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Abstract

We consider the problem of detecting Mahowald's family $\eta_i \in {}_2\pi^S_{2^i}$ in homology. This allows us to identify specific spherical classes in $H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ for $0 \leqslant k \leqslant 6$. We then identify the type of the subalgebras that these classes give rise to, and calculate the A-module and R-module structure of these subalgebras. We shall the discuss the relation of these calculations to the Curtis conjecture on spherical classes in $H_*Q_0S^0$.

1 Introduction and statement of results

Consider the problem of understanding the homology of spaces when loop beyond there connectivity. Of particular interest, is the problem of understanding the homology of the infinite loop spaces associated with desuspensions of the sphere spectrum, and describing a part of these homology algebras in a geometric way. Our work here provides an example in this direction by identifying specific subalgebras inside homology of iterated loop spaces associated with spheres of the form $\Omega^{n+k}S^n$.
The infinite loop space associated with S^{-k} is given by

$$QS^{-k} := \operatorname{colim}\ \Omega^{n+k}S^n$$

where $k \geqslant 0$. Notice that $QS^{-k} = \Omega QS^{-k+1}$. This provides a translation between stable unstable homotopy of spheres through the isomorphism

$${}_2\pi^S_{*+k} \simeq {}_2\pi_*QS^{-k}.$$

There is very little known on the homology algebras $H_*\Omega_0^{n+k}S^n$ and $H_*Q_0S^{-k}$ where H_* denotes $H_*(-;\mathbb{Z}/2)$; $\Omega_0^{n+k}S^n$ and Q_0S^{-k} denote the base point component of the related spaces. In fact, we only know about the homology algebras $H_*\Omega^{n+1}S^n$ and $H_*\Omega^{n+2}S^n$ due to Hunter [H89, Theorem 1.2, Corollary 1.3], as well as the algebras $H_*Q_0S^{-1}$ and $H_*Q_0S^{-2}$ due to Cohen and Peterson [CP89, Theorem 1.1, Theorem 1.2]. In both of these works the calculations are based on applying the Eilenberg-Moore spectral sequence machinery to the path loop fibrations over QS^{-k}. The problem remains open for $k > 2$.
Of course, for the purpose of applications in homotopy theory, the main motivation of studying these homology algebras is to understand the geometry of the spaces in question with the hope that they might give some information on the homotopy groups of spheres. However, in other direction, it appears that we can use the available geometric information to shed a light on the homology algebras $H_*\Omega^{n+k}S^n$ and H_*QS^{-k}. Recently, the second named author has provided a "geometric" description for the generators of $H_*Q_0S^{-1}$, as well as some information on $H_*Q_0S^{-2}$ [Z09, section 5.8]. In a separate work, the second named author has used the real and complex J-homomorphisms to detect infinite families of subalgebras inside $H_*Q_0S^{-n}$ [Z09a]. Such calculations are related to the Curtis conjecture on spherical classes $H_*Q_0S^0$ which is explained

in Section 5.
Our aim in this paper is to detect Mahowald's η_i family using the Hurewicz homomorphism, and to exhibit its implications on the homology of iterated loop spaces associated with spheres. This turns out to be fruitful as it provides examples of some cases that were not known before. In the rest of this section, we fix our notation and list our main results.
The $\eta_i \in {}_2\pi^S_{2^i}$ family was constructed by Mahowald in [M77, Theorem 2] as a stable composite

$$S^{2^i} \xrightarrow{f_i} X_i \xrightarrow{g_i} S^0$$

with $X_i = D_{2^i-3}(\mathbb{R}^2, S^7)$, $i \geqslant 3$, chosen to be one of pieces in the Snaith splitting for $\Omega^2 S^9$ [S74]. The complex X_i has the property that it is highly connected such that the mapping f_i can be assumed a genuine map. Moreover, the complex X_i has its top cell in a dimension less than 2^i which means that the mapping f_i is trivial in homology. The mapping g_i is clearly a stable mapping and can be realised as a genuine map after finitely many suspensions. These together imply that the stable adjoint of η_i may be reaslied as a mapping

$$S^{2^i} \xrightarrow{f_i} X_i \longrightarrow Q_0 S^0$$

where the component f_i is trivial in homology. This implies that the above composite is trivial in homology, i.e. the mapping η_i maps trivially under the Hurewicz homomorphism

$$h : {}_2\pi_{2^i} Q_0 S^0 \to H_{2^i} Q_0 S^0.$$

Despite the above observation, one might hope to detect it using the Hurewicz homomorphism, if keep adjointing down the mapping $S^{2^i} \to Q_0 S^0$. This is of course is a natural thing to expect. Our main results reads as following.

Main Theorem. *Let $\eta_i \in {}_2\pi^S_{2^i}$ denote Mahowald's family. This class is detected by the Hurewicz homomorphism*

$$h : {}_2\pi_6 Q_0 S^{-2^i+6} \to H_6 Q_0 S^{-2^i+6}.$$

The spherical class $[\eta_i]_6 = h\eta_i$ has the following property. Let $j_2^\infty : QS^{2^i-3} \to Q\Sigma^{2^i-3}P_{2^i-3}$ be the second stable James-Hopf invariant. We then have

$$(\Omega^{2^{i+1}-9} j_2^\infty)_*[\eta_i]_6 = (\Sigma^{-2^i+6} a_{2^i-3})^2 \neq 0$$

where $\Sigma^{-2^i+6} a_{2^i-3} \in H_3 Q\Sigma^{-2^i+6} P_{2^i-3}$ is the class given by the inclusion of the bottom cell $S^3 \to Q\Sigma^{-2^i+6} P_{2^i-3}$.

We note that the space $Q_0 S^{-2^i+6}$ is an infinite loop space and it is natural to think of the subalgebra of $H_* Q_0 S^{-2^i+6}$ generated by the classes of the form $Q^I[\eta_i]_6$. The problem becomes easier to answer when we consider the unstable case and replace infinite loop spaces with finite loop spaces. First we have the following observation which is an unstable version of our main theorem.

Theorem 1. *Let $\eta_i \in {}_2\pi^S_{2^i}$ denote Mahowald's family. This class is detected by the Hurewicz homomorphism*

$$h : {}_2\pi_6 \Omega_0^{2^{i+1}-8} S^{2^i-2} \to H_6 \Omega_0^{2^{i+1}-8} S^{2^i-2}.$$

The spherical class $[\eta_i]_6 = h\eta_i$ has the following property. Let $j_2 : \Omega S^{2^i-2} \to QS^{2^{i+1}-6}$ be the second James-Hopf invariant. We then have

$$(\Omega^{2^{i+1}-9} j_2)_*[\eta_i]_6 = g_3^2.$$

We may apply this to detect some subalgebras living in $H_* \Omega_0^{2^{i+1}-8} S^{2^i-2}$ and determine their algebraic structure. In fact we are able to detect polynomial subalgebras in $H_* \Omega_0^{2^{i+1}-8+k} S^{2^i-2}$ for $k = 0, 1, 2, 3$.

