. *If a group A acts on the variety $\mathbf{X}$, let $\mathcal{R}(A, \mathbf{X})$ denote the set of irreducible representations of A appearing in the homology $H_*(\mathbf{X})$, as in [16, p.118]. Let $\mathcal{R}_{top}(A, \mathbf{X})$ denote the set of irreducible representations of A appearing in the top homology of $\mathbf{X}$.*

The Springer correspondence yields a one-to-one correspondence

$$(x, \varrho) \mapsto \tau(x, \varrho) \tag{8}$$

between the set of $\mathfrak{G}(t)$-conjugacy classes of pairs (x, ϱ) formed by a nilpotent element $x \in \mathfrak{c}(t)$ and an irreducible representation ϱ of $A = A_x$ which occurs in $H_{d(x)}(\mathbf{B}_x, \mathbb{C})$ (that is, $\varrho \in \mathcal{R}_{top}(A_x, \mathbf{B}_x)$) and the set of isomorphism classes of irreducible representations of the Weyl group $W_{\mathfrak{G}(t)}$.

We now work with the Jacobson-Morozov theorem [4, p. 183]. Let e_0 be the standard nilpotent matrix in $\mathfrak{sl}(2, \mathbb{C})$:

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a rational homomorphism $\gamma : \mathrm{SL}(2, \mathbb{C}) \to \mathfrak{C}(t)$ such that its differential $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{c}(t)$ sends e_0 to x, see [4, §3.7.4].

Define

$$\Phi\colon \mathcal{W}_F^{\mathrm{ab}} \times \mathrm{SL}(2, \mathbb{C}) \to G, \qquad (w, \mathrm{Frob}, Y) \mapsto \widehat{\lambda}(w) \cdot t \cdot \gamma(Y) \tag{9}$$

$$\Upsilon\colon \mathcal{W}_F^{\mathrm{ab}} \times \mathrm{SL}(2, \mathbb{C}) \to H, \qquad (w, \mathrm{Frob}, Y) \mapsto \widehat{\lambda}(w) \cdot t \cdot \gamma(Y) \tag{10}$$

$$\Psi\colon \mathcal{W}_F^{\mathrm{ab}} \times \mathrm{SL}(2, \mathbb{C}) \to \mathfrak{G}(t), \qquad (w, \mathrm{Frob}, Y) \mapsto \widehat{\lambda}(w) \cdot t \cdot \gamma(Y) \tag{11}$$

$$\Xi\colon \mathcal{W}_F^{\rm ab} \times \mathrm{SL}(2,\mathbb{C}) \to \mathfrak{G}(t), \qquad (w, \mathrm{Frob}, Y) \mapsto \widehat{\lambda}(w)\cdot\gamma(Y). \tag{12}$$

where w is any element in $I_F^{\rm ab}$.

Note that $im\,\Phi \subset H$ (see [16, § 4.2]) and that $C(im\,\Psi) = C(im\,\Upsilon)$, for any element in $C(im\,\Upsilon)$ must commute with $\Upsilon(\mathrm{Frob}) = t$. We also have $C(im\,\Xi) = C(im\,\Psi) \subset \mathfrak{C}(t)$. Let

$$A_\Psi = \pi_0(C(im\,\Psi)), \qquad A_\Xi = \pi_0(C(im\,\Xi)).$$

Lemma 4.5. *We have*

$$A_x = A_\Xi = A_\Psi.$$

Proof. According to [4, §3.7.23], we have

$$Z_{\mathfrak{G}(t)}(x) = C(im\,\Xi)\cdot U$$

with U the unipotent radical of $Z_{\mathfrak{G}(t)}(x)$. Now U is contractible via the map

$$[0,1]\times U \to U, \qquad (\lambda, \exp Y) \mapsto \exp(\lambda Y)$$

for all $Y \in \mathfrak{n}$ with $\exp \mathfrak{n} = U$. □

Lemma 4.5 allows us to define

$$A := A_x = A_\Psi = A_\Xi.$$

Let $\mathcal{C}(t)$ denote a *predual* of $\mathfrak{G}(t)$, *i.e.,* $\mathfrak{G}(t)$ is the Langlands dual of $\mathcal{C}(t)$. Let $\mathbf{B}^\Psi$ (resp. $\mathbf{B}^\Xi$) denote the variety of the Borel subgroups of $\mathfrak{G}(t)$ which contain $S_\Psi := \Psi(\mathcal{W}_F \times B_2)$ (resp. $S_\Xi := \Xi(\mathcal{W}_F\times B_2) = \gamma(B_2)$).

Lemma 4.6. *We have*

$$\mathcal{R}_{top}(A, \mathbf{B}_x) = \mathcal{R}(A, \mathbf{B}^\Xi).$$

Proof. Let, as before, τ be an irreducible representation of $W_{\mathfrak{G}(t)}$. Let (x, ϱ) be the Springer parameter attached to τ by the inverse bijection of (8). Define Ξ as in Eqn.12. Note that Ξ depends on the morphism γ, which in turn depends on the nilpotent element $x \in \mathfrak{c}(t)$.

