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Geometric structure in the tempered dual of
the \( p \)-adic group \( \text{SL}(4) \)

Kuok Fai Chao and Roger Plymen

1 Introduction

In the representation theory of reductive \( p \)-adic groups, the issue of reducibility of induced representations is an issue of great intricacy. It is the contention of Aubert-Baum-Plymen, expressed as a conjecture [1, 2, 3, 4], that there exists a simple geometric structure underlying this intricate theory.

Let \( G \) be a reductive \( p \)-adic group. Let \( s \) be the point in the Bernstein spectrum of \( G \) which contains the cuspidal pair \( (M, \sigma) \). We will suppose that the irreducible cuspidal representation \( \sigma \) has unitary central character. Let \( \Psi^s(M) \) denote the set of unramified unitary characters of \( M \). Then \( \Psi^s(M) \) has the structure of a compact torus. Attached to the point \( s \) there is a compact torus \( E^s \):

\[
E^s := \{ \psi \otimes \sigma : \psi \in \Psi^s(M) \}
\]

Let \( W(M) \) denote the Weyl group of \( M \) and let \( W^s \) denote the isotropy subgroup \( \{ w \in W(M) : w \cdot s = s \} \). Let \( E^s//W^s \) denote the extended quotient, see §2. The extended quotient \( E^s//W^s \) is a compact Hausdorff space.

The reduced \( C^* \)-algebra of \( G \) is liminal, and its primitive ideal space is in canonical bijection with the tempered dual \( \text{Irr}^t(G) \) of \( G \). Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual, see [5, 3.1.1, 4.4.1, 18.3.2]. The set of tempered representations of \( G \) determined by \( s \) will be denoted \( \text{Irr}^t(G)^s \). We have

\[
\text{Irr}^t(G)^s \subset \text{Irr}^t(G)
\]

and, in the induced topology, \( \text{Irr}^t(G)^s \) is compact. The space \( \text{Irr}^t(G)^s \) is not necessarily Hausdorff.

In the context of the tempered dual, the ABP conjecture relates the two compact spaces \( E^s//W^s \) and \( \text{Irr}^t(G)^s \).
The evidence for the ABP conjecture begins to accumulate. The conjecture is true for $GL_n(F)$, see [1]; the proof uses the local Langlands correspondence for $GL(n)$. In [3], the authors prove that it is also true for the principal series of the exceptional group $G_2$. In [8], Jawdat and Plymen have proved the conjecture for the elliptic representations of the special linear group $SL_n(F)$. We must also mention the comprehensive article by Solleveld [13], in which the emphasis is placed on affine Hecke algebras.

Let $T$ denote a maximal torus in $SL_4(\mathbb{Q}_p)$. In this paper, we will prove part (3) of the ABP conjecture for $SL_4(\mathbb{Q}_p)$ when $s = [T, \sigma]_G$. The case $p = 2$ is especially interesting. In this case, there is a tetrahedron of reducibility in the tempered dual of $SL_4$ which does not occur when $p > 2$. The extended quotient performs a deconstruction: it creates the ordinary quotient and six unit intervals. The six intervals are then assembled into the six edges of a tetrahedron, and create a perfect model of reducibility.

By a cocharacter we shall mean a morphism $\mathbb{C}^\times \rightarrow T^\vee$ of algebraic groups, where $T^\vee$ is the dual torus in the Langlands dual $G^\vee$. The $q$-projection $\pi^{\sqrt{q}}$ is constructed from a finite set of cocharacters (depending on $s$), see [3, §1]. Let $inf.ch.$ denote the infinitesimal character.

**Theorem 1.1.** Let $G = SL_4(\mathbb{Q}_p)$. Let $s = [T, \sigma]_G$. There is a continuous bijection $\mu^s : E^s // W^s \rightarrow \text{Irr}^s(G)^s$ such that

$$\text{inf.ch.} \circ \mu^s = \pi^{s\sqrt{q}}$$  \hspace{1cm} (1)

This confirms, in a special case, part (3) of the conjecture in [2]. The cocharacters which enter the definition of the $q$-projection $\pi^{s\sqrt{q}}$ depend only on two-sided cells $c$.

There is an abundance of $L$-packets in the tempered dual of $SL_4$. There are, for example, $L$-packets in the tempered dual of $SL_4(\mathbb{Q}_2)$ which are parametrized by the 1-skeleton of a tetrahedron. The $L$-packets which occur in this article all conform to the $L$-packet conjecture in [4, §10].

**Acknowledgements.** We thank Paul Baum for helping us to resolve a crucial issue in §3, and Anne-Marie Aubert for her contributions to §4.

## 2 Geometric structure

We recall the definition of the extended quotient. Let $X$ be a Hausdorff topological space. Let $\Gamma$ be a finite group acting on $X$ as homeomorphisms. Let

$$\bar{X} = \{(x, \gamma) \in X \times \Gamma : \gamma x = x\}$$
with group action on $\tilde{X}$ given by
$$\alpha \cdot (x, \gamma) = (\alpha x, \alpha \gamma \alpha^{-1})$$
for $\alpha \in \Gamma$. Then the extended quotient is given by
$$X//\Gamma := \tilde{X}/\Gamma = \bigsqcup_{\gamma \in \Gamma} X^\gamma/\Gamma$$
with one $\gamma$ in each conjugacy class of $\Gamma$.

We fix the local field $F$ to be $\mathbb{Q}_p$. We have 5 conjugacy classes of Levi subgroups of $SL_4$, one for each partition of 4. Let $P = MU$ be a standard parabolic subgroup of $G = SL_4(F)$. Let $\tilde{M}$ be the corresponding Levi subgroup of $\tilde{G} = GL_4(F)$ so that $M = \tilde{M} \cap SL_4(F)$. We will use the framework, notation and results in [6]. Let $\sigma \in E_2(M)$ and $\pi_{\sigma} \in E_2(\tilde{M})$ with $\pi_{\sigma} \supset \sigma$.

Let $W(M)$ be the Weyl group of $M$. Let
$$L(\pi_{\sigma}) := \{ \eta \in \hat{F}^\times | \pi_{\sigma} \otimes \eta \simeq \pi_{\sigma} \text{ for some } w \in W \}$$
$$X(\pi_{\sigma}) := \{ \eta \in \hat{F}^\times | \pi_{\sigma} \otimes \eta \simeq \pi_{\sigma} \}$$

By [6, Theorem 2.4], the $R$-group of $\sigma$ is given by
$$R(\sigma) \cong L(\pi_{\sigma})/X(\pi_{\sigma}).$$

From now on, we will restrict ourselves to the case $M = T$ the standard maximal torus. For the Bernstein component $s = [T, \sigma]_G$, we let $\pi_{\sigma}|_T = \sigma$ where $\pi_{\sigma}$ is a unitary character of $M$. Then we write
$$\pi_{\sigma} = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$$
where $\pi_i$ is not an unramified twist of $\pi_j$ with $i \neq j$. In this section, we will discuss the extended quotient $E^s//W^s$ with respect to each Bernstein component $s = [T, \sigma]_G$ and prove the geometric conjecture for the principal series of $SL_4(F)$. Hence, we will construct the explicit bijection between $E^s//W^s$ and $\text{Irr}^s(G)$. From now on, we denote $(GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}) \cap SL_n$ by $n_1 + n_2 + \cdots + n_r$ where $\Sigma n_i = n$. For example, $1 + 1 + 1 + 1$ means $(GL_1 \times GL_1 \times GL_1 \times GL_1) \cap SL_4$. Recalling the definition of $E^s$, we see that $E^s$ may be identified with $T^4/T$, the maximal compact subgroup of the dual torus $T^\vee$ in the Langlands dual $G^\vee = PGL_4(\mathbb{C})$.