Notice that the space $\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ is a $(2^{i+1}-8+k)$-loop space, and admits operations [CLM76, Part III, Theorem 1.1]

$$Q_a : H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2} \to H_{a+2*}\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$$

for $a < (2^{i+1}-8+k)-1$. Hence, when $k=0$, we may consider to the subalgebra of $H_*\Omega_0^{2^{i+1}-8}S^{2^i-2}$ generated by the classes $Q_I[\eta_i]_6$ where $I=(i_1,\dots,i_r)$ is any sequence with $0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r < 2^{i+1}-9$. In fact we can do more. First, notice that realising η_i as an element in ${}_2\pi_0 Q_0 S^{-2^i}$ we know that this maps nontrivially under the Hurewicz homomorphism

$$h : {}_2\pi_0 QS^{-2^i} \to H_0 QS^{-2^i}.$$

Let $[\eta_i] = h\eta_i = (\eta_i)_* 1$ where $1 \in \overline{H}_0 S^0$ is the generator. One then may hope that this class will survive under the homology suspension finitely many times. Second, consider the Hurewicz homomorphism

$$h : {}_2\pi_j\Omega_0^{2^{i+1}-8+(6-j)}S^{2^i-2} \to H_j\Omega_0^{2^{i+1}-8+(6-j)}S^{2^i-2},$$

where $0 \leqslant j \leqslant 6$, and let

$$[\eta_i]_j = h(\eta_i).$$

This then implies that

$$\sigma_*[\eta_i]_j = [\eta_i]_{j+1}.$$

Note that the classes $[\eta_i]_j$ are A-annihilated and primitive as they are spherical. Observe that according to Theorem 1, we have $[\eta_i]_6 \neq 0$. This implies that $[\eta_i]_j \neq 0$ for $j < 6$. In particular, we have

$$[\eta_i]_5 \in H_5\Omega_0^{2^{i+1}-7}S^{2^i-2},$$

$$[\eta_i]_4 \in H_4\Omega_0^{2^{i+1}-6}S^{2^i-2},$$

$$[\eta_i]_3 \in H_*\Omega_0^{2^{i+1}-5}S^{2^i-2}.$$

Hence, we may consider to the subalgebra spanned by the classes of the form $Q_I[\eta_i]_5$, $Q_I[\eta_i]_4$ and $Q_I[\eta_i]_3$ living inside the correcponding algebras. Notice that we still don't know the structure of these algebras, nor even if the classes $Q_I[\eta_i]_j$ are nontrivial. We state the next theorem in terms the operations Q^i and their iterations. Recall that having a d-dimensional class ξ we have $Q^{i+d}\xi = Q_i\xi$. We call $I=(i_1,\dots,i_r)$ an admissible sequence if $i_j \leqslant 2i_{j+1}$. We also define the *excess* of I by $\mathrm{excess}(I) = i_1 - (i_2+\cdots+i_r)$. Our next result now reads as following.

Theorem 2. *The homology algebra $H_*\Omega_0^{2^{i+1}-8}S^{2^i-2}$ contains a primitively generated polynomial subalgebra given by*

$$\mathbb{Z}/2[Q^I[\eta_i]_6 : I \in \mathcal{I}_6, \mathrm{excess}(I) > 6, i_r < 2^{i+1}-3]$$

where $I=(i_1,\dots,i_r) \in \mathcal{I}_6$ if and only if it is admissible and all of its entries are even numbers. The action of the Steenrod algebra on this subalgebra is determined by the Nishida relations. Moreover, let the ideal $\underline{\mathfrak{a}}_6$ in $H_*\Omega_0^{2^{i+1}-8}S^{2^i-2}$ be given by

$$\underline{\mathfrak{a}}_6 = \langle Q^I[\eta_i]_6 : \mathrm{excess}(I) > 6, I \notin \mathcal{I}_6\rangle.$$

This ideal belongs to the kernel of $(\Omega^{2^{i+1}-9}j_2)_*$ where $j_2 : \Omega S^{2^i-2} \to QS^{2^{i+1}-6}$ is the second James-Hopf invariant.

In other cases, we have a similar statement.

Theorem 3. *For $k=1,2,3$, the homology algebra $H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ contains a primitively generated polynomial subalgebra given by*

$$\mathbb{Z}/2[Q^I[\eta_i]_{6-k} : I\in\mathcal{I}_{6-k}, \mathrm{excess}(I)>6-k, i_r<2^{i+1}-3],$$

with

$$\begin{array}{rcl}\mathcal{I}_5 &=& \{I : I \textit{ admissible }, Q^IQ^3\neq 0\},\\ \mathcal{I}_4 &=& \{I : I \textit{ admissible }, Q^IQ^3\neq 0, \textit{ or } I=4J\},\\ \mathcal{I}_3 &=& \{I : I \textit{ admissible }, Q^IQ^3\neq 0, \textit{ or } Q^IQ^2Q^1\neq 0\},\end{array}$$

where $I=4J$ means that I is an admissible sequence whose all entries are divisible by 4. The action of the Steenrod algebra on this subalgebra is determined by the Nishida relations. Moreover, let the ideal $\underline{\mathfrak{a}}_{6-k}$ in this algebra be given by

$$\underline{\mathfrak{a}}_{6-k}=\langle Q^I[\eta_i]_{6-k} : \mathrm{excess}(I)>6-k, I\not\in\mathcal{I}_{6-k}\rangle.$$

This ideal then belongs to the kernel of $(\Omega^{2^{i+1}-9+k}j_2)_*$ where $j_2:\Omega S^{2^i-2}\to QS^{2^{i+1}-6}$ is the second James-Hopf invariant.

Remark 4. The method of proving the above theorem can be applied to obtain a set of generators for certain subalgebras of $H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ for $k=4,5,6$. However, it does not tell anything about the algebraic structure of these subalgebras.

We have some comments on the above theorems. First, notice that having $Q^I[\eta_i]_{6-k}$ with $I\not\in\mathcal{I}_{6-k}$, where $k=0,1,2,3$, does not tell us much in the following sense. We don't know whether or not if these terms are trivial. Moreover, assuming that these classes are nontrivial the method of proof does not tell us about the subalgebras that they generate. Second, we note that there is some indeterminacy in determining the action of the Steenrod algebra on the stated polynomial algebras in the following sense. If we are given a class $Q^I[\eta_i]_{6-k}$ with $I\in\mathcal{I}_{6-k}$, then it is not clear at all if $Sq^r_*Q^I[\eta_i]_{6-k}=Q^J[\eta_i]_{6-k}$ for some $J\in\mathcal{I}_{6-k}$. Finally, notice that in general, calculating the homology algebras mentioned above will mostly depends on spectral sequence based arguments. However, our method firstly provides some information about a part of these algebras; and secondly gives geometric meaning to some of its generators.
It is almost certain that our theorems here, Theorem 2 and Theorem 3, do not calculate the homology algebras completely, nevertheless they shed light on some cases that have not been known previously, as well as they provide some knowledge about the algebraic structure of these algebras. In fact, they seem to detect a part of $H_*Q_0S^{-n}$ which is not detected by previous methods. We finish by stating a conjecture, which predicts the behavior of the class $[\eta_i]_6$ under the homology suspension. This reads as following.