Then Ξ is a real tempered L-parameter for the p-adic group $\mathcal{C}(t)$, see [2, 3.18]. According to several sources, see [11, §10.13], [2], there is a bijection between Springer parameters and Reeder parameters:

$$(d\gamma(e_0)), \varrho) \mapsto (\Xi, \varrho). \tag{13}$$

Now ϱ is an irreducible representation of A which appears simultaneously in $H_{d(x)}(\mathbf{B}_x, \mathbb{C})$ and $H_*(\mathbf{B}^\Xi, \mathbb{C})$. □

We will recall below a result of Ram and Ramagge, which is based on Clifford theoretic results developed by MacDonald and Green.

Let $\mathcal{H}$ be a finite dimensional $\mathbb{C}$-algebra and let $\mathcal{A}$ be a finite group acting by automorphisms on $\mathcal{H}$. If V is a finite dimensional module for $\mathcal{H}$ and $a \in \mathcal{A}$, let ${}^aV$ denote the $\mathcal{H}$-module with the action $f \cdot v := a^{-1}(f)v$, $f \in \mathcal{H}$ and $v \in V$. Then V is simple if and only if ${}^aV$ is. Let V be a simple $\mathcal{H}$-module. Define the inertia subgroup of V to be

$$\mathcal{A}_V := \{a \in \mathcal{A} \,:\, V \simeq {}^aV\}.$$

Let $a \in \mathcal{A}_V$. Since both V and ${}^aV$ are simple, Schur's lemma implies that the isomorphism $V \to {}^aV$ is unique up to a scalar multiple. For each $a \in \mathcal{A}_V$ we fix an isomorphism

$$\phi_a \colon V \to {}^{a^{-1}}V.$$

Then, as operators on V,

$$\phi_a v = a(r)\phi_a, \quad \text{and} \quad \phi_a\phi_{a'} = \eta_V(a,a')^{-1}\phi_{aa'},$$

where $\eta_V(a,a') \in \mathbb{C}^\times$. The resulting function

$$\eta_V \colon \mathcal{A}_V \times \mathcal{A}_V \to \mathbb{C}^\times,$$

is a cocycle. The isomorphism class of η_V is independent of the choice of the isomorphism ϕ_a.

Let $\mathbb{C}[\mathcal{A}_V]_{\eta_V}$ be the algebra with basis $\{c_a \,:\, a \in \mathcal{A}_V\}$ and multiplication given by

$$c_a \cdot c_{a'} = \eta_V(a,a')c_{aa'}, \quad \text{for } a, a' \in \mathcal{A}_V.$$

Let ψ be a simple $\mathbb{C}[\mathcal{A}_V]_{\eta_V}$-module. Then putting

$$(fa)\cdot(v \otimes z) = f\phi_a v \otimes c_a z, \quad \text{for } f \in \mathcal{H},\ a \in \mathcal{A}_V,\ v \in V,\ z \in \psi,$$

defines an action of $\mathcal{H} \rtimes \mathcal{A}_V$ on $V \otimes \psi$. Define the induced module

$$V \rtimes \psi := \mathrm{Ind}_{\mathcal{H}\rtimes\mathcal{A}_V}^{\mathcal{H}\rtimes\mathcal{A}}(V \otimes \psi).$$

Theorem 4.7. (Ram-Ramagge, [14, Theorem A.6], Reeder, [16, (1.5.1)]) *The induced module $V \rtimes \psi$ is a simple $\mathcal{H} \rtimes \mathcal{A}$-module, every simple $\mathcal{H} \rtimes \mathcal{A}$-module occurs in this way, and if $V \rtimes \psi \simeq V' \rtimes \psi'$, then V, V' are $\mathcal{A}$-conjugate, and $\psi \simeq \psi'$ as $\mathbb{C}[\mathcal{A}_V]_{\eta_V}$-modules.*

One the other hand, it follows from Lemma 4.3 that the isotropy group of t in W_H admits the following semidirect product decomposition:

$$W_H(t) = W_{\mathfrak{G}(t)} \rtimes A_H(t) \quad \text{with } A_H(t) := \pi_0(C_H(t)).$$

Hence the group algebra $\mathbb{C}[W_H(t)]$ is a crossed-product algebra

$$\mathbb{C}[W_H(t)] = \mathbb{C}[W_{\mathfrak{G}(t)}] \rtimes A_H(t).$$

By applying Theorem 4.7 with $\mathcal{H} = \mathbb{C}[W_{\mathfrak{G}(t)}]$ and $\mathcal{A} = A_H(t)$, we see that the irreducible representations of $W_H(t)$ are the

$$\tau(x, \varrho) \rtimes \psi,$$

with ψ any simple $\mathbb{C}[A_\tau]_{\eta_\tau}$-module and $\tau = \tau(x, \varrho)$.