2.1 Case 1: $\pi_{\sigma} = \pi \otimes \pi \otimes \pi \otimes \pi$

The group $W^s$ is the symmetric group $S_4$. This group has five conjugacy classes, one for each cycle type. The following is the structure of each component in the extended quotient.
\[ \gamma = (abcd) = 1, \quad E^\gamma / Z(\gamma) = E^g / W^s \]

\[ \gamma = (acbd), \quad E^\gamma = \{(a, b, b, d) : a, c, d \in \mathbb{T}\} / \mathbb{T} \cong \{(z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2 \]

\[ E^\gamma / Z(\gamma) \cong \mathbb{T}^2 \]

\[ \gamma = (bcad), \quad E^\gamma = \{(a, a, a, b) : a, b \in \mathbb{T}\} / \mathbb{T} \cong \{(z, z, 1) : z \in \mathbb{T}\} \cong \mathbb{T} \]

\[ E^\gamma / Z(\gamma) \cong \mathbb{T} \]

\[ \gamma = (bade), \quad E^\gamma = \{(a, a, b, b), (a, -a, b, -b) : a, b \in \mathbb{T}\} / \mathbb{T} \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in \mathbb{T}\} \cong \mathbb{T} \sqcup \mathbb{T} \]

\[ E^\gamma / Z(\gamma) \cong \mathbb{T} \sqcup \mathbb{T} \]

\[ \gamma = (bcda), \quad E^\gamma = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i), (1, -i, -1, i)\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \]

\[ E^\gamma / Z(\gamma) \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \]

Hence, we have

\[ E^g / W^s = E^g / W^g \sqcup \mathbb{T}^2 \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \quad (3) \]

Now we identify each element in the compact subspace \( \text{Irr}^4(G)^s \). The induced representations

\[ \{\text{Ind}^4_{(2+1+1)}(z_1^\text{val} \cdot \text{St}_2 \otimes z_2^\text{val} \otimes 1) : z_1, z_2 \in \mathbb{T}\} \]

are irreducible, tempered, their infinitesimal characters lie in \( \mathfrak{s} \), and they are parametrized by \( \mathbb{T}^2 \).

The induced representations

\[ \{\text{Ind}^4_{(3+1)}(z \cdot \text{St}_3 \otimes 1) : z \in \mathbb{T}\} \]

are irreducible, tempered, have central characters in \( \mathfrak{s} \) and are parametrized by \( \mathbb{T} \).
The induced representations
\[ \{ \text{Ind}_{2+2}^4(z \cdot \text{St}_2 \otimes \text{St}_2) : z \in \mathbb{T} \} \]
have central characters in \( \mathfrak{s} \), and are parameterized by \( \mathbb{T} \). They are irreducible except when \( z = -1 \). The \( R \)-group is as follows:
\[ R((-1)^{\text{val}} \cdot \text{St}_2 \otimes \text{St}_2) = < (-1)^{\text{val}} >. \]

There are two irreducible components, denoted by \( \rho^+ \) and \( \rho^- \). We will locate \( \rho^- \) in the second copy of \( \mathbb{T} \) and identify \( \rho^+ \) by \( \text{pt}_1 \).

The Steinberg representation \( \text{St}(\text{SL}_4) \) has central character in \( \mathfrak{s} \). We identify this representation by \( \text{pt}_2 \).

The unramified unitary principal series of \( \text{SL}_4 \) contains points of reducibility. In fact, there is a circle of reducibility, as we now proceed to explain. Let \( t = (z, -z, 1, -1) \) except \( z = i \) and let \( \chi_t \) be the corresponding unramified unitary character. Then the representation \( \chi_t \otimes \pi \) is given by
\[ z^{\text{val}} \pi \otimes (-z)^{\text{val}} \pi \otimes \pi \otimes (-1)^{\text{val}} \pi. \]

Then \( (-1)^{\text{val}} \pi \) is an element in \( \tilde{L}(\chi_t \otimes \pi) \) and \( X(\chi_t \otimes \pi) = 1 \). Hence
\[ R(\chi_t \otimes \pi) \cong \mathbb{Z}/2\mathbb{Z} \]
and the induced representation
\[ \lambda(t) := \text{Ind}_T^G(\chi_t \otimes \pi) \]
is reducible and admits two irreducible subrepresentations:
\[ \lambda(t) = \lambda(t)^+ \oplus \lambda(t)^-. \]

We assign \( \lambda(t)^+ \) to \( [z, -z, 1, -1] \in E^s/W^s \) and \( \lambda(t)^- \) to \( z \in \mathbb{T} \).

Now, we turn to the point \( t = (i, -i, 1 - 1) \). Then
\[ \tilde{L}(\chi_t \otimes \pi) = < i^{\text{val}} > \]
and \( X(\pi) = 1 \). Hence, we know \( R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z} \). The induced representation \( \tau = \text{Ind}_T^G(\chi_t \otimes \sigma) \) is reducible with 4 irreducible constituents \( \tau_1, \tau_2, \tau_3, \tau_4 \). We locate \( \tau_1 \) to \( [i, -i, 1, -1] \in E^s/W^s \) and \( \tau_2 \) to the point \( i \) in the third copy of \( \mathbb{T} \) and identify \( \tau_3 \) and \( \tau_4 \) by \( \text{pt}_3 \) and \( \text{pt}_4 \) respectively.

For \( t = (z_1, z_2, z_3, 1) \in E^s/W^s \) except \( t = (z, -z, 1, -1) \), the induced representation \( \text{Ind}_M^G(\chi_t \otimes \sigma) \) is irreducible.
We build a map \( \mu : E^{s//W^{s}} \rightarrow \text{Irr}^1(G)^s \)

and here are the details:

<table>
<thead>
<tr>
<th>Point in ( E^{s//W^{s}} )</th>
<th>Irreducible representation</th>
<th>Cocharacter ( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (z_1, z_2) \in T^2 )</td>
<td>( \text{Ind}_{\Gamma}^{G}(z_1 \cdot \text{St}_2 \otimes z_2^{\text{val}} \otimes \pi) )</td>
<td>( (t, t^{-1}, 1, 1) )</td>
</tr>
<tr>
<td>( z \in T )</td>
<td>( \text{Ind}_{\Gamma}^{G}(z \cdot \text{St}_3 \otimes \pi) )</td>
<td>( (t^2, 1, t^{-2}, 1) )</td>
</tr>
<tr>
<td>( z \in T )</td>
<td>( \text{Ind}_{\Gamma}^{G}(z \cdot \text{St}_2 \otimes \text{St}_2) )</td>
<td>( (t, t^{-1}, t^{-1}) )</td>
</tr>
<tr>
<td>( \rho^+ )</td>
<td>( \lambda(t)^+ )</td>
<td>1</td>
</tr>
<tr>
<td>( \rho^+ )</td>
<td>( \rho )</td>
<td>( (t, t^{-1}, t^{-1}) )</td>
</tr>
<tr>
<td>( \rho^+ )</td>
<td>( \rho )</td>
<td>( (t^2, t^{-1}, t^{-3}) )</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( \tau_4 )</td>
<td>1</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( \tau_4 )</td>
<td>1</td>
</tr>
<tr>
<td>( \tau_3 )</td>
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<td>1</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( \tau_4 )</td>
<td>1</td>
</tr>
</tbody>
</table>

It is clear that Eqn.(1) is satisfied. We note that the compact space \( \text{Irr}^1(G)^s \) is non-Hausdorff. One connected component contains a double-point, and another connected component contains a double-circle (and a quadruple point), see [9].

Hence, we have

**Lemma 2.1.** Part (3) of the geometric conjecture is true for

\[ s = [T, \pi \otimes \pi \otimes \pi \otimes \pi]_G. \]

### 2.2 Case 2: \( \pi_{\sigma} = \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2 \)

For this case, the isotropy group is the symmetric group \( \mathfrak{S}_3 \). There are three conjugacy classes. They are \( \{\gamma_1\}, \{\gamma_2, \gamma_3, \gamma_6\}, \{\gamma_4, \gamma_5\} \). Now we choose \( \gamma_1, \gamma_2 \) and \( \gamma_4 \) as the representatives for their own conjugacy classes. Now we analysis case by case.

- \( \gamma = (abcd) = 1, E^\gamma / Z(\gamma) = E^s / W^s \)
- \( \gamma = (bacd), E^\gamma = \{(a, a, c, d) : a, c, d \in T\} / T \cong \{(z_1, z_1, z_2) : z_1, z_2 \in T\} \cong T^2 \)
- \( \gamma = (bcad), E^\gamma = \{(a, a, c) : a, c \in T\} / T \cong \{(z, z, 1) : z \in T\} \cong T \)

Hence, we have

\[ E^s / W^s = E^s / W^s \sqcup T^2 \sqcup T \]  (4)
The representations

\[ \{ \text{Ind}^4_{(2+1+1)}(z \cdot St_2(\pi_1) \otimes z_2^{\text{valdet}} \pi_1 \otimes \pi_2) : z_1, z_2 \in T \} \]

are irreducible, tempered, and have infinitesimal character in \( s \). These representations are parametrized by \( T^2 \) and we set the cocharacter by

\[ h_c(t) = (t, t^{-1}, 1, 1). \]

The representations

\[ \{ \text{Ind}^4_{(3+1)}(z \cdot St_3(\pi_1) \otimes \pi_2) : z \in T \} \]

are irreducible, tempered and have infinitesimal character in \( s \). These representations are parametrized by \( T \). For this component, we set the cocharacter:

\[ h_c(t) = (t^2, 1, t^{-2}, 1). \]

Now we consider the point \( t = (z_1, z_2, z_3, 1) \in E^s/W^s \) and \( \chi_t \) is the unramified unitary character determined by \( t \). The induced representation \( \text{Ind}^G_T(\chi_t \otimes \sigma) \) is irreducible. Each unramified unitary character determines a tempered representation of \( G \) with respect to \( s \).