Conjecture. *The class $[\eta_i]_6\in H_6Q_0S^{-2^i+6}$ dies under the homology suspension $\sigma_*:H_*Q_0S^{-2^i+6}\to H_{*+1}Q_0S^{-2^i+7}$. Consequently, the subalgebra of $H_*Q_0S^{-2^i+6}$ generated by $Q^I[\eta_i]_6$ belong to $\ker\sigma_*$.*

Finally we note that techniques to prove the above results maybe applied in a wider generality. For instance, we may use the classical Hopf invariant one elements to do a similar job. Notice that the Hopf invariant one elements map nontrivially under the Hurewicz homomorphism $h:{}_2\pi_*Q_0S^0\to H_*Q_0S^0$. We state the following and leave the proof to reader.

Theorem 5. *Let $i=0,1,2,3$ and consider $\nu\in{}_2\pi_3^S$, and let $[\nu]_i\in H_iQ_0S^{-3+i}$ be the image of ν under the Hurewicz homomorphism*

$$h:{}_2\pi_*Q_0S^{-3+i}\to H_*Q_0S^{-3+i}.$$

This class pulls back to a spherical class $[\nu]_i\in H_i\Omega^{7-i}S^4$. This class gives rise to a primitively generated polynomial algebra inside $H_*\Omega^{7-i}S^4$ given by

$$\mathbb{Z}/2[Q^I[\nu]_i : I\in \textit{ admissible }, \mathrm{excess}(I)>0, i_r<6].$$

The action of the Steenrod algebra on this polynomial algebra is completely determined by the Nishida relations. Moreover, this subalgebra maps monomorphically under $(\Omega^{6-i}j_2)_*$ *where* $j_2 : \Omega S^4 \to QS^6$ *is the second James-Hopf invariant.*

The key fact in the proof will be that ν maps to the identity element under $j_2 : {}_2\pi_6\Omega S^4 \to {}_2\pi_6 QS^6$. A similar statement can be made about $\sigma \in {}_2\pi_7^S$, and the outcome seems to be more interesting as we get more loops!

Theorem 6. *Let* $i = 0, 1, 2, \ldots, 7$ *and consider* $\sigma \in {}_2\pi_7^S$, *and let* $[\sigma]_i \in H_iQ_0S^{-7+i}$ *be the image of* ν *under the Hurewicz homomorphism*

$$h : {}_2\pi_*Q_0S^{-7+i} \to H_*Q_0S^{-7+i}.$$

This class pulls back to a spherical class $[\sigma]_i \in H_i\Omega^{15-i}S^4$. *This class gives rise to a primitively generated polynomial algebra inside* $H_*\Omega^{15-i}S^4$ *given by*

$$\mathbb{Z}/2[Q^I[\sigma]_i : I \in \ \textit{admissible}\ , \mathrm{excess}(I) > 0, i_r < 14].$$

The action of the Steenrod algebra on this polynomial algebra is completely determined by the Nishida relations. Moreover, this subalgebra maps monomorphically under $(\Omega^{14-i}j_2)_*$ *where* $j_2 : \Omega S^4 \to QS^6$ *is the second James-Hopf invariant.*

Note 7. We have detected polynomial subalgebras inside the homology algebras $H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ for $k = 4, 5, 6$. The application of the Steenrod operations then will detect infinitely many other terms inside these algebras that give rise to polynomial subalgebras like the work of Cohen and Peterson in [CP89, Lemma 6.3]. If we have a class $Q^I[\eta_i]_j$ such as given by previous theorems, and a class ξ such that $Sq^r_*\xi = Q^I[\eta_i]_j$, then we know that $\xi \neq 0$. The relations such as

$$Sq^{2r}_*\xi^2 = (Sq^r_*\xi)^2 = (Q^I[\eta_i]_j)^2 \neq 0$$

show that $\xi^{2^t} \neq 0$. Therefore, the class ξ, as well as classes of the form $Q^I\xi$ for suitable choices of I, will give rise to polynomial subalgebra inside $H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$ for $k = 4, 5, 6$.

The rest of this paper is devoted to the proof of these observations and related calculation. We start by proving our results in the unstable case. We shall then provide the reader with the proof of our main result. We note that in other sections the numbering of theorems is done by sections, where here we used a single numbering to single out our main results.

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2 Proof of Theorem 1

The proof of our theorems are based on two basic observations. The first observation is an equivalence between two definitions of the Hopf invariant. We recall the following result of Eccles [E93, Proposition 4.4].

Lemma 2.1. *Let $\alpha \in {}_2\pi_{2m}QX$. Then $h\alpha = x_m^2$ with $x_m \in H_mX$ if and only if the stable adjoint of α is detected by Sq^{m+1} on x_m in its stable mapping cone. Here h is the Hurewicz homomorphism*

$$h : {}_2\pi_*QX \to H_*QX.$$

We also need to fix our terminology. By the statement that "$f : S^t \to Y$ is detected by Sq^r on $y \in H_*Y$" we mean that $Sq_*^r g_{t+1} = y$ in $C_f = Y \cup_f e^{t+1}$, the mapping cone of f, where $g_{t+1} \in H_{t+1}C_f$ is a generator given by the attached $(t+1)$-cell provided that $f_* = 0$. Here $Sq_*^r : H_*C_f \to H_{*-r}C_f$ is the operation dual to Sq^r.
The second observation is provided by the fact that the class $\eta_i \in {}_2\pi_{2^i}^S$ pulls back to ${}_2\pi_{2^{i+1}-2}S^{2^i-2}$, i.e.

$$S^{2^{i+1}-2} \to S^{2^i-2},$$

and maps to $\nu \in {}_2\pi_3^S$ under the second James-Hopf invariant [M77]. Here by the 2nd James-Hopf invariant we mean

$$j_2 : \Omega\Sigma X \to Q(X \wedge X),$$

where in our case $X = S^{2^i-3}$. In this case, the fact that η_i has Hopf invariant ν means that $j_2\eta_i = \nu$. This implies that as an unstable mapping ν is given by the following composite

$$S^{2^{i+1}-3} \xrightarrow{\eta_i} \Omega S^{2^i-2} \xrightarrow{j_2} QS^{2^{i+1}-6}.$$

Here $\eta_i : S^{2^{i+1}-3} \to \Omega S^{2^i-2}$ is the adjoint to the mapping $S^{2^{i+1}-2} \to S^{2^i-2}$. The mapping ν is detected by Sq^4 on $g_{2^{i+1}-6}$ in its mapping cone, where $g_{2^{i+1}-6} \in H_{2^{i+1}-6}QS^{2^{i+1}-6}$ is the generator given by the inclusion $S^{2^{i+1}-6} \to QS^{2^{i+1}-6}$. We may adjoint down the above composite to obtain the following composite

$$\nu : S^7 \xrightarrow{\eta_i} \Omega_0^{2^{i+1}-9}S^{2^i-2} \xrightarrow{\Omega^{2^{i+1}-10}j_2} QS^4.$$

This composite it detected by Sq^4 on $g_4 \in H_4QS^4$ in its mapping cone. Applying Lemma 2.1 implies that if adjoint down once more, we then obtain a mapping which is detected by homology. More precisely, the composite

$$\widetilde{\nu}_6 : S^6 \xrightarrow{\eta_i} \Omega_0^{2^{i+1}-8}S^{2^i-2} \xrightarrow{\Omega^{2^{i+1}-9}j_2} QS^3 \qquad (1)$$

is detected by

$$\widetilde{\nu}_{6*}g_6 = g_3^2$$

where $\widetilde{\nu}_6$ denotes adjoint of ν, and $g_3 \in H_3QS^3$ is a generator given by the inclusion $S^3 \to QS^3$. Setting $[\eta_i]_6 = h\eta_i$ we then have $[\eta_i]_6 \neq 0$ in $H_6\Omega_0^{2^{i+1}-8}S^{2^i-2}$ and that

$$(\Omega^{2^{i+1}-7}j_2)_*[\eta_i]_6 = g_3^2$$

where $h : {}_2\pi_6\Omega_0^{2^{i+1}-8}S^{2^i-2} \to H_6\Omega_0^{2^{i+1}-8}S^{2^i-2}$ denotes the Hurewicz homomorphism.

