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Abstract

For a group G and X a subset of G the commuting graph of G on X, denoted by $\mathcal{C}(G, X)$, is the graph whose vertex set is X with $x, y \in X$ joined by an edge if $x \neq y$ and x and y commute. If the elements in X are involutions, then $\mathcal{C}(G, X)$ is called a commuting involution graph. This paper studies $\mathcal{C}(G, X)$ when G is a 4-dimensional projective symplectic group and X a G-conjugacy class of involutions, determining the diameters and structure of the discs of these graphs.

1 Introduction

For G a group and X a subset of G, the commuting graph of G on X, $\mathcal{C}(G, X)$, is the graph whose vertex set is X with $x, y \in X$ joined whenever $x \neq y$ and $xy = yx$. In effect commuting graphs first appeared in the paper of Brauer and Fowler [13], famous for containing a proof that up to isomorphism only finitely many non-abelian simple groups can have a given centralizer of an involution. The commuting graphs considered in [13] had $X = G \setminus \{1\}$ - such graphs have played an important role in recent work related to the Margulis–Platanov conjecture (see [24]). The complement of this type of commuting graph appeared in [22] where B.H. Neumann solved a problem posed by Erdös. Various kinds of commuting graphs have been deployed in the study of finite groups, particularly the non-abelian simple groups. For example, the analysis and subsequent construction by Fischer [18] of the three simple Fischer groups used the commuting graph on the conjugacy class of 3-transpositions. While a computer-free uniqueness proof of the Lyons simple group by Aschbacher and Segev [6] employed a commuting graph where the vertices consisted of the 3-central subgroups of order 3. For G either a symmetric group, or more generally a finite Coxeter group, or a projective special linear group and X a certain conjugacy class of G, the structure of $\mathcal{C}(G, X)$ has been investigated at length by Bundy [14], Bates, Bundy, Hart, Perkins and Rowley [8], [9], [10], [11] and [12]. Infinite Coxeter groups have also been studied in Perkins [23]. A different flavour of graph (also called a commuting graph) has been examined in Akbari, Mohammadian, Radjavi, Raja [3] and Iranmanesh, Jafarzadeh [20]. There, for a group G, the vertex set is $G \setminus Z(G)$ with two distinct elements being joined if they commute. Recently there has been work on commuting graphs for rings (see, for example, [1], [2]).

This paper investigates $\mathcal{C}(G, X)$ when G is a finite 4-dimensional projective symplectic group and X is a G-conjugacy class of involutions. Such graphs are referred to as commuting involution graphs. From now on H will denote the symplectic group $Sp(4, q)$, $q = p^a$ and p a prime. Let V be the natural (symplectic) $GF(q)H$-module, and set $G = H/Z(H)$. So $G \cong PSp(4, q)$ and $G \cong H$ when $p = 2$. In the case when $p = 2$, G has three conjugacy classes of involutions. Recalling that for an involution x of G, $V(x) = \{v \in V | (v, v^x) = 0\}$ these three classes X_1, X_2, X_3 may be described thus (see [7])

$$\begin{aligned}
X_1 &= \{x \in G \big| x^2 = 1,\ \dim\ C_V(x) = 3\};\\
X_2 &= \{x \in G \big| x^2 = 1,\ \dim\ C_V(x) = 2,\ \dim\ V(x) = 3\};\ \text{and}\\
X_3 &= \{x \in G \big| x^2 = 1,\ \dim\ C_V(x) = 2,\ V(x) = V\}.
\end{aligned}$$

For $t \in X_i$, we define

$$\Delta_i(t) = \{x \in X_i |\, d(t, x) = i\}$$

where d is the standard distance metric on $\mathcal{C}(G, X_i)$. Our four main theorems are as follows.

Theorem 1.1 *Suppose that $p = 2$ and $i = 1, 3$. Then $\mathcal{C}(G, X_i)$ is connected of diameter 2 with the disc sizes being*

$$\begin{aligned}
|\Delta_1(t)| &= q^3 - 2;\ \textit{and}\\
|\Delta_2(t)| &= q^3(q - 1).
\end{aligned}$$

Theorem 1.2 *Suppose that $p = 2$. Then $\mathcal{C}(G, X_2)$ is connected of diameter 4, the disc sizes being*

$$\begin{aligned}
|\Delta_1(t)| &= q^2(2q - 3);\\
|\Delta_2(t)| &= 2q^2(q - 1)^2;\\
|\Delta_3(t)| &= 2q^3(q - 1)^2;\ \textit{and}\\
|\Delta_4(t)| &= q^4(q - 1)^2.
\end{aligned}$$

Turning to the case when p is odd, we have that there are two G-involution conjugacy classes Y_1 and Y_2. We shall let Y_1 denote the G-conjugacy class whose elements are the images of an involution in H, and Y_2 to denote the G-conjugacy class whose elements are the image of an element of H of order 4 which square to the non-trivial element of $Z(H)$.

Theorem 1.3 *If p is odd, then $\mathcal{C}(G, Y_1)$ is connected of diameter 2 with disc sizes*

$$\begin{aligned}
|\Delta_1(t)| &= \frac{1}{2}q(q^2 - 1);\ \textit{and}\\
|\Delta_2(t)| &= \frac{1}{2}(q^4 - q^3 + q^2 + q - 2).
\end{aligned}$$

Theorem 1.4 *(i) If* $q \equiv 3 \pmod 4$ *then* $\mathcal{C}(G, Y_2)$ *is connected of diameter 3. Furthermore,*

$$\begin{aligned}
|\Delta_1(t)| &= \frac{1}{2}q(q^2+2q-1);\\
|\Delta_2(t)| &= \frac{1}{16}(q+1)(3q^5-2q^4+8q^3-30q^2+13q-8); \text{ and}\\
|\Delta_3(t)| &= \frac{1}{16}(q-1)(5q^5-4q^4-2q^3+4q^2+5q+5).
\end{aligned}$$

(ii) If $q \equiv 1 \pmod 4$ *then* $\mathcal{C}(G, Y_2)$ *is connected of diameter 3. Furthermore,*

$$\begin{aligned}
|\Delta_1(t)| &= \frac{1}{2}q(q^2+1);\\
|\Delta_2(t)| &= \frac{1}{16}(q-1)(3q^5-6q^4+32q^3-10q^2-27q-8); \text{ and}\\
|\Delta_3(t)| &= \frac{1}{16}(q-1)(5q^5+22q^4-8q^3+34q^2+51q+24).
\end{aligned}$$

Theorems 1.1 and 1.2 are established in Section 2. While in Section 3 we give a proof of Theorem 1.3. The structure and properties of $\mathcal{C}(G, Y_2)$, in Section 4, are a much tougher nut to crack than the other four cases. The reason for this is that for $\mathcal{C}(G, X_i)$, $(i = 1, 2, 3)$ and $\mathcal{C}(G, Y_1)$ the graph can be studied effectively by working in $H = Sp(4, q)$ and looking at certain configurations in the natural symplectic module V involving $C_V(x)$ for various $x \in X$ ($X = X_i$, $i = 1, 2, 3$ or $XZ(H)/Z(H) = Y_1$). The key point being that, in these four cases for $x \in X$, $C_V(x)$ is a non-trivial subspace of V whereas, for x of order 4 and squaring into $Z(H)$, $C_V(x)$ is trivial. If we change tack and look at G acting on the projective symplectic space things are not much better. When $q \equiv 3 \pmod 4$ elements of Y_2 fix no projective points, while in the case $q \equiv 1 \pmod 4$ they fix $2q + 2$ projective points. However, even in the latter case, the fixed projective points didn't appear to be of much assistance. It is the isomorphism $PSp(4, q) \cong O_5(5, q)$ that comes to our rescue. If now V is the 5-dimensional orthogonal module and $x \in Y_2$, then $\dim C_V(x) = 3$. Even so, probing $\mathcal{C}(G, Y_2)$ turns out to be a lengthy process. Fix $t \in Y_2$. Then by Lemma 4.3, $Y_2 \subseteq \bigcup_{U \in \mathcal{U}_1} C_G(U)$ where $\mathcal{U}_1$ is the set of all 1-subspaces of $C_V(t)$ and as a result, by Lemma 4.4, $\mathcal{C}(G, Y_2)$ may be viewed as the union of commuting involution graphs for various subgroups of G. Up to isomorphism there are three of these commuting involution graphs (called $\mathcal{C}(G^-, Y^-)$, $\mathcal{C}(G^+, Y^+)$ and $\mathcal{C}(G^0, Y^0)$ in Section 4). After studying these three commuting involution graphs in Theorems 4.6, 4.8 and 4.14 it follows immediately (Theorem 4.15) that $\mathcal{C}(G, Y_2)$ is connected and has diameter at most 3. Using the sizes of the discs in $\mathcal{C}(G^-, Y^-)$, $\mathcal{C}(G^+, Y^+)$ and $\mathcal{C}(G^0, Y^0)$ we then complete the proof of Theorem 1.4. This "patching together" of the discs is quite complicated - for example we must confront such issues as t and x in Y_2 being of distance 3 in each of the commuting involution subgraphs which contain both t and x, yet they have distance 2 in $\mathcal{C}(G, Y_2)$ (see Lemmas 4.29 to 4.34).
Our group theoretic notation is standard as given, for example, in [5] or [19].

2 Structure of $\mathcal{C}(G, X_i), i = 1, 2, 3$

We begin looking at $G_0 = Sp_{2n}(q)$ where $n \geq 2$, $q = p^a$ and $p = 2$. Let V_0 denote the $GF(q)G_0$-symplectic module of dimension $2n$ and let t_0 be an involution in G_0 for which $\dim C_V(t_0) = 2n-1$. Put $X_0 = t_0^{G_0}$, the G_0-conjugacy class of t_0.

Theorem 2.1 *$\mathcal{C}(G_0, X_0)$ is connected and has diameter 2.*

Proof For $x \in X_0$,

$$C_{G_0}(x) \leq Stab_{G_0}(C_{V_0}(x))$$

with $Stab_{G_0}(C_{V_0}(x))$ having shape $q^{2n-1}SL_{2n-2}(q)(q-1)$. Set $K_x = O^{2'}(Stab_{G_0}(C_{V_0}(x)))$. Then $K_x \sim q^{2n-1}SL_{2n-2}(q)$ and $C_{G_0}(x) = K_x$. Let $x \in X_0 \setminus \{t_0\}$. If $C_{V_0}(t_0) = C_{V_0}(x)$, then $x \in K_{t_0}$ and so $x \in \Delta_1(t_0)$. Now suppose that $C_{V_0}(t_0) \neq C_{V_0}(x)$. Then $\dim(C_{V_0}(t_0) \cap C_{V_0}(x)) = 2n-2$. Let U be a 1-dimensional subspace of $C_{V_0}(t_0) \cap C_{V_0}(x)$. Since $[V_0, t_0]$ is a 1-space and G_0 acts transitively on the 1-spaces of V_0, there exists $y \in X_0$ such that $[V_0, y] = U$. So $[V_0, y] \leq C_{V_0}(t_0) \cap C_{V_0}(x)$ and hence y leaves both $C_{V_0}(t_0)$ and $C_{V_0}(x)$ invariant. Thus $y \in K_{t_0} \cap K_x = C_{G_0}(t_0) \cap C_{G_0}(x)$ and so $d(t_0, x) \leq 2$ and we see that $\mathcal{C}(G_0, X_0)$ is connected. Since $\mathcal{C}(G_0, X_0)$ cannot have diameter 1 (as then $\langle X_0 \rangle$ would be abelian), the theorem follows. $\square$

The remainder of this section is devoted to establishing Theorems 1.1 and 1.2. So we have $G = Sp(4, q)$ with $q = p^a$ and $p = 2$. For V, the natural $GF(q)$ module for G, we choose the symplectic basis $\{v_1, v_2 \big| v_3, v_4\}$ with $(v_1, v_4) = (v_2, v_3) = 1$. Thus the matrix defining this form is

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and we may suppose that $G = \{A \in GL(4, q) \big| A^T J A = J\}$. We further define

$$S = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & ad+b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c, d \in GF(q) \right\},$$

$$Q_1 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in GF(q) \right\} \text{ and } Q_2 = \left\{ \begin{pmatrix} 1 & 0 & b & c \\ 0 & 1 & d & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, b, c, d \in GF(q) \right\}.$$

Lemma 2.2 *(i)* $S \in Syl_2 G$.
(ii) $S = Q_1 Q_2$ *with* $Q_1^{\#} \cup Q_2^{\#}$ *consisting of all the involutions of* S.

Proof It is straightforward to check that S is a subgroup of G. Since $|G| = q^4(q^2-1)(q^4-1)$ and $|S| = q^4$, we have part (i). Part (ii) is an easy calculation. $\square$

The following three involutions are elements of G.

$$t_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ t_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ t_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 2.3 *(i) For* $i = 1, 2, 3, t_i \in X_i$.
(ii) $C_G(t_1) \sim q^3 SL(2, q)$ *with* $O_2(C_G(t_1)) = Q_1$ *of order* q^3.
(iii) $C_G(t_2) = S$.
(iv) $|X_1| = q^4 - 1$.
(v) $|X_2| = (q^2-1)(q^4-1)$.

Proof (i) Let $v = (\alpha, \beta, \gamma, \delta) \in V$. Then $v^{t_1} = (\alpha, \beta, \gamma, \alpha + \delta)$, $v^{t_2} = (\alpha, \beta, \alpha + \gamma, \alpha + \beta + \delta)$ and $v^{t_3} = (\alpha, \alpha + \beta, \gamma, \gamma + \delta)$. Hence $[v, t_1] = (0, 0, 0, \alpha)$, $[v, t_2] = (0, 0, \alpha, \alpha + \beta)$ and $[v, t_3] = (0, \alpha, 0, \gamma)$. Consequently dim $[V, t_1] = 1$ and dim $[V, t_2] = 2 =$ dim $[V, t_3]$. Thus $t_1 \in X_1$. Now

$$(v, v^{t_2}) = \alpha(\alpha + \beta + \delta) + \beta(\alpha + \gamma) + \gamma\beta + \delta\alpha = \alpha^2 = 0$$

implies that $\alpha = 0$ and so dim $V(t_3) = 3$. Therefore $t_2 \in X_2$. Turning to t_3 we have that

$$(v, v^{t_3}) = \alpha(\gamma + \delta) + \beta\gamma + \gamma(\alpha + \beta) + \delta\alpha = 0$$

implies that $V(t_2) = V$, as v is an arbitrary vector of V. Hence $t_3 \in X_3$, and we have (i).
(ii) By direct calculation we see that

$$C_G(t_1) = \left\{ \begin{pmatrix} 1 & b & c & d \\ 0 & f & g & h \\ 0 & k & m & n \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} b, c, d, f, g, h, k, m, n \in GF(q) \\ gk + fm = 1 \\ b + hk + fn = 0 \\ c + mh + gn = 0 \end{array} \right\}.$$

Moreover

$$SL_2(q) \cong R = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & 0 \\ 0 & k & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} f,g,k,m \in GF(q) \\ fg+km=1 \end{array} \right\} \leq C_G(t_1).$$

with Q_1 a normal elementary abelian subgroup of $C_G(t_1)$ and $|Q_1| = q^3$. So $C_G(t_1) = RQ_1$. Thus (ii) holds.
(iii) This is a routine calculation.
From parts (ii) and (iii) $|C_G(t_1)| = q^4(q^2-1)$ and $|C_G(t_2)| = q^4$. Combining this with $|G| = q^4(q^2-1)(q^4-1)$ yields (iv) and (v). □

Lemma 2.4 $|C_G(t_1) \cap X_1| = q^3 - 1$.

Proof Let s be an involution in S. Then, by Lemma 2.2(ii), $s \in Q_1^\# \cup Q_2^\#$. Let $v = (\alpha, \beta, \gamma, \delta)$ be a vector in V. Assume for the moment that $s \in Q_1$. Then

$$s = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in GF(q)$. So $v^s = (\alpha, a\alpha+\beta, b\beta+\gamma, c\alpha+b\beta+a\gamma+\delta)$. Suppose that at least one of a and b is non-zero. If $v \in C_V(s)$, then we have $a\alpha = b\beta = c\alpha+b\beta+a\gamma = 0$. If, say, $a \neq 0$ then this gives $\alpha = 0$ and $b\beta + a\gamma = 0$. Hence $\gamma = \lambda\beta$ for some $\lambda \in GF(q)$. Thus $\dim C_V(s) = 2$, with the same conclusion if $b \neq 0$.
When $a = b = 0$ we see that $\dim C_V(s) = 3$. Therefore we conclude that

$$|Q_1 \cap X_1| = q - 1. \tag{2.4.1}$$

Now we suppose $s \in Q_2 \setminus Q_1$. Then

$$s = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in GF(q)$ and $c \neq 0$. Here $v^s = (\alpha, \beta, a\alpha+c\beta+\gamma, b\alpha+a\beta+\delta)$ and so, if $v \in C_V(s)$, $a\alpha + c\beta = b\alpha + a\beta = 0$. Suppose that $a = 0$ and $b \neq 0$. Then $c\beta = b\alpha = 0$ which yields $\alpha = 0 = \beta$. Hence $\dim C_V(s) = 2$. Likewise, when $a \neq 0$ and $b = 0$ we get $\dim C_V(s) = 2$. On the other hand, $a = 0 = b$ gives $\dim C_V(s) = 3$.

