Hysteretic regime switching diffusions and resource extraction

Moriarty, John and Evatt, Geoffrey W. and Johnson, Paul V. and Duck, Peter W.

2011

MIMS EPrint: **2011.19**

ISSN 1749-9097

Hysteretic regime switching diffusions and resource extraction

John Moriarty, Geoffrey W. Evatt, Paul V. Johnson, Peter W. Duck
School of Mathematics, Univeristy of Manchester, Oxford Road, Manchester UK, M13 9PL

We calculate the probability that an extraction project will be abandoned, directly from a real options model closely related the seminal work of Brennan and Schwartz (1985). We assume that the resource is extracted at two alternative rates, with a capital cost for switching, and with an option to abandon due to unsatisfactory market prices. The abandonment probability is expressed as a hitting probability for a regime switching diffusion with hysteresis, which is shown to be the unique solution of a system of coupled boundary value problems. Our work lends itself to use as a quantitative and easily interpreted measure of risk in the planning of extraction projects. Numerical results show that the abandonment probability may be non-monotone with respect to the volatility of the price process, in contrast with project valuations. In the one-dimensional stationary case, the stochastic process is a hysteretic system with noise in the sense of Freidlin et al. (2000), and we obtain a closed-form expression for the hitting or abandonment probability in this case.

Key words: Partial Differential Equations, Regime Switching Diffusion, Hysteresis, Hitting Probability.
History:

1. Introduction

An understanding of patterns in the abandonment of resource extraction projects lends itself to use in a number of related contexts. As noted in Brennan and Schwartz (1985), it is relevant to policymakers concerned with the social costs of layoffs in cyclical industries, and with policies to avert them; similar considerations apply to environmental costs. Abandonment probability is also an easily interpreted measure of project risk. Although the net present value (NPV) of an extraction project is influenced by the possibility of abandonment, the external considerations indicated above are not in general reflected in the NPV. Moel and Tufano (2002) present an empirical study of mine closings among American gold mines over a ten-year period, and conclude that most of the predictions of the real options model of Brennan and Schwartz (1985) are borne out in the observed pattern of closings and reopenings. In the present paper we derive the abandonment probability for a resource extraction project directly from a real options model. These values provide a set of predictions for use in the planning of resource extraction projects.

Since the lifetime of a resource extraction project, such as a mine, may be of the order of decades, the output price of the commodity extracted can vary significantly. In the model of Brennan and Schwartz (1985), a mine operator responds to fluctuations in output price by controlling the extraction regime. By incurring a capital cost, the operator may switch between three available states for the mine: open, closed, or even abandoned if the output price is sufficiently low. The closed state incurs fixed maintenance costs, but may be reopened; abandonment is permanent, but is assumed to incur no further costs.

In the present paper we assume that, when the mine is open, the extraction rate may be varied between a finite number of admissible states via capital investment, which may be reversed at cost. One of these rates may be set equal to zero, in order to represent closure, and there is also an option to abandon the extraction project. Both the resource and the lifetime of the project are assumed to be finite, and so the economic system is three-dimensional: output price S, remaining resource size Q, and time t. There is a growing literature on optimal control strategies for reversible investment (see, for example, Merhi and Zervos (2007) and references therein). We assume that a control strategy has been chosen, and make the further (weak) assumption that it is of threshold type.

By considering a hitting time for an Itô diffusion, and exploiting the link with partial differential equations (PDEs), Evatt et al. (2011) obtain the probability that a mine will be abandoned under a simplified real options model, in which the mine either operates at a fixed rate or is abandoned. Our contribution in this paper is to obtain the abandonment probability under a variable extraction rate model. For simplicity of exposition, we assume just two alternative extraction rates (higher and lower); the methodology extends in a straightforward manner to any finite number of rates.

We aim to demonstrate that the probability of abandonment gives an additional perspective to that provided by NPVs and financial risks alone.