As we mentioned earlier, the homology of the space $\Omega_0^{2^{i+1}-8}S^{2^i-6}$ admits operations

$$Q_a : H_*\Omega_0^{2^{i+1}-8+k}S^{2^i-2} \to H_{a+2*}\Omega_0^{2^{i+1}-8+k}S^{2^i-2}$$

for $a < (2^{i+1}-8)-1$. We like to investigate the R-module spanned by $[\eta_i]_6$, i.e. the module spanned by elements of the form $Q_I[\eta_i]_6$ with $I = (i_1, \ldots, i_r)$ such that

$$0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r < 2^{i+1}-9.$$

The mapping

$$\Omega^{2^{i+1}-9}j_2:\Omega_0^{2^{i+1}-8}S^{2^i-2}\to QS^6$$

is a $(2^{i+1}-9)$-fold loop map. This implies that $(\Omega^{2^{i+1}-9}j_2)_*$ commutes with all classes of the form $Q_I[\eta_i]_6$ with $i_r<(2^{i+1}-9)-1=2^{i+1}-9$, i.e. having $Q_I[\eta_i]_6$ with $0<i_1\leqslant\cdots\leqslant i_r<2^{i+1}-9$ then we have

$$(\Omega^{2^{i+1}-9}j_2)_*Q_I[\eta_i]_6=Q_I(\Omega^{2^{i+1}-9}j_2)_*[\eta_i]_6=Q_Ig_3^2.$$

Let us write $I=2K$ if $K=(k_1,\ldots,k_r)$ with $i_j=2k_j$ for any $1\leqslant j\leqslant r$. We then have

$$(\Omega^{2^{i+1}-9}j_2)_*Q_I[\eta_i]_6=\begin{cases}0 & \text{if } i_j \text{ is odd for some } j\\ (Q_Kg_3)^2 & \text{if } I=2K.\end{cases}$$

This implies that if $I=2K$, then $Q_I[\eta_i]_6\neq 0$. On the other hand notice that $\Omega^{2^{i+1}-7}j_2$ is an iterated loop map, which in particular implies $(\Omega^{2^{i+1}-7}j_2)_*$ is a multiplicative map. Also, notice that H_*QS^3 is a polynomial algebra. Hence, if we have an arbitrary pair of terms $Q_I[\eta_i]_6$, $Q_L[\eta_i]_6$ which map nontrivially under $(\Omega^{2^{i+1}-7}j_2)_*$ then their product will map nontrivially under this homomorphism. This then implies that

$$\mathbb{Z}/2[Q_I[\eta_i]_6:I=2K\text{ increasing}, i_1>0, i_r<2^{i+1}-9]$$

is a polynomial algebra living in $H_*\Omega_0^{2^{i+1}-8}S^{2^i-2}$. Recall that for a d-dimensional class ξ we have $Q_a\xi=Q^{a+d}\xi$. Hence, we may rewrite the above polynomial algebra as

$$\mathbb{Z}/2[Q^I[\eta_i]_6:I=2K\text{ admissible } i_1>0, i_r<2^{i+1}-3].$$

We note that $I=2K$ are all of the sequences living in $\mathcal{I}_6$. Finally notice that the class $[\eta_i]_6$ is an A-annihilated class. Hence, to describe the action of Steenrod operations Sq^t_* on $Q^I[\eta_i]_6$ we only need to apply Nishida relations. This completes the proof of Theorem 1.

3 Proof of Theorem 2

The proof of this result is similar to the proof of Theorem 1. We like to draw reader's attention to the following table, where the left hand side denotes the mapping ν, suspended down, and the right hand side denotes the Hurewicz image of the corresponding mapping

$$\begin{array}{ll}
\widetilde{\nu}_6:S^6\to QS^3 & h\widetilde{\nu}_6=Q^3g_3,\\
\widetilde{\nu}_5:S^5\to QS^2 & h\widetilde{\nu}_5=Q^3g_2,\\
\widetilde{\nu}_4:S^4\to QS^1 & h\widetilde{\nu}_4=Q^3g_1+Q^2Q^1g_1,\\
\widetilde{\nu}_3:S^3\to Q_0S^0 & h\widetilde{\nu}_3=x_3+Q^2x_1+D,\\
\widetilde{\nu}_2:S^2\to Q_0S^{-1} & h\widetilde{\nu}_2=w_2',\\
\widetilde{\nu}_1:S^1\to Q_0S^{-2} & h\widetilde{\nu}_1=p_1^{S^{-2}},
\end{array}$$

where D denotes a sum of decomposable terms, $w_2'\in H_2Q_0S^{-1}$ is an A-annihilated primitive class with $\sigma_*w_2'=p_3'=x_3+Q^2x_1+D$, and $p_1^{S^{-2}}\in H_1Q_0S^{-2}$ is an A-annihilated primitive class with $\sigma_*p_1^{S^{-2}}=w_2'$. Recall that (1) provided us with a decomposition for $\widetilde{\nu}_6$. This allows us to have the following decompositions for $\widetilde{\nu}_5$, $\widetilde{\nu}_4$, and $\widetilde{\nu}_3$ respectively

$$\widetilde{\nu}_5 : S^5 \xrightarrow{\eta_i} \Omega_0^{2^{i+1}-7} S^{2^i-2} \xrightarrow{\Omega^{2^{i+1}-8} j_2} QS^2,$$

$$\widetilde{\nu}_4 : S^4 \xrightarrow{\eta_i} \Omega_0^{2^{i+1}-6} S^{2^i-2} \xrightarrow{\Omega^{2^{i+1}-7} j_2} QS^1,$$

$$\widetilde{\nu}_3 : S^3 \xrightarrow{\eta_i} \Omega_0^{2^{i+1}-5} S^{2^i-2} \xrightarrow{\Omega^{2^{i+1}-6} j_2} Q_0 S^0.$$