Now consider the case when $a \neq 0 \neq b$ and $a^2+bc=0$. From $a\alpha+c\beta=0$ we obtain $\beta = a\alpha c^{-1}$ and so $0 = b\alpha + a\beta = b\alpha + a^2c^{-1}\alpha = (b+a^2c^{-1})\alpha$. Since $a^2+bc=0$, this equation holds for all $\alpha \in GF(q)$ and consequently $\dim C_V(s) = 3$. Similar considerations show that $\dim C_V(s) = 2$ when $a \neq 0 \neq b$ and $a^2+bc \neq 0$. So, to summarize, for $s \in Q_2 \setminus Q_1$, $s \in X_1$ when either $a = 0 = b$ or $a \neq 0 \neq b$ and $a^2+bc=0$. For the former, there are $q-1$ such involutions (as $c \neq 0$). For the latter, there are $q-1$ choices for each of b and c and in each case a is uniquely determined (as $GF(q)^{\#}$ is cyclic of odd order), so giving $(q-1)^2$ involutions. Therefore

$$|(X_1 \cap S) \setminus Q_1| = |X_1 \cap (Q_2 \setminus Q_1)| = q(q-1). \tag{2.4.2}$$

Since any two distinct Sylow 2-subgroups of $SL(2,q)$ have trivial intersection and $SL(2,q)$ possesses $q+1$ Sylow 2-subgroups, Lemma 2.3(ii) together with (2.4.1) and (2.4.2) yields that

$$\begin{aligned}|C_G(t_1) \cap X_1| &= (q-1) + q(q-1)(q+1)\\ &= (q-1)(1+q^2+q) = q^3-1.\end{aligned}$$

This proves Lemma 2.4. $\square$

Proof of Theorem 1.1 As is well-known – see for example [16] – G has an outer automorphism arising from the Dynkin diagram of type $C_2 = B_2$. This outer automorphism interchanges the two involution conjugacy classes X_1 and X_3 and as a consequence $\mathcal{C}(G, X_1)$ and $\mathcal{C}(G, X_3)$ are isomorphic graphs. Thus we need only consider $\mathcal{C}(G, X_1)$. From Lemma 2.4, as $\Delta_1(t) = (C_G(t_1) \cap X_1) \setminus \{t_1\}$,

$$|\Delta_1(t_1)| = (q^3-1) - 1 = q^3 - 2.$$

By Theorem 2.1, $\mathcal{C}(G, X_1)$ has diameter 2. Hence, by Lemma 2.3(iv),

$$|\Delta_2(t_1)| = |X_1| - (q^3-1) = (q^4-1) - (q^3-1) = q^4 - q^4 = q^3(q-1),$$

so proving Theorem 1.1.

Before moving on to prove Theorem 1.2 we need additional preparatory material. If W is a subspace of V, then $W^{\perp}$ denotes the subspace of V defined by

$$W^{\perp} = \{v \in V \,|\, (v,w) = 0 \text{ for all } w \in W\}$$

and we recall that $\dim W + \dim W^{\perp} = \dim V = 4$.
By Lemma 2.3(i),(iii) we see that $C_V(C_G(t_2)) = \{(0,0,0,\alpha) | \alpha \in GF(q)\}$ is 1-dimensional. For $x \in X_2$ set $U_1(x) = C_V(C_G(x))$ and $U_2(x) = C_V(x)$. So $\dim U_1(x) = 1$ and $\dim U_2(x) = 2$ (with the subscripts acting as a reminder). We denote the stabilizer in G of $U_1(t_2)$, respectively $U_2(t_2)$, by P_1, respectively P_2. Then $P_i \sim q^3SL_2(q)(q-1)$ for $i = 1,2$. Also $Q_i = O_2(P_i)$ with $C_{P_i}(Q_i) = Q_i$ for $i = 1,2$.
We start analyzing $\mathcal{C}(G, X_2)$ by determining $\Delta_1(t_2)$. For $x \in X_2$ we let $Z_{C_G(x)}$ denote $Z(C_G(x)) \cap X_2$.

Lemma 2.5 $X_2 = \dot{\bigcup}_{R \in \mathrm{Syl}_2 G} Z_R$.

Proof Clearly $X_2 = \bigcup_{R \in \mathrm{Syl}_2 G} Z_R$ by Lemma 2.3(iii). If $Z_R \cap Z_T = \varnothing$ for $R, T \in \mathrm{Syl}_2 G$, then we have some $x \in Z(R) \cap Z(T) \cap X_2$ whence, using Lemma 2.3(iii), $R = C_G(x) = T$. So the lemma holds. $\square$

Lemma 2.6 *Let $R, T \in \mathrm{Syl}_2 G$. If there exists $x \in Z_R$ and $y \in Z_T$ such that $[x, y] = 1$ then $[Z_R, Z_T] = 1$.*

Proof Since $xy = yx$, $y \in C_G(x) = R$. Hence $Z(R) \leq C_G(y) = T$ and so $[Z_R, Z_T] = 1$. $\square$

Let Δ be the building for G and $\mathcal{C}(\Delta)$ denote the chamber graph of Δ. We may view the vertices (chambers) of $\mathcal{C}(\Delta)$ as being $\{ N_G(R) | R \in \mathrm{Syl}_2 G\}$ with two distinct chambers $N_G(R)$ and $N_G(T)$ being adjacent whenever $\langle N_G(R), N_G(T) \rangle \leq P_i^g$ for some $g \in G$ and some $i \in \{1, 2\}$. We use $d^{\mathcal{C}}$ to denote the standard distance metric in $\mathcal{C}(\Delta)$ and for a chamber c put $\Delta_j^{\mathcal{C}}(c) = \left\{ d \in \mathcal{C}(\Delta) | d^{\mathcal{C}}(c, d) = j \right\}$. The structure of $\mathcal{C}(\Delta)$ is well-known.

Lemma 2.7 *$\mathcal{C}(\Delta)$ has diameter 4 and $\left|\Delta_1^{\mathcal{C}}(c)\right| = 2q$; $\left|\Delta_2^{\mathcal{C}}(c)\right| = 2q^2$; $\left|\Delta_3^{\mathcal{C}}(c)\right| = 2q^3$; and $\left|\Delta_4^{\mathcal{C}}(c)\right| = q^4$.*

Proof A straightforward calculation. $\square$

We now introduce a graph $\mathcal{Z}$ whose vertex set is $V(\mathcal{Z}) = \{ Z_R | R \in \mathrm{Syl}_2 G\}$ with $Z_R, Z_T \in V(\mathcal{Z})$ joined if $Z_R \neq Z_T$ and $[Z_R, Z_T] = 1$.

Lemma 2.8 *The graphs $\mathcal{Z}$ and $\mathcal{C}(\Delta)$ are isomorphic.*

Proof Define $\varphi : V(\mathcal{Z}) \to V(\mathcal{C}(\Delta))$ by $\varphi : Z_R \mapsto N_G(R)$ $(R \in \mathrm{Syl}_2 G)$. If $\varphi(Z_R) = \varphi(Z_T)$ for $R, T \in \mathrm{Syl}_2 G$, then $N_G(R) = N_G(T)$ and so $R = T$ and then $Z_R = Z_T$. Thus φ is a bijection between $V(\mathcal{Z})$ and $V(\mathcal{C}(\Delta))$. Suppose $N_G(R)$ and $N_G(T)$ are distinct, adjacent chambers in $\mathcal{C}(\Delta)$. Without loss of generality we may assume $T = S$. Then $N_G(R), N_G(S) \leq P_i$ for $i \in \{1, 2\}$. The structure of P_i then forces $Z(R), Z(S) \leq Q_i$. Since Q_i is abelian, we deduce that $[Z_R, Z_S] = 1$. So Z_R and Z_S are adjacent in $\mathcal{Z}$. Conversely, suppose Z_R and Z_S are adjacent in $\mathcal{Z}$. Then $[Z_R, Z_S] = 1$ with, by Lemma 2.5, $Z_R \cap Z_S = \varnothing$. Hence $Z_R \subseteq S$ and so by Lemma 2.2(ii), $Z_R \subseteq Q_1 \cup Q_2$. Now $Q_1 \cap Q_2 \cap X_2 = Z_S$ and so we must have $Z_R \subseteq Q_i$ for $i \in \{1, 2\}$. The structure of P_i now gives $N_G(R) \leq P_i$ and therefore $N_G(R)$ and $N_G(S)$ are adjacent in $\mathcal{C}(\Delta)$, which proves the lemma. $\square$

Proof of Theorem 1.2

Since for all $x_1, x_2 \in X$, $[x_1, x_2] = 1$ if and only if $[Z_{C_G(x_1)}, Z_{C_G(x_2)}] = 1$ by Lemma 2.5, then for $i > 1$, $d^{\mathcal{C}}(x_1, x_2) = i$ if and only if $d^{\mathcal{Z}}(Z_{C_G(x_1)}, Z_{C_G(x_2)}) = i$ (where $d^{\mathcal{Z}}$ denotes the distance in $\mathcal{Z}$). Note that if $d^{\mathcal{C}}(x_1, x_2) = 1$, then either $Z_{C_G(x_1)} = Z_{C_G(x_2)}$ or $d^{\mathcal{Z}}(Z_{C_G(x_1)}, Z_{C_G(x_2)}) = 1$. Since X_2 is a disjoint union of the elements of $\mathcal{Z}$, then $\mathcal{C}(G, X_2)$ is connected of diameter 4. Now

$$\Delta_1(t) = \bigcup_{\substack{R \in \mathrm{Syl}_2 G \\ [Z_S, Z_R] = 1}} Z_R \qquad \text{and} \qquad \Delta_i(t) = \bigcup_{\substack{R \in \mathrm{Syl}_2 G \\ d^{\mathcal{Z}}(Z_S, Z_R) = i}} Z_R, \quad i > 1$$

and so $|\Delta_1(t)| = |Z_S| + 2q\,|Z_S| - 1$. From $|Z_S| = (q-1)^2$ we get $|\Delta_1(t)| = (q-1)^2 + 2q(q-1)^2 - 1 = q^2(2q-3)$. The remaining disc sizes are immediate from the structure of the chamber graph $\mathcal{C}(\Delta)$.

3 Structure of $\mathcal{C}(G, Y_1)$

This section is devoted to the proof of Theorem 1.3. In order to investigate the disc structure of $\mathcal{C}(G, Y_1)$ it is advantageous for us to work in $H = Sp_4(q)$ (and so $\overline{H} = H/Z(H) \cong G$). We assume that $\{v_1, v_2, v_3, v_4\}$ is a hyperbolic basis for V with $(v_2, v_1) = (v_4, v_3) = 1$. Thus if J is the matrix defining this form then

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and J has two diagonal blocks J_0 where $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We may suppose that for $t \in Y_1$, we have $\overline{s} = t$ where $s = \left(\begin{array}{c|c} -I_2 & \\ \hline & I_2 \end{array}\right)$. Put $X = s^H$. Then $Y_1 = \{\overline{x} \,|\, x \in X\}$. For $x \in X$, set $N_x = N_H(\langle x, Z(H) \rangle)$. Evidently, for $\overline{x_1}, \overline{x_2} \in Y_1$ (where $x_1, x_2 \in X$) $\overline{x_1}$ and $\overline{x_2}$ commute if and only if $x_1 \in N_{x_2}$ (or equivalently $x_2 \in N_{x_1}$). Now N_s consists of $g \in H$ for which $s^g = s$ or $s^g = -s$. Letting $g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ where A, B, C and D are 2×2 matrices over $GF(q)$, direct calculation reveals that either $B = C = 0$ or $A = D = 0$. Also, as $g \in H$, we must have $A^T J_0 A = D^T J_0 D = J_0$ and therefore

$$\begin{aligned} N_s &= \left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right), \left(\begin{array}{c|c} & A \\ \hline B & \end{array}\right) \,\middle|\, A, B \in SL_2(q) \right\} \\ &\sim (SL_2(q) \times SL_2(q)) : 2 \end{aligned}$$

Lemma 3.1 $|\Delta_1(t)| = \frac{1}{2} q(q^2 - 1)$.

Proof Since $X = s^H$ consists of all the involutions in $H \setminus Z(H)$, a quick calculation gives

$$X \cap N_s = \left\{ \left(\begin{array}{c|c} & A \\ \hline A^{-1} & \end{array} \right) \middle| A \in SL_2(q) \right\} \cup \{s, -s\}.$$

Under the natural homomorphism to G, for $x \in X$ $\overline{x} = \overline{-x}$, and so $|\Delta_1(t)| = \frac{1}{2}|SL_2(q)| = \frac{1}{2}q(q^2-1)$. □

Put $E = \langle v_3, v_4 \rangle$. Then $E^\perp = \langle v_1, v_2 \rangle$ and we note that $C_V(s) = E$. Furthermore we have that $\mathrm{Stab}_H(\{E, E^\perp\}) = N_s$. Put $\Sigma = \left\{ \{F, F^\perp\} \middle| F \text{ is a hyperbolic 2-subspace of } V \right\}$. Now let $\beta \in GF(q)$ and set $U_\beta = \langle (1,0,1,0), (0,\beta,0,-\beta-1) \rangle$. Then U_β is a hyperbolic 2-subspace of V and so $\{U_\beta, U_\beta^\perp\} \in \Sigma$. The N_s-orbit of $\{U_\beta, U_\beta^\perp\}$ will be denoted by Σ_β.

Lemma 3.2 *Let F be a hyperbolic 2-subspace of V with $F \neq E$ or $E^\perp$. Then $\{F, F^\perp\} \in \Sigma_\beta$ for some $\beta \in GF(q)$. Moreover, for $\beta \in GF(q)$, $\Sigma_\beta = \Sigma_{-\beta-1}$.*

Proof Since $F \neq E$ or $E^\perp$, we may find $w_1 \in F$ with $w_1 = (\alpha_1, \beta_1, \gamma_1, \delta_1)$ and $\{\alpha_1, \beta_1\} \neq \{0\} \neq \{\gamma_1, \delta_1\}$. Now N_s contains two $SL_2(q)$ subgroups for which $\langle v_1, v_2 \rangle$ and $\langle v_3, v_4 \rangle$ are natural $GF(q)SL_2(q)$-modules. Because $SL_2(q)$ acts transitively on the non-zero vectors of such modules, we may suppose $w_1 = (1,0,1,0)$. Now choose $w_2 \in F$ such that $(w_1, w_2) = 1$ (and so $\langle w_1, w_2 \rangle = F$). Then if $w_2 = (\alpha, \beta, \gamma, \delta)$ we must have $\beta + \delta = -1$ and so $w_2 = (\alpha, \beta, \gamma, -\beta-1)$. The matrices in N_s fixing w_1 are

$$C_{N_s}(w_1) = \left\{ \left(\begin{array}{cc|cc} 1 & & & \\ a_1 & 1 & & \\ \hline & & 1 & \\ & & a_2 & 1 \end{array} \right), \left(\begin{array}{cc|cc} & & 1 & \\ & & a_1 & 1 \\ \hline 1 & & & \\ a_2 & 1 & & \end{array} \right) \middle| a_1, a_2 \in GF(q) \right\}.$$

Let $g = \left(\begin{array}{cc|cc} 1 & & & \\ a_1 & 1 & & \\ \hline & & 1 & \\ & & a_2 & 1 \end{array} \right)$ where $a_1, a_2 \in GF(q)$. Then $w_1^g = w_1$.

We single out the cases $\beta = 0$ and $\beta = -1$ for special attention. If, say, $\beta = 0$, then $w_2 = (\alpha, 0, \gamma, -1)$. Hence $w_2 - \alpha w_1 = (0, 0, \gamma - \alpha, -1)$ and $F = \langle w_1, w_2 - \alpha w_1 \rangle$. Since $(0, 0, \gamma - \alpha, -1)g = (0, 0, (\gamma - \alpha) - a_2, -1)$ and choosing $a_2 = -\gamma + \alpha$, we obtain $Fg = U_0$. For $\beta = -1$ a similar argument works (using $w_2 - \gamma w_1$ instead of $w_2 - \alpha w_1$). So we may assume that $\beta \neq 0, -1$. From

$$w_2 g = (\alpha, \beta, \gamma, -\beta - 1) = (\alpha + \beta a_1, \beta, \gamma + (-\beta - 1)a_2, -\beta - 1)$$

by a suitable choice of a_1 and a_2, as $\beta \neq 0, -1$, we get $w_2 g = (0, \beta, 0, -\beta - 1)$, whence $Fg = U_\beta$. Thus we have shown $\{F, F^\perp\} \in \Sigma_\beta$ for some $\beta \in GF(q)$. Finally, for $\beta \in GF(q)$, $\Sigma_\beta = \Sigma_{-\beta-1}$

follows from

$$(0,\beta,0,-\beta-1)\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)=(0,-\beta-1,0,\beta).$$

□

Let $\phi : GF(q)\setminus\{-1\}\to GF(q)$ be defined by

$$\phi(\lambda)=-(1+(\lambda+1)^{-2}(1-\lambda^2))^{-1}\qquad(\lambda\in GF(q)).$$

There is a possibility that this is not well-defined should $1+(\lambda+1)^{-2}(1-\lambda^2)=0$. This would then give $(\lambda+1)^2+(1-\lambda^2)=0$ from which we infer that $\lambda=-1$. So we conclude that ϕ is well-defined.