We turn to build up the map \( \mu \) satisfying the geometric conjecture, i.e.

\[ \mu : E^s/W^s \sqcup T^2 \sqcup T \rightarrow \text{Irr}^4(G)^s \]

The details of this map are as follows:

<table>
<thead>
<tr>
<th>Point in ( E^s/W^s )</th>
<th>Irreducible representation</th>
<th>Cocharacter ( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((z_1, z_2) \in T^2)</td>
<td>( \text{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2^{\text{valdet}} \pi_1 \otimes \pi_2) )</td>
<td>((t, t^{-1}, 1, 1))</td>
</tr>
<tr>
<td>(z \in T)</td>
<td>( \text{Ind}^4_{(3+1)}(z \cdot St_3(\pi_1) \otimes \pi_2))</td>
<td>((t^2, 1, t^{-2}, 1))</td>
</tr>
<tr>
<td>(t \in E^s/W^s)</td>
<td>( \text{Ind}^G_T(\chi_t \otimes \sigma) )</td>
<td>1</td>
</tr>
</tbody>
</table>

It is clear that Eqn.(1) holds. Hence, we have

**Lemma 2.2.** Part (3) of the geometric conjecture is true for

\[ s = [M, \pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2]_G. \]
2.3 Case 3: $\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$

There are two distinct cases to be considered. In case 3.1, the corresponding $\pi_\sigma$ is $\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$. The R-group $R(\sigma)$ is trivial and the isotropy subgroup $W^\s$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now we investigate the extended quotient $E^\s/W^\s$ respect to $\s$. Since $W^\s$ is abelian in this case, then we know that each element contribute one conjugacy class and the centralizer $Z(\gamma_i)$ of $\gamma_i$ is $W$ itself. Then, we have

$$E^\s/W^\s = E^{\gamma_1}/W^\s \sqcup E^{\gamma_2}/W^\s \sqcup E^{\gamma_3}/W^\s \sqcup E^{\gamma_4}/W^\s$$

We will analysis case by case. Now we compute each component in extended quotient $E^\s/W^\s$.

- $\gamma = (abcd) = 1$, $E^\gamma/Z(\gamma) = E^\s/W^\s$.
- $\gamma = (bacd)$, $E^\gamma = \{(a, a, c, d) : a, c, d \in T\}/T \cong \{(1, 1, z_1, z_2) : z_1, z_2 \in T\} \cong T^2$. Then we have
  $$E^\gamma/Z(\gamma) \cong T^2.$$
- $\gamma = (abdc)$, $E^\gamma = \{(a, b, c, c) : a, b, c \in T\}/T \cong \{(z_1, z_2, 1, 1) : z_1, z_2 \in T\} \cong T^2$.
  $$E^\gamma/Z(\gamma) \cong T^2.$$
- $\gamma = (badc)$, $E^\gamma = \{(a, a, c, c) : a, c \in T\} = \{1, -1, z, -z) : z \in T\} \cong T \sqcup T$.
  $$E^\gamma/Z(\gamma) \cong T \sqcup T.$$

This leads to the equation

$$E^\s/W^\s = E^\s/W^\s \sqcup T^2 \sqcup T^2 \sqcup T \sqcup T$$

and we now construct the bijection $\mu$ between $E^\s/W^\s$ and the set $\text{Irr}^1(G)^\s$.

The representations

$$\{\text{Ind}^4_4(2_{2+1+1}) (z_1 \cdot St_2(\pi_1) \otimes z_2^{val_{det}} \pi_2 \otimes \pi_2) : z_1, z_2 \in T\}$$

are tempered, irreducible and have inf.character in $\s$. Parameter space is $T^2$.

The representations

$$\{\text{Ind}^4_4(1_{1+1+2}) (\pi_1 \otimes z_1^{val_{det}} \pi_1 \otimes z_2 \cdot St_2(\pi_2)) : z_1, z_2 \in T\}$$
are irreducible, tempered, have inf.ch. in \( s \), parameter space \( \mathbb{T}^2 \).

The representations
\[
\text{Ind}^4_{(2+2)}(z \cdot St_2(\pi_1) \otimes St_2(\pi_2))
\]
are irreducible, tempered, have inf.ch. in \( s \), parameter space \( \mathbb{T} \).

Let \( t = (1, -1, z, -z) \) and \( \chi_t \) be the corresponding character. We have \( \chi_t \otimes \pi_\sigma = \pi_1 \otimes (\chi_1 \otimes \chi_2) \otimes z^{\text{val}} \) and \( X(\chi_t \otimes \pi_\sigma) = 1 \). Thus, \( R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z} \). It leads the induced representation \( \text{Ind}^G_M(\chi_t \otimes \sigma) \) is reducible and there are two irreducible subrepresentations \( \pi(t)^+ \) and \( \pi(t)^- \) of \( G \), i.e.
\[
\text{Ind}^G_M(\chi_t \otimes \sigma) = \pi(t)^+ \oplus \pi(t)^-
\]
Hence, \( \pi(t)^+ \) and \( \pi(t)^- \) are tempered. They will contribute the elements in \( \text{Irr}^t(G)^s \). Such induced representations will be identified by \( \mathbb{T} \) and we set the character
\[
h_c(t) = 1.
\]
Indeed, each unitary unramified twist of \( \sigma \) except the type \( t = (1, -1, z, -z) \) will be an irreducible tempered representation. In other word, every point \( t \) in \( E^s/W^s \), which does not belong to \( (1, -1, z, -z) \), will generate an unitary unramified character of \( \mathbb{T} \), i.e. \( \chi_t = (\chi_1, \chi_2, \chi_3, \chi_4) \) and the representation \( \text{Ind}^G_M(\chi_t \otimes \sigma) \) induced by \( \chi_t \otimes \sigma \) is irreducible. Thus such induced representation contributes an element in \( \text{Irr}^t(G)^s \). The detail of this map is as follow:

<table>
<thead>
<tr>
<th>Point in ( E^s/W^s )</th>
<th>Irreducible representation</th>
<th>Cocharacter ( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((z_1, z_2) \in \mathbb{T}^2)</td>
<td>( \text{Ind}^4_{(2+1+1)}(z_1 \cdot St_2(\pi_1) \otimes z_2 \cdot \pi_2 \otimes \pi_2) )</td>
<td>((t, t^{-1}, 1, 1))</td>
</tr>
<tr>
<td>((z_1, z_2) \in \mathbb{T}^2)</td>
<td>( \text{Ind}^4_{(1+1+2)}(\pi_1 \otimes z_1 \cdot \pi_1 \otimes z_2 \cdot St_2(\pi_2)) )</td>
<td>((1, 1, t, t^{-1}))</td>
</tr>
<tr>
<td>( z \in \mathbb{T} )</td>
<td>( \text{Ind}^4_{(2+2)}(z \cdot St_2(\pi_1) \otimes St_2(\pi_2)) )</td>
<td>((t, t^{-1}, t, t^{-1}))</td>
</tr>
<tr>
<td>( z \in \mathbb{T} )</td>
<td>( \pi(t)^+ )</td>
<td>1</td>
</tr>
<tr>
<td>( t \in E^s/W^s )</td>
<td>( \text{Ind}^G_T(\chi_t \otimes \sigma) )</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, Eqn.(1) is satisfied. We conclude

**Lemma 2.3.** Part (3) of the geometric conjecture is true for
\[
s = [T, \pi_1 \otimes \pi_2 \otimes \pi_2]_G.
\]

The next we will discuss the case 3.2. In this case, the representation \( \pi_\sigma \) is of the form \( \pi \otimes \pi \otimes \eta \pi \otimes \eta \). The \( R \)-group is \( \mathbb{Z}/2\mathbb{Z} \) and the isotropy group is \( \mathbb{Z}/\mathbb{Z} \) generated by \( < \gamma_2, \gamma_3, \gamma_4 > \) where \( \gamma_2, \gamma_3, \gamma_4 \) are explicitly given by the table below:
There are five conjugacy classes:

\{\gamma_1\}, \{\gamma_2, \gamma_3\}, \{\gamma_4, \gamma_8\}, \{\gamma_5\}, \{\gamma_6, \gamma_7\}

From above, there are five conjugacy classes. The extended quotient \( E_s // W_s \) is given by

\[ E_s // W_s = \bigsqcup_{\gamma} E_{\gamma} / Z(\gamma) \]

and we choose \( \gamma_1, \gamma_2, \gamma_4, \gamma_5 \) and \( \gamma_6 \) to be the representatives in their own conjugacy classes and their centralizers are as follows:

\[
Z(\gamma_1) = W_s, \quad Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_3\}, \quad Z(\gamma_4) = \{\gamma_1, \gamma_4, \gamma_5, \gamma_8\} \\
Z(\gamma_5) = W_s, \quad Z(\gamma_6) = \{\gamma_1, \gamma_6, \gamma_7\}
\]

Now we analyze case by case.