Now we can complete proof of Theorem 2. We only do one case and leave the other cases to the reader. Consider $\widetilde{\nu}_4 : S^4 \to QS^1$ with $h\widetilde{\nu}_4 = Q^3 g_1 + Q^2 Q^1 g_1$, see for example [E80, Proposition 3.4]. This then implies that

$$[\eta_i]_4 = h\eta_i \neq 0.$$

Moreover, this shows that

$$(\Omega^{2^{i+1}-7} j_2)_* [\eta_i]_4 = Q^3 g_1 + Q^2 Q^1 g_1.$$

Next, we like to consider the subalgebra of $H_*\Omega^{2^{i+1}-6} S^{2^i-2}$ generated by classes $Q^I[\eta_i]_4$. The homology of the space $\Omega^{2^{i+1}-6} S^{2^i-2}$ admits operations

$$Q_a : H_*\Omega_0^{2^{i+1}-6} S^{2^i-2} \to H_{a+2*}\Omega_0^{2^{i+1}-6} S^{2^i-2}$$

with $a < (2^{i+1}-6)-1$. This then implies that the mapping $(\Omega^{2^{i+1}-7} j_2)_*$ commutes with $Q_I[\eta_i]_4$ where $I = (i_1, \ldots, i_r)$ such that $0 < i_1 \leqslant \cdots \leqslant i_r < 2^{i+1}-7$. Notice that written with operations Q^I we then look for the subalgebra generated by the classes of the form $Q^I[\eta_i]_4$ with I admissible and $i_r < 2^{i+1}-3$. This yields the following

$$\begin{aligned}
(\Omega^{2^{i+1}-7} j_2)_* Q^I [\eta_i]_4 &= Q^I (\Omega^{2^{i+1}-7} j_2)_* [\eta_i]_4 \\
&= Q^I (Q^3 g_1 + Q^2 Q^1 g_1) \\
&= Q^I Q^3 g_1 + Q^I Q^2 Q^1 g_1.
\end{aligned}$$

Notice that in the above sum the second term is of the form $Q^I g_1^4$. Therefore, the above sum is nontrivial only if either $Q^I Q^3 \neq 0$, or all entries of I are divisible by 4. Notice that this characterises the set of sequences belonging to $\mathcal{I}_4$. The fact that H_*QS^1 is a polynomial algebra, combined with the fact that $(\Omega^{2^{i+1}-7} j_2)_*$ is a multiplicative map, implies that the subalgebra of $H_*\Omega_0^{2^{i+1}-6} S^{2^i-2}$ generated by classes of the form $Q^I[\eta_i]_4$ is a polynomial algebra, i.e. we have a primitively generated subalgebra sitting inside $H_*\Omega^{2^{i+1}-6} S^{2^i-2}$ determined by

$$\mathbb{Z}/2[Q^I[\eta_i]_4 : I \in \mathcal{I}_4, \text{excess}(I) > 4, i_r < 2^{i+1}-3].$$

Notice that if $I \notin \mathcal{I}_4$ then $(\Omega^{2^{i+1}-7} j_2)_* Q^I [\eta_i]_4 = 0$. This means that the ideal $\underline{\mathfrak{a}}_4 \subseteq H_*\Omega_0^{2^{i+1}-6} S^{2^i-2}$ generated by such classes belongs to the kernel of $(\Omega^{2^{i+1}-7} j_2)_*$, i.e.

$$\underline{\mathfrak{a}}_4 = \langle Q^I[\eta_i]_4 : \text{excess}(I) > 4, I \notin \mathcal{I}_4 \rangle \subseteq \ker(\Omega^{2^{i+1}-7} j_2)_*.$$

This completes the proof of Theorem 2.

4 Stablisation: The Main Theorem

We like to restate our results when the finite loop spaces are replaced with infinite loop spaces. More precisely, notice that there is a mapping

$$E:\Omega S^{2^i-2}\to QS^{2^i-3}.$$

Applying the iterated loop functor $\Omega^{2^{i+1}-9}$ to this mapping we obtain

$$\Omega^{2^{i+1}-9}E:\Omega^{2^{i+1}-8}S^{2^i-2}\to QS^{-2^i+6}$$

where restricting to base point components yields

$$\Omega_0^{2^{i+1}-8}S^{2^i-2}\to Q_0S^{-2^i+6}.$$

We then may consider the mapping

$$(\Omega^{2^{i+1}-9}E)_*:H_*\Omega_0^{2^{i+1}-8}S^{2^i-2}\to H_*Q_0S^{-2^i+6}$$

and the image of the polynomials identified by Theorem 2.
Previously, we used James-Hopf invariant $j_2:\Omega S^{2^i-2}\to QS^{2^{i+1}-6}$ and its iterated loop. In the stable case, we consider the stable James-Hopf invariant

$$j_2^\infty:QS^{2^i-3}\to Q\Sigma^{2^i-3}P_{2^i-3}$$

where the upper index ∞ is used to note that this is a map associated with infinite loop spaces. Applying $\Omega^{2^{i+1}-9}$ to j_2^∞ we obtain

$$Q_0S^{-2^i+6}\to Q_0\Sigma^{-2^i+6}P_{2^i-3}.$$

We recall that there is a commutative diagram given by

$$\begin{array}{ccc} \Omega S^{2^i-2} & \xrightarrow{j_2} & QS^{2^{i+1}-6} \\ \downarrow E & & \downarrow i \\ QS^{2^i-3} & \xrightarrow{j_2^\infty} & Q\Sigma^{2^i-3}P_{2^i-3}. \end{array} \qquad (2)$$

In particular, the mapping $S^{2^{i+1}-6}\to QS^{2^{i+1}-6}\to Q\Sigma^{2^i-3}P_{2^i-3}$ may be viewed as the inclusion of the bottom cell, and is nontrivial in homology. Applying $\Omega^{2^{i+1}-9}$ to this diagram we obtain

$$\begin{array}{ccc} \Omega_0^{2^{i+1}-8}S^{2^i-2} & \longrightarrow & QS^3 \\ \downarrow & & \downarrow \\ Q_0S^{-2^i+6} & \longrightarrow & Q_0\Sigma^{-2^i+6}P_{2^i-3}. \end{array}$$

We like to study the composite

$$j_2^\infty\eta_i:S^{2^{i+1}-3}\to QS^{2^i-3}\to Q\Sigma^{2^i-3}P_{2^i-3}.$$

This allows us to restrict our attention to

$$j_2^\infty\eta_i:S^{2^{i+1}-3}\to QS^{2^i-3}\to Q\Sigma^{2^i-3}P_{2^i-3}^{2^i}. \qquad (3)$$

The fact that η_i maps to $\nu \in {}_2\pi_3^S$ under the Hopf invariant implies that (3) should be detected by Sq^4 on the bottom cell, i.e. by Sq^4 on $\Sigma^{2^i-3}a_{2^i-3}$. Like the proof of Theorem 2, adjointing down, $(2^{i+1}-10)$-times, we obtain

$$S^7 \longrightarrow QS^{-2^i+7} \xrightarrow{\Omega^{2^{i+1}-10}j_2^\infty} Q\Sigma^{-2^i+7}P_{2^i-3}\ . \tag{4}$$