As argued in Dixit and Pindyck (1994), switching between operational regimes exhibits hysteresis due to the capital cost of switching. Assuming that the economic system is a stochastic process, we are led to consider a regime switching diffusion which switches upon crossing between overlapping subsets of its state space. In this paper we call such a stochastic process a *hysteretic* diffusion; an example arises in the analysis by Brekke and Øksendal (1994). Abandonment of the mine corresponds to the hitting of the abandonment threshold by the hysteretic diffusion. We establish that the abandonment probability is the unique solution to a system of boundary value problems on the overlapping domains. If the state space is one dimensional, as in the stationary case solved in section 3 below, the stochastic process is a diffusion on a graph in the sense of Freidlin et al. (2000). A review of recent work in the area of hysteretic nonlinearities driven by diffusion processes is presented in Mayergoyz and Dimian (2005). In Chen and Forsyth (2010) and references therein, regime switching diffusions with Markovian switching are used to model economic cycles; an alternative approach is to use mean reverting stochastic processes with jumps, as in Thompson et al. (2004). Such regime switching models differ from that in the present paper, where the process of regimes is non-Markovian. It resembles the model of Flood and Garber (1983), in that switching is driven by the endogenous variable of interest, rather than by a hidden process; however, in the present paper we allow the regime to switch repeatedly, rather than just once.

The remainder of the paper is structured as follows: in section 2 we obtain the coupled system of Dirichlet problems, and prove the necessary uniqueness theorem. In section 3 we derive the hitting probability in closed form, in the one-dimensional stationary case. We apply the methodology to mining operations in section 4, and give explicitly the general system of PDEs which must be solved. In section 5 we present numerical solutions for a particular gold mining operation. Our conclusions are presented in section 6.

2. Main results

Itô diffusions are solutions of stochastic differential equations in $\mathbf{R}^d$ of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{1}$$

where $(B_t)_{t\geq 0}$ is a standard Wiener process in $\mathbf{R}^d$, and the instantaneous drift and volatility are given by $\mu(X_t) \in \mathbb{R}^d$ and $\sigma(X_t) \in \mathbb{R}^{d\times d}$ respectively. Itô diffusions are Markovian; however, if there is a process $i = (i_t)_{t\geq 0}$ taking values in $\{0,1\}$ and

$$dX_t = \mu(i_t, X_t)dt + \sigma(i_t, X_t)dB_t \tag{2}$$

then the solution X is a *regime switching* diffusion (and we may call i the *regime process*), and is in general not Markovian. In this paper we consider a regime process constructed from the random sequence of times at which X crosses between two overlapping domains, calling the jointly Markovian process $(X_t, i_t)_{t\geq 0}$ a *hysteretic* diffusion.

In order to state our main result we introduce some notation. Let two domains $L_0, L_1 \subset \mathbb{R}^d$ have nonempty intersection and let the process $(\nu_t)_{t\geq 0}$ record the sequence of times at which the hysteretic diffusion X crosses from one domain to the other:

$$\nu_0 = 0 \tag{3}$$

$$\nu_{2k+1} = \inf\{t > \nu_{2k} : X_t \notin L_0\} \tag{4}$$

$$\nu_{2k+2} = \inf\{t > \nu_{2k+1} : X_t \notin L_1\} \tag{5}$$

These exit times may be used to define the hysteretic regime process $i = (i_t)_{t\geq 0}$ by

$$i_t = \begin{cases} 0, \ \nu_{2k} \leq t < \nu_{2k+1}, \\ 1, \ \nu_{2k+1} \leq t < \nu_{2k+2}. \end{cases} \tag{6}$$

Finally, let $E_{x,k}$ denote the expectation operator associated with the law of $(X_t, i_t)_{t\geq 0}$ when $(X_0, i_0) = (x, k)$ almost surely.