Let $\mathcal{I}$ be a standard Iwahori subgroup in $\mathcal{C}(t)$, and let $\mathcal{H}(\mathcal{C}(t), \mathcal{I})$ denote the corresponding Iwahori-Hecke algebra. Recall that $x = d\gamma(e_0)$. We will denote by $V = V(x, \varrho)$ the real tempered simple module of $\mathcal{H}(\mathcal{C}(t), \mathcal{I})$ which corresponds to (x, ϱ). Here "real" means that the central character of V is real.

By applying Theorem 4.7 with $\mathcal{H} = \mathcal{H}(\mathcal{C}(t), \mathcal{I})$ and $\mathcal{A} = A_H(t)$, we obtain the following subset of simple modules for $\mathcal{H}(\mathcal{C}(t), \mathcal{I}) \rtimes A_H(t)$:

$$V(x, \varrho) \rtimes \psi,$$

with ψ any simple $\mathbb{C}[A_V]_{\eta_V}$-module and $V = V(x, \varrho)$.

Lemma 4.8. *We have*

$$A_{\tau(x,\varrho)} = A_{V(x,\varrho)}.$$

Moreover, the cocycles $\eta_{\tau(x,\varrho)}$ and $\eta_{V(x,\varrho)}$ can be chosen to be equal.

Proof. Recall that the *closure order on nilpotent adjoint orbits* is defined as follows

$$\mathcal{O}_1 \le \mathcal{O}_2 \quad \text{when } \mathcal{O}_1 \subset \overline{\mathcal{O}_2}.$$

$$\mathcal{O}_1 \le \mathcal{O}_2 \quad \text{when } \mathcal{O}_1 \subset \overline{\mathcal{O}_2}.$$

For x a nilpotent element of $\mathfrak{c}(t)$, we will denote by $\mathcal{O}_x$ the nilpotent adjoint orbit which contains x. Then as in [2, (6.5)], we define a *partial order on the representations of* $W_{\mathfrak{G}(t)}$ by

$$\tau(x_1, \varrho_1) \le \tau(x_2, \varrho_2) \quad \text{when } \mathcal{O}_{x_1} \le \mathcal{O}_{x_2}. \tag{14}$$

In this partial order, the trivial representation of $W(t)$ is a minimal element and the sign representation of $W(t)$ is a maximal element.

The $W_{\mathfrak{G}(t)}$-structure of $V(x, \varrho)$ is

$$V(x, \varrho)|_{W_{\mathfrak{G}(t)}} = \tau(x, \varrho) \oplus \bigoplus_{\substack{(x_1,\varrho_1) \\ \tau(x,\varrho) < \tau(x,\varrho_1)}} m_{(x_1,\varrho_1)}\, \tau(x_1, \varrho_1), \tag{15}$$

where the $m_{(x_1,\varrho_1)}$ are non-negative integers. (In case $\mathcal{C}(t)$ has connected centre, (15) is implied by [2, Theorem 6.3 (1)], the proof in the general case follows the same lines.) In particular, it follows from (15) that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{W_{\mathfrak{G}(t)}}(\tau(x,\varrho), V(x,\varrho)) = 1. \tag{16}$$

Let $a \in A_H(t)$. Since the action of $A_H(t)$ on $W_{\mathfrak{G}(t)}$ comes from its action on the root datum, we have (see [16, 2.6.1, 2.7.3]):

$$^a\tau(x,\varrho) = \tau(a\cdot x, {}^a\varrho).$$

Then

$$^aV(x,\varrho)|_{W_{\mathfrak{G}(t)}} = \tau(a\cdot x, {}^a\varrho) \oplus \bigoplus_{\substack{(x_1,\varrho_1)\\ \tau(x,\varrho)\le\tau(x_1,\varrho_1)}} m_{(x_1,\varrho_1)}\, \tau(a\cdot x, {}^a\varrho_1).$$

Since $\tau(x,\varrho) \le \tau(x_1,\varrho_1)$ if and only if $\chi(a\cdot x, {}^a\varrho) \le \tau(a\cdot x_1, {}^a\varrho_1)$, it follows that $^aV(x,\varrho)$ corresponds to the $\mathfrak{G}(t)$-conjugacy class of $(a\cdot x, {}^a\varrho)$ via the bijection induced by (13).

Hence

$$^aV(x,\varrho) \simeq V(x,\varrho) \quad \text{if and only if} \quad {}^a\tau(x,\varrho) \simeq \tau(x,\varrho).$$

The equality of the inertia subgroups

$$A_H(t)_{V(x,\varrho)} = A_H(t)_{\tau(x,\varrho)} =: A_H(t)_{x,\varrho}$$

follows.