Our claim then is that this mapping is detected by Sq^4 on a 4-dimensional homology class, say $\Sigma^{-2^i+7}a_{2^i-3} \in H_4Q\Sigma^{-2^i+7}P_{2^i-3}$. It is not difficult to see that there is a such homology class. Applying the iterated loop functor $\Omega^{2^{i+1}-10}$ to diagram (2) and taking homology results the following commutative diagram

$$\begin{array}{ccc} H_*QS^{2^{i+1}-6} & \xrightarrow{i_*} & H_*Q\Sigma^{2^i-3}P_{2^i-3} \\ \uparrow{\scriptstyle \sigma_*^{2^{i+1}-10}} & & \uparrow{\scriptstyle \sigma_*^{2^{i+1}-10}} \\ H_*QS^4 & \xrightarrow{(\Omega^{2^{i+1}-10}i)_*} & H_*Q\Sigma^{-2^i+7}P_{2^i-3}. \end{array}$$

Here we have used $\sigma_*^{2^{i+1}-10}$ to denote the iterated homology suspension. Notice that

$$\sigma_*^{2^{i+1}-10}(\Omega^{2^{i+1}-10}i)_*g_4 = i_*\sigma_*^{2^{i+1}-10}g_4 = i_*g_{2^{i+1}-6} = \Sigma^{2^i-3}a_{2^i-3}.$$

This allows us to define

$$\Sigma^{-2^i+7}a_{2^i-3} = (\Omega^{2^{i+1}-10}i)_*g_4$$

with the property that

$$\sigma_*^{2^{i+1}-10}\Sigma^{-2^i+7}a_{2^i-3} = \Sigma^{2^i-3}a_{2^i-3}.$$

Here $g_4 \in H_4QS^4$ is the generator given by $S^4 \to QS^4$. Similarly, we may define $\Sigma^{-2^i+6}a_{2^i-3} \in H_3Q\Sigma^{-2^i+6}P_{2^i-3}$ by

$$\Sigma^{-2^i+6}a_{2^i-3} = (\Omega^{2^{i+1}-9}i)_*g_3.$$

The observation that $g_3 \in H_3QS^3$ and $g_4 \in H_4QS^4$ are spherical implies that the classes $\Sigma^{-2^i+6}a_{2^i-3}$, $\Sigma^{-2^i+7}a_{2^i-3}$ are also spherical classes in the respective homology groups. Notice that these are quite natural to expect, as for instance $\Sigma^{-2^i+6}a_{2^i-3}$ corresponds to the bottom call of $\Sigma^{-2^i+6}P_{2^i-3}$ whereas we know that bottom cells always give rise spherical classes.
Now we are ready to prove our Main Theorem. We recall the statement that we want to prove.

Main Theorem. *Let $\eta_i \in {}_2\pi_{2^i}^S$ denote Mahowald's family. This class is detected by the Hurewicz homomorphism*

$$h : {}_2\pi_6Q_0S^{-2^i+6} \to H_6Q_0S^{-2^i+6}.$$

The spherical class $[\eta_i]_6 = h\eta_i$ has the following property. Let $j_2^\infty : QS^{2^i-3} \to Q\Sigma^{2^i-3}P_{2^i-3}$ be the second stable James-Hopf invariant. We then have

$$(\Omega^{2^{i+1}-9}j_2^\infty)_*[\eta_i]_6 = (\Sigma^{-2^i+6}a_{2^i-3})^2 \neq 0$$

where $\Sigma^{-2^i+6}a_{2^i-3} \in H_3Q_0\Sigma^{-2^i+6}P_{2^i-3}$ is the class given by the inclusion of the bottom cell $S^3 \to Q_0\Sigma^{-2^i+6}P_{2^i-3}$.

Here we use $[\eta_i]_6$ to denote this spherical class as we like to remember that it is the class given by the mapping

$$\Omega_0^{2^{i+1}-8}S^{2^i-2} \to Q_0S^{-2^i+6}.$$

To complete the proof, we need a more general version of Lemma 2.1. The result is as following.

Lemma 4.1. *Suppose $f: S^{2m} \to \Omega X$ is given with X having its bottom call in dimension $m+1$. Then the adjoint mapping $S^{2m+1} \to X$ is detected by Sq^{m+1} on $\sigma_* x_m$ if and only if $hf = x_m^2 \neq 0$ where $x_m \in H_*\Omega X$.*

We leave the proof of this lemma to another section.

Proof of the Main Theorem. We have already done a part of the proof above. To complete the proof, notice that the composite

$$S^7 \longrightarrow QS^{-2^i+7} \xrightarrow{\Omega^{2^{i+1}-10} j_2^\infty} Q\Sigma^{-2^i+7} P_{2^i-3}^{2^i}$$

is detected by Sq^4 on a 4-dimensional homology class, say $\Sigma^{-2^i+7} a_{2^i-3} \in H_4 Q\Sigma^{-2^i+7} P_{2^i-3}$. Moreover, we know that

$$\Sigma^{-2^i+7} a_{2^i-3} = \sigma_* \Sigma^{-2^i+6} a_{2^i-3}.$$

This then implies that adjointing down once, we have

$$S^6 \xrightarrow{\eta_i} QS^{-2^i+6} \xrightarrow{\Omega^{2^{i+1}-9} j_2^\infty} Q\Sigma^{-2^i+6} P_{2^i-3}^{2^i}.$$

Lemma 4.1 now implies that the above composite is detected by

$$(\Omega^{2^{i+1}-9} j_2^\infty \eta_i)_* g_3 = (\Sigma^{-2^i+6} a_{2^i-3})^2 \neq 0.$$

This completes the proof. □

According to the above proof, there are some classes in $H_* Q\Sigma^{-2^i+6} P_{2^i-3}$ with nontrivial square, e.g. $(\Sigma^{-2^i+6} a_{2^i-3})^2 \neq 0$. However, this does not imply that the subalgebra generated by such classes is a polynomial algebra as one still has to eliminate the possible truncations, i.e. we don't know if $(\Sigma^{-2^i+6} a_{2^i-3})^{2^t} \neq 0$ for all $t > 1$.
Moreover, we don't know if all classes of the form $Q^I \Sigma^{-2^i+6} a_{2^i-3}$ are nontrivial. However, there are two families of such classes where it is easy to show they are not trivial. First, notice that

$$\sigma_*^{2^i-6} \Sigma^{-2^i+6} a_{2^i-3} = a_{2^i-3} + \text{ other terms}.$$

This implies that if we choose I such that excess$(I) \geqslant 2^i - 3$ then $Q^I \Sigma^{-2^i+6} a_{2^i-3} \neq 0$. This comes easy from the fact that

$$\sigma_*^{2^i-6} Q^I \Sigma^{-2^i+6} a_{2^i-3} = Q^I a_{2^i-3}.$$

Hence, we may consider the subalgebra spanned by such elements. Second, we use the method suggested by Note 7. That is, if we have a class ξ with $Sq_*^r \xi = Q^I \Sigma^{-2^i+6} a_{2^i-3}$ for some I with $Q^I \Sigma^{-2^i+6} a_{2^i-3} \neq 0$, then we know that $\xi \neq 0$ as well as $\xi^2 \neq 0$. This then gives infinite number of generators inside $H_* Q\Sigma^{-2^i+6} P_{2^i-3}$ whose generators has are of minimum height 2.