- \( \gamma = \gamma_1 \), \( E_{\gamma} / W_s = E / W_s \).
- \( \gamma = \gamma_2 \), \( E_{\gamma} = \{(a, a, c, d) : a, c, d \in T\} / T \cong \{(z_1, z_2, 1) : z_1, z_2 \in T\} \cong T^2 \).
- \( \gamma = \gamma_4 \), \( E_{\gamma} = \{(a, b, a, b), (a, -a, -b) : a, b \in T\} / T \cong \{(1, z, 1, -z) : z \in T\} \cong T \sqcup T \).
- \( \gamma = \gamma_5 \), \( E_{\gamma} = \{(a, a, b, b), (a, -a, b, -b) : a, b \in T\} / T \cong \{(1, 1, z, z) : z \in T\} \cong T \sqcup T \).
- \( \gamma = \gamma_6 \), \( E_{\gamma} = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i) : z_1, z_2 \in T\} \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \).

This leads to the equation

\[ E_s // W_s = E_s / W_s \sqcup T^2 \sqcup T \sqcup T \sqcup T \sqcup T \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \]  

(8)

The representations

\[ \{ \text{Ind}_{(2+1+1)}^1 (z_1 \cdot St_2(\pi) \otimes z_2^\pi \eta \pi \otimes \eta \pi) : z_1, z_2 \in T\} \]
are irreducible, tempered, have inf.ch. in $s$, parameter space $\mathbb{T}^2$.

The representations

$$\text{Ind}^4_{(2+2)}(z \cdot St_2(\pi) \otimes St_2(\eta \pi))$$

are irreducible, tempered, have inf.ch. in $s$, parameter space $\mathbb{T}$.

In the following, we consider the point $t = (z, 1, -z, -1)$. Thus, $\chi_t \otimes \pi_\sigma = z^{\text{val} \pi} \otimes \pi \otimes (-1)^{\text{val} \eta \pi} \otimes (-1)^{\text{val} \eta \pi}$. We have $\hat{L}(\chi_t \otimes \pi_\sigma) = < -(-1)^{\text{val} \eta} >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\rho(z)$ is reducible and there are two irreducible constituents $\rho(z)^+$, $\rho(z)^-$, i.e.

$$\rho(z) = \text{Ind}^G_{T}(z^{\text{val} \text{det} \pi} \otimes \pi \otimes (-z)^{\text{val} \eta \pi} \otimes (-1)^{\text{val} \eta \pi} = \rho(z)^+ \oplus \rho(z)^-$$

We identify these representations by $T$.

Then we consider the point $t = (z, 1, z, 1)$. We get $\chi_t \otimes \pi_\sigma = z^{\text{val} \pi} \otimes \pi \otimes z^{\text{val} \eta \pi} \otimes \eta \pi$. We have $\hat{L}(\chi_t \otimes \pi_\sigma) = < \eta >$ and $X(\pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\varrho(z)$ is reducible. There are two irreducible constituents $\varrho(z)^+$, $\varrho(z)^-$, i.e.

$$\varrho(z) = \text{Ind}^G_{T}(z^{\text{val} \pi} \otimes \pi \otimes z^{\text{val} \eta \pi} \otimes \eta \pi = \varrho(z)^+ \oplus \varrho(z)^-$$

We identify these representations by $T$.

Finally, we consider the point $t = (1, -1, z, -z)$ except $z = 1, i$. Thus, $\chi_t \otimes \pi_\sigma = \pi \otimes (-1)^{\text{val} \pi} \otimes (z)^{\text{val} \eta \pi} \otimes (-z)^{\text{val} \eta \pi}$. We have $\hat{L}(\chi_t \otimes \pi_\sigma) = < (-1)^{\text{val} \eta} >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Thus $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z}$. It leads the induced representation $\vartheta(z)$ is reducible. There are two irreducible constituents $\vartheta(z)^+$, $\vartheta(z)^-$, i.e.

$$\vartheta(z) = \text{Ind}^G_{M}(\pi \otimes (-1)^{\text{val} \pi} \otimes (z)^{\text{val} \eta \pi} \otimes (-z)^{\text{val} \eta \pi} = \vartheta(z)^+ \oplus \vartheta(z)^-$$

We identify these representations by $T$.

Now we still consider the point $t = (1, -1, z, -z)$ and fix $z = 1$. We have $t = (1, -1, 1, -1)$. Then $\hat{L}(\chi_t \otimes \pi_\sigma) = < (-1)^{\text{val} \text{det} \eta} >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then the induced representation $\xi = \text{Ind}^G_{M}(\chi_t \otimes \pi_\sigma)$ is reducible and there are 4 irreducible constituents $\xi_1$, $\xi_2$, $\xi_3$ and $\xi_4$. We locate $\xi_1$ to $E^5/W^5$ and $\xi_2$ to component $c_5$ and identify $\xi_3$ and $\xi_4$ by $p_t \xi_1$ and $p_t \xi_2$ respectively.

In the next, we fix $z = i$. We have $t = (1, -1, i, -i)$. Then $\hat{L}(\chi_t \otimes \pi_\sigma) = < i^{\text{val} \text{det} \eta} >$ and $X(\chi_t \otimes \pi_\sigma) = 1$. Hence, we know $R(\chi_t \otimes \sigma) = \mathbb{Z}/4\mathbb{Z}$. The order of $R(\chi_t \otimes \sigma)$ is 4. Then the induced representation $\tau = \text{Ind}^G_{M}(\chi_t \otimes \sigma)$ is reducible and there are 4 irreducible constituents $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$. We locate...
\(\tau_1\) to \(E^s/W^s\) and \(\tau_2\) to component \(c_5\) and identify \(\tau_3\) and \(\tau_4\) by \(pt_3\) and \(pt_4\) respectively.

<table>
<thead>
<tr>
<th>Point in (E^s/W^s)</th>
<th>Irreducible representation</th>
<th>Cocharacter (h(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((z_1, z_2) \in \mathbb{T}^2)</td>
<td>(\text{Ind}_{4}^{(4)}(z_1 \cdot \text{St}_2(\pi) \otimes z_2 \text{val det} \eta \pi \otimes \eta \pi))</td>
<td>((t, t^{-1}, 1, 1))</td>
</tr>
<tr>
<td>(z \in \mathbb{T})</td>
<td>(\rho(z)^+)</td>
<td>((t, t^{-1}, t, t^{-1}))</td>
</tr>
<tr>
<td>(z \in \mathbb{T})</td>
<td>(\theta(t)^+)</td>
<td>1</td>
</tr>
<tr>
<td>(z \in \mathbb{T})</td>
<td>(\theta(z)^+)</td>
<td>1</td>
</tr>
<tr>
<td>(t \in E^s/W^s)</td>
<td>(\text{Ind}_{4}^{G}(\chi_t \otimes \sigma))</td>
<td>1</td>
</tr>
<tr>
<td>(pt_1)</td>
<td>(\xi_3)</td>
<td>1</td>
</tr>
<tr>
<td>(pt_2)</td>
<td>(\xi_4)</td>
<td>1</td>
</tr>
<tr>
<td>(pt_3)</td>
<td>(\tau_3)</td>
<td>1</td>
</tr>
<tr>
<td>(pt_4)</td>
<td>(\tau_4)</td>
<td>1</td>
</tr>
</tbody>
</table>