PROPOSITION 1. *Assume that ν, the first exit time of X from $L = L_0 \cup L_1$, is almost surely finite.*

a) Let $X = (X_t)_{t\geq 0}$ be a strong solution of (2)-(6). Then provided L_0, L_1 and f are sufficiently regular, the function

$$u(x,k) = E_{x,k}\left[f(X_\nu)\right] \tag{7}$$

satisfies the coupled boundary value problems

$$k = 0,1$$

$$u_k(x) := u(x,k) \in C^2(L_k)$$

$$\mathcal{L}_x^{(k)} u(x,k) = 0 \textit{ for } x \in L_k,$$

$$u(x,k) \to f(y) \qquad \textit{as } x \in L_k \to y \in \partial L,$$

$$u(x,k) \to u(y,1-k) \quad \text{as} \quad x \in L_k \to y \in \partial L_k \cap L_{1-k} \tag{8}$$

where

$$\mathcal{L}_x^{(k)} \equiv \sum_{i,j=1}^{n} a_{ij}^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{(k)} \frac{\partial}{\partial x_i} \tag{9}$$

and $[b_i^{(k)}] = \mu(k,x)$, $[a_{ij}^{(k)}] = \frac{1}{2}\sigma(k,x)\sigma(k,x)'$.

b) If a bounded solution u exists to equations (8), then it is given by (7).

Proof. For necessary background on stochastic differential equations we refer the reader to Øksendal (2003). To establish part a), consider the case when $k = 0$. By conditioning on X_{ν_1} and the strong Markov property, we obtain

$$u(x,0) = E_{x,0}[E_{x,0}\left[f(X_\nu)|X_{\nu_1}\right]] \tag{10}$$

$$= E_{x,0}[u(X_{\nu_1},1)] \tag{11}$$

and by the existence theorem for stochastic solutions of the Dirichlet problem (Theorem 9.2.14 of Øksendal (2003)) and (7), we conclude that equations (8)-(9) hold for $k = 0$. The case $k = 1$ is proved similarly.

For part b), it is sufficient to observe that (X, i) is a Feller process on the metric space

$$\Gamma = (\{(x,k) : x \in L_k\}, d((x,k),(y,j)) = |x-y|)$$

equipped with Lebesgue measure. A standard uniqueness proof for stochastic solutions of the Dirichlet problem (Corollary 9.1.2 of Øksendal (2003)) then gives the required result.

3. Hitting probabilities for one-dimensional hysteretic diffusions

From Proposition 1 we may obtain analytic expressions for hitting probabilities for one-dimensional hysteretic diffusions in the stationary (time-independent) case. This is related to Freidlin et al. (2000), where a hysteretic system with noise is modeled by composing a rectangular loop operator γ with a one-dimensional Itô diffusion X, and the joint process $(X, \gamma(X))$ is considered as a diffusion on a graph. As an example, for constants $a < l < u$ let

$$L_0 = (a, u),\; L_1 = (l, \infty) \tag{12}$$

(here the level a represents 'abandonment'; u and l are respectively the upper and lower boundaries used to define the regime process). For $k \in \{0, 1\}$ define the hitting probability $P^k(x)$ as follows. Given that X is initiated at $x \in L_k$ and the regime process i is initiated at k, we denote by $P^k(x)$ the probability that X_t decreases to the level a in finite time — that is, that $X_t = a$ for some $t \in (0, \infty)$.We obtain from Proposition 1 the coupled boundary value problems

$$\frac{1}{2}\sigma^2(0,x)\frac{d^2P^1}{dx^2} + \mu(0,x)\frac{dP^1}{dx} = 0 \qquad \text{on } L_0, \tag{13}$$

$$P^1(x) \to 1 \text{ as } x \to a, \quad P^1(x) \to P^2(u) \text{ as } x \to u$$

and

$$\frac{1}{2}\sigma^2(1,x)\frac{d^2P^2}{dx^2} + \mu(1,x)\frac{dP^2}{dx} = 0 \qquad \text{on } L_1, \tag{14}$$

$$P^2(x) \to 0 \text{ as } x \to \infty, \quad P^2(x) \to P^1(l) \text{ as } x \to l.$$