Lemma 3.3 *ϕ is injective.*

Proof Suppose $\phi(\lambda)=\phi(\mu)$ for $\lambda,\mu\in GF(q)\setminus\{-1\}$ with $\lambda\neq\mu$. Hence

$$(1+(\lambda+1)^{-2}(1-\lambda^2))^{-1}=(1+(\mu+1)^{-2}(1-\mu^2))^{-1}.$$

Simplifying and using the fact that q is odd gives

$$\mu^2+\mu-\mu\lambda^2-\lambda^2-\lambda+\lambda\mu^2=0,$$

and then

$$(\mu+\lambda)(\mu-\lambda)+(\mu-\lambda)+\lambda\mu(\mu-\lambda)=0.$$

Hence $(\mu-\lambda)(\mu+\lambda+1+\lambda\mu)=0$. Since $\mu\neq\lambda$, we get $\mu+\lambda+1+\lambda\mu=0$ from which we deduce that either $\lambda=-1$ or $\mu=-1$, a contradiction. So the lemma holds. □

Proof of Theorem 1.3

We first show that Diam $\mathcal{C}(G,Y_1)=2$. So let $x\in X$ be such that $x\notin\{t\}\cup\Delta_1(t)$. Now $\left\{C_V(x),C_V(x)^\perp\right\}\in\Sigma$ as $C_V(x)\neq E$ or $E^\perp$ (otherwise $x\in\{s,-s\}$ and then $\overline{x}=t$). Hence $\left\{C_V(x),C_V(x)^\perp\right\}\in\Sigma_\mu$ for some $\mu\in GF(q)$ by Lemma 3.2. Let $y=\left(\begin{array}{c|c} & I_2\\ \hline I_2 & \end{array}\right)\in X\cap N_s$. Then $\overline{y}\in\Delta_1(t)$. Our aim is to choose an $x_\lambda\in N_y\cap X$ (so $\overline{x_\lambda}\in\Delta_1(\overline{y})$) for which $\left\{C_V(x_\lambda),C_V(x_\lambda)^\perp\right\}\in\Sigma_\mu$. Since Σ_μ is an N_s-orbit, there exists $h\in N_s$ such that $\left\{C_V(x_\lambda),C_V(x_\lambda)^\perp\right\}^h=\left\{C_V(x),C_V(x)^\perp\right\}$. As a consequence either $x=x_\lambda^h$ or x_λ^{-1h} and therefore $\overline{x}=\overline{x_\lambda}^h$, whence $d(t,\overline{x})\leq 2$.

We first look at the case when $\mu=-2^{-1}$. Then $\mu=-\mu-1$ and hence

$$U_{-2^{-1}}=\langle(1,0,1,0),(0,1,0,1)\rangle.$$

Observing that $U_{-2^{-1}} = C_V(y)$, we see that for $\mu = -2^{-1}$, $\overline{x} \in \Delta_1(y)$, which we are not concerned with here. So we may assume $\mu \neq -2^{-1}$.

Let $x_\lambda = \left(\begin{array}{c|c} \lambda I_2 & -B \\ \hline B & -\lambda I_2 \end{array}\right)$ where $\lambda \in GF(q) \setminus \{0\}$ and such that B has zero trace and determinant $1-\lambda^2$. So $x_\lambda \in X \cap N_y$. We now move onto the case when $\mu = 0$ (or equivalently $\mu = -1$). Here we take $\lambda = 1$ and $B = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$, noting that B satisfies the conditions to ensure that $\overline{x_1} \in \Delta_1(\overline{y})$. Let $v = (\alpha, \beta, \gamma, \delta) \in V$. Then $v \in C_V(x_1)$ precisely when

$$\begin{aligned} 2\gamma + 2\delta &= 0; & -2\gamma - 2\delta &= 0; \\ -2\alpha - 2\beta - \gamma &= \gamma; & 2\alpha + 2\beta - \delta &= \delta; \end{aligned}$$

and thus the only conditions we get are $\gamma = -\beta - \alpha$ and $\alpha + \beta = \delta$. Thus

$$\begin{aligned} C_V(x_1) &= \{(\alpha, \beta, -\alpha - \beta, \alpha + \beta)\} \\ &= \langle (1,0,-1,1), (0,1,-1,1) \rangle . \end{aligned}$$

It is straightforward to check that $\left\{C_V(x_1), C_V(x_1)^\perp\right\} \in \Sigma_0$. Therefore we may also assume that $\mu \neq 0, -1$. Choosing $B = \begin{pmatrix} \lambda & \lambda^{-1} \\ -\lambda & -\lambda \end{pmatrix}$ we see that the requisite conditions are satisfied. Take $v = (\alpha, \beta, \gamma, \delta) \in V$ and calculating v^{x_λ} gives the relations

$$\begin{aligned} (\lambda - 1)\alpha + \gamma\lambda - \delta\lambda &= 0; & (\lambda - 1)\beta + \gamma\lambda^{-1} - \delta\lambda &= 0; \\ -\lambda\alpha + \lambda\beta - (\lambda + 1)\gamma &= 0; & -\lambda^{-1}\alpha + \lambda\beta - (\lambda+1)\delta &= 0; \end{aligned}$$

which, after rearranging gives

$$\begin{aligned} \alpha &= \lambda(\lambda-1)^{-1}(\delta - \gamma); & \beta &= \lambda(\lambda-1)^{-1}\delta - \lambda^{-1}(\lambda-1)^{-1}\gamma; \\ \gamma &= \lambda(\lambda+1)^{-1}(\beta - \alpha); & \delta &= \lambda(\lambda+1)^{-1}\alpha - \lambda^{-1}(\lambda+1)^{-1}\alpha; \end{aligned}$$

and note that the relations for γ and δ are satisfied after substitution for α and β. Hence

$$\begin{aligned} C_V(x_\lambda) &= \left\{\left(\alpha, \beta, \lambda(\lambda+1)^{-1}(\beta - \alpha), \lambda(\lambda+1)^{-1}\beta - \lambda^{-1}(\lambda+1)^{-1}\alpha\right)\right\} \\ &= \left\langle \left(1, 0, -\lambda(\lambda+1)^{-1}, -\lambda^{-1}(\lambda+1)^{-1}\right), \left(0, 1, \lambda(\lambda+1)^{-1}, \lambda(\lambda+1)^{-1}\right) \right\rangle . \end{aligned} \tag{3.3.1}$$

We want to determine which N_s-orbit, Σ_β, that $C_V(x_\lambda)$ lies in. Our representative, U_β, for Σ_β has $w_1 = (1,0,1,0)$ as one component of the hyperbolic pair, so we need an element of N_s to send the first generator in (3.3.1) to w_1. We need to find conditions on $C, D \in SL_2(q)$ such that

$$\left(1, 0, -\lambda(\lambda+1)^{-1}, -\lambda^{-1}(\lambda+1)^{-1}\right) \left(\begin{array}{c|c} C & \\ \hline & D \end{array}\right) = (1,0,1,0)$$

and so without loss of generality we can take $C = I_2$. This reduces to solving

$$\left(-\lambda(\lambda+1)^{-1}, -\lambda^{-1}(\lambda+1)^{-1}\right)\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = (1,0)$$

and after multiplying out, we get that $d_3 = -(d_1+1)\lambda^2 - \lambda$ and $d_4 = -d_2\lambda^2$. Since D has determinant 1, we find that $d_2 = \lambda^{-1}(\lambda+1)^{-1}$ and so $d_4 = -\lambda(\lambda+1)^{-1}$. Without loss of generality, by taking $d_1 = 1$ we have that

$$D = \begin{pmatrix} 1 & \lambda^{-1}(\lambda+1)^{-1} \\ -2\lambda^2 - \lambda & -\lambda(\lambda+1)^{-1} \end{pmatrix}$$

and a quick check shows that the first generator in (3.3.1) is mapped to w_1. Using the same matrix, by multiplying on the right of the second generator in (3.3.1), we get

$$\left(0, 1, \lambda(\lambda+1)^{-1}, \lambda(\lambda+1)^{-1}\right)\left(\begin{array}{c|c} I_2 & \\ \hline & D \end{array}\right) = \left(*, 1, *, (\lambda+1)^{-2}(1-\lambda^2)\right) = u'$$

and $\langle w_1, u' \rangle$ is a hyperbolic 2-subspace conjugate to some U_β. Recall that for a fixed $\beta \in GF(q)$, N_s is transitive on $\{(\alpha, \beta, \gamma, -\beta-1) \mid \alpha, \gamma \in GF(q)\}$. Hence, we need only find the hyperbolic pair representing such a conjugate of U_β, to determine β. This is found by requiring that some multiple of u' has inner product 1 with w_1, that is

$$\beta \cdot 1 = -1 - \beta\left((\lambda+1)^{-2}(1-\lambda^2)\right)$$

for some $\beta \in GF(q)$. By expanding, we get that $\beta = -\left(1 + (\lambda+1)^{-2}(1-\lambda^2)\right)^{-1}$ and so $C_V(x_\lambda) \in \Sigma_\beta$. By Lemma 3.3, $\phi : \lambda \mapsto -(1 + (\lambda+1)^{-2}(1-\lambda^2))^{-1}$ is an injective map from $GF(q) \setminus \{-1\}$ into $GF(q)$. Since $\mu \neq -2^{-1}$, $\mu \neq -\mu - 1$ and therefore there exists $\lambda \in GF(q) \setminus \{-1\}$ such that $\phi(\lambda) = \mu$ or $-\mu - 1$. Bearing in mind that $U_\mu = U_{-\mu-1}$ by Lemma 3.2, we conclude that $\left\{C_V(x_\lambda), C_V(x_\lambda)^\perp\right\} \in \Sigma_\mu$. Consequently we have proved that $\mathrm{Diam}\, \mathcal{C}(G, Y_1) = 2$.
From $|G| = \frac{q^4}{2}(q^2-1)(q^4-1)$ and $|C_G(t)| = q^2(q^2-1)^2$ we see that $|Y_1| = \frac{q^2}{2}(q^2+1)$. Using Lemma 3.1 then gives

$$|\Delta_2(t)| = \frac{1}{2}(q^4 - q^3 + q^2 + q - 2),$$

which completes the proof of Theorem 1.3.

4 Structure of $\mathcal{C}(G, Y_2)$

In this section we present a proof of Theorem 1.4. The uncovering of the disc structures of $\mathcal{C}(G, Y_2)$ will be a long haul. As discussed in Section 1, it will be advantageous for us to use the well known isomorphism that $PSp_4(q) \cong O_5(q)$ (see Corollary 12.32 of [26]). So we take $G = O_5(q)$ and from now on V will denote the 5-dimensional $GF(q)$ orthogonal module for G. Thus the elements of G are 5×5 orthogonal matrices with respect to the orthogonal form $(\ ,\ )$ which have spinor norm a square in $GF(q)$. We may assume that the Gram matrix with respect to $(\ ,\ )$ is

$$J = \left(\begin{array}{cc|ccc} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & 0 & 0 & 1 \\ & & 0 & -2 & 0 \\ & & 1 & 0 & 0 \end{array}\right).$$

Let

$$t = \left(\begin{array}{c|ccc} I_2 & & & \\ \hline & 0 & 0 & 1 \\ & 0 & -1 & 0 \\ & 1 & 0 & 0 \end{array}\right).$$

Then $t \in G$ and $Y_2 = t^G$. Let $\delta = \pm 1$ where $q \equiv \delta \pmod 4$.

Lemma 4.1 *(i)* $\dim(C_V(t)) = 3$.
(ii) $C_V(t)^\perp = [V, t]$ *is a 2-subspace of* V *of* δ*-type.*
(iii) $V = C_V(t) \perp C_V(t)^\perp$.

Proof An easy calculation. $\square$

Put $L_t = C_G(t) \cap C_G([V, t])$.

Lemma 4.2 *(i) Let* $x \in Y_2$*. Then* $t = x$ *if and only if* $C_V(t) = C_V(x)$*.*
(ii) $C_G(t) = \mathrm{Stab}_G(C_V(t)) \sim (L_2(q) \times \frac{q-\delta}{2}).2^2$.
(iii) L_t *acts faithfully on* $C_V(t)$ *and* $L_t \cong L_2(q)$*.*

Proof

(i) Suppose $C_V(x) = C_V(t)$. Then, using Lemma 4.1 (ii), $[V, x] = C_V(x)^\perp = C_V(t)^\perp = [V, t]$. Hence by Lemma 4.1(iii), tx acts trivially on V and thus $tx = 1$. Therefore $t = x$ and (i) holds.

(ii) Plainly $C_G(t) \leq \mathrm{Stab}_G(C_V(t))$, and if $g \in \mathrm{Stab}_G(C_V(t))$, then $C_V(t) = C_V(t)^g = C_V(t^g)$. Hence, and $t^g \in Y_2$, $t = t^g$ by part (i). So $g \in C_G(t)$ and thus $C_G(t) = \mathrm{Stab}_G(C_V(t))$. That $\mathrm{Stab}_G(C_V(t)) \sim (L_2(q) \times \frac{q-\delta}{2}).2^2$ can be read off from Proposition 4.1.6 of [21].

(iii) For any $g \in C_G(t)$, we have $[V,t]^g = C_V(t)^{\perp g} = C_V(t^g)^{\perp} = C_V(t)^{\perp} = [V,t]$ and so $C_G(t) \leq \mathrm{Stab}_G[V,t]$. If any element in L_t acts trivially on $C_V(t)$, then it would act trivially on V and thus be the identity. Hence L_t acts faithfully on $C_V(t)$. Let $v \in C_V(t)$ and by Lemma 4.1(iii), we have $[V,t] \leq \langle v\rangle^{\perp}$. Hence $\langle v\rangle^{\perp} = [V,t] \oplus W$ where $W \leq C_V(t)$. But since $\dim(\langle v\rangle^{\perp}) = 4$, we have $\dim(W) = 2$ and so $C_V(t) \nleq \langle v\rangle^{\perp}$. Therefore for all $u \in C_V(t)$, $(v,u) = 0$ if and only if $v = 0$ and thus $(\ ,\ )$ is non-degenerate on restriction to $C_V(t)$. Hence we have $L_t \hookrightarrow GO(C_V(t)) \sim GO_3(q)$ as L_t fixes $[V,t]$ pointwise, by definition. Since $L_t \leq G$ and acts as determinant 1 on $[V,t]$, then it must act as determinant 1 on $C_V(t)$. In addition, as L_t fixes $[V,t]$ pointwise, when the elements of L_t are decomposed as products of refections, the vectors reflected will lie in $C_V(t)$. Since the spinor norm of the elements of L_t are a square in $GF(q)$ and the vectors reflected lie in $C_V(t)$, then the spinor norm doesn't change on restriction to $C_V(t)$. Hence, $L_t \sim O_3(q) \sim L_2(q)$ proving (iii).

□

Let $\mathcal{U}_i$ denote the set of i-dimensional subspaces of $C_V(t)$, $i = 1,2$. In proving Theorem 1.4, our divide and conquer strategy is based on the following observation.

Lemma 4.3 $Y_2 \subseteq \displaystyle\bigcup_{U \in \mathcal{U}_1 \cup \mathcal{U}_2} C_G(U)$.

Proof Let $x \in Y_2 \setminus \{t\}$ and set $U = C_V(t) \cap C_V(x)$. By Lemmas 4.1(i) and 4.2(i), $U \in \mathcal{U}_1 \cup \mathcal{U}_2$. Since $t, x \in C_G(U)$, we have Lemma 4.3. □

The three cases we must chase down are presaged by our next result.

Lemma 4.4 *(i) Let U_0 be an isotropic 1-subspace of $C_V(t)$. Then $C_G(U_0) \sim q^3 : L_2(q)$.*

(ii) Let U_ε be a 1-subspace of $C_V(t)$, such that $U_\varepsilon^{\perp} \cap C_V(t)$ is a 2-space of ε-type ($\varepsilon = \pm 1$). Then

$$C_G(U_\varepsilon) \sim \begin{cases} SL_2(q) \circ SL_2(q)) & \delta = \varepsilon \\ L_2(q^2) & \delta = -\varepsilon. \end{cases}$$

Proof Let U_0 be an isotropic 1-subspace of $C_V(t)$. From Proposition 4.1.20 of [21], we know that $\mathrm{Stab}_G(U_0) \sim C_0 : (C_1 \times C_2)\langle r\rangle$ where C_1 acts as scalars on U_0, r a reflection of U_0 and $C_0 \sim q^3$, $C_2 \sim L_2(q)$ fixing U_0 pointwise. Hence $C_G(U_0) \sim q^3 : L_2(q)$, so proving (i).
If $\delta = 1$, then $[V,t]$ is a 2-subspace of V of $+$-type, and hence $U_+^{\perp} = (U_+^{\perp} \cap C_V(t)) \perp [V,t]$ is a 4-subspace of $+$-type. Similarly, $U_-^{\perp} = (U_-^{\perp} \cap C_V(t)) \perp [V,t]$ is a 4-space of $-$-type. If $\delta = -1$, then $[V,t]$ is a 2-subspace of V of $-$-type, and the results for when $\delta = 1$ interchange. Let W_+ and W_- be 4-subspaces of V of $+$ and $-$-type respectively, such that $W_+^{\perp}$ and $W_-^{\perp}$

are 1-subspaces of $C_V(t)$, observing that $\mathrm{Stab}_G(W_\pm) = \mathrm{Stab}_G(W_\pm^\perp)$. From Proposition 4.1.6 of [21], we have

$$\begin{aligned}\mathrm{Stab}_G(W_+) &\sim C_+ \langle s_+\rangle\\ \mathrm{Stab}_G(W_-) &\sim C_- \langle s_-\rangle\end{aligned}$$

where $C_+ \sim SL_2(q) \circ SL_2(q)$ fixes $W_+^\perp$ pointwise, $C_- \sim L_2(q^2)$ fixes $W_-^\perp$ pointwise and s_+, s_- are reflections of $W_+^\perp$ and $W_-^\perp$ respectively. This proves (ii) and hence the lemma. □

Lemma 4.5 *(i) Let U_0 be a 2-subspace of $C_V(t)$ such that $U_0^\perp \cap C_V(t)$ is an isotropic 1-space. Then $C_G(U_0) \cong q^2 : \frac{q-\delta}{2}$.*
(ii) Let U_ε be a 2-subspace of $C_V(t)$ of ε-type ($\varepsilon = \pm 1$). Then $C_G(U_\varepsilon) \sim L_2(q)$.