Let $\{\phi_a^V : a \in A_H(t)_{x,\varrho}\}$ (resp. $\{\phi_a^\tau : a \in A_H(t)_{x,\varrho}\}$) a family of isomorphisms for $V = V(x,\varrho)$ (resp. $\tau = \tau(x,\varrho)$) which determines the cocycle η_V (resp. η_τ). We have

$$\mathrm{Hom}_{W_{\mathfrak{G}(t)}}(\tau, V) \overset{\phi_a^V}{\to} \mathrm{Hom}_{W_{\mathfrak{G}(t)}}(\tau, {}^{a^{-1}}V) \overset{\phi_a^\tau}{\to} \mathrm{Hom}_{W_{\mathfrak{G}(t)}}({}^{a^{-1}}\tau, {}^{a^{-1}}V).$$

The composed map is given by a scalar, since by Eqn. (16) these spaces are one-dimensional. We normalize ϕ_a^V so that this scalar equals to one. This forces η_V and η_τ to be equal. □

Lemma 4.9. *There is a bijection between Springer parameters and Reeder parameters for the group $C_H(t)$:*

$$(x,\varrho,\psi) \mapsto (\Xi,\varrho,\psi).$$

Proof. Lemma 4.8 allows us to extend the bijection (13) from $\mathfrak{G}(t)$ to $C_H(t)$. □

Lemma 4.10. *We have*

$$\mathbf{B}^{\Psi} = \mathbf{B}^{\Xi}.$$

Proof. We note that

$$S_{\Psi} =< t > \gamma(B_2), \qquad S_{\Xi} = \gamma(B_2)$$

Let $\mathfrak{b}$ denote a Borel subgroup of the reductive group $C_H(t)$. Since $\mathfrak{b}$ is maximal among the connected solvable subgroups of $C_H(t)$, we have $\mathfrak{b} \subset \mathfrak{G}(t)$. Then we have $\mathfrak{b} = T_{\mathfrak{b}} U_{\mathfrak{b}}$ with $T_{\mathfrak{b}}$ a maximal torus in $\mathfrak{G}(t)$, and $U_{\mathfrak{b}}$ the unipotent radical of $\mathfrak{b}$. Note that $T_{\mathfrak{b}} \subset \mathfrak{G}(t)$. Therefore $yt = ty$ for all $y \in T_{\mathfrak{b}}$. This means that t centralizes $T_{\mathfrak{b}}$, i.e. $t \in Z(T_{\mathfrak{b}})$. In a connected Lie group such as $\mathfrak{G}(t)$, we have

$$Z(T_{\mathfrak{b}}) = T_{\mathfrak{b}}$$

so that $t \in T_{\mathfrak{b}}$. Since $T_{\mathfrak{b}}$ is a group, it follows that $< t > \subset T_{\mathfrak{b}}$.

As a consequence, we have

$$\mathfrak{b} \supset < t > \gamma(B_2) \iff \mathfrak{b} \supset \gamma(B_2).$$

□

Let $S_{\Upsilon} = \Upsilon(\mathcal{W}_F \times B_2)$, a solvable subgroup of H. Let $\mathbf{B}^{\Upsilon}$ denote the variety of Borel subgroups of H containing S_{Υ}.

Lemma 4.11. *We have*

$$\mathcal{R}(A, \mathbf{B}^{\Upsilon}) = \mathcal{R}(A, \mathbf{B}^{\Psi})$$

Proof. We denote the Lie algebra of $\mathfrak{G}(t)$ by $\mathfrak{g}(t)$, and the Lie algebra of $C_H(t)$ by $\mathfrak{c}_H(t)$ so that

$$\mathfrak{g}(t) = \mathfrak{c}_H(t).$$

We note that the codomain of Ψ is $\mathfrak{G}(t)$.

Let $\mathbf{B}^t$ denote the variety of all Borel subgroups of G which contain t. Let $B \in \mathbf{B}^t$. Then $B \cap \mathfrak{G}(t)$ is a Borel subgroup of $\mathfrak{G}(t)$.

The proof in [4, p.471] depends on the fact that $\mathfrak{G}(t)$ is connected, and also on a triangular decomposition of Lie($\mathfrak{G}(t)$):

$$\mathrm{Lie}\,\mathfrak{G}(t) = \mathfrak{n}^t \oplus \mathfrak{t} \oplus \mathfrak{n}^t_-$$

from which it follows that $\mathrm{Lie}\,B \cap \mathrm{Lie}\,\mathfrak{G}(t) = \mathfrak{n}^t \oplus \mathfrak{t}$ is a Borel subalgebra in $\mathrm{Lie}\,\mathfrak{G}(t)$. The superscript "$t$" stands for the centralizer of t.