Since both (13) and (14) are linear ordinary differential equations, defining

$$K_i(\epsilon) = \int^{\epsilon} \frac{2\mu(i,\xi)}{\sigma^2(i,\xi)} d\xi, \quad f_i(x) = \int^{x} e^{-K_i(\epsilon)} d\epsilon$$

for $i = 1, 2$, we have

$$P^1(x) \;=\; A_1 + B_1 f_1(x) \;\; \text{for } x \in (e, u),$$

$$P^2(x) = B_2 f_2(x) \quad \text{for } x \in (l, \infty) \tag{15}$$

where

$$\begin{aligned} A_1 &= 1 - B_1 f_1(e) \\ B_1 &= \frac{B_2 f_2(u) - 1}{f_1(u) - f_1(e)} \\ B_2 &= \frac{f_1(l) - f_1(u)}{f_2(u)[f_1(l) - f_1(e)] - f_2(l)[f_1(u) - f_1(e)]} \end{aligned} \tag{16}$$

This solution confirms that, in general, $P^1(x) \neq P^2(x)$ for $x \in (l, u)$ due to hysteresis.

4. Application to the planning of mines

In this section we solve a problem with application to the planning of mines, using the methodology presented in Section 2. The problem is to calculate the probability that the extraction of a finite resource will be abandoned before its successful completion. The economic system is modeled by a hysteretic diffusion $X_t = (t, S_t, Q_t)$ as in (2)-(6), where the variable t represents time, S_t the output price, and Q_t the quantity of resource remaining. The regime process governs the extraction rate, so that the domains L_0, L_1 in (4)-(5) correspond to lower and higher extraction rate respectively. These two regimes, which correspond to the values $k = 0$ and $k = 1$ in equations (8), are therefore distinguished only by their differing extraction rates

$$dQ_t = -q_k dt \text{ when } i_t = k, \tag{17}$$

with $0 \leq q_0 < q_1$.

Note that the dynamics (17) of the natural resource may also be chosen to be nondeterministic; for an example in the exploitation of fish stocks, see Nøstbakken (2006). As in Brennan and Schwartz (1985) and Brekke and Øksendal (1994), we assume a lognormal process for S,

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{18}$$

although any other Itô diffusion could equally be chosen. We suppose that the mine's operation expires at the fixed time T and that τ time units currently remain before expiry. We write $x =$

(t,s,q) for the current state $X_{T-\tau}$ of the system, and $(X_t)_{t\in(T-\tau,T)}$ for its evolution from the current time until expiry. Let $A\subset L$ be the abandonment threshold in our chosen control strategy. Setting

$$f(y)=\begin{cases}1 & \text{if } y\in A,\\ 0 & \text{otherwise,}\end{cases}$$

Proposition 1 gives the coupled boundary value problems

$$\frac{1}{2}\sigma^2 s^2\frac{\partial^2 P^k}{\partial s^2}-\frac{\partial P^k}{\partial \tau}-q_k\frac{\partial P^k}{\partial q}+\mu s\frac{\partial P^k}{\partial s}=0 \qquad \text{on } L_k,$$

$$P^k(x)=0 \quad \text{on} \quad q=0,$$

$$P^k(x)=0 \quad \text{when} \quad t=T,$$

$$P^k(x)\to 0 \quad \text{as} \quad s\to\infty,$$

$$P^k(x)=1 \quad \text{when} \quad x\in A,$$

$$P^k(x)=P^{1-k}(x) \quad \text{when} \quad x\in\partial L_k\cap L_{1-k}, \tag{19}$$

which we solve numerically in the next section. Since the associated PDEs have two advective terms (in q and τ), we calculate the results using the semi-Lagrangian finite difference technique. This scheme permits a solution which might exhibit jumps across the characteristic given by (17). For a detailed justification of this scheme see Chen and Forsyth (2007); related applications are given in Chen and Forsyth (2010) and Evatt et al. (2011). The main technical difference in our analysis is that two semi-Lagrangian schemes are now required, one for each of the two PDEs. Because of the coupling in (19), the numerical solution is iterated until convergence. The sets L_0, L_1 and A are calculated numerically from empirical data.

5. Results

We now apply our model to a gold mine with a maximum operating lifetime of $T=4.9$ years, which can process $q_0=1,000,000$ tonnes of ore-bearing material per year, which may be increased to $q_1=1,250,000$ tonnes. The initial output price of processed gold ore has been scaled to a reference price of \$1 gr^{-1}, and the parameters of the price dynamics in (18) are given by $\mu=0.5\%\ \text{yr}^{-1}$, and $\sigma=40\%\ \text{yr}^{-\frac{1}{2}}$, unless otherwise stated.