Proof See Propositions 4.1.6 and 4.1.20 of [21]. □

Define the following subsets of $\mathcal{U}_i$, $i = 1, 2$.

$$\begin{aligned}\mathcal{U}_1^+ &= \{U \in \mathcal{U}_1 |\, C_G(U) \sim SL_2(q) \circ SL_2(q)\}\\ \mathcal{U}_1^- &= \left\{U \in \mathcal{U}_1 |\, C_G(U) \sim L_2(q^2)\right\}\\ \mathcal{U}_1^0 &= \left\{U \in \mathcal{U}_1 |\, C_G(U) \sim q^3 : L_2(q)\right\}\\ \mathcal{U}_2^+ &= \{U \in \mathcal{U}_2 |\, U \text{ is of } +\text{-type}\}\\ \mathcal{U}_2^- &= \{U \in \mathcal{U}_2 |\, U \text{ is of } -\text{-type}\}\\ \mathcal{U}_2^0 &= \left\{U \in \mathcal{U}_2 \,\middle|\, C_G(U) \sim q^2 : \frac{q-\delta}{2}\right\}.\end{aligned}$$

In the notation of Lemma 4.4, $\mathcal{U}_1^+$ is the case $\delta = \varepsilon$ while $\mathcal{U}_1^-$ is when $\delta = -\varepsilon$. Note by Lemmas 4.4 and 4.5 that $\mathcal{U}_i = \mathcal{U}_i^0 \cup \mathcal{U}_i^+ \cup \mathcal{U}_i^-$, $i = 1, 2$. We now study $C_G(U) \cap Y_2$ for $U \in \mathcal{U}_1$. By Lemma 4.4 there are three possibilities for the structure of $C_G(U)$. First we look at the case $U \in \mathcal{U}_1^-$, and set $G^- = C_G(U)$. Then $G^- \cong L_2(q^2)$ by definition of $\mathcal{U}_1^-$. Define $\Delta_i^-(t) = \{x \in G^- \cap Y_2 |\, d^-(t, x) = i\}$ where $i \in \mathbb{N}$ and d^- is the distance metric on the commuting graph $\mathcal{C}(G^-, G^- \cap Y_2)$.

Theorem 4.6 *If $q \neq 3$ then $\mathcal{C}(G^-, G^- \cap Y_2)$ is connected of diameter 3 with*

$$\begin{aligned}\left|\Delta_1^-(t)\right| &= \frac{1}{2}(q^2-1);\\ \left|\Delta_2^-(t)\right| &= \frac{1}{4}(q^2-1)(q^2-5); \text{ and}\\ \left|\Delta_3^-(t)\right| &= \frac{1}{4}(q^2-1)(q^2+7).\end{aligned}$$

Proof Since $q^2 \equiv 1 \pmod 4$ and $q \neq 3$ implies $q^2 > 13$, this follows from Theorem 1.1(iii) of [10]. $\square$

We move on to analyze $G^+ = C_G(U)$ where $U \in \mathcal{U}_1^+$. Hence, by definition of $\mathcal{U}_1^+$, $G^+ \sim L_1 \circ L_2$ where $L_1 \sim SL_2(q) \sim L_2$ (with the central product identifying $Z(L_1)$ and $Z(L_2)$). Set $Y^+ = G^+ \cap Y_2$. We begin by describing Y^+.

Lemma 4.7 *$Y^+ = \{x_1x_2 |\, x_i \in L_i$ and x_i has order 4, $i = 1, 2\}$.*

Proof Apart from the central involution z of G^+, all other involutions of G^+ are of the form g_1g_2 where $g_i \in L_i$ $(i = 1, 2)$ has order 4. Since all involutions in $L_i/Z(G^+)$ are conjugate, it quickly follows that $\{g_1g_2 |\, g_i \in L_i$ and g_i has order $4,\ i = 1, 2\}$ is a G^+-conjugacy class. Now z acts as -1 on $U^\perp$ and thus dim $C_V(z) = 1$. Therefore $t \neq z$ whence, as $t \in G^+$, the lemma holds. $\square$

Let d^+ denote the distance metric on the commuting graph $\mathcal{C}(G^+, Y^+)$ and, for $i \in \mathbb{N}$, $\Delta_i^+(t) = \{x \in Y^+ |\, d^+(t, x) = i\}$.

Theorem 4.8 *Assume that $q \notin \{3, 5, 9, 13\}$. Then $\mathcal{C}(G^+, Y^+)$ is connected of diameter 3 with*

$$\begin{aligned}
\left|\Delta_1^+(t)\right| &= \frac{1}{2}(q-\delta)^2 + 1;\\
\left|\Delta_2^+(t)\right| &= \frac{1}{8}(q-\delta)^3(q-4-\delta) + (q-\delta)(q-2-\delta);\ \textit{and}\\
\left|\Delta_3^+(t)\right| &= \frac{3}{8}q^4 + \frac{1}{2}(1+3\delta)q^3 - \frac{1}{4}(7+6\delta)q^2 + \frac{7}{2}(1+\delta)q - \frac{1}{8}(29+20\delta).
\end{aligned}$$

Proof Let $\overline{G^+} = G^+/Z(G^+)$ $(= \overline{L_1} \times \overline{L_2})$. Note that for $x_1x_2 \in Y^+$, $x_1^{-1}x_2 = x_1x_2^{-1}$ and $x_1x_2 = x_1^{-1}x_2^{-1}$ and so the inverse image of $\overline{x_1x_2}$ contains two elements of Y^+. Let $d^{(i)}$ denote the distance metric on the commuting graph of $\overline{L_i}$ and $\Delta_j^{(i)}(\overline{x_i})$ the j^{th} disc of $\overline{x_i}$ in the commuting graph of $\overline{L_i}$. By Lemma 4.7, $t = t_1t_2$ where, for $i = 1, 2$, $t_i \in L_i$ has order 4. Let $x = x_1x_2 \in Y^+$ with $x \neq t$. Then $tx = xt$ if and only if tx has order 2. So, bearing in mind that $Y^+ \cup \{z\}$ (where $\langle z \rangle = Z(G^+)$) are all the involutions of G^+, we have that $tx = xt$ if and only if one of the following holds:- $x_1 = t_1,\ x_2 = t_2^{-1}$; $x_1 = t_1^{-1}$, $x_2 = t_2$; $\overline{x_1} \in \Delta_1^{(1)}(\overline{t_1})$ and $\overline{x_2} \in \Delta_1^{(2)}(\overline{t_2})$. Thus

$$\Delta_1^+(t) = \left\{ x_1x_2 \,\middle|\, \overline{x_i} \in \Delta_1^{(i)}(\overline{t_i}),\ i = 1, 2 \right\} \cup \left\{ t_1t_2^{-1} \right\}. \tag{4.8.1}$$

Hence, using [10],

$$\left|\Delta_1^+(t)\right| = 2\left(\frac{1}{2}(q-\delta)\right)^2 + 1 = \frac{1}{2}(q-\delta)^2 + 1. \tag{4.8.2}$$

Next we examine $\Delta_2^+(t)$. Let $x \in Y^+$. Assume that $x = x_1t_2$ or $x_1t_2^{-1}$ where $\overline{x_1} \in \Delta_1^{(1)}(\overline{t_1})$. Then $x \in \Delta_1^+(t_1t_2^{-1})$ (recall $t_1t_2^{-1} = t_1^{-1}t_2$) which implies, by (4.8.1), that $x \in \Delta_2^+(t)$. If $x = t_1x_2$ or $t_1^{-1}x_2$ where $\overline{x_2} \in \Delta_1^{(2)}(\overline{t_2})$, we similarly get $x \in \Delta_2^+(t)$. Therefore

$$\left\{x_1x_2 \,\middle|\, \overline{x_1} \in \Delta_1^{(1)}(\overline{t_1}),\ \overline{x_2} = \overline{t_2}\right\} \cup \left\{x_1x_2 \,\middle|\, \overline{x_2} \in \Delta_1^{(2)}(\overline{t_2}),\ \overline{x_1} = \overline{t_1}\right\} \subseteq \Delta_2^+(t). \tag{4.8.3}$$

Now suppose $x = x_1x_2$ where $\overline{x_1} \in \Delta_2^{(1)}(\overline{t_1})$ and $\overline{x_2} \in \Delta_1^{(2)}(\overline{t_2})$. So there exists $\overline{y_1} \in \overline{L_1}$ such that $(\overline{t_1}, \overline{y_1}, \overline{x_1})$ is a path of length 2 in the commuting graph for $\overline{L_1}$. Then $(t = t_1t_2,\ y_1x_2^{-1},\ x_1x_2 = x)$ is a path of length 2 in $\mathcal{C}(G^+, Y^+)$. Thus, by (4.8.1), $x \in \Delta_2^+(t)$. If, on the other hand, $\overline{x_1} \in \Delta_1^{(1)}(\overline{t_1})$ and $\overline{x_2} \in \Delta_2^{(2)}(\overline{t_2})$ we obtain the same conclusion. Should we have $\overline{x_1} \in \Delta_2^{(1)}(\overline{t_1})$ and $\overline{x_2} \in \Delta_2^{(2)}(\overline{t_2})$, similar arguments also give $x \in \Delta_2^+(t)$. So

$$\begin{aligned}\left\{x_1x_2 \,\middle|\, \overline{x_1} \in \Delta_2^{(1)}(\overline{t_1}),\ \overline{x_2} \in \Delta_1^{(2)}(\overline{t_2})\right\} &\cup \left\{x_1x_2 \,\middle|\, \overline{x_1} \in \Delta_1^{(1)}(\overline{t_1}),\ \overline{x_2} \in \Delta_2^{(2)}(\overline{t_2})\right\} \\ &\cup \left\{x_1x_2 \,\middle|\, \overline{x_1} \in \Delta_2^{(1)}(\overline{t_1}),\ \overline{x_2} \in \Delta_2^{(2)}(\overline{t_2})\right\} \subseteq \Delta_2^+(t).\end{aligned} \tag{4.8.4}$$

Since $x = x_1x_2 \in \Delta_2^+(t)$ implies $d^{(i)}(\overline{t_i}, \overline{x_i}) \le 2$ for $i = 1, 2$, $\Delta_2^+(t)$ is the union of the two sets in (4.8.3) and (4.8.4). Thus, employing [10],

$$\left|\Delta_2^+(t)\right| = \frac{1}{8}(q-\delta)^3(q-4-\delta) + (q-\delta)(q-2-\delta). \tag{4.8.5}$$

Now, as $q \notin \{3, 5, 9, 13\}$, by [10] the commuting graph for $\overline{L_i}$ is connected of diameter 3. Arguing as above we deduce that $\mathcal{C}(G^+, Y^+)$ is also connected with diameter 3. Because $|Y^+| = 2\left|\overline{t_1}^{\overline{L_1}}\right|\left|\overline{t_2}^{\overline{L_2}}\right| = \frac{1}{2}q^2(q+\delta)^2$, combining (4.8.2) and (4.8.5) we may determine $\left|\Delta_3^+(t)\right|$ to be as stated, so completing the proof of Theorem 4.8. $\square$

Finally we look at $C_G(U)$ where $U \in \mathcal{U}_1^0$. This will prove to be trickier than the other two cases. Put $G^0 = C_G(U)$. So $G^0 \sim q^3 : L_2(q)$. We require an explicit description of G^0 which we now give. Let $Q = \{(\alpha, \beta, \gamma) \,|\, \alpha, \beta, \gamma \in GF(q)\}$ and

$$L = \left\{ \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \,\middle|\, \begin{array}{c} a, b, c, d \in GF(q) \\ ad - bc = 1 \end{array} \right\}.$$

with L acting on Q by right multiplication. Then $Q \sim q^3$ and $L \sim L_2(q)$. Since Q is the 3-dimensional $GF(q)L$-module (see the description on page 15 of [4]), $G^0 \cong Q \rtimes L$. We will identify this semidirect product with G^0, writing $G^0 = QL$. Any $g \in G^0$ has a unique expression $g = g_Qg_L$ where $g_Q \in Q$ and $g_L \in L$ - in what follows we use such subscripts to describe this expression. Set $Y^0 = G^0 \cap Y_2$, let d^0 denote the distance metric and $\Delta_i^0(t)$ the

i^{th} disc of the commuting graph $\mathcal{C}(G^0, Y^0)$. In determining the discs of $\mathcal{C}(G^0, Y^0)$ we make use of the commuting involution graph of $L \cong L_2(q)$ (as given in [10]). So we shall use d^L to denote the distance metric on $\mathcal{C}(L, L \cap Y^0)$ and for $x \in L \cap Y^0$ and $i \in \mathbb{N}$, $\Delta_i^L(x) = \{ y \in L \cap Y^0 | \, d^L(x,y) = i \}$. It is straightforward to check that

$$L \cap Y^0 = \left\{ \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & bc - a^2 & -ab \\ c^2 & -2ac & a^2 \end{pmatrix} \middle| \begin{array}{c} a, b, c \in GF(q) \\ a^2 + b^2 = -1 \end{array} \right\}$$

and, as G^0 has one conjugacy class of involutions, $Y^0 = \{ x_Q x_L | \, x_L \in L \cap Y^0 \text{ and } x_L \text{ inverts } x_Q \}$. Without loss of generality, we take

$$t = t_L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and, up until Theorem 4.14, we will assume that $q \notin \{3, 5, 9, 13\}$. Thus the diameter of $\mathcal{C}(L, L \cap Y^0)$ is 3.

Lemma 4.9 *(i)* $Qt \cap Y^0 = \{ (\alpha, \beta, -\alpha)t | \, \alpha, \beta \in GF(q) \}$ *and* $|Qt \cap Y^0| = q^2$.
(ii) $Qt \cap \Delta_1^0(t) = \varnothing$.

Proof A straightforward calculation. $\square$

Lemma 4.10 *We have*

$$\Delta_1^0(t) = \left\{ x \middle| x_Q = (\alpha, 0, \alpha), \; x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}, \; a^2 + b^2 = -1 \right\},$$

and $|\Delta_1^0(t)| = \frac{1}{2} q(q - \delta)$.

Proof Let $x, y \in Y^0$. If $[x, y] = 1$ then clearly $[x_L, y_L] = 1$. From [10] we have

$$\Delta_1^L(t) = \left\{ \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix} \middle| \, a^2 + b^2 = -1 \right\}.$$

If $x_Q = (\alpha, \beta, \gamma)$ and $x_L \in \Delta_1^L(t)$ then $[t, x] = 1$ implies $\alpha = \gamma$ and $\beta = 0$. Moreover, every $x = (\alpha, 0, \alpha)x_L$, where $x_L \in \Delta_1^L(t)$, is in Y^0. Hence, $\Delta_1^0(t)$ is as described above. By [10], for any involution $x_L \in L$ we have $\left| \Delta_1^L(x_L) \right| = \frac{1}{2}(q - \delta)$ and there are q possible values that α can take for a fixed such x_L, proving the lemma. $\square$

Lemma 4.11 *Let* $x \in Y^0$ *with* $x_L \in \Delta_1^L(t)$. *If* $x \notin \Delta_1^0(t)$, *then* $x \in \Delta_2^0(t)$.