Eqn. (1) is satisfied. Hence, we have

**Lemma 2.4.** Part (3) of the geometric conjecture is true for

\[s = [T, \pi \otimes \pi \otimes \eta \pi \otimes \eta \pi]_G.\]

### 2.4 Case 4: \(\pi_\sigma = \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3\)

In this section, we will consider the case \(\pi_\sigma \cong \pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3\). Hence, we know the isotropy subgroup \(W^s = \mathbb{Z}/2\mathbb{Z}\). Analysis case by case:

- \(\gamma = (abcd) = 1\), \(E^\gamma/Z(\gamma) = E^s/W^s\).
- \(\gamma = (bacd)\), \(E^\gamma = \{(a, a, c, d) : a, c, d \in \mathbb{T}\}/\mathbb{T} \cong \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \cong \mathbb{T}^2\).

\[E^\gamma/Z(\gamma) \cong \mathbb{T}^2\]

Then, we have

\[E^s/W^s = E^s/W^s \sqcup \mathbb{T}^2\]

(9)

The representations

\[\{(\text{Ind}_{4}^{(4)}(z_1 \cdot \text{St}_2(\pi_1) \otimes z_2 \text{val det} \pi_2 \otimes \pi_3) : z_1, z_2 \in \mathbb{T}\}\]

are irreducible, tempered, have \(inf.ch.\) in \(s\), parameter space \(\mathbb{T}^2\).

We construct the bijection:

\[\mu : E^s/W^s \sqcup \mathbb{T}^2 \rightarrow \text{Irr}^4(G)^s\]
in accordance with the following table:

<table>
<thead>
<tr>
<th>Point in $E^s/W^s$</th>
<th>Irreducible representation</th>
<th>Cocharacter $h(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(z_1, z_2) \in T^2$</td>
<td>$\text{Ind}<em>{G}^{T</em>{3+1+1}}(z_1 \cdot \text{St}_2(\pi_1) \otimes z_2 \cdot \pi_2 \otimes \pi_3)$</td>
<td>$(t, t^{-1}, 1, 1)$</td>
</tr>
<tr>
<td>$t \in E^s/W^s$</td>
<td>$\text{Ind}_T(\chi_t \otimes \sigma)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Once again, Eqn.(1) is satisfied. Hence, we have

**Lemma 2.5.** Part (3) of the geometric conjecture is true for $s = [T, \pi_1 \otimes \pi_2 \otimes \pi_3]_G$.

### 2.5 Case 5: $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

In this section, we will discuss the case when $\pi_\sigma \cong \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$. In fact, from the table of $R$-group above, we have known that there are four types in this case. First of all, we focus on the case 5.1. Indeed, $\pi_\sigma$ is given by the form $\pi \otimes \eta \pi \otimes \eta^2 \pi \otimes \eta^3 \pi$ where $\eta$ is ramified. We have proved that the $R$-group with respect to this is the cyclic group $\mathbb{Z}/4\mathbb{Z}$. Furthermore, the isotropy group $W^s$ is given by $\mathbb{Z}/4\mathbb{Z}$. In the following, we figure out the extended quotient with respect to $W^s$. The cyclic group is abelian, each element comprises a single conjugacy classes and the centralizer of each element is the cyclic group itself. Then we can immediately get the extended quotient

$$E^s/W^s = E^{\gamma_1}/W^s \sqcup E^{\gamma_2}/W^s \sqcup E^{\gamma_3}/W^s \sqcup E^{\gamma_4}/W^s$$

We analyze case by case.

- $\gamma = (abcd) = 1$, $E^\gamma/W^s = E^s/W^s$.
- $\gamma = (bcda)$, $E^\gamma = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i), (1, i, -1, -i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$. Hence, we have
  $$E^\gamma/W^\gamma \cong pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4$$
- $\gamma = (cdab)$, $E^\gamma = \{(a, b, -a, b), (a, b, a, b) : a, b \in T\}/T \cong \{(1, z, -1, -z), (1, z, 1, z) : z \in T\} \cong T \sqcup T$. Then
  $$E^\gamma/W^\gamma \cong T \sqcup T$$
- $\gamma = (dabc)$, $E^\gamma = \{(1, 1, 1, 1), (1, -1, 1, -1), (1, i, -1, -i), (1, i, -1, -i)\} \cong pt \sqcup pt \sqcup pt \sqcup pt$. Then
  $$E^\gamma/W^\gamma \cong pt_5 \sqcup pt_6 \sqcup pt_7 \sqcup pt_8$$
Then we have the decomposition

\[
E^s // W^s = E^s / W^s \sqcup \mathbb{T} \sqcup \mathbb{T} \sqcup pt_1 \sqcup pt_2 \sqcup pt_3 \sqcup pt_4 \sqcup pt_5 \sqcup pt_6 \sqcup pt_7 \sqcup pt_8
\]

(10)

It coincides the result in [8]. By [8, Theorem 5.3], we have

**Lemma 2.6.** Part (3) of the conjecture is true for \( s = [M, \pi \otimes \eta \pi \otimes \eta^2 \pi \otimes \eta^3 \pi] \).

Then, we will concentrate on the case 5.2. Recall that the representation \( \pi_\sigma \) is given by \( \pi \otimes \chi \pi \otimes \eta \pi \otimes \chi \eta \pi \) where \( \chi \) and \( \eta \) are ramified quadratic characters. In fact, we consider the field \( F = \mathbb{Q}_p \), \( p \neq 3 \). There are two ramified quadratic characters, the Legendre symbol \( \left( \frac{a}{p} \right) \) and its twist with \( (-1)^{\text{val}} \). For convenience, we denote the Legendre symbol by \( \lambda \) and its twist by \( (-1)^{\text{val}} \lambda \). Since \( \eta \) and \( \chi \) have to be distinct, we set

\[
\eta = \lambda, \ \chi = (-1)^{\text{val}} \lambda.
\]

Indeed, this case would belong to

\[
\pi \otimes (-1)^{\text{val}} \lambda \pi \otimes \lambda \pi \otimes (-1)^{\text{val}} \pi
\]

It means it should be in case 3.2. Hence, this case does not exist when \( F = \mathbb{Q}_3 \).

Then we turn to the case 5.3. We know that \( \pi_\sigma = \pi_1 \otimes \eta \pi_1 \otimes \pi_2 \otimes \eta \pi_2 \). We have \( W^s = \mathbb{Z}/2\mathbb{Z} \). Analysis case by case:

- \( \gamma = (abcd) = 1 \), \( E^\gamma / W^s = E / W^s \).
- \( \gamma = ((badc), E^\gamma = \{(a, a, c, c), (a, -a, c, -c) : a, c \in T\} / T \cong \{(1, 1, z, z), (1, -1, z, -z) : z \in T\} \cong \mathbb{T} \sqcup \mathbb{T} \).

Hence, we have the decomposition

\[
E^s // W^s = E^s / W^s \sqcup \mathbb{T} \sqcup \mathbb{T}
\]

(11)

From now, we try to exhaust the tempered dual with respect to \( s \). First of all, we consider \( t \) in the form of \( (1, 1, z, z) \) and \( (1, -1, z, -z) \). For the point \( t = (1, 1, z, z) \), \( \chi_t \) corresponds to the character \( \chi_t = (1, 1, z^{\text{val}}, z^{\text{val}}) \). After twisting ,we have the representation \( \chi_t \otimes \sigma \). Now we compute the \( R \)-group for this representation. This implies we can consider \( \chi_t \otimes \pi_\sigma \cong \pi_1 \otimes \eta \pi_1 \otimes z^{\text{val}} \pi_2 \otimes z^{\text{val}} \eta \pi_2 \). By computation, we know that the trivial character and \( \eta \) are also contained in \( \tilde{L}(\chi_t \otimes \pi_\sigma) \). Then we have \( R(\chi_t \otimes \sigma) = < 1, \eta > \)
and $X(\chi_t \otimes \pi_\sigma) = 1$. We note that the character $\eta$ is ramified and of order 2. This leads us to $R(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$. This implies the representation $\lambda(t)$ induced by $\chi_t \otimes \sigma$ is reducible and can be decomposed as two parts:

$$\lambda(t) = \lambda^+ \oplus \lambda^-$$

Indeed, $\lambda^+$ and $\lambda^-$ are tempered.