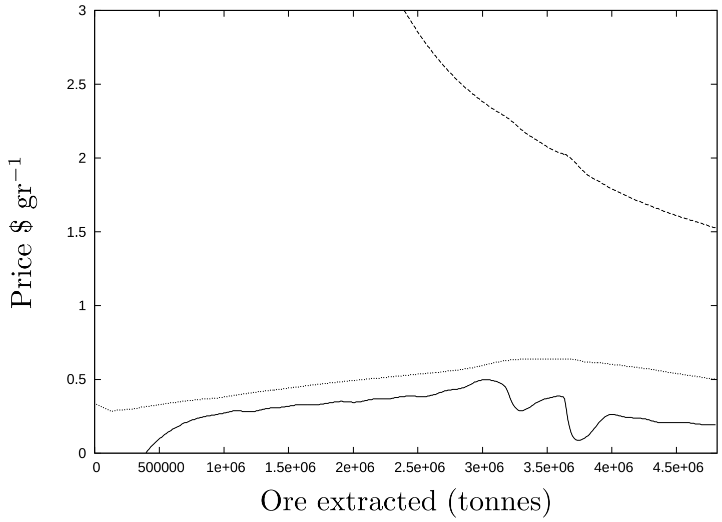

Figure 1 The optimal price thresholds for each management decision: expansion (top line, dashed); contraction (middle line, dotted); abandonment (bottom line, solid). These thresholds are used as input in Figures 2, 3 and 5.

Figure 1 shows the boundaries ∂L_0 (dashed line), ∂L_1 (dotted line) and ∂A (solid line), calculated from empirical ore-grade data as in Evatt et al. (2011). The ore-grade is non-monotonic over the course of the planned extraction schedule, and it is this feature, rather than numerical instability, that gives rise to the convoluted boundaries shown. To quantify the effect of current regime on abandonment probability, Figure 2 compares the probability of abandonment from the lower rate (solid line) and the higher rate (dashed line); the separation of the two solutions is caused by hysteresis. At the reference price, the probability of abandonment is around 15% less from the higher rate, compared with the lower rate.

To examine how the probability of abandonment varies as extraction proceeds, Figure 3 shows this relationship from current output price \$0.8 gr^{-1} (top pair of curves), \$1 gr^{-1}(middle pair), and \$1.2 gr^{-1} (bottom pair), each pair again representing the lower and higher current rate. Since abandonment risk reduces as the resource nears complete extraction, a higher initial extraction rate tends to reduce abandonment risk, and this is borne out uniformly in Figures 2 and 3.

In Figure 4 we quantify the effect of the real option to switch to the higher extraction rate. The abandonment probability is shown against current output price, for a mine without this option (dotted line), and a mine with this option, from the lower rate (solid line) and the higher rate

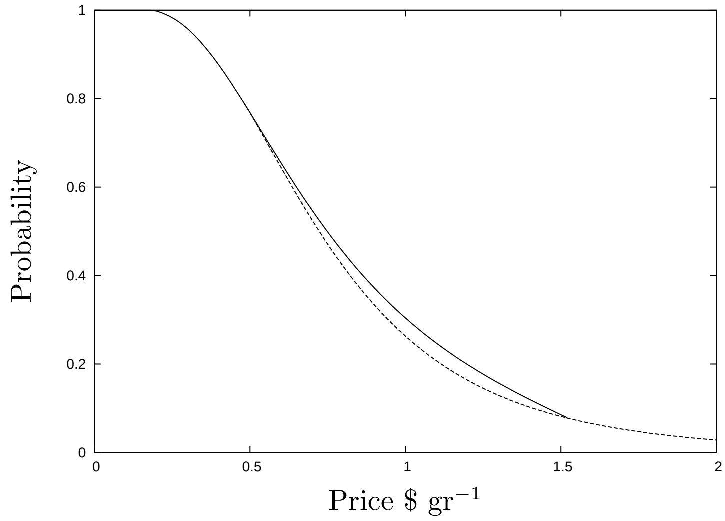

Figure 2 The probability of abandonment versus initial output price, from the lower rate (solid line) and higher rate (dashed line) of extraction.