Proof Suppose $x \in Y^0$ where $x_Q = (\alpha, \beta, \gamma)$ and

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}.$$

Then x_L inverts x_Q if and only if

$$\begin{aligned} a^2\alpha + 2ab\beta + b^2\gamma &= -\alpha \\ ab\alpha + (b^2 - a^2)\beta - ab\gamma &= -\beta \\ b^2\alpha - 2ab\beta + a^2\gamma &= -\gamma. \end{aligned} \tag{4.11.1}$$

Suppose first that $\delta = -1$. Then, since -1 is not square in $GF(q)$, we must have $a, b \neq 0$. Rearranging the first equation gives $\alpha = 2ab^{-1}\beta + \gamma$ and (4.11.1) remains consistent. Note that when $\beta = 0$, we have $\alpha = \gamma$ and so $x \in \Delta_1^0(t)$. So assume $\beta \neq 0$. Let $y \in \Delta_1^0(t)$ where $y_Q = (ab^{-1}\beta + \gamma, 0, ab^{-1}\beta + \gamma)$ and

$$y_L = \begin{pmatrix} b^2 & -2ab & a^2 \\ -ab & a^2 - b^2 & ab \\ a^2 & 2ab & b^2 \end{pmatrix}.$$

It is a routine calculation to show that $[x, y] = 1$, proving the lemma for $\delta = -1$. Now assume $\delta = 1$. If $a, b \neq 0$ then the argument from the previous case still holds, so assume first that $a = 0$, and hence b is the unique element in $GF(q)$ that squares to -1. Then (4.11.1) simplifies to $\alpha = \gamma$, and so $x_Q = (\alpha, \beta, \alpha)$. Let $z \in \Delta_1^0(t)$ where $z_Q = (\alpha, 0, \alpha)$ and

$$z_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An easy calculation shows that $[x, z] = 1$. Similarly, assuming $b = 0$ then a is the unique element of $GF(q)$ squaring to -1 and (4.11.1) simplifies to $\beta = 0$. Then $x_Q = (\alpha, 0, \gamma)$ and if $w \in \Delta_1^0(t)$ where $w_Q = (2^{-1}(\alpha + \gamma), 0, 2^{-1}(\alpha + \gamma))$ and

$$w_L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then an easy check shows that $[x, w] = 1$, proving the lemma for $\delta = 1$. $\square$

Lemma 4.12 *We have $Qt \cap Y^0 \subseteq \{t\} \cup \Delta_2^0(t) \cup \Delta_3^0(t)$. Moreover,*

$$\begin{aligned} \left|Qt \cap \Delta_2^0(t)\right| &= \frac{1}{2}(q^2 - (1+\delta)q + \delta); \text{ and} \\ \left|Qt \cap \Delta_3^0(t)\right| &= \frac{1}{2}(q^2 + (1+\delta)q - (2+\delta)). \end{aligned}$$

Proof If $x \in Qt \cap Y^0$ and $x \neq t$ then $x_Q = (\alpha, \beta, -\alpha)$ and $x \notin \Delta_1^0(t)$ by Lemma 4.9. Let $y \in \Delta_1^0(t)$ where $y_Q = (\gamma, 0, \gamma)$ and

$$y_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

with $a^2 + b^2 = -1$. Then $[x, y] = 1$ if and only if $-a^2\alpha = ab\beta$ and $-b^2\beta = ab\alpha$.
Assume first that $\delta = -1$. Since -1 is not square in $GF(q)$, we have $a, b \neq 0$ and so $\alpha = -a^{-1}b\beta$. Hence if $y \in Qt$ is such that $y_Q = (-a^{-1}b\beta, \beta, a^{-1}b\beta)$, then $y \in \Delta_2^0(t)$. By looking at $\Delta_1^L(t)$, we see there are $q+1$ ordered pairs (a, b) that satisfy $a^2+b^2 = -1$. However, if $(a, b) \neq (c, d)$ where $a^2 + b^2 = c^2 + d^2 = -1$ and $a^{-1}b = c^{-1}d$, then an easy calculation shows that $(c, d) = (-a, -b)$. Hence there are $\frac{1}{2}(q+1)$ distinct values of $a^{-1}b$ satisfying the relevant conditions. If $\beta = 0$ then $x = t$ and if $\beta \neq 0$ there are $\frac{1}{2}(q^2 - 1)$ elements in $Qt \cap \Delta_2^0(t)$.
Assume now that $\delta = 1$. If $a, b \neq 0$ then the arguments of the previous case still hold, with the exception that there are now $q - 1$ ordered pairs (a, b) that satisfy $a^2 + b^2 = -1$. However, as $a, b \neq 0$ we exclude the pairs $(\pm i, 0)$ and $(0, \pm i)$ where i is the unique element of $GF(q)$ squaring to -1. Hence there are $q - 5$ ordered pairs (a, b) satisfying $a^2 + b^2 = -1$, $a, b \neq 0$ and thus $\frac{1}{2}(q - 5)$ distinct values of $a^{-1}b$. Hence there are $\frac{1}{2}(q - 5)(q - 1)$ elements $z \in Qt \cap \Delta_2^0(t)$ such that $z_Q = (-a^{-1}b\beta, \beta, a^{-1}b\beta)$ where $\beta \neq 0$ (note that if $\beta = 0$, then $z = t$). Suppose $a = 0$, then $b \neq 0$ and so $\beta = 0$. Hence $x_Q = (\alpha, 0, -\alpha)$ and all such x lie in $\Delta_2^0(t)$ if $\alpha \neq 0$. Similarly, if $b = 0$ then $a \neq 0$ and $x_Q = (0, \beta, 0)$ where $\beta \neq 0$ and all such x lie in $\Delta_2^0(t)$. Therefore, $|Qt \cap \Delta_2^0(t)| = \frac{1}{2}(q - 5)(q - 1) + 2(q - 1) = \frac{1}{2}(q - 1)^2$ as required.
Hence it suffices to show that these remaining involutions all lie in $\Delta_3^0(t)$. Let $w \in Qt$ be such that $w_Q = (\gamma, \varepsilon, -\gamma)$. Choose $s \in Y^0$ such that $s_Q = (ab\varepsilon - b^2\gamma, ab\gamma - a^2\varepsilon, b^2\gamma - ab\varepsilon)$ with $ab\gamma \neq a^2\varepsilon$ and

$$s_L = \begin{pmatrix} b^2 & -2ab & a^2 \\ -ab & a^2 - b^2 & ab \\ a^2 & 2ab & b^2 \end{pmatrix},$$

with $a^2 + b^2 = -1$. It is an easy check to show that $s \in \Delta_2^0(t)$, and moreover $[w, s] = 1$. This accounts for the remaining involutions in Qt, thus proving the lemma. $\square$

Lemma 4.13 *Suppose $x \in Y^0$ with $x_L \in \Delta_2^L(t)$. Then $x \in \Delta_2^0(t)$.*

Proof It can be shown (see Remark 2.3 of [10], noting the result holds for any odd q) that for a fixed $a, b \in GF(q)$ such that $a^2 + b^2 = -1$,

$$C_L\left(\begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}\right) = \left\{ \begin{pmatrix} c^2 & 2cd & d^2 \\ ce & de - c^2 & -cd \\ e^2 & -2ce & c^2 \end{pmatrix} \;\middle|\; \begin{array}{c} c^2 + de = -1 \\ b(e + d) = -2ac \end{array} \right\}.$$

Let $y \in Y^0$ be such that $y_Q = (\alpha, \beta, \gamma)$ and

$$y_L = \begin{pmatrix} c^2 & 2cd & d^2 \\ ce & de - c^2 & -cd \\ e^2 & -2ce & c^2 \end{pmatrix} \in \Delta_2^L(t).$$

So there exists $a, b \in GF(q)$ such that $a^2 + b^2 = -1$ and $b(e + d) = -2ac$ with $d \neq e$. Since y_L inverts y_Q, we have

$$\begin{aligned} c^2\alpha + 2cd\beta + d^2\gamma &= -\alpha \\ ce\alpha + (de - c^2)\beta - cd\gamma &= -\beta \\ e^2\alpha - 2ce\beta + c^2\gamma &= -\gamma. \end{aligned} \tag{4.13.1}$$

Assume first that $\delta = -1$. Since -1 is not square in $GF(q)$, then $d, e \neq 0$ and any $a, b \in GF(q)$ such that $b(d + e) = -2ac$ and $a^2 + b^2 = -1$ must also be non-zero. Moreover, if $c = 0$ then $d = -e^{-1}$ and $b(d - d^{-1}) = 0$ implying that $d = -1$. But then $y_L = t \notin \Delta_2^L(t)$, so $c \neq 0$. The system (4.13.1) now simplifies to $\alpha = 2ce^{-1}\beta + de^{-1}\gamma$. Let $x \in \Delta_1^0(t)$ be such that $x_Q = (\varepsilon, 0, \varepsilon)$ and

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

where $\varepsilon = -abc^{-1}e^{-1}(\gamma + (d-e)^{-1}(2c + a^{-1}be - ab^{-1}e - (ab)^{-1}e)\beta)$. Using the `PolynomialAlgebra` command in Magma [15] we verify that $[x, y] = 1$ and so $y \in \Delta_2^0(t)$.

Assume now that $\delta = 1$. Let $a, b \in GF(q)$ be such that $a^2 + b^2 = -1$ and $b(d + e) = -2ac$. Suppose $c, d, e \neq 0$ and $d \neq -e$. Then $b(d + e) = -2ac \neq 0$ and so $a, b \neq 0$. The argument for the case when $\delta = -1$ then holds. Suppose then $c, d, e \neq 0$ and $d = -e$. Then $b(d + e) = -2ac = 0$ and since $c \neq 0$ we must have $a = 0$ and $b^2 = -1$. The system (4.13.1) then becomes $\alpha = 2ce^{-1}\beta - \gamma$. If $x \in \Delta_1^0(t)$ is such that $x_Q = (-c^{-1}e^{-1}\beta, 0, -c^{-1}e^{-1}\beta)$ and

$$x_L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then a routine check shows that $[x, y] = 1$.

Now assume $c \neq 0$ and $d = 0$. Since $y_L \in \Delta_2^L(t)$, we must have $e \neq 0$ and so $c^2 = -1$. The system (4.13.1) becomes $\alpha = 2ce^{-1}\beta$ and using Magma [15] we deduce that if $x \in \Delta_1^0(t)$ where $x_Q = (\varepsilon, 0, \varepsilon)$,

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

and $\varepsilon = (ce^{-1}(1-a^2)-ab)\beta - 2^{-1}b^2\gamma$, then $[x,y]=1$. Similarly, if $c \neq 0$ and $e=0$, then $d \neq 0$ and $c^2=-1$. The system (4.13.1) becomes $\beta = 2^{-1}cd\gamma$ and [15] will verify that if $x \in \Delta_1^0(t)$ where $x_Q = (\varepsilon, 0, \varepsilon)$,

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2-a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

and $\varepsilon = 2^{-1}(\gamma - b^2\alpha + abcd\gamma - a^2\gamma)$, then $[x,y]=1$.
Finally, if $c=0$ then $d=-e^{-1}$ and so $a^2=-1$ and $b=0$ satisfies the relevant conditions. Note that if $d=\pm 1$ then $y_L = t$, so we may assume $d \neq \pm 1$. The system (4.13.1) becomes $\alpha = d^2\gamma$, so if $x \in \Delta_1^0(t)$ where $x_Q = (2d^2\gamma(1-d^2)^{-1}, 0, 2d^2\gamma(1-d^2)^{-1})$ and

$$x_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then a routine check again shows that $[x,y]=1$. Therefore, for all $y \in Y^0$ such that $y_L \in \Delta_2^L(t)$, there exists $x \in \Delta_1^L(t)$ such that $[x,y]=1$, so proving the lemma. $\square$

Theorem 4.14 *If $q \notin \{3,5,9,13\}$, then $\mathcal{C}(G^0, Y^0)$ is connected of diameter 3, with disc sizes*

$$\begin{aligned} \left|\Delta_1^0(t)\right| &= \frac{1}{2}q(q-\delta); \\ \left|\Delta_2^0(t)\right| &= \frac{1}{4}(q^4-(2\delta+2)q^3+(1+2\delta)q^2-2q+2\delta); \ \textit{and} \\ \left|\Delta_3^0(t)\right| &= \frac{1}{4}(q^4+2(1+2\delta)q^3-(3+2\delta)q^2+2(1+\delta)q-2(2+\delta)). \end{aligned}$$

Proof It is known that $\mathcal{C}(L, L\cap Y^0)$ has diameter 3. Hence, for any $h_i \in \Delta_i^L(t)$, there exists $h_{i\pm1} \in \Delta_{i\pm1}^L(t)$ that commutes with h_i, $i=1,2$. Therefore for any $x \in Y^0$ where $x_L \in \Delta_i^L(t)$, there exists $y \in Y^0$ with $y_L \in \Delta_{i\pm1}^L(t)$ and such that $[x,y]=1$. Since any $z \in Y^0$ where $z_L \in \Delta_3^L(t)$ must commute with some $w \in Y^0$ with $w_L \in \Delta_2^L(t)$ (which lies in $\Delta_2^0(t)$ by Lemma 4.13), $z \in \Delta_3^0(t)$. This finally covers all possible involutions in Y^0 and so the diameter of $\mathcal{C}(G^0, Y^0)$ is 3. Now for each $x_L \in L\cap Y^0$, $|Qx_L \cap Y^0| = q^2$ by Lemma 4.9, and therefore there are $\frac{1}{2}q^2(q-\delta)$ involutions $y \in Y^0$ such that $y_L \in \Delta_1^L(t)$. From Lemma 4.10, $|\Delta_1^0(t)| = \frac{1}{2}q(q-\delta)$. Therefore

$$\left| \bigcup_{x_L \in \Delta_1^L(t)} Qx_L \cap \Delta_2^0(t) \right| = \frac{1}{2}q^2(q-\delta) - \frac{1}{2}q(q-\delta) = \frac{1}{2}q(q-1)(q-\delta).$$

There are $q^2\left|\Delta_2^L(t)\right|$ involutions $z \in Y^0$ such that $z_L \in \Delta_2^L(t)$, which is known to be $\frac{1}{4}q^2(q-\delta)(q-4-\delta)$ (see [10]). Also, by Lemma 4.12, $|Qt \cap \Delta_2^0(t)| = \frac{1}{2}(q^2-(1+\delta)q-\delta)$. Hence

$$\begin{aligned}\left|\Delta_2^0(t)\right| &= \left|Qt \cap \Delta_2^0(t)\right| + \left|\bigcup_{x_L \in \Delta_1^L(t)} Qx_L \cap \Delta_2^0(t)\right| + q^2\left|\Delta_2^L(t)\right| \\ &= \frac{1}{4}(q^4-(2\delta+2)q^3+(1+2\delta)q^2-2q+2\delta).\end{aligned}$$

Finally, there are $|Y^0| = q^2\,|L \cap Y^0| = \frac{1}{2}q^3(q+\delta)$ involutions in G^0 and therefore

$$\begin{aligned}\left|\Delta_3^0(t)\right| &= \left|Y^0\right| - \left|\Delta_2^0(t)\right| - \left|\Delta_1^0(t)\right| - 1 \\ &= \frac{1}{4}(q^4+2(1+2\delta)q^3-(3+2\delta)q^2+2(1+\delta)q-2(2+\delta))\end{aligned}$$

which proves Theorem 4.14. □

Theorem 4.15 *$\mathcal{C}(G, Y_2)$ is connected of diameter at most 3.*

Proof For $q \le 13$, this is easily checked using MAGMA [15], so assume $q > 13$. Combining Lemma 4.3 with Theorems 4.6, 4.8 and 4.14 yields the theorem. □

We now focus on finding the disc sizes of $\mathcal{C}(G, Y_2)$. First, we need the following four lemmas.

Lemma 4.16 *The sets $\mathcal{U}_1^+$, $\mathcal{U}_1^-$ and $\mathcal{U}_1^0$ are single $C_G(t)$-orbits. Moreover,*

$$\begin{aligned}\left|\mathcal{U}_1^0\right| &= q+1; \\ \left|\mathcal{U}_1^+\right| &= \frac{1}{2}q(q+\delta); \text{ and} \\ \left|\mathcal{U}_1^-\right| &= \frac{1}{2}q(q-\delta).\end{aligned}$$

Proof Since $C_G(t)$ acts orthogonally on $C_V(t)$, the first statement is immediate. Recall the Gram matrix J for V with respect to $(\ ,\ )$ and the basis $\{v_i\}$. Observe that $C_V(t) = \{(\alpha,\beta,\gamma,0,\gamma) \mid \alpha,\beta,\gamma \in GF(q)\}$ and so a basis for $C_V(t)$ is $\{v_1, v_2, v_3+v_5\}$. Let $v = (\alpha,\beta,\gamma,0,\gamma)$ be a non-zero vector in $C_V(t)$ and so $(v,v) = 2\alpha\beta + 2\gamma^2$.
Suppose v is isotropic, so $C_G(\langle v\rangle) \sim q^3 : L_2(q)$ and $(v,v) = 2\alpha\beta+2\gamma^2 = 0$. If $\gamma = 0$, then $\alpha\beta = 0$ and so either $\alpha = 0$ or $\beta = 0$ (but not both as $v \neq 0$). Hence there are $2(q-1)$ such vectors with $\gamma = 0$. If $\gamma \neq 0$, then $\alpha = -\beta^{-1}\gamma^2$ and there are $(q-1)^2$ such vectors satisfying this. Hence there are $2(q-1)+(q-1)^2 = (q-1)(q+1)$ non-zero isotropic vectors contained

in $C_V(t)$ and thus $q+1$ isotropic 1-subspaces of $C_V(t)$.
Suppose now v is $C_G(t)$-conjugate to v_3+v_5, which is non-isotropic. Note that $\langle v_3+v_5\rangle^\perp \cap C_V(t)$ is a 2-subspace of V of $+$-type. If $\delta = 1$, then by Lemma 4.1(ii), $\langle v_3+v_5\rangle^\perp$ is a 4-subspace of V of $+$-type and so $C_G(\langle v_3+v_5\rangle) \sim SL_2(q) \circ SL_2(q)$. While $\delta = -1$ gives that $\langle v_3+v_5\rangle^\perp$ is a 4-subspace of V of $-$-type and so $C_G(\langle v_3+v_5\rangle) \sim L_2(q^2)$. A quick check shows that $(v_3+v_5, v_3+v_5) = 2$ and so $(v,v) = 2\alpha\beta + 2\gamma^2 = 2\lambda^2$ for some $\lambda \in GF(q)^*$. Thus, $\alpha\beta+\gamma^2 = \lambda^2$ for some $\lambda \in GF(q)^*$. If $\gamma = 0$, then $\alpha = \beta^{-1}\lambda^2$ and so there are $q-1$ such vectors that satisfy this. If $\gamma = \pm\lambda$, then $\alpha\beta = 0$ and so for both values of γ, there are $2(q-1)+1$ vectors that satisfy this. Finally, if $\gamma \in GF(q) \setminus \{0, \lambda, -\lambda\}$, then $\alpha\beta = 1-\gamma^2 \neq 0$ and so $\alpha = \beta^{-1}(1-\gamma^2)$. There are $(q-1)(q-3)$ such vectors that satisfy this. Hence for any given λ, there exist $(q-1)+4(q-1)+2+(q-1)(q-3) = q(q+1)$ vectors that satisfy $\alpha\beta+\gamma^2 = \lambda^2$. Since there are $\frac{1}{2}(q-1)$ squares in $GF(q)$, there are $q(q+1)(q-1)$ vectors that are $C_G(t)$-conjugate to v_3+v_5 and hence $\frac{1}{2}(q+1)$ 1-subspaces of $C_V(t)$ that are $C_G(t)$-conjugate to v_3+v_5.
This leaves the remaining orbit $\mathcal{U}_1^-$. Recall there are q^2+q+1 subspaces of $C_V(t_1)$ of dimension 1, and hence the size of the remaining orbit is $q^2+q+1-(q+1)-\frac{1}{2}q(q+1) = \frac{1}{2}q(q-1)$, so proving the lemma. $\square$

Corollary 4.17 *The sets $\mathcal{U}_2^+$, $\mathcal{U}_2^-$ and $\mathcal{U}_2^0$ are single $C_G(t)$-orbits. Moreover,*

$$\begin{aligned} \left|\mathcal{U}_2^0\right| &= q+1; \\ \left|\mathcal{U}_2^+\right| &= \frac{1}{2}q(q+1); \text{ and} \\ \left|\mathcal{U}_2^-\right| &= \frac{1}{2}q(q-1). \end{aligned}$$

Proof Since $C_V(t)$ is 3-dimensional, $U^\perp \cap C_V(t) \in \mathcal{U}_1$ for any $U \in \mathcal{U}_2$, and so the result is immediate by Lemma 4.16. $\square$

Lemma 4.18 *Let $U, U' \in \mathcal{U}_2$ be such that $U \neq U'$. Then $C_G(U) \cap C_G(U') \cap Y_2 = \{t\}$.*

Proof Suppose $x \in C_G(U) \cap C_G(U') \cap Y_2$. Since $U \neq U'$ and x fixes each 2-subspace pointwise, $U + U' = C_V(t)$ and so x fixes $C_V(t)$ pointwise. That is to say, $C_V(x) = C_V(t)$ and so $t = x$ by Lemma 4.2(i). $\square$

Lemma 4.19 *Let $U_0 \in \mathcal{U}_2^0$, and $G^0 = QL$, Y^0 be as defined in the discussion prior to Lemma 4.9. Let $\rho : C_G(U_0^\perp \cap C_V(t)) \to G^0$ be an isomorphism such that*

$$t^\rho = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $C_G(U_0)$ is totally disconnected and $(C_G(U_0) \cap Y_2)^\rho = Qt \cap Y^0$.