Similarly, we consider the point in the form of $t = (1, -1, z, -z)$ and $\chi_t$ corresponds to the character $\chi_t = (1, (-1)^{v\text{al}}, z^{v\text{al}}, (-z)^{v\text{al}})$. After twisting, we have the representation $\chi_t \otimes \sigma$. Then we can consider $\chi_t \otimes \pi_\sigma \cong \pi \otimes (-1)^{v\text{al}}\eta \pi_1 \otimes z^{v\text{al}}\pi_2 \otimes (-z)^{v\text{al}}\eta \pi_2$. We obtain $R(\chi_t \otimes \sigma) = \langle (-1)^{v\text{al}}\eta \rangle = \mathbb{Z}/2\mathbb{Z}$. This implies that the representation $\rho(t)$ induced by $\chi_t \otimes \sigma$ has two irreducible constituents:

$$\rho(t) = \rho^+ \oplus \rho^-$$

where $\rho^+$ and $\rho^-$ are tempered representations of $G$.

The definition of the map

$$\mu : E^s/W^s \sqcup T \sqcup T \rightarrow \text{Irr}^4(G)^s$$

is as follows:

- $T \mapsto \lambda^+ \mapsto \text{Ind}^G_M(\pi_1 \otimes \eta \pi_1 \otimes z^{v\text{al}}\pi_2 \otimes z^{v\text{al}}\eta \pi_2)$
- $T \mapsto \rho^+ \mapsto \text{Ind}^G_M(\pi \otimes (-1)^{v\text{al}}\eta \pi_1 \otimes z^{v\text{al}}\pi_2 \otimes (-z)^{v\text{al}}\det \eta \pi_2)$
- $t \mapsto \text{Ind}^G_M(\chi_t \otimes \sigma)$

It is easy to verify that Eqn.(1) holds. Hence, we have

**Lemma 2.7.** Part (3) of the conjecture is true for $s = [T, \pi \otimes \eta \pi \otimes \eta^2 \pi \otimes \eta^3 \pi]$.

Case 5.4 is next. The isotropy group $W^s$ is trivial. Indeed each induced representation by unitary twist with $\sigma$ is irreducible. This implies every induced representation is irreducible. From the side of extended quotient $E^s/W^s$, we know that $E^s/W^s = E^s/W^s$ because $W^s = 1$. The bijection $\mu$ between $E^s/W^s$ and $\text{Irr}^4(G)^s$

$$\mu : E^s/W^s \rightarrow \text{Irr}^4(G)^s$$

is given by

$$t \in E^s/W^s \mapsto \text{Ind}(\chi_t \otimes \sigma)$$

**Lemma 2.8.** Part 3 of the conjecture is true for $s = [T, \sigma]$ where $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$

**Theorem 2.9.** Part (3) of the geometric conjecture is true for $s = [T, \sigma]$.

**Proof.** Combine 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 above.
3 A tetrahedron of reducibility

We exhibit a tetrahedron of reducibility in the tempered dual of SL$_4(\mathbb{Q}_2)$ which does not occur in the tempered dual of SL$_4(\mathbb{Q}_p)$ when $p > 2$. This confirms a special case of the recent conjecture in [1, 2, 3], and also has independent interest. Let $F$ denote the $p$-adic field $\mathbb{Q}_p$, and let $U_F^n := 1 + p^n \mathbb{Z}_p$, $n \geq 1$ denote the standard congruence unit groups. Let $U_F = \sigma_F^\times$. For $p = 2$ define homomorphisms $\eta, \chi : U/U^3 \rightarrow \mathbb{Z}/2\mathbb{Z}$ as in [11, p. 18]:

- $\eta(x) = 0, x \equiv 1 \mod 4$
- $\eta(x) = 1, x \equiv -1 \mod 4$
- $\chi(x) = 0, x \equiv \pm 1 \mod 8$
- $\chi(x) = 1, x \equiv \pm 5 \mod 8$

The map $\eta$ defines an isomorphism of $U/U^2$ onto $\mathbb{Z}/2\mathbb{Z}$ and the map $\chi$ defines an isomorphism of $U^2/U^3$ onto $\mathbb{Z}/2\mathbb{Z}$. The level of a character $\psi$ of $F^\times$ is the least integer $n \geq 0$ such that $\psi$ is trivial on $U_F^{n+1}$. Then we have $\eta$ is level 1 and $\chi$ is level 2. The product $\eta \cdot \chi$ is also level 2.

The three ramified quadratic characters of $\mathbb{Q}_2^\times$ create a unitary character of the standard Borel subgroup in SL$_4(\mathbb{Q}_2)$:

\[
\tau : \begin{bmatrix} x_1 & * & * & * \\
0 & x_2 & * & * \\
0 & 0 & x_3 & * \\
0 & 0 & 0 & x_4 \end{bmatrix} \mapsto \eta(x_2)\chi(x_3)(\eta \cdot \chi)(x_4)
\]

We will twist this quadratic character by an unramified unitary character $\psi$ and form the induced representation $\text{Ind}_B^G(\psi^\tau)$. Let $D$ be the irreducible component in the Bernstein variety containing $\tau$, let $E \subset D$ be the corresponding compact manifold. The subgroup of the Weyl group which fixes $E$ is the finite group $W := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have the standard projection

\[
\pi : E//W \rightarrow E/W
\]

of the extended quotient onto the ordinary quotient. The extended quotient $E//W$ is the disjoint union of 6 unit intervals $a, b, c, d, e, f$ and the ordinary quotient $E/W$. In the projection $\pi$, these 6 intervals assemble themselves into the 6 edges of a tetrahedron in $E/W$. The cardinality of each fibre of $\pi$ creates a perfect model of reducibility. The locus of reducibility is the 1-skeleton $\mathcal{R}$ of a tetrahedron, and we have

\[
|\pi^{-1}(\psi^\tau)| = |\text{Ind}_B^G(\psi^\tau)|
\]
for all unramified unitary characters \( \psi \) of \( T \). On the interior of each edge \( \pi(a), \ldots, \pi(f) \) of \( \mathcal{R} \), each induced representation admits 2 distinct irreducible constituents; on each vertex of \( \mathcal{R} \), each induced representation admits 4 distinct irreducible components.

In this section, we intend to discuss a special case when \( F = \mathbb{Q}_2 \). Recall that the case 5.2 for \( \text{SL}_4(\mathbb{Q}_p) \), \( p \geq 3 \) does not exists since it cannot admit two ramified quadratic characters which is not the twist with an unramified character each other. Recall that the representation \( \pi_\sigma \) is given by \( \pi \otimes \chi \pi \otimes \eta \pi \otimes \chi \eta \pi \). In this case, the \( R \)-group is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and we also know that the isotropy group is \( \mathcal{W} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

- \( \gamma = (abcd) = 1 \), \( E^\gamma/W^\gamma = E/W^\gamma \).
- \( \gamma = (badc), E^\gamma = \{(1,1,z,z), (1,-1,z,-z) : z \in T\}. \)
  \[ E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I} \]
- \( \gamma = (cdab), E^\gamma = \{(1,z,1,z), (1,z,-1,-z) : z \in T\}. \)
  \[ E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I} \]
- \( \gamma = (dcab), E^\gamma = \{(1,z,z,1), (1,z,-z,-1) : z \in T\}. \)
  \[ E^\gamma/W^\gamma \cong \mathbb{I} \sqcup \mathbb{I} \]

Therefore, we can decompose the extended quotient as follows:

\[ E^\sigma/W^\sigma \cong E^\sigma/W^\sigma \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \sqcup \mathbb{I} \]

where \( \mathbb{I} \) is the unit interval in complex plane.

We start from the two disjoint varieties \( (1,1,z,z) \) and \( (1,-1,z,-z) \). We assume \( z \in \{1,-1\} \). Firstly, we check the character \( \chi_t \) generated by \( t = (1,1,z,z) \). Then we have \( \chi_t = (1,1,z^{\text{val}},z^{\text{val}}) \) and \( \chi_t \otimes \pi_\sigma \cong \pi \otimes \pi \otimes \eta z^{\text{val}} \pi \otimes \chi \eta z^{\text{val}} \pi \). In fact, \( \chi \) is the element of \( \tilde{L}(\chi_t \otimes \pi_\sigma) \) except the trivial character. Roughly speaking, the reason is, in this component, the variety fixes the terms which twists with character \( z^{\text{val}} \). Easily, we have \( R(\chi_t \otimes \pi_\sigma) = < \chi > = \mathbb{Z}/2\mathbb{Z} \). This implies for each \( t = (1,1,z,z) \), the representation induced \( \delta_t(z) \) by \( \chi_t \otimes \pi_\sigma \) is reducible.