5 Relations to spherical classes homology of $Q_0 S^0$

The type of spherical classes in $H_* Q_0 S^0$ is predicted by a conjecture due to Curtis [C75, Thoerem 7.1]. This predicts that only the Hopf invariant one elements $\eta, \nu, \sigma \in {}_2\pi_{2^i-1} Q_0 S^0$ and the potential Kervaire invariant one elements $\theta_i \in {}_2\pi_{2^{i+1}-2} Q_0 S^0$ map nontrivially under the Hurewicz homomorphism

$$h: {}_2\pi_* Q_0 S^0 \to H_* Q_0 S^0.$$

Observe that the Hopf invariant one elements are the only elements which have Adams filtration one. Moreover, the only elements which have Adams filtration at least 2 are the potential Kervaire invariant one elements and the classes $\eta_i \in {}_2\pi_{2^i}Q_0S^0$. We recall from the introduction that the elements η_i map trivially under the Hurewicz homomorphism $h : {}_2\pi_*Q_0S^0 \to H_*Q_0S^0$. Hence, the Curtis conjecture states that if $\alpha \in {}_2\pi_*^S$ has Adams filtration at least 3 then its stable adjoint viewed as an element of ${}_2\pi_*Q_0S^0$ maps trivially under the Hurewicz homomorphism ${}_2\pi_*Q_0S^0 \to H_*Q_0S^0$.
Now, we can ask two related questions. First, notice that having $\alpha \in {}_2\pi_*Q_0S^0$ we may consider to adjoint of α as elements of ${}_2\pi_{*-k}Q_0S^{-k}$ under the suspension isomorphism

$${}_2\pi_{*-k}^S S^{-k} \simeq {}_2\pi_{*-k}Q_0S^{-k} \to {}_2\pi_*Q_0S^0 \simeq {}_2\pi_*^S.$$

We then may ask what is the least k such that the adjoint of α maps nontrivially under the Hurewicz homomorphism

$${}_2\pi_{*-k}Q_0S^{-k} \to H_*Q_0S^{-k}.$$

This paper then provides an example by calculating an upper bound for k, a lower bound for $-k$, when $\alpha = \eta_i$.
Second, if we assume that the Curtis conjecture fails then how we can calculate the Hurewicz image of those elements of which their Adams filtration is at least 3? Our calculation here then suggest that some knowledge on the (iterated) Hopf invariant of α will be very useful in this regard. This then suggests that an EHP-approach is the right approach to deal with these questions.
On the other hand, it is an immediate consequence of the homology of the J-homomorphism $SO \to Q_0S^0$ that apart from the Hopf invariant one elements any other element of ${}_2\pi_*Q_0S^0$ which belongs to im$J$ maps trivially under $h : {}_2\pi_*Q_0S^0 \to H_*Q_0S^0$ [EZ09, Main Theorem], where the implications on of this observation are investigated in [Z09a]. According to this observations, the Curtis conjecture then reduces to the following statement.

The Curtis Conjecture. *Suppose $\alpha \in {}_2\pi_*Q_0S^0$ which maps nontrivially under the projection ${}_2\pi_*Q_0S^0 \to {}_2\pi_*cokerJ$ and does not correspond to a Kervaire invariant one element. Then α maps trivially under the Hurewicz homomorphism*

$$h : {}_2\pi_*Q_0S^0 \to H_*Q_0S^0.$$

One then may do a similar job, as we did here, for those elements of ${}_2\pi_*Q_0S^0$ that belong to ${}_2\pi_*$coker$J$ and determine the type of subalgebras inside $H_*Q_0S^{-n}$ that they give rise to. **Notice** that the η_i family does not belong to im$J$, i.e. it belongs to ${}_2\pi_*$coker$J$, i.e. we have already provided an example of how such a calculation maybe carried out.

6 Proof of Lemma 4.1

Here we like to give a proof of Lemma 4.1. The following observation, which is a corollary of the Freudenthal's suspension theorem, will be used in the proof of lemma.

Lemma 6.1. *Let X_n^i denote a cell complex with bottom cell at dimension n and top cell at dimension i. If $i < 2n$ then X_n^i admits at least one desuspension, i.e.*

$$X_n^i \simeq \Sigma Y_{n-1}^{i-1}.$$

Proof. The proof is based on induction. If $i = n$, then X_n^i is a wedge of spheres and hence desuspends. Assume that the statement is true for X_n^i, and we prove it for X_n^{i+1}. Let $f : S^i \to X_n^i \simeq \Sigma Y_{n-1}^{i-1}$ denote the attaching map of an $(i+1)$-cell. Observe that $f \in \pi_i \Sigma Y^{i-1}_{n-1}$ with $i < 2n$. According to the suspension theorem, f desuspends to $\pi_{i-1} Y^{i-1}_{n-1}$. The fact that f desuspends implies that the cofibre of $X^i_n \cup_f e^{i+1}$ also desuspends. Finally the fact that X^{i+1}_n is obtained by attaching some $(i+1)$-dimensional cells through a map from a wedge of spheres to X^i_n shows that X^{i+1}_n also admits a desuspension. This completes the proof. □

Now we proceed with the proof of Lemma 4.1.

Proof of Lemma 4.1. First, let $f : S^{2m+1} \to X$ be given such that X has its bottom cell at dimension $m+1$. Assume that f is detected by Sq^{m+1} on $x_{m+1} \in H_{m+1}X$. We like to show that the adjoint of f, say $g : S^{2m} \to \Omega X$ is detected in homology by $hg = y_m^2 \neq 0$ with $y_m \in H_m \Omega X$ such that $\sigma_* y_m = x_{m+1}$.
Notice that f pulls back to the $(2m+1)$-skeleton of X, i.e. it is in the image of

$$i_\# : \pi_{2m+1} X^{2m} \to \pi_{2m+1} X$$

where $i : X^{2m+1} \to X$ denotes the inclusion. We may apply Lemma 8 to X^{2m+1} to observe that there exists a homotopy equivalence

$$X^{2m+1} \xrightarrow{\simeq} \Sigma Y^{2m}$$

where Y^{2m} has its bottom cell at dimension m and top cell at dimension $2m$. Now we may adjoint f to obtain a mapping $g : S^{2m} \to \Omega X$ where according to the above observation it pulls back to a map $S^{2m} \to \Omega X^{2m+1} \simeq \Omega \Sigma Y^{2m}$, i.e. we have the following commutative diagram

$$\begin{array}{ccccc} S^{2m} & \xrightarrow{g} & \Omega X & & \\ & \searrow^{g'} & \uparrow{\scriptstyle \Omega i} & & \\ & & \Omega X^{2m+1} & \xrightarrow{\simeq} & \Omega \Sigma Y^{2m}. \end{array}$$