Proof Since $U_0^\perp \cap C_V(t)$ is isotropic, it must lie inside of U_0 and so $C_G(U_0) \leq C_G(U_0^\perp \cap C_V(t))$. As t fixes U_0 pointwise, $t^\rho \in (C_G(U_0))^\rho \sim q^2 : \frac{q-\delta}{2}$ by Lemma 4.5(i). The subgroup of L with shape $\frac{q-\delta}{2}$ contains one single involution which must necessarily be t^ρ. For all $x \in Y^0$, we have $x_L^2 = 1$ and x_L inverts x_Q, so $(C_G(U_0) \cap Y_2)^\rho \subseteq Qt \cap Y^0$. By comparing the orders of both sides, we get equality. By Lemma 4.9(ii) $C_G(U_0) \cap C_G(t) \cap Y_2 = \{t\}$, hence $C_G(U_0)$ is totally disconnected. $\square$

Lemma 4.20 $|\Delta_1(t)| = \frac{1}{2}q(q^2 + (1-\delta)q + \delta)$.

Proof Clearly, $x \in \Delta_1(t)$ if and only if $x \in \Delta_1(t) \cap C_G(U)$ for $U = C_V(t) \cap C_V(x)$, so

$$\Delta_1(t) = \bigcup_{U \in \mathcal{U}_1 \cup \mathcal{U}_2} (\Delta_1(t) \cap C_G(U)).$$

If $W, W' \in \mathcal{U}_1$ with $W \neq W'$, then $W \oplus W' \in \mathcal{U}_2$ and if $y \in C_G(W) \cap C_G(W')$ then $y \in C_G(W \oplus W')$ and hence $y \in C_G(W'')$ for any 1-subspace W'' of $W \oplus W'$. Since there are $q+1$ subspaces of W'' of dimension 1, any such y will lie in exactly $q+1$ such $C_G(U)$ for $U \in \mathcal{U}_1$. Together with $C_G(W'')$ and Lemma 4.18,

$$|\Delta_1(t)| = \sum_{U \in \mathcal{U}_1} |\Delta_1(t) \cap C_G(U)| - q \sum_{U \in \mathcal{U}_2} |\Delta_1(t_1) \cap C_G(U)|.$$

Combining Lemmas 4.16, 4.19 and Corollary 4.17 with Theorems 4.6, 4.8, 4.14 and [10], we have

$$\begin{aligned}
|\Delta_1(t)| = &\frac{1}{2}q(q+1)(q-\delta) + \frac{1}{2}q(q+\delta)\left[\frac{1}{2}(q-\delta)^2 + 1\right] + \frac{1}{4}q(q-\delta)(q^2-1) \\
&- \frac{1}{2}q(q-\delta)\left[\frac{1}{2}q(q+1) + \frac{1}{2}q(q-1)\right] \\
= &\frac{1}{2}q(q^2 + (1-\delta)q + \delta)
\end{aligned}$$

as required. $\square$

We now consider the second disc $\Delta_2(t)$. Here, we must be careful as elements that are distance 2 from t in some subgroup $C_G(U)$ may not be distance 2 from t in another subgroup $C_G(U')$. Moreover, there may be elements that are distance 3 from t in every such subgroup centralizing an element of $\mathcal{U}_1$, but actually are distance 2 from t in G. We introduce the following notation. Let $\Delta_2^K(t)$ be the second disc in the commuting involution graph $\mathcal{C}(K, K \cap Y_2)$ and

$$\Gamma_i(K) = \left\{ x \in \Delta_2^K(t) \middle| \dim C_V(\langle t, x\rangle) = i \right\}$$

for $K = C_G(U)$, $U \in \mathcal{U}_1 \cup \mathcal{U}_2$. Clearly, $\Delta_2(t) = \Gamma_1(G)\dot{\cup}\Gamma_2(G)$. A full list of cases with corresponding notation is found in Table 1. Also we use the following notation: for any $U \leq C_V(t)$, define $\mathcal{U}_i(U)$ to be the totality of i-dimensional subspaces of U and $\mathcal{W}_i(U)$ to be the totality of i-dimensional subspaces of $C_V(t)$ containing U. Note that $\mathcal{U}_i = \mathcal{U}_i(C_V(t))$.

Lemma 4.21 *(i) If $W \in \mathcal{U}_2^0$, then $|\mathcal{U}_1^0 \cap \mathcal{U}_1(W)| = 1$ and $\left|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)\right| = q$.*
(ii) If $W \in \mathcal{U}_2^+$, then $|\mathcal{U}_1^0 \cap \mathcal{U}_1(W)| = 2$ and $\left|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)\right| = \left|\mathcal{U}_1^- \cap \mathcal{U}_1(W)\right| = \frac{q-1}{2}$.
(iii) If $W \in \mathcal{U}_2^-$, then $\left|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)\right| = \left|\mathcal{U}_1^- \cap \mathcal{U}_1(W)\right| = \frac{q+1}{2}$.

Proof Recall the Gram matrix J, with respect to the ordered basis $\{v_i\}$, $i = 1, \dots, 5$. Suppose $W^\perp \cap C_V(t) = U_0 \in \mathcal{U}_1^0$. Without loss of generality, choose $W = \langle v_1, v_3 + v_5\rangle$. Clearly $\langle v_1\rangle \in \mathcal{U}_1^0$, and $\langle v_3 + v_5\rangle^\perp \cap C_V(t) \in \mathcal{U}_2^+$. Since

$$(v_1 + \lambda(v_3 + v_5), v_1 + \lambda(v_3 + v_5)) = \lambda^2(v_3 + v_5),$$

$v_1 + \lambda(v_3 + v_5)$ lies in the same $C_G(t)$-orbit as $v_3 + v_5$ and so $\langle v_1 + \lambda(v_3 + v_5)\rangle^\perp \cap C_V(t) \in \mathcal{U}_2^+$, proving (i).
Suppose now $W \in \mathcal{U}_2^+$. Without loss of generality, choose $W = \langle v_1, v_2\rangle$. Clearly $\langle v_1\rangle$, $\langle v_2\rangle \in \mathcal{U}_1^0$. Let $U_\lambda = v_1 + \lambda v_2$ for $\lambda \neq 0$ and note that $(v_1 + \lambda v_2, v_1 + \lambda v_2) = 2\lambda = \mu \neq 0$. Since the type of $U_\lambda^\perp$ is determined by whether μ is a square or a non-square in $GF(q)$, and there are $\frac{q-1}{2}$ of each, it is clear that there exist $\frac{q-1}{2}$ such U_λ for which $U_\lambda^\perp$ is of $+$-type, and similarly for $-$-type, proving (ii).
Finally suppose $W \in \mathcal{U}_2^-$, so for all $v \in W$, $(v, v) \neq 0$. The simple orthogonal group on W is cyclic of order $\frac{q+1}{2}$ and acts on the 1-subspaces of W in exactly two orbits with representatives $\langle u_1\rangle$ and $\langle u_2\rangle$ where (u_1, u_1) is a square and (u_2, u_2) is a non-square in $GF(q)$. Since $|\mathcal{U}_1(W)| = q+1$, both orbits must be of size $\frac{q+1}{2}$. This proves (iii) and hence the lemma follows. □

Corollary 4.22 *Let $U \in \mathcal{U}_1$. Then,*

(i) $|\mathcal{W}_2(U)| = q + 1$
(ii) If $U \in \mathcal{U}_1^0$, then $|\mathcal{U}_2^0 \cap \mathcal{W}_2(U)| = 1$ and $\left|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)\right| = q$.
(iii) If $U \in \mathcal{U}_1^\delta$, then $|\mathcal{U}_2^0 \cap \mathcal{W}_2(U)| = 2$ and $\left|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)\right| = \left|\mathcal{U}_2^- \cap \mathcal{W}_2(U)\right| = \frac{q-1}{2}$.
(iv) If $U \in \mathcal{U}_2^{-\delta}$, then $\left|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)\right| = \left|\mathcal{U}_2^- \cap \mathcal{W}_2(U)\right| = \frac{q+1}{2}$.

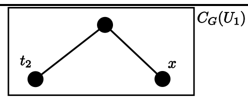
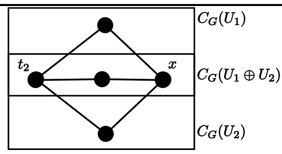
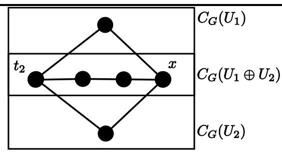
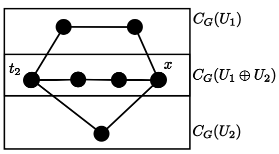
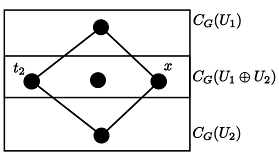
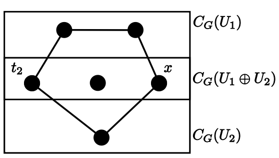
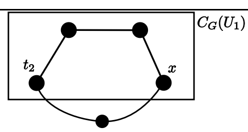
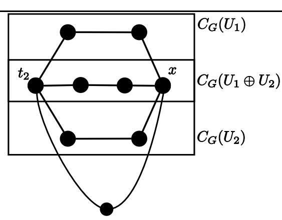
Case	Configuration	Properties	Description as Set
1		$x \in \Delta_2^{C_G(U_1)}(t)$ $U_1 = C_V(\langle t, x\rangle) \in \mathcal{U}_1$	$\dot{\bigcup}_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$
2		$x \in \Delta_2^{C_G(U_1 \oplus U_2)}(t)$ $U_1 \oplus U_2 = C_V(\langle t, x\rangle)$ $U_i \in \mathcal{U}_1$	$\dot{\bigcup}_{W \in \mathcal{U}_2} \Gamma_2(C_G(W))$
3	   	$x \in \Delta_2^{C_G(U_2)}(t)$ for some $U_2 \leq C_V(\langle t, x\rangle)$ $x \notin \Delta_2^{C_G(U_1 \oplus U_2)}(t)$ $U_1 \oplus U_2 = C_V(\langle t, x\rangle)$ $U_i \in \mathcal{U}_1$	$\bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U)) \setminus \dot{\bigcup}_{W \in \mathcal{U}_2} \Gamma_2(C_G(W))$
4		$x \in \Delta_2^{G}(t)$ $x \notin \Delta_2^{C_G(U_1)}(t)$ $U_1 = C_V(\langle t, x\rangle) \in \mathcal{U}_1$	$\Gamma_1(G) \setminus \dot{\bigcup}_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$
5		$x \in \Delta_2^{G}(t)$ $x \notin \Delta_2^{C_G(U_i)}(t)$ for any $U_i \leq C_V(\langle t, x\rangle)$ $U_1 \oplus U_2 = C_V(\langle t, x\rangle)$ $U_i \in \mathcal{U}_1$	$\Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$

Table 1: List of cases in $\Delta_2(t)$

Proof Let $U \leq W \leq C_V(t)$. Then $W^\perp \cap C_V(t) \leq U^\perp \cap C_V(t) \leq C_V(t)$. The result follows from Lemma 4.21. $\square$

Lemma 4.23 *Let $U \in \mathcal{U}_1^0$ and $W \in \mathcal{U}_2^+ \cap \mathcal{W}_2(U)$. If $x \in Y_2 \cap C_G(W)$ is such that $d^{C_G(W)}(t,x) = 3$, then $d^{C_G(U)}(t,x) = 3$. Moreover,*

$$|\Gamma_1(C_G(U))| = \begin{cases} \frac{1}{4}q(q-3)(q-1)^2 & q \equiv 1 \pmod 4 \\ \frac{1}{4}q(q-1)^2(q+1) & q \equiv -1 \pmod 4). \end{cases}$$

Proof Recall that $C_G(U) = QL \sim G^0$ where G^0 is defined as in the discussion prior to Lemma 4.9. By conjugacy, we may assume $L = C_G(W)$. Now $C_G(U) \cap C_G(t) = Q_0C_L(t) \sim q : Dih(q-\delta)$ where $Q_0 \leq Q$ is elementary abelian of order q. Let $x \in Q_0C_L(t) \cap Y_2$, so $x_L^2 = 1$ and x_L inverts x_Q. Clearly, $x_L^{x_Q} = x_Lx_Q^2 \notin L$ since Q_0 is of odd order. Hence, $C_L(t)$ is self-normalizing in $Q_0C_L(t)$ and thus there are q distinct conjugates of $C_L(t)$ in $Q_0C_L(t)$. Let $g \in Q_0C_L(t) \setminus C_L(t)$, so $C_L(t)^g \neq C_L(t)$. Now $[C_L(t), t] = [C_L(t)^g, t] = 1$ and so $\langle C_L(t), C_L(t)^g\rangle$ centralizes t. If $C_L(t)$, $C_L(t)^g \leq L^h$ for some $h \in QL$, then $\langle C_L(t), C_L(t)^g\rangle \leq L^h$. However, $C_L(t) \lneqq \langle C_L(t), C_L(t)^g\rangle \leq C_L(t)$, a contradiction. Hence every conjugate of $C_L(t)$ lies in a different conjugate of L and so there are q distinct $Q_0C_L(t)$-conjugates of L. Therefore, $\mathcal{U}_2^+ \cap \mathcal{W}_2(U)$ is contained in the same $C_G(U) \cap C_G(t)$-orbit, and $\left|U_2^+ \cap \mathcal{W}_2(U)\right| = q$ by Corollary 4.22. There are exactly q $+$-type 2-subspaces of $C_V(t)$ containing U, all of which lie in the same $C_G(U) \cap C_G(t)$ orbit.

Let $x \in C_G(W) \cap Y_2$ be such that $d^{C_G(W)}(t,x) = 3$. Suppose $W^g \in \mathcal{U}_2^+ \cap \mathcal{W}_2(U)$ for some $g \in C_G(U) \cap C_G(t)$, $W \neq W^g$. If $d^{C_G(U)}(t,x) = 2$ then $d^{C_G(U)}(t^g, x^g) = d^{C_G(U)}(t, x^g) = 2$, and $d^{C_G(W)}(t,x) = d^{C_G(W')}(t,x^g) = 3$. Hence it suffices to prove the lemma for $C_G(W)$. By Theorem 4.14, any involution distance 3 away from t in L is necessarily distance 3 away from t in $C_G(U)$, proving the first statement.