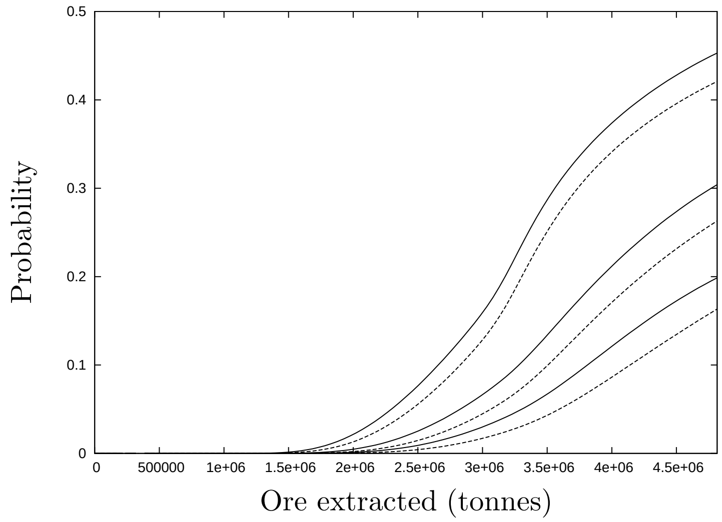

Figure 3 The probability of abandonment, versus the amount of ore extracted from the mine. The three pairs of curves show the abandonment probability from the lower level (solid line) and higher level (dashed line) of extraction, and from current output prices \$0.8 gr^{-1} (top pair of curves), \$1 gr^{-1}(middle pair), and \$1.2 gr^{-1} (bottom pair).

(dashed line). It can be seen that, in this example, the option to expand extraction as much as halves the abandonment probability, depending on the current output price.

To examine the sensitivity of the results to the percentage drift μ and volatility σ, we compare three sets of results for each parameter in Figure 5 (it should be noted that boundaries in Figure

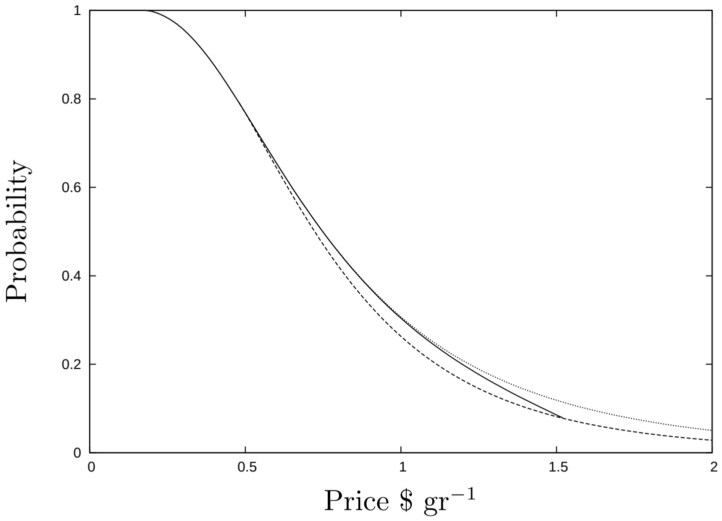

Figure 4 Abandonment probability versus current output price, for a mine without the option to extract at the higher rate (dotted line), and a mine with this option and currently extracting at the lower rate (solid line) and the higher rate (dashed line).