Similarly, the character \( \chi_t \) generated by \( t = (1,-1,z,-z) \) is given by

\[ \chi_t = (1,(-1)^{\text{val}},z^{\text{val}},(-z)^{\text{val}}). \]
We have $\chi_t \otimes \pi_\sigma \simeq \pi \otimes (-1)^{\text{val}} \chi \pi \otimes \eta \pi \otimes \chi(-z)^{\text{val}}$. Then we know $\bar{L}(\chi_t \otimes \pi_\sigma) = \{1, (-1)^{\text{val}}\chi\}$. Indeed, $R(\chi_t \otimes \pi) = \mathbb{Z}/2\mathbb{Z}$. This means for each $t = (1, -1, z, -z)$ the representation induced by $\chi_t \otimes \pi_\sigma$ is reducible. $(1, z, 1, z)$ and $(1, z, -1, -z)$ are two disjoint varieties. Following the similar method, we know

$$t = (1, z, 1, z) \rightarrow \chi_t = (1, z^{\text{val}}, z^{\text{val}})$$

and we have $R(\chi_t \otimes \pi) = \langle \eta \rangle = \mathbb{Z}/2\mathbb{Z}$.

For $t = (1, z, -1, -z)$, we get the corresponding unitary character $\chi_t$ as follows

$$t = (1, z, -1, -z) \rightarrow \chi_t = (1, z^{\text{val}}, (-1)^{\text{val}}, (-z)^{\text{val}})$$

and we will have $R(\chi_t \otimes \pi) = \langle (-1)^{\text{val}}\eta \rangle = \mathbb{Z}/2\mathbb{Z}$.

In this component, $(1, z, z, 1)$ and $(1, z, -z, -1)$ are two disjoint varieties. Following the similar method, we know

$$t = (1, z, z, 1) \rightarrow \chi_t = (1, z^{\text{val}}, z^{\text{val}}, 1)$$

and we have $R(\chi_t \otimes \pi) = \langle \chi\eta \rangle = \mathbb{Z}/2\mathbb{Z}$.

For $t = (1, z, -z, -1)$, we get the corresponding unitary character $\chi_t$ as follows

$$t = (1, z, -z, -1) \rightarrow \chi_t = (1, z^{\text{val}}, (-z)^{\text{val}}, (-1)^{\text{val}})$$

and we will have $R(\chi_t \otimes \pi) = \langle (-1)^{\text{val}}\chi\eta \rangle = \mathbb{Z}/2\mathbb{Z}$.

For convenience, we denote

$$(1, 1, z, z) - (a)$$

$$(1, -1, z, -z) - (b)$$

$$(1, z, 1, z) - (c)$$

$$(1, z, -1, -z) - (d)$$

$$(1, z, z, 1) - (e)$$

$$(1, z, -z, -1) - (f)$$

Now, we investigate the points

$$(1, 1, 1, 1)$$

$$(1, -1, 1, -1)$$

$$(1, 1, -1, -1)$$

$$(1, -1, -1, 1)$$

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It is easy to check that

\[(1, 1, 1, 1) \in (a), (c), (e)\]
\[(1, -1, 1, -1) \in (b), (c), (f)\]
\[(1, 1, -1, -1) \in (a), (d), (f)\]
\[(1, -1, -1, 1) \in (b), (d), (e)\]

In fact, for such points, the $R$-group $R(\chi_t \otimes \sigma)$ is given by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

This implies, for each $\text{Ind}_M^G(\chi_t \otimes \sigma)$, there are 4 irreducible constituents. The extended quotient is the disjoint union of the ordinary quotient and six unit intervals. The six intervals are sent to the edges of a tetrahedron by the canonical projection

\[\pi : E^s/W^s \to E^s/W^s\]

The preimage of the interior of one edge is the union of two open intervals (the one corresponding to the given edge and one in the ordinary quotient), replicating the fact that the $R$-group has order 2, while the preimage of a vertex is the union of three endpoints of intervals and one point in the ordinary quotient, replicating the fact that the $R$-group has order 4 here. The 1-skeleton of the tetrahedron is perfect model of reducibility and confirms the ABP-conjecture in this case.

4 Decomposition according to cells

Let $G = \text{SL}_4(F)$. Let $T = (F^* \times F^* \times F^* \times F^*) \cap G$ be the standard maximal torus in $G$. Let $s = [T, \sigma]_G$ be a Bernstein component with respect to a character $\sigma$ of $T$. In this section, we denote by $W_s$ the isotropy group of $s$. We let $\pi_\sigma|_T = \sigma$ where $\pi_\sigma$ is a (unitary) character of the standard maximal torus $\tilde{T} = F^* \times F^* \times F^* \times F^*$ of $\tilde{G} = \text{GL}_4(F)$. We write $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$.

We denote by $W^0_s$ the isotropy of $\tilde{s} = [\tilde{T}, \pi_\sigma]_{\tilde{G}}$. The group $W^0_s$ is a finite Weyl group. Let $\Phi_s$ denote a root system for $W^0_s$, and let $\Phi^+_s = \Phi_s \cap \Phi^+$, where $\Phi^+$ is a positive root system for the Weyl group of $G$. Then $\Phi^+_s$ is a positive system in $\Phi_s$. The group $W_s$ is not a Weyl group in general. However, we have the following relation (see for instance [7, Prop. 2.3]):

\[W_s = W^0_s \times C_s, \tag{12}\]

where

\[C_s = \{ w \in W^0_s : w \cdot \Phi^+_s = \Phi^+_s \} \,.

In the table we list all the possibilities for $\pi_\sigma$ due to the relations among the $\pi_i$. The conditions for the $\pi_i$ in the first column of the table are that $\pi_i$
The determination of the group $G$ results are listed in the fourth column of Table 1.

We set the dual torus of $T$ be the group of characters of $T$. We have

$$X(T) \simeq \{ (l_1, l_2, l_3, l_4) \in \mathbb{Z}^4 : l_1 + l_2 + l_3 + l_4 = 0 \}.$$ 

We set

$$W_s^e = W_s^0 \rtimes X(T).$$

Then $W_s^e$ is the extended affine Weyl group of the $p$-adic group $H_s^0$ described in the fifth column of the table. The group $H_s^0$ arises from $[10, \S \,8]$. Let $\Phi_s^\vee$ denote the set of coroots of the root system $\Phi_s$. The quadruple $(X(T), \Phi_s, X(T^\vee), \Phi_s^\vee)$ is the root datum of $H_s^0$. We set $H_s := H_s^0 \rtimes C_s$. The unipotent classes of $H_s^0$ will be easy to figure out. We will attach a unipotent class to each cocharacter. In particular, the minimal (for the usual order) unipotent class (this is the trivial unipotent class) should correspond to the trivial cocharacter. In other words, all the connected components of the compact extended quotient which are attached in $\S \,2$ to a trivial cocharacter should correspond to the minimal unipotent class. When the group $W_s^0 = \{1\}$, there is only one unipotent class and all the cocharacters are trivial as proved in $\S \,2$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\pi_s$</th>
<th>$W_s^0$</th>
<th>$C_s$</th>
<th>$H_s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi \otimes \pi \otimes \pi \otimes \pi$</td>
<td>$S_4$</td>
<td>1</td>
<td>$\text{SL}_4(F)$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi_1 \otimes \pi_1 \otimes \pi_1 \otimes \pi_2$</td>
<td>$S_3$</td>
<td>1</td>
<td>$(\text{GL}_2(F) \times F^\times) \cap G$</td>
</tr>
<tr>
<td>3.1</td>
<td>$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_2$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
<td>1</td>
<td>$(\text{GL}_2(F) \times \text{GL}_2(F)) \cap G$</td>
</tr>
<tr>
<td>3.2</td>
<td>$\pi \otimes \pi \otimes \eta \pi \otimes \eta \pi$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$(\text{GL}_2(F) \times \text{GL}_2(F)) \cap G$</td>
</tr>
<tr>
<td>4</td>
<td>$\pi_1 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>1</td>
<td>$(\text{GL}_2(F) \times F^\times \times F^\times) \cap G$</td>
</tr>
<tr>
<td>5.1</td>
<td>$\pi \otimes \eta \pi \otimes \eta \pi \otimes \eta \pi$</td>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>$\pi \otimes \chi \pi \otimes \eta \pi \otimes \eta \lambda \pi$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>5.3</td>
<td>$\pi_1 \otimes \eta \pi_1 \otimes \pi_2 \otimes \eta \pi_2$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>$\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_4$</td>
<td>1</td>
<td>1</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Table 1: Table of the groups $W_s = W_s^0 \rtimes C_s$ and $H_s = H_s^0 \rtimes C_s$.
Case 1. The Langlands dual group of $\text{SL}_4(F)$ is the complex Lie group $\text{PGL}_4(\mathbb{C})$. There are five unipotent classes in $\text{PGL}_4(\mathbb{C})$:

$$u_0 \leq u_3 \leq u_2 \leq u_1 \leq u_e,$$

which are respectively parametrized by the following partitions of 4:

$$(1^4) \leq (2, 1^2) \leq (2^2) \leq (3, 1) \leq (4).$$

They correspond (see for instance [12]) to the two-sided cells

$$c_0 \leq c_3 \leq c_2 \leq c_1 \leq c_e.$$

We write

$$pt_1 = (1, 1, 1, 1) \quad pt_2 = (1, -1, 1, -1)$$

$$pt_3 = (1, i, -1, i) \quad pt_4 = (1, -i, -1, i)$$

and define

$$(E_s//W_s)_0 := E_s/W_s \sqcup \{(z, -z, 1, -1) : z \in T\} \sqcup pt_3 \sqcup pt_4 \simeq E_s/W_s \sqcup T \sqcup pt_3 \sqcup pt_4$$

$$(E_s//W_s)_3 := \{(z_1, z_1, z_2, 1) : z_1, z_2 \in T\} \simeq T^2$$

$$(E_s//W_s)_2 := \{(z, z, 1, 1) : z \in T\} \sqcup pt_1 \simeq T \sqcup pt_1$$

$$(E_s//W_s)_1 := \{(z, z, z, 1) : z \in T\} \simeq T$$

$$(E_s//W_s)_e := pt_2.$$

From Eqn.(3), we get the following cell-decomposition of $E_s//W_s$:

$$E_s//W_s = (E_s//W_s)_0 \sqcup (E_s//W_s)_3 \sqcup (E_s//W_s)_2 \sqcup (E_s//W_s)_1 \sqcup (E_s//W_s)_e,$$

with cocharacters

$$h_0 = 1, \quad h_3(t) = (t, t^{-1}, 1, 1), \quad h_2(t) = (t, t^{-1}, t, t^{-1})$$

$$h_1(t) = (t^2, 1, t^{-2}, 1), \quad h_e(t) = (t^3, t, t^{-1}, t^{-3}).$$

We have included $pt_1$ in the subset $(E_s//W_s)_2$ in order to attach the two elements $\rho^+$ and $\rho^-$ (defined in §2) to the same unipotent class. It should be a general fact that all the elements in a given $L$-packet are attached to the same unipotent class.

Case 2. The Langlands dual group of $H_s^0$ is $(\text{GL}_3(\mathbb{C}) \times \mathbb{C}^x)/\mathbb{C}^x$. There are three unipotent classes in it:

$$u_0 \leq u_1 \leq u_e,$$
which are respectively parametrized by the following partitions of 3:

\[(1^3) \leq (2, 1) \leq (3).\]

They correspond to the two-sided cells

\[c_0 \leq c_1 \leq c_e.\]

We define

\[(E_s//W_s)_0 := E_s/W_s,\]

\[(E_s//W_s)_1 := \{(z, z, z, z) : z \in \mathbb{T}\} \simeq \mathbb{T},\]

\[(E_s//W_s)_e := \{(z_1, z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2.\]

From Eqn.(4), we get the following cell-decomposition of \(E_s//W_s\):

\[E_s//W_s = (E_s//W_s)_0 \sqcup (E_s//W_s)_1 \sqcup (E_s//W_s)_e,\]

with cocharacters

\[h_0 = 1, \quad h_1(t) = (t^2, 1, t^{-2}, 1), \quad h_e(t) = (t, t^{-1}, 1, 1).\]

**Case 3.** The Langlands dual group of \(H_0^s\) is \((\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))/\mathbb{C}^\times\).

There are four unipotent classes in it: \(u_0 \leftrightarrow (2, 2), u_1 \leftrightarrow (2, 1^2), u'_1 \leftrightarrow (1^2, 2), u_e \leftrightarrow (1^2, 1^2)\). The closure order on unipotent classes is the following:

\[
\begin{array}{c}
\text{u}_0 \\
\text{u}_1 \\
\text{u}_e \\
\text{u'}_1
\end{array}
\]

**Case 3.1.** We define

\[(E_s//W_s)_0 := E_s/W_s \sqcup \{(1, -1, z, -z) : z \in \mathbb{T}\} \simeq E_s/W_s \sqcup \mathbb{T},\]

\[(E_s//W_s)_1 := \{(1, 1, z_1, z_2) : z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2,\]

\[(E_s//W_s)'_1 := \{(z_1, z_2, 1, 1) : z_1, z_2 \in \mathbb{T}\} \simeq \mathbb{T}^2,\]

\[(E_s//W_s)_e := \{(1, 1, z, z) : z \in \mathbb{T}\} \simeq \mathbb{T}.\]

From Eqn.(5), we get the following cell-decomposition of \(E_s//W_s\):

\[E_s//W_s = (E_s//W_s)_0 \sqcup (E_s//W_s)_1 \sqcup (E_s//W_s)'_1 \sqcup (E_s//W_s)_e,\]

with cocharacters

\[h_0 = 1, \quad h_1(t) = (t, t^{-1}, 1, 1), \quad h'_1(t) = (1, 1, t, t^{-1}), \quad h_e(t) = (t, t^{-1}, t, t^{-1}).\]
In this case, $H_s$ is disconnected, and it seems that one has to consider unipotent classes in the disconnected complex Lie group

$$((\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})) / \mathbb{C}^\times) \times \mathbb{Z}/2\mathbb{Z}.$$ 

This group can be considered as “the Langlands dual group of $H_s$” (see [10, top of page 395]).

This should give only three unipotent classes

$$u_0 \leq (u_1 \cup u'_1) \leq u_e.$$ 

We set

$$T_0 := \{(z, 1, -z, -1) : z \in T\} \simeq T$$

$$T'_0 := \{(z, 1, z, 1) : z \in T\} \simeq T$$

$$T''_0 := \{(1, -1, z, -z) : z \in T\} \simeq T$$

We define

$$(E_s / W_s)_0 := E_s / W_s \sqcup T_0 \sqcup T'_0 \sqcup T''_0 \sqcup \text{pt}_1 \sqcup \text{pt}_2 \sqcup \text{pt}_3 \sqcup \text{pt}_4,$$

where pt$_1$, pt$_2$, pt$_3$, pt$_4$ are defined as in Eqn. (13),

$$(E_s / W_s)_1 := \{(z_1, z_1, z_2, 1) : z_1, z_2 \in T\} \simeq T^2$$

$$(E_s / W_s)_e := \{(1, 1, z, z) : z \in T\} \simeq T.$$ 

From Eqn.(8), we get the following cell-decomposition of $E_s / W_s$:

$$E_s / W_s = (E_s / W_s)_0 \sqcup (E_s / W_s)_1 \sqcup (E_s / W_s)_e,$$

with cocharacters

$$h_0 = 1, \quad h_1(t) = (t, t^{-1}, 1, 1), \quad h_e(t) = (t, t^{-1}, t, t^{-1}).$$ 

**Case 4.** The Langlands dual group of $H_s^0$ is $(\text{GL}_2(\mathbb{C}) \times \mathbb{C}^\times \times \mathbb{C}^\times) / \mathbb{C}^\times$.

This group admits two unipotent classes:

$$u_0 \leftrightarrow (1^2, 1, 1) \leq u_e \leftrightarrow (2, 1, 1).$$

We define

$$(E_s / W_s)_0 := E_s / W_s$$

$$(E_s / W_s)_e := \{(z_1, z_1, z_2, 1) : z_1, z_2 \in T\} \simeq T^2.$$ 

From Eqn.(9), we have the following cell-decomposition:

$$(E_s / W_s) = (E_s / W_s)_0 \sqcup (E_s / W_s)_e.$$
with cocharacters
\[ h_0 = 1, \quad h_e(t) = (t, t^{-1}, 1, 1). \]

Cases 5.1, 5.2, 5.3, 5.4. The Langlands dual group of \( H_0^0 = T \) is the complex torus \( T^\vee \) in \( \text{PGL}_4(\mathbb{C}) \). There is only one unipotent class in \( T^\vee \): the trivial class \( u_0 \). Hence we set
\[ (E_s//W_s)_0 = E_s//W_s. \]

There only one cocharacter, the trivial cocharacter.

References


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