There is a canonical map

$$\mathbf{B}^t \to \operatorname{Flag}\mathfrak{G}(t), \quad B \mapsto B \cap \mathfrak{G}(t) \tag{17}$$

Now $\mathfrak{G}(t)$ acts by conjugation on $\mathbf{B}^t$. We have

$$\mathbf{B}^t = \mathbf{B}_1 \sqcup \mathbf{B}_2 \sqcup \cdots \sqcup \mathbf{B}_m \tag{18}$$

a disjoint union of $\mathfrak{G}(t)$-orbits, see [4, Prop. 8.8.7]. These orbits are the connected components of $\mathbf{B}^t$, and the irreducible components of the projective variety $\mathbf{B}^t$. The above map (17), restricted to any one of these orbits, is a bijection from the $\mathfrak{G}(t)$-orbit onto $\operatorname{Flag}\mathfrak{G}(t)$ and is $\mathfrak{G}(t)$-equivariant. It is then clear that

$$\mathbf{B}_j^{\Upsilon} \simeq \operatorname{Flag}\mathfrak{G}(t)^{\Psi}$$

for each $1 \le j \le m$. We also have $t \in S_{\Upsilon} = S_{\Psi}$. Now

$$\mathbf{B}^{\Upsilon} = (\mathbf{B}^t)^{\Upsilon} = (\mathbf{B}^t)^{\Psi}$$

and then

$$H_*(\mathbf{B}^{\Upsilon}, \mathbb{C}) = H_*(\mathbf{B}_1^{\Psi}, \mathbb{C}) \oplus \cdots \oplus H_*(\mathbf{B}_m^{\Psi}, \mathbb{C})$$

a direct sum of *equivalent* A-modules. Hence ϱ occurs in $H_*(\mathbf{B}^{\Upsilon}, \mathbb{C})$ if and only if it occurs $H_*(\mathbf{B}^{\Psi}, \mathbb{C})$. □

Recall that x is a nilpotent element in $\mathfrak{c}(t)$ (the Lie algebra of $\mathfrak{G}(t)$). Define

$$A^+ := \pi_0(Z_{C_H(t)}(x)).$$

Lemma 4.12. *We have*

$$\mathcal{R}(A, \mathbf{B}^{\Upsilon}) = \mathcal{R}(A^+, \mathbf{B}^{\Upsilon}).$$

Proof. Choose an isogeny $\iota\colon \widetilde{H} \to H$ with $\widetilde{H}_{\mathrm{der}}$ simply connected (as in [16, Theorem 3.5.4]) such that $H = \widetilde{H}/Z$ where Z is a finite subgroup of the centre of $\widetilde{H}$ (see [16, § 3]). Let $\tilde{t}$ be a lift of t in $\widetilde{H}$, that is, $\iota(\tilde{t}) = t$. Then we have (see [16, § 3.1]):

$$\iota(C_{\widetilde{H}}(\tilde{t})) = C_H^0(t) = \mathfrak{G}(t). \tag{19}$$

Let $u := \exp(x)$, a unipotent element in $\mathfrak{G}(t)$. It follows from Eqn. (19) that there exists $\tilde{u} \in C_{\widetilde{H}}(\tilde{t})$ such that $u = \iota(\tilde{u})$. Recall that $A = \pi_0(Z_{\mathfrak{G}(t)}(x))$. Then

$$A \simeq \pi_0(Z_{\mathfrak{G}(t)}(u)) = \pi_0(Z_{\iota(C_{\widetilde{H}}(\tilde{t}))}(\iota(\tilde{u}))) \simeq \pi_0(Z_{C_{\widetilde{H}}}(\tilde{t}, \tilde{u})),$$

and A is a subgroup of $\pi_0(Z_{C_H(t)}(u)) \simeq A^+$ (see [16, § 3.2–3.3]).

Recall from [16, Lemma 3.5.3] that

$$(\tilde{t}, \tilde{u}, \varrho, \psi) \mapsto (t, u, \rho)$$

induces a bijection between G-conjugacy classes of quadruples $(\tilde{t}, \tilde{u}, \varrho, \psi)$ and G-conjugacy classes of triples (t, u, ρ), where $\rho \in \mathcal{R}(A^+, \mathbf{B}^\Upsilon)$ is such that the restriction of ρ to A contains ϱ. □

Lemma 4.13. *We have*

$$\mathcal{R}(A^+, \mathbf{B}^\Upsilon) = \mathcal{R}(A^+, \mathbf{B}^\Phi).$$

Proof. It follows from [16, Lemma 4.4.1]. □

The proof can be reversed. Here is the reason for this claim: Lemmas 4.5, 4.6, 4.8 4.10 – 4.13 are all equalities, and Lemma 4.9 is a bijection.