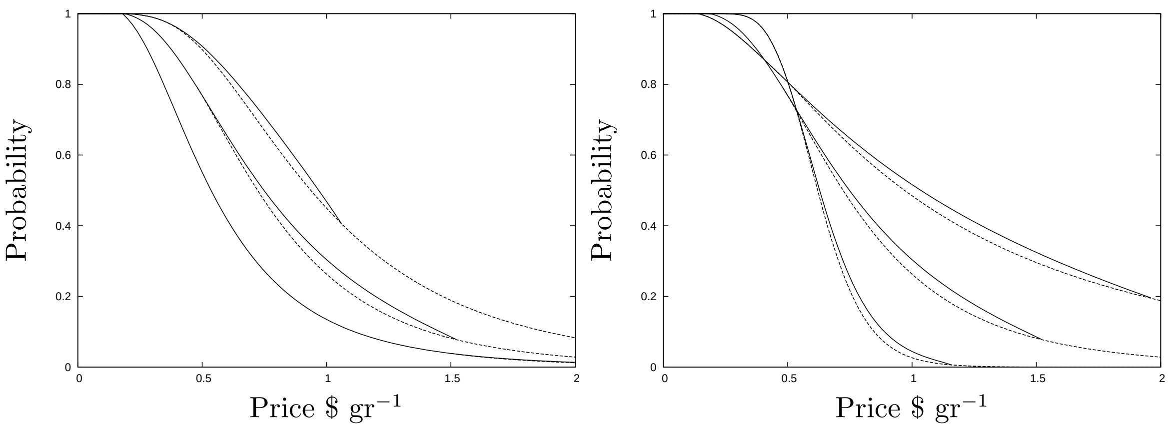

Figure 5 The probability of abandonment versus current output price, where the drift, μ and volatility, σ are varied, from both the lower extraction rate (dashed line) and the higher rate (solid line). In the left hand graph, from top to bottom μ is -10%, 0% and 10%. In the right hand graph, from top to bottom (at a price of \$ 1 gr^{-1}), σ is 60%, 40% and 20%.

1 are themselves functions of μ and σ). It is clear from Figure 5 and by construction that an increased value of μ decreases the probability of abandonment. However, there is no such monotonic relationship for the parameter σ. From large current gold prices, the probability of abandonment increases with σ: however, from lower current prices this relationship reverses, and the probability of abandonment decreases with higher σ.

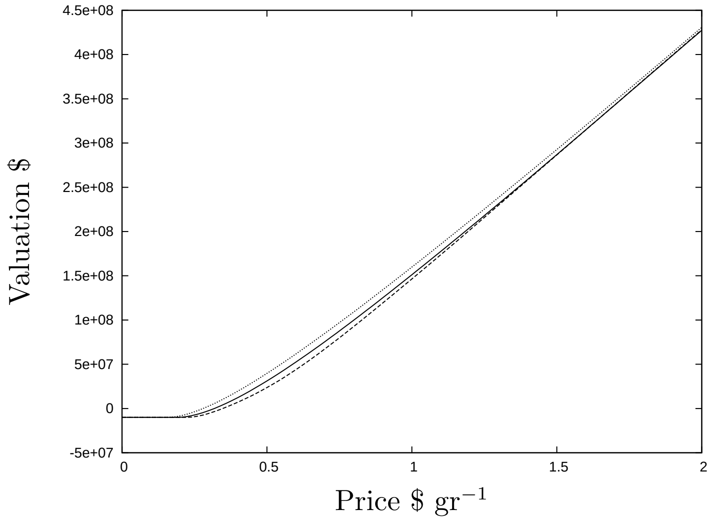

Figure 6 Mine valuation against current output price. Valuations are shown for the parameter values $\sigma = 60\%$ (dotted line), $\sigma = 40\%$ (solid line) and $\sigma = 20\%$ (dashed line).

Our final result is shown in Figure 6, which shows the mine's valuation for three different values of σ: 60% (dotted line), 40% (solid line) and 20% (dashed line). The method used to calculate the valuation is that of Evatt et al. (2011), using a Feynman-Kac type extension of Proposition 1 whose details we omit. Since the valuation increases monotonically with σ we see that, in this example, abandonment probability can have qualitatively different properties to the valuation, depending on the volatility of the output price.

6. Conclusions

The probability of abandonment is an easily interpreted measure of the risk inherent in a resource extraction project under uncertain future output price, and lends itself to applications in planning and policy. We have shown that, for projects with two available extraction rates and capital costs for switching, the abandonment probability can be calculated using coupled Dirichlet boundary value problems. In this way, the reduction in abandonment risk conferred by the real option to switch extraction rate may be quantified. We have demonstrated that the abandonment probability is not uniformly monotone with respect to the volatility of the output price. When viewed alongside valuations, the abandonment probability thus offers an additional perspective in real options analysis.