If we assume that f is detected by Sq^{m+1} on $x_{m+1} \in H_{m+1}X$, this then also implies that the pull back of f to X^{2m+1} is also detected by Sq^{m+1} on $x_{m+1} = \Sigma y_m$ where $y_{2m} \in H_m Y^{2m}$. Lemma 6 then implies that the mapping

$$g' : S^{2m} \to \Omega X^{2m+1} \simeq \Omega \Sigma Y^{2m}$$

is detected by homology, i.e. $hg' = y_m^2$ where we have used y_m to denote the preimage of y_m under the isomorphism $H_m \Omega X^{2m+1} \to H_m Y^{2m}$. The class y_m has the property that $\sigma_* y_m = x_{m+1}$.
To complete the proof, we need to show that $hg = (\Omega i)_* y_m^2 \neq 0$. This is straightforward once we consider the pair $(\Omega X, \Omega X^{2m+1})$ and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{2m+1}(\Omega X, \Omega X^{2m+1}) & \xrightarrow{\partial} & \pi_{2m} \Omega X^{2m+1} & \xrightarrow{(\Omega i)_\#} & \pi_{2m} \Omega X & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h(\simeq)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \\ \cdots & \longrightarrow & H_{2m+1}(\Omega X, \Omega X^{2m+1}) & \xrightarrow{\partial} & H_{2m} \Omega X^{2m+1} & \xrightarrow{(\Omega i)_*} & H_{2m} \Omega X & \longrightarrow & \cdots. \end{array}$$

If we assume that $(\Omega i)_* hg' = (\Omega i)_* y_m^2 = 0$, then y_m^2 pulls back to $H_{2m+1}(\Omega X, \Omega X^{2m+1})$. One may use homotopy excision property to show that

$$H_{2m+1}(\Omega X, \Omega X^{2m+1}) \simeq \pi_{2m+1}(\Omega X, \Omega X^{2m+1}),$$

i.e. g' belongs to the image of $\partial : \pi_{2m+1}(\Omega X, \Omega X^{2m+1}) \to \pi_{2m} \Omega X^{2m+1}$. This then implies that $(\Omega i)_\# g' = 0$. However, we know that $0 \neq g = (\Omega i)_\# g$. This gives a contradiction to the assumption that $(\Omega i)_* y_m^2 = 0$. Hence, $(\Omega i)_* y_m^2 \neq 0$ and the proof is complete.
The proof in the other direction is in a similar way and we leave it to the reader. □

7 One application and a conjecture

Consider the case when $i = 3$. In this case case we obtain spherical classes $[\eta_3]_6 \in H_6Q_0S^{-2}$ corresponding to η_3. A quick observation is that this class dies under the homology suspension $\sigma_* : H_*Q_0S^{-2} \to H_*Q_0S^{-1}$, and hence the subalgebra of $H_*Q_0S^{-2}$ generated by $Q^I[\eta_3]_6$ belongs to $\ker\sigma_*$. This is easy to see from the following fact.

Lemma 7.1. *A spherical class $\xi_{-1} \in H_*Q_0S^{-1}$ survives under the homology suspension $\sigma_* : H_*Q_0S^{-1} \to H_*Q_0S^0$.*

Proof. Recall from [CP89, Theorem 1.1] that the homology suspension $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism where Q is the indecomposable quotient module functor, and P is the primitive submodule functor. Moreover, the homology algebra $H_*Q_0S^{-1}$ is an exterior given by

$$H_*Q_0S^{-1} \simeq E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*Q_0S^0).$$

Notice that a spherical class is primitive. This implies that a spherical class in $H_*Q_0S^{-1}$ cannot be a decomposable, as if this happens this it must be a square which is trivial in the exterior algebra. Hence, a given spherical class $\xi_{-1} \in H_*Q_0S^{-1}$ does not die under the suspension. This proves the lemma. □

Now assuming that $\sigma_*[\eta_3]_6 \neq 0$ would imply that η_i gives a spherical class in $H_*Q_0S^{-1}$ and hence to a spherical class in $H_*Q_0S^0$. But this is a contradiction, as we observed at the beginning of the paper that η_i does not give rise to a spherical class in $H_*Q_0S^0$. Hence, $\sigma_*[\eta_3]_6 = 0$. In particular this detects a part of $H_*Q_0S^{-2}$ which does not come from pull back of any class in $H_*Q_0S^{-1}$. We note that the existing literature on the calculation of $H_*Q_0S^{-2}$ has not detected this bit. This motivates the following conjecture.

Conjecture. *the class $[\eta_i]_6 \in H_6Q_0S^{-2^i+6}$ dies under the homology suspension $\sigma_* : H_*Q_0S^{-2^i+6} \to H_{*+1}Q_0S^{-2^i+7}$. Consequently, the subalgebra of $H_*Q_0S^{-2^i+6}$ generated by $Q^I[\eta_i]_6$ belong to* $\ker\sigma_*$.

8 A note on odd primary versions

This short section is dedicated to the problem of understanding $H_*(\Omega_0^{n+k}S^n;\mathbb{Z}/p)$ and $H_*(Q_0S^{-k};\mathbb{Z}/p)$ when p is an odd prime. As the reader has observed the essence of our calculations seems to be applicable in odd primes, provided that one has the right machinery that we have used. We found mappings $S^t \to Y$ with a factorisation through $\Omega^{n+k}S^n$, i.e.

$$S^t \to \Omega^{n+k}S^n \to Y$$

with nontrivial homology, for some choices of Y and mappings $\Omega^{n+k}S^n \to Y$. In our case, the spaces Y and the mapping were provided by the James-Hopf invariants.

It is likely that one can do a similar job at odd primes. In the level of homotopy, assume that $\alpha \in {}_p\pi_k^S$ pulls back to ${}_p\pi_{n+k+1}S^{n+1}$. We then may consider the James-Hopf invariant of the EHP-sequence at prime p, and use the p-th James-Hopf invariant

$$j_p : \Omega\Sigma S^n \to QS^{np}$$

inducing

$${}_p\pi_{n+k}\Omega\Sigma S^n \to {}_p\pi_{n+k}QS^{np}.$$

Of course it is not known whether or not the p-analogue of Mahowald's family exists in odd primary parts of the stable homotopy groups, when at any odd prime p the η_i is defined to be a class which is detected by h_0h_i in the Adams spectral sequence.

Remark 8.1. It is debatable what is the "right" analogous for Mahowald's family, as it is observed by Hunter and Kuhn [HK99] that there could be potential families in odd primes with "more similar" behavior to the well known family at prime 2.

Assuming that the η_i exists at the odd prime p, Minami [M00, Theorem 4.1] has proved that η_i pulls back to ${}_p\pi_{(q(p^i+1)-2)+2(p^i-p)+3}S^{2(p^i-p)+3}$, $q=2(p-1)$, and maps to a multiple of a desuspension of $\beta_1 \in {}_p\pi^2_{qp-2}$ under the James-Hopf invariant j_p, i.e.

$$j_p\eta_i = k\beta_1$$

for some k with $p \nmid k$. Hence, detecting the class β_1, and in general β_t, in $H_*(-\mathbb{Z}/p)$ as a map into a loop space will be useful in this direction with to possible applications: as in one direction it may reveal information on the homology algebras $H_*(\Omega^{n+k}S^k;\mathbb{Z}/p)$, and it other direction it may provide some obstruction to the existence of η_i in odd primes.

Finally, we note that at prime 2 a key tool for us was provided by Lemmata 2.1 and 4.1 which allowed us to translate between "detecting a class by a primary operation" and "detecting the adjoint mapping in homology". It would be very interesting to see an odd primary version of any of these observations. We postpone further work on this to a future work.

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