This creates a canonical bijection between the extended quotient of the second kind $(T//W^{\mathfrak{s}})_2$ and $\mathfrak{Q}(G)_{\widehat{\lambda}}$:

$$\mu\colon (T//W^{\mathfrak{s}})_2 \longrightarrow \mathfrak{Q}(G)_{\widehat{\lambda}}, \qquad (t, x, \varrho, \psi) \mapsto (\Phi, \rho). \tag{20}$$

This in turn creates a bijection

$$T//W^{\mathfrak{s}} \longrightarrow \mathfrak{Q}(G)_{\widehat{\lambda}}. \tag{21}$$

This bijection is not canonical in general, depending as it does on a choice of bijection between the set of conjugacy classes in $W_H(t)$ and the set of irreducible characters of $W_H(t)$. When $G = \mathrm{GL}(n)$, the finite group $W_H(t)$ is a product of symmetric groups: in this case there is a canonical bijection between the set of conjugacy classes in $W_H(t)$ and the set of irreducible characters of $W_H(t)$, by the classical theory of Young tableaux.

To close this section, we will consider the case of $\mathrm{GL}(n, F)$), and the Iwahori point $\mathfrak{i}$ in the Bernstein spectrum of $\mathrm{GL}(n, F)$. The Langlands dual of $\mathrm{GL}(n, F)$ is $\mathrm{GL}(n, \mathrm{C})$, and we will take T to be the standard maximal torus in $\mathrm{GL}(n, \mathrm{C})$. The Weyl group is the symmetric group S_n. We will denote our bijection, in this case canonical, as follows:

$$\mu_F^{\mathfrak{i}} : T//W \to \mathfrak{P}(\mathrm{GL}(n, F))$$

Let E/F be a finite Galois extension of the local field F. According to [12, Theorem 4.3], we have a commutative diagram

$$\begin{array}{ccc} T//W & \xrightarrow{\mu_F^{\mathfrak{i}}} & \mathfrak{P}(\mathrm{GL}(n, F)) \\ \big\downarrow & & \big\downarrow \mathrm{BC}_{E/F} \\ T//W & \xrightarrow{\mu_E^{\mathfrak{i}}} & \mathfrak{P}(\mathrm{GL}(n, E)) \end{array}$$

In this diagram, the right vertical map $\mathrm{BC}_{E/F}$ is the standard base change map sending one Reeder parameter to another as follows:

$$(\Phi, 1) \mapsto (\Phi_{|W_E}, 1).$$

Let

$$f = f(E, F)$$

denote the residue degree of the extension E/F. We proceed to describe the left vertical map. We note that the action of W on T is as automorphisms of the algebraic group T. Since T is a group, the map

$$T \to T, \quad t \mapsto t^f$$

is well-defined for any positive integer f. The map

$$\widetilde{T} \to \widetilde{T}, \quad (t, w) \mapsto (t^f, w)$$

is also well-defined, since

$$w \cdot t^f = wt^f w^{-1} = wtw^{-1}wtw^{-1} \cdots wtw^{-1} = t^f.$$

Since

$$\alpha \cdot (t^f) = (\alpha \cdot t)^f$$

for all $\alpha \in W$, this induces a map

$$T//W \to T//W$$

which is an endomorphism (as algebraic variety) of the extended quotient $T//W$. We shall refer to this endomorphism as the *base change endomorphism of degree f*. The left vertical map is the base change endomorphism of degree f, according to [12, Theorem 4.3]. That is, our bijection $\mu^{\mathfrak{i}}$ is compatible with base change for $\mathrm{GL}(n)$.

When we restrict our base change endomorphism from the extended quotient $T//W$ to the ordinary quotient T/W, we see that the commutative diagram containing $\mathrm{BC}_{E/F}$ is consistent with [5, Lemma 4.2.1].

5 Interpolation

We will now provide details for the interpolation procedure described in §1. We will focus on the Iwahori point $\mathfrak{i} \in \mathfrak{B}(\mathcal{G})$, *i.e.,* on the smooth irreducible representations of $\mathcal{G}$ which admit nonzero Iwahori fixed vectors. To simplify notation, we will write $\mu = \mu^{\mathfrak{i}}$. Let $\mathfrak{P}(G)$ denote the set of conjugacy classes in G of Kazhdan-Lusztig parameters. For each $s \in \mathbb{C}^\times$, we construct a commutative diagram:

$$\begin{array}{ccc} T/\!/W & \xrightarrow{\mu} & \mathfrak{P}(G) \\ {\scriptstyle \pi_s}\downarrow & & \downarrow{\scriptstyle i_s} \\ T/W & = & T/W \end{array}$$

in which the map μ is bijective. In the top row of this diagram, the set $T/\!/W$, the set $\mathfrak{P}(G)$, and the map μ are independent of the parameter s.