Let $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$, so $C_G(W_0) \sim q^2 : \frac{q-\delta}{2}$. By Lemma 4.19, $\Delta_2^{C_G(U)}(t) \cap C_G(W_0) = Qt \cap \Delta_2^{C_G(U)}(t)$. Let W_i, $i = 1, \ldots, q$ be the subspaces in $\mathcal{U}_2^+ \cap \mathcal{W}_2(U)$. From Lemma 4.18, $C_G(W_i) \cap C_G(W_j) \cap Y_2 = \{t\}$ if and only if $i = j$. Using Corollary 4.22(i) with [10], we have

$$\left|\bigcup_{i=1}^{q} \Delta_2^{C_G(W_i)}(t)\right| = \frac{1}{4}q(q-\delta)(q-4-\delta). \tag{4.23.1}$$

Combining Lemma 4.12 with (4.23.1),

$$\begin{aligned}|\Gamma_2(C_G(U))| &= \left|\bigcup_{i=1}^{q} \Delta_2^{C_G(W_i)}(t)\right| + \left|\Delta_2^{C_G(U)}(t) \cap C_G(W_0)\right| \\ &= \frac{1}{4}(q^3 - 2(1+\delta)q^2 + (2\delta-1)q + 2\delta). \end{aligned} \tag{4.23.2}$$

Together, (4.23.2) and Theorem 4.14 give

$$\begin{aligned}|\Gamma_1(C_G(U))| &= \left|\Delta_2^{C_G(U)}(t)\right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{4}q(q^3 - (2\delta+3)q^2 + (4\delta+3)q - 2\delta - 1)\end{aligned}$$

as required. $\square$

Lemma 4.24 *Let $t, x \in L_2(q)$. Then $d^{L_2(q)}(t,x) \leq 2$ if and only if the order of tx divides $\frac{1}{2}(q-\delta)$.*

Proof See Lemma 2.11 of [10]. $\square$

Lemma 4.25 *Let $U \in \mathcal{U}_1^+$, and $W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U)$.*

(i) If $\delta = 1$ and $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$, then $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$.
(ii) If $x \in Y_2 \cap C_G(W)$ is such that $d^{C_G(W)}(t,x) = 3$, then $d^{C_G(U)}(t,x) = 3$ and

$$|\Gamma_1(C_G(U))| = \begin{cases} \frac{1}{8}(q-1)(q-3)(q^2-6q+13) & q \equiv 1 \pmod 4 \\ \frac{1}{8}(q^2-1)(q^2-2q+5) & q \equiv -1 \pmod 4. \end{cases}$$

Proof Recall that $C_G(U) \sim G^+ \sim L_1 \circ L_2$ for $L_1 \sim SL_2(q) \sim L_2$. Suppose $y \in C_G(W)$ is such that $d^{C_G(W)}(t,y) = 3$. Since $C_G(W) \sim L_2(q)$ is simple, then $y = gg^\varphi$ for some $g \in L_1$ and $\varphi : L_1 \to L_2$. Since $t \in C_G(W)$, write $t = ss^\varphi$ for some $s \in L_1$. Then $d^{L_1}(s,g) = 3$, so $d^{C_G(U)}(t,y) = 3$ by Theorem 4.8, and thus

$$\Delta_3^{C_G(W)}(t) \subseteq \Delta_3^{C_G(U)}(t) \qquad \text{for all } W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U). \tag{4.25.1}$$

If $\delta = -1$, then $\mathcal{U}_2^0 \cap \mathcal{W}_2(U) = \varnothing$ by Corollary 4.22. If $\delta = 1$, there exists $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$. Recall that $W_0^\perp \cap C_V(t) \in \mathcal{U}_1^0$ so $C_G(W_0) \leq C_G(W_0^\perp \cap C_V(t)) \sim G^0 = QL$. By Lemma 4.19, if $x \in C_G(W_0) \cap Y_2$ then $x = x_Q t$ and x_Q is inverted by t and has order p. Since x_Q also lies in $C_G(U)$, we can write $x_Q = hh^\varphi$ for some $h \in L_1$. Now $x_Q^{-1} = h^{-1}h^{-1\varphi}$ and so $x_Q^t = x_Q^{ss^\varphi} = h^s(h^\varphi)^{s^\varphi} = h^{-1}h^{-1\varphi}$. Therefore, $h^s = h^{-1}$ and $h^{\varphi s^\varphi} = h^{-1\varphi}$. Moreover, $x = x_Q t = (hs)(hs)^\varphi$ where $hs \in L_1$ is an element of order 4 squaring to the non-trivial element of $Z(L_1)$, and $h = (hs)s$ has order p. By Lemma 4.24 and [10], $d^{L_1}(hs,s) = 3$ and so $d^{C_G(U)}(t, x_Q t) = 3$ by Theorem 4.8. Therefore,

$$C_G(W_0) \cap \Delta_2^{C_G(U)}(t) = \varnothing \qquad \text{for all } W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U). \tag{4.25.2}$$

Hence combining (4.25.1) with Lemma 4.21, [10] and, if $\delta = 1$, (4.25.2) we get

$$\left| \bigcup_{U \leq W} \Delta_2^{C_G(W)}(t) \right| = |\Gamma_2(C_G(U))| = \frac{1}{4}(q-\delta)^2(q-4-\delta).$$

This, together with Theorem 4.8 yields

$$\begin{aligned} |\Gamma_1(C_G(U))| &= \left|\Delta_2^{C_G(U)}(t)\right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{8}(q-1)(q-1-2\delta)(q^2-(4+2\delta)q+9+4\delta), \end{aligned}$$

which proves the lemma. $\square$

Lemma 4.26 *Let $U \in \mathcal{U}_1^-$, and $W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U)$.*

(i) If $\delta = -1$ and $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$, then $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$.
(ii) We have

$$\left| \Gamma_2(C_G(U)) \setminus \dot{\bigcup_{W \in \mathcal{W}_2(U)}} \Gamma_2(C_G(W)) \right| = \frac{1}{4}(q-2+\delta)(q^2-1)$$

and $|\Gamma_1(C_G(U))| = \frac{1}{4}(q-1)^3(q+1)$.

Proof First assume $\delta = -1$, and consider $C_G(W_0)$. By Lemma 4.19, every involution in $C_G(W_0)$ can be written as xt where x has order p. But $(xt)t = x$ has order p, which does not divide $\frac{1}{2}(q^2-1)$, and hence $d^{C_G(U)}(xt,t) = 3$. In other words, $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$, so proving (i).
Consider then $C_G(W) \sim L_2(q)$. We utilize the character table of $L_2(q)$ from Chapter 38 of [17] (see also Schur [25]). Recall that $L_2(q)$ contains one conjugacy class of involutions, and two conjugacy classes of elements of order p. The remaining conjugacy classes partition into two cases: those whose order divides $\frac{1}{2}(q-1)$ and those whose order divides $\frac{1}{2}(q+1)$. Let C be a conjugacy class of elements in $C_G(W)$ and define $X_C = \{x \in Y_2 \cap C_G(W) | \, tx \in C\}$. It is a well-known character theoretic result (see, for example, Theorem 4.2.12 of [19]) that

$$|X_C| = \frac{|C|}{|C_G(t)|} \sum_{\substack{\chi \\ \text{Irreducible}}} \frac{\chi(x)\, |\chi(t)|^2}{\chi(1)} \tag{4.26.1}$$

and all X_C are pairwise disjoint. Let $x \in Y_2 \cap C_G(W)$. If the order of tx divides $\frac{1}{2}(q^2-1)$ but not $\frac{1}{2}(q-\delta)$ then it must necessarily divide $\frac{1}{2}(q+\delta)$. Hence, if C is a conjugacy class of elements of order dividing $\frac{q+\delta}{2}$, then any $y \in X_C$ has the property that $d^{C_G(W)}(t,y) = 3$ but $d^{C_G(U)}(t,y) = 2$, by Lemma 4.24. Recall that $\Gamma_2(C_G(U)) \setminus \dot{\bigcup}_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W))$ is the set consisting of all such involutions. Therefore, it suffices to calculate the sizes of all such relevant X_C. We use $\mathcal{F}$ to denote to be the set of all conjugacy classes of elements with order dividing $\frac{q+\delta}{2}$.

By [17], we see that for any $C \in \mathcal{F}$, $|C| = q(q-\delta)$ and so for any $x \in C$, $\left|C_{C_G(W)}(x)\right| = (q-\delta)$. Hence (4.26.1) and [17] gives $|X_C| = q - \delta$. Now if $\delta = 1$, then $|\mathcal{F}| = \frac{q-1}{4}$ by [17]. If $\delta = -1$, then $|\mathcal{F}| = \frac{q-3}{4}$. Since $\left|\Delta_3^{C_G(W)}(t) \cap \Delta_2^{C_G(U)}\right| = |X_C|\,|\mathcal{F}|$, and by Corollary 4.22, $\left|\mathcal{W}_2(U) \cap (\mathcal{U}_2^+ \cup \mathcal{U}_2^-)\right| = q + \delta$, we obtain

$$\left|\Gamma_2(C_G(U)) \setminus \dot{\bigcup_{W \in \mathcal{W}_2(U)}} \Gamma_2(C_G(W))\right| = \left|\mathcal{W}_2(U) \cap (\mathcal{U}_2^+ \cup \mathcal{U}_2^-)\right| |X_C|\,|\mathcal{F}|$$

$$= \begin{cases} \frac{1}{4}(q-1)(q^2-1) & q \equiv 1 \pmod 4 \\ \frac{1}{4}(q-3)(q^2-1) & q \equiv -1 \pmod 4 \end{cases}$$

which proves the first part of (ii). We now prove the last part of (ii). Recall that

$$\left|\dot{\bigcup_{W \in \mathcal{W}_2(U)}} \Gamma_2(C_G(W))\right| = (q+\delta)\left|\Delta_2^{C_G(W)}(t)\right| = \frac{1}{4}(q^2-1)(q-4-\delta)$$

by [10] and Corollary 4.22. Together with the above statement, we have

$$\begin{aligned}|\Gamma_2(C_G(U))| &= \left|\dot{\bigcup_{W \in \mathcal{W}_2(U)}} \Gamma_2(C_G(W))\right| + \left|\Gamma_2(C_G(U)) \setminus \dot{\bigcup_{W \in \mathcal{W}_2(U)}} \Gamma_2(C_G(W))\right| \\ &= \frac{1}{4}(q^2-1)(q-4-\delta) + \frac{1}{4}(q^2-1)(q-2+\delta) \\ &= \frac{1}{2}(q^2-1)(q-3).\end{aligned}$$

Hence

$$\begin{aligned}|\Gamma_1(C_G(U))| &= \left|\Delta_2^{C_G(U)}(t)\right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{4}(q-1)^3(q+1),\end{aligned}$$

and Lemma 4.26 holds. □

Lemma 4.27 $$\left|\dot{\bigcup_{U \in \mathcal{U}_1}} \Gamma_1(C_G(U))\right| = \begin{cases} \frac{1}{16}q(q^2-1)(3q^3-11q^2+21q-29) & q \equiv 1 \pmod 4 \\ \frac{1}{16}q(q^2-1)(q-1)(3q^2+2q+7) & q \equiv -1 \pmod 4. \end{cases}$$

Proof Since $\mathcal{U}_1 = \mathcal{U}_1^0 \dot\cup \mathcal{U}_1^+ \dot\cup \mathcal{U}_1^-$, with each orbit size given in Lemma 4.16, the result follows immediately from Lemmas 4.23, 4.25 and 4.26. $\square$

Recall the list of cases in Table 1. The next lemma concerns Cases 2 and 3, in other words, $\bigcup_{U\in\mathcal{U}_1} \Gamma_2(C_G(U))$.

Lemma 4.28 $\left|\bigcup_{U\in\mathcal{U}_1} \Gamma_2(C_G(U))\right| = \frac{1}{2}(q-\delta)(q^3-2q^2-1)$.

Proof By Lemmas 4.12 and 4.19, for any $W_0 \in \mathcal{U}_2^0$ we have $\left|\Delta_2^{C_G(U)}(t) \cap C_G(W_0)\right| = \frac{1}{2}(q-1)(q-\delta)$ for some $U \in \mathcal{U}_1(W_0)$. Additionally, for any $W \in (\mathcal{U}_2^+ \dot\cup \mathcal{U}_2^-)$ we have

$$\begin{aligned}\left|\Delta_2^{C_G(U)}(t) \cap C_G(W)\right| &= \left|\Delta_2^{C_G(W)}\right| + \left|\Delta_3^{C_G(W)}(t) \cap \Delta_2^{C_G(U)}(t)\right| \\ &= \frac{1}{2}(q-\delta)(q-3),\end{aligned}$$

for some $U \in \mathcal{U}_1(W)$, by [10] and Lemma 4.26. Since $\mathcal{U}_2 = \mathcal{U}_2^0 \dot\cup \mathcal{U}_2^+ \dot\cup \mathcal{U}_2^-$, with the orbit sizes given in Corollary 4.17, this covers every involution in $\bigcup_{U\in\mathcal{U}_1} \Gamma_2(C_G(U))$, and the lemma follows. $\square$

We now concern ourselves with the final two cases. These concern involutions that are distance 3 from t in every $C_G(U)$ that they appear in, but actually are distance 2 from t in G. Recall that for any involution $y \in Y_2$, $C_G(y) = \mathrm{Stab}_G C_V(y) = L_y K_y$ where $L_y = C_G(y) \cap C_G([V,y]) \sim L_2(q)$ and $|K_y| = 2(q-\delta)$. Also note that $L_y \trianglelefteq C_G(y)$ acts faithfully on $C_V(y)$, and $\mathrm{Syl}_p C_G(y) = \mathrm{Syl}_p L_y$. The following three lemmas concern Case 5.

Lemma 4.29 *Let $W \in \mathcal{U}_2^0 \cup \mathcal{U}_2^{-\delta}$ and $x \in C_G(W)$ be such that $d^{C_G(U)}(t,x) = 3$ for all $U \in \mathcal{U}_1(W)$. Then $d(t,x) = 3$.*

Proof If $W \in \mathcal{U}_2^0$, then any involution in $C_G(W)$ can be written as $x = x_Q t$ where $x_Q = xt$ has order p. If $W \in \mathcal{U}_2^{-\delta}$, then, from Lemma 4.26, any involution $x \in C_G(W)$ such that tx has order dividing $\frac{1}{2}(q^2-1)$ must be distance 2 from t in $C_G(U)$ for some $U \in \mathcal{U}_1(W)$. Hence, any x satisfying the hypothesis must have the property that the order of tx is p.
Let $W \in \mathcal{U}_2^0 \cup \mathcal{U}_2^{-\delta}$ and suppose $d(t,x) = 2$, then there exists $y \in Y_2$ such that $t, x \in C_G(y) = L_y K_y$. Since tx has order p, $tx \in L_y$ and so $tx \in C_G([V,y])$. As L_y acts faithfully on $C_V(y)$, any element of order p must fix a 1-subspace of $C_V(y)$, say U_y. Therefore, $tx \in C_G(U_y \oplus [V,y])$. But $tx \in C_G(W + [V,y])$ and since $[V,y] \in \mathcal{U}_2^{\delta}$, we have $W \neq [V,y]$. Set $W + [V,y] = U_y \oplus [V,y]$.

Suppose $U_y \leq W$. Then $t, x, y \in C_G(U_y)$ and so $d^{C_G(U_y)}(t,x) = 2$, contradicting our assumption. Hence $U_y \nleq W$ and so $U_y = \langle u_1 + u_2 \rangle$ for $u_1 \in W \setminus [V,y]$ and $u_2 \in [V,y]$. Since $y \in C_G(y)$, $(u_1+u_2)^y = u_1+u_2$. However, $(u_1+u_2)^y = u_1^y+u_2^y = u_1^y-u_2$ and so $u_2 = -2^{-1}u_1+2^{-1}u_1^y$. Thus $u_1 + u_2 = 2^{-1}(u_1 + u_1^y)$ and so $U_y = \langle u_1 + u_1^y \rangle$. Recall that $t, x \in C_G(y)$ and $u_1 \in W \setminus [V,y]$, so $u_1^t = u_1^x = u_1$. Hence $u_1 + u_1^y$ is centralized by both t and x and so $U_y \leq W = C_V(\langle t, x \rangle)$, a contradiction. Therefore, $d(t,x) \neq 2$ and the lemma holds. $\square$

Lemma 4.30 *Let $W \in \mathcal{U}_2^\delta$. Then $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2(t)$. In particular,*

$$\left| \Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U)) \right| = \begin{cases} q(q^2-1) & q \equiv 1 \pmod 4 \\ 0 & q \equiv -1 \pmod 4. \end{cases}$$

Proof We deal first with the case when $\delta = -1$. From Lemma 4.26, the number of involutions distance 3 from t in $C_G(W)$ that are actually distance 2 from t in some $U \in \mathcal{U}_1(W)$ is $\frac{1}{4}(q+1)(q-3) = \left|\Delta_3^{C_G(W)}(t)\right|$. That is to say all elements in $\Delta_3^{C_G(W)}(t)$ are distance 2 from t in $C_G(U)$ for some $U \in \mathcal{U}_2(W)$. This occurs for every such $W \in \mathcal{U}_2^\delta$ and so $\Gamma_2(G) = \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$.
Assume now that $\delta = 1$. As before, any element x in $\Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$ must have the property that the order of tx is p. Suppose $d(t,x) = 2$, and so there exists $y \in Y_2$ such that $t, x \in C_G(y)$. If $W \neq [V,y]$ then the argument from Lemma 4.29 holds and results in a contradiction. So we must have $W = [V,y]$. Since $\mathrm{Stab}_G C_V(y) = \mathrm{Stab}_G[V,y] = C_G(y)$, $C_G([V,y]) \leq C_G(y)$ and so any element in $C_G([V,y]) = C_G(W)$ centralizes y. In particular, $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2(t)$, establishing the first statement. By Lemma 4.26, the number of involutions distance 3 from t in $C_G(W)$ that are actually distance 2 from t in some $U \in \mathcal{U}_1(W)$ is $\frac{1}{4}(q-1)^2$. By [10], $\left|\Delta_3^{C_G(W)}(t)\right| = \frac{1}{4}(q-1)(q+7)$ and so by subtracting the two, there are $2(q-1)$ involutions in $\Delta_3^{C_G(W)}(t)$ that are distance 3 from t in $C_G(U)$ for all $U \in \mathcal{U}_1(W)$, but are actually distance 2 from t in $\mathcal{C}(G, Y_2)$. Since $\left|\mathcal{U}_2^\delta\right| = \frac{1}{2}q(q+\delta)$ by Corollary 4.17, the lemma follows. $\square$

Finally we turn to Case 4, $\Gamma_1(G) \setminus \dot{\bigcup}_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$.