Acknowledgments

This project was aided by funding from the University of Manchester Knowledge Transfer Account and the SPRIng and CICADA projects, awarded by the Engineering and Physical Sciences Research Council, UK. The authors are grateful to Gemcom Software International who supplied the ore-grade data, and for valuable discussions on mining operations.

References

Brekke, K.A., B. Øksendal. 1994. Optimal switching in an economic activity under uncertainty. *SIAM Journal on Control and Optimization* **32**(4) 1021–1036. doi:10.1137/S0363012992229835. URL `http://link.aip.org/link/?SJC/32/1021/1`.

Brennan, M. J., E.S. Schwartz. 1985. Evaluating natural resource investments. *The Journal of Business* **58**(2) pp. 135–157. URL `http://www.jstor.org/stable/2352967`.

Chen, Z., P.A. Forsyth. 2007. A semi-lagrangian approach for natural gas storage valuation and optimal operation. *SIAM J. Sci. Comput.* **30** 339–368. doi:10.1137/060672911. URL `http://portal.acm.org/citation.cfm?id=1350656.1350671`.

Chen, Z., P.A. Forsyth. 2010. Implications of a regime-switching model on natural gas storage valuation and optimal operation. *Quantitative Finance* **10**(2) 159–176. URL `http://ideas.repec.org/a/taf/quantf/v10y2010i2p159-176.html`.

Dixit, A.K., R.S. Pindyck. 1994. *Investment under uncertainty*. Princeton University Press. URL `http://books.google.co.uk/books?id=i5o6RgAACAAJ`.

Evatt, G. W., P.V. Johnson, P.W. Duck, S.D. Howell, J. Moriarty. 2011. The expected lifetime of an extraction project. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **467** 244–263. doi:10.1098/rspa.2010.0247. URL `http://rspa.royalsocietypublishing.org/content/early/2010/07/22/rspa.2010.0247.abstract`.

Flood, R.P., P.M. Garber. 1983. A model of stochastic process switching. *Econometrica* **51**(3) pp. 537–551. URL `http://www.jstor.org/stable/1912144`.

Freidlin, M. I., I.D. Mayergoyz, R. Pfeiffer. 2000. Noise in hysteretic systems and stochastic processes on graphs. *Phys. Rev. E* **62**(2) 1850–1855. doi:10.1103/PhysRevE.62.1850.

Mayergoyz, I.D., M. Dimian. 2005. Stochastic aspects of hysteresis. *Journal of Physics: Conference Series* **22**(1) 139. URL `http://stacks.iop.org/1742-6596/22/i=1/a=009`.

Merhi, Amal, Mihail Zervos. 2007. A model for reversible investment capacity expansion. *SIAM Journal on Control and Optimization* **46**(3) 839–876. doi:10.1137/050640758. URL `http://link.aip.org/link/?SJC/46/839/1`.

Moel, A., P. Tufano. 2002. When Are Real Options Exercised? An Empirical Study of Mine Closings. *Review of Financial Studies* **15**(1) 35–64. doi:10.1093/rfs/15.1.35. URL `http://rfs.oxfordjournals.org/content/15/1/35.abstract`.

Nøstbakken, L. 2006. Regime switching in a fishery with stochastic stock and price. *Journal of Environmental Economics and Management* **51**(2) 231 – 241. doi:DOI:10.1016/j.jeem.2005.07.001. URL `http://www.sciencedirect.com/science/article/B6WJ6-4HC6KN8-1/2/fe06af441315ff42252a9c57c68ab280`.

Øksendal, B.K. 2003. *Stochastic differential equations: an introduction with applications*. Universitext (1979), Springer. URL `http://books.google.co.uk/books?id=kXw9hB4EEpUC`.

Thompson, M., M. Davison, H. Rasmussen. 2004. Valuation and optimal operation of electric power plants in competitive markets. *OPERATIONS RESEARCH* **52**(4) 546–562. doi:10.1287/opre.1040.0117. URL `http://or.journal.informs.org/cgi/content/abstract/52/4/546`.