We start by defining the vertical maps i_s, π_s in the diagram. Let $s \in \mathbb{C}^\times$. We will define

$$i_s : \mathfrak{P}(G) \to T/W, \quad (\Phi, \rho) \mapsto \Phi(\mathrm{Frob}, T_s) \tag{22}$$

$$\pi_s : T/\!/W \to T/W, \quad (t, w) \mapsto t \cdot \gamma(T_s) \tag{23}$$

where (Φ, ρ) is a Reeder parameter, and $(t, w) \in T/\!/W$. We note that

$$\Phi(\mathrm{Frob}, T_s) = t \cdot \gamma(T_s)$$

so that the diagram is commutative.

• Let $s = 1$, and assume, for the moment, that $C_H(t)$ is connected. The map μ in Theorem 4.1 sends (t, τ) to (Φ, ρ). We note that

$$t = \Phi(\mathrm{Frob}, T_1) = \Phi(\mathrm{Frob}, 1).$$

The map μ determines the map

$$(t, \tau) \mapsto (t, \Phi(1, u_0), \rho)$$

which, in turn, determines the map

$$\tau \mapsto (\exp(x), \rho)$$

which is the Springer correspondence for the Weyl group $W_H(t)$.

• Now let $s = \sqrt{q}$ where q is the cardinality of the residue field k_F of F. We now link our result to the representation theory of the p-adic group $\mathcal{G}$ as follows. As in §3, let

$$\sigma := \Phi(\mathrm{Frob}, T_{\sqrt{q}}), \qquad u := \Phi(1, u_0).$$

Then we have

$$\sigma u \sigma^{-1} = u^q$$

and the triple (σ, u, ρ) is a Kazhdan-Lusztig triple.

The correspondence $\sigma \mapsto \chi_\sigma$ between points in T and unramified quasicharacters of $\mathcal{T}$ can be fixed by the relation

$$\chi_\sigma(\lambda(\varpi_F)) = \lambda(\sigma)$$

where ϖ_F is a uniformizer in F, and $\lambda \in X_*(\mathcal{T}) = X^*(T)$. The Kazhdan-Lusztig triples (σ, u, ρ) parametrize the irreducible constituents of the (unitarily) induced representation

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_\sigma \otimes 1).$$

Note that

$$i_{\sqrt{q}} : (\Phi, \rho) \mapsto \sigma$$

so that $i_{\sqrt{q}}$ is the *infinitesimal character*. The infinitesimal character is denoted **Sc** in [15, VI.7.1.1] (**Sc** for *support cuspidal*)

Since μ is bijective and the diagram is commutative, the number of points in the fibre of the q-projection $\pi_{\sqrt{q}}$ equals the number of inequivalent irreducible constituents of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_\sigma \otimes 1)$:

$$|\pi_{\sqrt{q}}^{-1}(\sigma)| = |\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi_\sigma \otimes 1)| \tag{24}$$

The q-projection $\pi_{\sqrt{q}}$ is a model of the infinitesimal character **Sc**.

Our formulation leads to Eqn.(24), which appears to have some predictive power. Note that the definition of the q-projection $\pi_{\sqrt{q}}$ depends only on the L-parameter Φ. An L-parameter determines an L-packet, and does not determine the number of irreducible constituents of the L-packet.

6 Examples

Example 1. *Realization of the ordinary quotient T/W*. Consider an L-parameter Φ for which $\Phi|_{\mathrm{SL}(2,\mathbb{C})} = 1$. Let $t = \Phi(\mathrm{Frob})$. Then

$$G_\Phi := C(im\,\Phi) = C(t)$$

so that G_Φ is connected and acts trivially in homology. Therefore ρ is the unit representation 1.

Now t is a semisimple element in G, and all such semisimple elements arise. Modulo conjugacy in G, the set of such L-parameters Φ is parametrized by the quotient T/W. Explicitly, let

$$\mathfrak{P}_1(G) := \{\Phi \in \mathfrak{P}(G) : \Phi|_{\mathrm{SL}(2,\mathbb{C})} = 1\}.$$

Then we have a canonical bijection

$$\mathfrak{P}_1(G) \to T/W, \qquad (\Phi, 1) \mapsto \Phi(\text{Frob}, 1)$$

which fits into the commutative diagram

$$\begin{array}{ccc} \mathfrak{P}_1(G) & \longrightarrow & T/W \\ \downarrow & & \downarrow \\ \mathfrak{P}(G) & \longrightarrow & T//W \end{array}$$

where the vertical maps are inclusions.

Example 2. *The general linear group.* Let $\mathcal{G} = \text{GL}(n), G = \text{GL}(n, \mathbb{C})$. Let

$$\Phi = \chi \otimes \tau(n)$$

where χ is an unramified quasicharacter of $\mathcal{W}_F$ and $\tau(n)$ is the irreducible n-dimensional representation of $\text{SL}(2, \mathbb{C})$. By local classfield theory, the quasicharacter χ factors through $F^\times$. In the local Langlands correspondence for $\text{GL}(n)$, the image of Φ is the unramified twist $\chi \circ \det$ of the Steinberg representation $\text{St}(n)$.