Lemma 4.31 *Let $U \in \mathcal{U}_1^- \cup \mathcal{U}_1^0$ and $x \in C_G(U)$ be such that $C_V(\langle t, x \rangle) = U$ and $d^{C_G(U)}(t,x) = 3$. Then $d(t,x) = 3$.*

Proof Assume first that $U \in \mathcal{U}_1^-$. By Lemma 4.24, tx has order p or divides $\frac{1}{2}(q^2+1)$. Suppose $d(t,x) = 2$, then there exists $y \in Y_2$ such that $t, x \in C_G(y)$. Since $\frac{1}{2}(q^2+1)$ is coprime to $|C_G(y)| = q(q^2-1)(q-\delta)$, tx must have order p. Indeed, clearly $\frac{1}{2}(q^2+1)$ is coprime to both q and q^2-1, and any factor dividing $q - \delta$ must divide $q^2 - 1$ and so $\frac{1}{2}(q^2+1)$ is coprime to

$q-\delta$. Since tx has order p, then $tx \in L_y$.
Assume now that $U \in \mathcal{U}_1^0$. Let x be an involution in $C_G(U) = QL \sim G^0$ as defined in the discussion prior to Lemma 4.9. Then $tx \in Qtx_L$ which has order n dividing $\frac{1}{2}(q+\delta)$ in QL/L. Therefore, $(Qtx_L)^n \in Q$ and so $(tx)^n$ has order p. Therefore, tx has order dividing $\frac{1}{2}q(q+\delta)$. Suppose $d(t,x) = 2$. Then there exists $y \in Y_2$ such that $t, x \in C_G(y)$. By the structure of $C_G(y) \sim (L_2(q) \times \frac{q-\delta}{2}) : 2^2$, the order of tx forces $tx \in L_y$.
We may now assume $U \in \mathcal{U}_1^- \cup \mathcal{U}_1^0$, so $tx \in L_y = C_G([V,y])$ and hence $tx \in C_G(U + [V,y])$. Suppose $U \not\leq [V,y]$, then $tx \in C_G(U \oplus [V,y])$ Also, $tx \in C_G(U_y \oplus [V,y])$ for some $U_y \leq C_V(y)$. However, if $U = U_y$ then $t, x, y \in C_G(U)$ and $d^{C_G(U)}(t,x) = 2$. While $U_y \neq U$ results in a contradiction using an analogous argument from Lemma 4.29. Hence $U \leq [V,y]$.
As $t, x \in C_G(y) = \mathrm{Stab}_G([V,y])$, $tx \in L_y = C_G([V,y])$ and $[V,y] = U \perp U'$ where $U' = U^\perp \cap [V,y]$. Then for $u \in [V,y]$ we have $u^{tx} = u$ and so $u^t = u^x$. In particular, if $u \in U'$ then $u^t = u^x = -u$. Hence $[V,y] = U \perp ([V,t] \cap [V,x])$. If $C_V(\langle t,y\rangle)$ is 1-dimensional, then $C_V(y) = C_V(\langle t,y\rangle) \perp [V,t]$ since t stabilizes $C_V(y)$. However, then $[V,t] \oplus ([V,t] \cap [V,x])$ is 3-dimensional, a contradiction. A similar argument holds for $C_V(\langle x,y\rangle)$. Therefore both $C_V(\langle t,y\rangle)$ and $C_V(\langle x,y\rangle)$ are 2-dimensional. But since $\dim C_V(y) = 3$, this means $C_V(\langle t,y\rangle)$ and $C_V(\langle x,y\rangle)$ intersect non-trivially, that is $C_V(\langle t,x,y\rangle) \neq 0$, contradicting our assumption. Therefore, $d(t,x) \neq 2$, and consequently $d(t,x) = 3$. $\square$

The final case when $U \in \mathcal{U}_1^+$ is slightly trickier. Recall the definition of Y_1. For any $z \in Y_1$, we have $C_G(z) \sim SL_2(q) \circ SL_2(q) : 2$ and $C_V(z)$ is 1-dimensional. We choose z such that $t \in C_G(z)$ and $C_V(z) = U$, and return to work in the setting of $Sp(4,q)/\langle -I_4\rangle = G^\tau \sim G$. We denote the image of any subgroup $K \leq G$ by K^τ. Choose

$$z = \left(\begin{array}{c|c} -I_2 & \\ \hline & I_2 \end{array}\right) \in G^\tau$$

and note that $C_{G^\tau}(z) \sim C_G(U) : 2$. Hence,

$$C_G(U) \sim \left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right) \middle| \, A, B \in SL_2(q) \right\} / \langle -I_4\rangle = C_G(U)^\tau.$$

Let t^τ be the image of t in G^τ. We start with a preliminary lemma concerning the commuting involution graph $\mathcal{C}(L_2(q), X)$ where X is the sole conjugacy class of involutions. Denote by $L \sim L_2(q)$ and $\widehat{L} \sim PGL_2(q)$.

Lemma 4.32 *Let x be an involution in L. Then $\Delta_3^L(x)$ splits into $\frac{1}{4}(q+2+5\delta)$ $C_L(x)$-orbits of length $q-\delta$. Moreover, every $C_L(x)$-orbit in $\Delta_3^L(x)$ is $C_{\widehat{L}}(x)$-invariant.*

Proof Assume first that $\delta = -1$. Choose $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $x_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$ for some $\lambda \in GF(q) \setminus \{\pm 1\}$. There are two possibilities for an element of $C_L(x)$:

$$g_1 = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} b_1 & b_2 \\ b_2 & -b_1 \end{pmatrix}.$$

By direct calculation, if $g_1^{-1} x_\lambda g_1 = x_\mu$ for some $\lambda, \mu \in GF(q) \setminus \{\pm 1\}$ then $(-\lambda^{-1} + \lambda) a_1 a_2 = 0$. Note that since $\lambda \neq \pm 1$, then $\lambda \neq \lambda^{-1}$. If $a_1 = 0$ then $a_2^2 = 1$, and so $\mu = \lambda^{-1}$. On the other hand, if $a_2 = 0$ then $a_1^2 = 1$ and so $\mu = \lambda$. Note that in the case of g_2, neither b_1 or b_2 can be 0 and so $g_2^{-1} x_\lambda g_2 = x_\mu$ requires $xy(\lambda - \lambda^{-1}) = 0$, a contradiction. Hence for $\lambda, \mu \in GF(q) \setminus \{\pm 1\}$, x_λ and x_μ lie in different $C_L(x)$ orbits if and only if $\mu \notin \{\lambda, \lambda^{-1}\}$. As we work modulo $\langle -I_4 \rangle$, there are at least $\frac{1}{4}(q-3)$ $C_L(x)$-orbits in $\Delta_3^L(x)$. However for any $\lambda \neq \pm 1$, $C_L(x, x_\lambda) = 1$ and so, each $C_L(x)$-orbit containing an x_λ is of length $q+1$. But $\left|\Delta_3^L(x)\right| = \frac{1}{4}(q-3)(q+1)$ and so all involutions in $\Delta_3^L(x)$ are accounted for. Hence the first statement holds for $\delta = -1$, and each $C_L(x)$-orbit has representative x_λ for some $\lambda \neq \pm 1$. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \widehat{L} \setminus L$$

and note that $C_{\widehat{L}}(x) = \langle e \rangle \, C_L(x)$, and an easy check shows $[e, x_\lambda] = 1$ for all $\lambda \neq \pm 1$. Let $y \in \Delta_3^L(x)$, then $y = x_\lambda^s$ for some $s \in C_L(x)$. Let $g = er \in C_{\widehat{L}}(x)$ for some $r \in C_L(x)$. Then $y^g = x_\lambda^{s^e r}$ and since $C_L(x) \trianglelefteq C_{\widehat{L}}(x)$, $s^e r \in C_L(x)$. That is, every $C_L(x)$-orbit in $\Delta_3^L(x)$ is $C_{\widehat{L}}(x)$-invariant.

Assume now that $\delta = 1$. Choose $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ where $i^2 = -1$ and let $y = \begin{pmatrix} \sigma & \mu\tau \\ \tau & \sigma \end{pmatrix}$ for some $\sigma, \mu, \tau \in GF(q)$, $\sigma \neq 0$ and μ a non-square in $GF(q)$. By [10], $y \in \Delta_3^L(x)$. There are two possibilities for an element of $C_L(x)$:

$$g_1 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}.$$

By direct calculation, if $g_1^{-1} y g_1 = y$ then $a = 1$. Note that $g_2^{-1} y g_2 \neq y$ as $\pm b^2 \neq \mu$ for any non-square μ. Hence $C_L(\langle x, y \rangle) = 1$. Since y was arbitrary, each $C_L(x)$-orbit has length $q-1$. Now $\left|\Delta_3^L(x)\right| = \frac{1}{4}(q+7)(q-1)$ and so the first statement holds for $\delta = 1$. Let

$$e_\nu = \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix} \in \widehat{L} \setminus L$$

and note that $C_{\widehat{L}}(x) = \langle e_\nu \rangle \, C_L(x)$ for any non-square ν. It is easy to check that $y^{e_\mu} = y$. Let $g = e_\mu r \in C_{\widehat{L}}(x)$ for some $r \in C_L(x)$. Then $y^g = y^{e_\mu r} = y^r$ and since y was arbitrary and $r \in C_L(x)$, every $C_L(x)$-orbit in $\Delta_3^L(x)$ is $C_{\widehat{L}}(x)$-invariant. $\square$

Lemma 4.33 $\left|\Delta_3^{C_G(U)}(t) \cap \Gamma_1(G)\right| = \frac{1}{4}(q-\delta)^2(q+2+5\delta)$.

Proof We first work in the setting of G^τ. Choose

$$t^\tau = \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} J_0 & \\ \hline & J_0 \end{array}\right).$$

By direct calculation, it is easily seen that

$$C_{G^\tau}(t^\tau) \subseteq \left\{ \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right) \middle| \; A_i^{-1} J_0 A_i = J_0 \pmod{\langle -I_4 \rangle} \right\}$$

and any involution $y \in C_{G^\tau}(t^\tau)$ has the additional properties that

$$\begin{array}{rl} & \det A_1 + \det A_3 = \det A_2 + \det A_4 = 1 \\ \text{and} & A_1^2 + A_2A_3 = A_3A_2 + A_4^2 = -I_2. \end{array} \tag{4.33.1}$$

Recall that if $x \in C_G(U)^\tau$ then $x = \left(\begin{array}{c|c} A & \\ \hline & B \end{array}\right)$ for some $A, B \in SL_2(q)$ and by Theorem 4.8, $x \in \Delta_3^{C_G(U)^\tau}(t^\tau)$ if and only if A, B are involutions in L and either $d^L(A, J_0) = 3$ or $d^L(B, J_0) = 3$. So without loss of generality, set $A = B_i$ where $d^L(B_i, J_0) = i$ and choose $B \in \Delta_3^L(J_0)$.
If $x \in \Delta_2^{G^\tau}(t^\tau)$ then there exists $y = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right) \in C_{G^\tau}(t^\tau)$ such that $y^2 = 1$ and $[x, y] = 1$. Suppose $\det A_2 = 0$. Then $\det A_4 = 1$ by (4.33.1), and so $A_4 \in C_L(J_0)$. As $[x, y] = 1$, $[A_4, B] = 1$. However $C_L(\langle J_0, B\rangle) = 1$, by Lemma 4.32 and so $A_4 = \pm I_2$. But then $A_3A_2 = -2I_2$ by (4.33.1), which is impossible as $\det A_2 = 0$. An analogous argument holds for $\det A_3$. Hence $\det A_2, \det A_3 \neq 0$. Since $[x, y] = 1$, $B_iA_2B = \pm A_2$ and so B_i and B must be $C_{\widehat{L}}(J_0)$-conjugate. In other words, if B_i and B are not $C_{\widehat{L}}(J_0)$-conjugate, then $[x, y] \neq 1$. By Lemma 4.32, every $C_L(J_0)$ orbit is an $C_{\widehat{L}}(J_0)$-orbit and so if $[x, y] = 1$ then B_i and B must be $C_L(J_0)$-conjugate. Assume then B_i and B are $C_L(J_0)$-conjugate and let $A \in C_L(J_0)$ be such that $B_i^A = B$. Hence if $y_A = \left(\begin{array}{c|c} & A \\ \hline -A^{-1} & \end{array}\right) \in C_{G^\tau}(t^\tau)$, then $[y_A, x] = 1$ and so $d^{G^\tau}(t^\tau, x) = 2$. By Lemma 4.32, each $C_L(J_0)$-orbit of $\Delta_3^L(J_0)$ is of length $q - \delta$, and there are $\frac{1}{4}(q + 2 + 5\delta)$ such orbits. Moreover, for any involution $x_0 \in C_G(U)^\tau$ conjugate to t^τ, zx_0 is also an involution in $C_G(U)^\tau$ conjugate to t^τ which has not been accounted for. Therefore, the number of involutions in $\Delta_3^{C_G(U)^\tau}(t^\tau)$ that are actually distance 2 from t^τ in G^τ is $\frac{1}{2}(q - \delta)^2(q + 2 + 5\delta)$.
We now return to the setting of G, and first assume that $\delta = -1$ and so by Corollary 4.22(i), $|\mathcal{W}_2(U)| = q + 1$, and for every $W \in \mathcal{W}_2(U)$, $C_G(W) \sim L_2(q)$. For each W, there exists $U_W \in \mathcal{U}_1^+$ such that $C_G(W) \leq C_G(U_W) \sim L_2(q^2)$ by Lemma 4.21, and $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2^{C_G(U_W)}(t)$ by Lemma 4.30. Hence, there are $\frac{1}{4}(q + 1)^2(q - 3)$ involutions already counted (from Case 3)

and the remaining involutions do not fix a 2-subspace of $C_V(t)$. Therefore

$$\begin{aligned}\left|\Delta_3^{C_G(U)}(t)\cap\Gamma_1(G)\right| &= \frac{1}{2}(q+1)^2(q-3)-\frac{1}{2}(q+1)^2(q-3)\\ &= \frac{1}{4}(q+1)^2(q-3),\end{aligned}$$

as required. Now assume that $\delta=1$ and so by Corollary 4.22. For each W, there exists $U_W\in\mathcal{U}_1^-$ such that $C_G(W)\le C_G(U_W)\sim L_2(q^2)$ by Lemma 4.21 and $\left|\Delta_3^{C_G(W)}(t)\cap\Delta_2^{C_G(U_W)}(t)\right|=\frac{1}{4}(q-1)^2$ by Lemma 4.26. Since $\left|\mathcal{W}_2(U)\cap(\mathcal{U}_1^+\cup\mathcal{U}_1^-)\right|=q-1$ by Corollary 4.22(iii), this accounts for $\frac{1}{4}(q-1)^3$ involutions. Suppose now $W_0\in\mathcal{W}_2(U)\cap\mathcal{U}_2^0$. By Lemma 4.21, there exists $U_0\in\mathcal{U}_1^0$ such that $C_G(W_0)\le C_G(U_0)$. From Lemmas 4.12 and 4.19, $\left|C_G(W)\cap\Delta_2^{C_G(U_0)}(t)\right|\frac{1}{2}(q-1)^2$. Since $|\mathcal{W}_2(U)\cap\mathcal{U}_2^0|=2$ by Corollary 4.22(iii), this yields a further $(q-1)^2$ involutions. Finally, if $W\in\mathcal{U}_2^+$, then by Lemma 4.30, $\Delta_3^{C_G(W)}(t)\subseteq\Delta_2(t)$ and there are $2(q-1)$ involutions in $\Delta_3^{C_G(W)}(t)$ not already enumerated. Now $\left|\mathcal{U}_2^+\cap\mathcal{W}_2(U)\right|=\frac{1}{2}(q-1)$ by Corollary 4.22(iii), and this yields another $(q-1)^2$ involutions. Hence, there are $\frac{1}{4}(q-3)^2+2(q-1)^2=\frac{1}{4}(q-1)^2(q+7)$ involutions already counted (from Cases 3 and 5) and the remaining involutions do not fix a 2-subspace of $C_V(t)$. Consequently

$$\begin{aligned}\left|\Delta_3^{C_G(U)}(t)\cap\Gamma_1(G)\right| &= \frac{1}{2}(q-1)^2(q+7)-\frac{1}{2}(q-1)^2(q+7)\\ &= \frac{1}{4}(q-1)^2(q+7),\end{aligned}$$

as required. $\square$

Corollary 4.34 $\left|\Gamma_1(G)\setminus\dot{\bigcup}_{U\in\mathcal{U}_1}\Gamma_1(C_G(U))\right|=\frac{1}{8}q(q-\delta)(q^2-1)(q+2+5\delta)$.

Proof Since $\left|\mathcal{U}_1^+\right|=\frac{1}{2}q(q+\delta)$, the result holds by Lemmas 4.32 and 4.33. $\square$

Lemma 4.35 *If* $q\equiv 3 \pmod 4$, *then*

(i) $|\Delta_2(t)|=\frac{1}{16}(q+1)(3q^5-2q^4+8q^3-30q^2+13q-8)$*; and*

(ii) $|\Delta_3(t)|=\frac{1}{16}(q-1)(5q^5-4q^4-2q^3+4q^2+5q+5)$.

If $q\equiv 1 \pmod 4$, *then*

(iii) $|\Delta_2(t)|=\frac{1}{16}(q-1)(3q^5-6q^4+32q^3-10q^2-27q-8)$*; and*

(iv) $|\Delta_3(t)|=\frac{1}{16}(q-1)(5q^5+22q^4-8q^3+34q^2+51q+24)$.

Proof The cases listed in Table 1 are disjoint. Hence $|\Delta_2(t)|$ is determined by summing the values calculated in Lemmas 4.27, 4.28, 4.30 and 4.34. By Theorem 4.15, $\mathcal{C}(G, Y_2)$ has diameter 3 and so $|\Delta_3(t)| = |Y_2| - |\Delta_1(t)| - |\Delta_2(t)|$. Since $|G| = \frac{1}{2}q^4(q^2-1)(q^4-1)$ and $|C_G(t)| = q(q^2-1)(q-\delta)$, $|Y_2| = \frac{1}{2}q^3(q+\delta)(q^2+1)$. Together with Lemma 4.20, this proves the lemma. $\square$

Together, Theorem 4.15 and Lemmas 4.20 and 4.35 complete the proof of Theorem 1.4.

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