The sign representation *sgn* of the Weyl group W has Springer parameters $(\mathcal{O}_{prin}, 1)$, where $\mathcal{O}_{prin}$ is the principal orbit in $\mathfrak{gl}(n, \mathbb{C})$. In the *canonical* correspondence between irreducible representations of S_n and conjugacy classes in S_n, the trivial representation of W corresponds to the conjugacy class containing the n-cycle $w_0 = (123 \cdots n)$.

Now $G_\Phi = C(im\, \Phi)$ is connected [4, §3.6.3], and so acts trivially in homology. Therefore ρ is the unit representation 1. The image $\Phi(1, u_0)$ is a regular nilpotent, i.e. a nilpotent with one Jordan block (given by the partition of n with one part). The corresponding conjugacy class in W is $\{w_0\}$. The corresponding irreducible component of the extended quotient is

$$T^{w_0}/Z(w_0) = \{(z, z, \ldots, z) : z \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times.$$

This is our model, in the extended quotient picture, of the complex 1-torus of all unramified twists of the Steinberg representation $\text{St}(n)$. The map from L-parameters to pairs $(w, t) \in T//W$ is given by

$$\chi \otimes \tau(n) \mapsto (w_0, \chi(\text{Frob}), \ldots, \chi(\text{Frob})).$$

Among these representations, there is one real tempered representation, namely $\text{St}(n)$, with L-parameter $1 \otimes \tau(n)$, attached to the principal orbit $\mathcal{O}_{prin} \subset G$.

More generally, let

$$\Phi = \chi_1 \otimes \tau(n_1) \oplus \cdots \oplus \chi_k \otimes \tau(n_k)$$

where $n_1+\cdots+n_k = n$ is a partition of n. This determines the unipotent orbit $\mathcal{O}(n_1, \ldots, n_k) \subset G$. There is a conjugacy class in W attached canonically to this orbit: it contains the product of disjoint cycles of lengths $n_1, \ldots, n_k$. The fixed set is a complex torus, and the component in $T//W$ is a product of symmetric products of complex 1- tori.

EXAMPLE 3. *The exceptional group of type* G_2. This example contains a Reeder parameter (Φ, ρ) with $\rho \neq 1$. The torus $\mathcal{T}$ is identified with $F^\times \times F^\times$. We take $\lambda = \chi \otimes \chi$ where χ is a nontrivial quadratic character of $\mathfrak{o}_F^\times$.

Here we have $H = \mathrm{SO}(4, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. This complex reductive Lie group is neither simply-connected nor of adjoint type. We have $W^{\mathfrak{s}} = W_H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We will write

$$\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow H^{\mathfrak{s}}, \quad (x, y) \mapsto [x, y],$$

$$T_{s,s'} = [T_s, T_{s'}], \quad s, s' \in \mathbb{C}^\times.$$

We have

$$\mathfrak{Q}(G)_{\hat{\lambda}} \to T//W_H \simeq \mathbb{A}^1 \sqcup \mathbb{A}^1 \sqcup pt_1 \sqcup pt_2 \sqcup pt_* \sqcup T/W_H,$$

where

- one $\mathbb{A}^1$ corresponds to $(\Phi, 1)$ with $\Phi(\mathrm{Frob}, 1) = [I, T_s]$ and $\Phi(1, u_0) = [u_0, I]$,
- the other $\mathbb{A}^1$ corresponds to $(\Phi, 1)$ with $\Phi(\mathrm{Frob}, 1) = [T_s, I]$ and $\Phi(1, u_0) = [I, u_0]$,
- pt_1 corresponds to $(\Phi, 1)$ with $\Phi(\mathrm{Frob}, 1) = T_{1,1}$ and $\Phi(1, u_0) = [u_0, u_0]$,
- pt_2 corresponds to $(\Phi, 1)$ with $\Phi(\mathrm{Frob}, 1) = T_{1,-1}$ and $\Phi(1, u_0) = [u_0, u_0]$,
- T/W_H corresponds to $(\Phi, 1)$ with $\Phi(\mathrm{Frob}, 1) = T_{s,s'}$ $s, s' \in \mathbb{C}^\times$, and $\Phi(1, u_0) = [I, I]$,
- pt_* corresponds to (Φ, sgn) with $\Phi(\mathrm{Frob}, 1) = T_{i,i}$, $i = \sqrt{-1}$ and $\Phi(1, u_0) = [I, I]$.

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Anne-Marie Aubert, Institut de Mathématiques de Jussieu, U.M.R. 7586 du C.N.R.S., Paris, France
Email: aubert@math.jussieu.fr
Paul Baum, Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA
Email: baum@math.psu.edu
Roger Plymen, School of Mathematics, Alan Turing building, Manchester University, Manchester M13 9PL, England
Email: plymen@manchester.ac.uk