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Infinitely divisible cylindrical measures on Banach spaces

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Abstract

In this work infinitely divisible cylindrical probability measures on arbitrary Banach spaces are introduced. The class of infinitely divisible cylindrical probability measures is described in terms of their characteristics, a characterisation which is not known in general for infinitely divisible Radon measures on Banach spaces. Furthermore, continuity properties and the relation to infinitely divisible Radon measures of infinitely divisible cylindrical probability measures are considered.

1 Introduction

Probability theory in Banach spaces has been extensively studied since 1960 and several monographs are dedicated to this field of mathematics, e.g. Araujo and Giné [5], Heyer [7], Ledoux and Talagrand [10], Linde [11] and Vakhania et al [20]. In general, probability theory in Banach spaces is more complicated than in Euclidean spaces or in infinite-dimensional Hilbert spaces and many of the phenomena which occur in probability theory in Banach spaces are related to the theory of Banach space geometry.

Cylindrical Wiener processes, which appear rather in the Hilbert space setting, can also be considered in the more general situation of Banach spaces. Defining cylindrical Wiener processes by cylindrical probability measures yields even that most of the results in the Hilbert space situation can be easily rewritten for the Banach space situation, see Kallianpur and Xiong [9], Metivier and Pellaumail [13] or Riedle [16]. Cylindrical probability measures are finitely additive set functions which have projections to Euclidean spaces that are always bona fide probability measures. In Applebaum and Riedle [1] the cylindrical approach

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enables us to introduce cylindrical Lévy processes as a natural generalisation of cylindrical Wiener process, a concept which is even new in the Hilbert space situation.

The definition of cylindrical Lévy processes in Applebaum and Riedle [1] relies on the introduction of infinitely divisible cylindrical measures. However, since the work [1] is focused on cylindrical Lévy processes and their stochastic integral, no further properties of infinitely divisible cylindrical measures are derived. In this work we give a rigorous introduction of infinitely divisible cylindrical measures and derive some fundamental properties of them. Some of the results give also a new insight on genuine infinitely divisible Radon measures on Banach spaces.

On a Hilbert space H it is well known that the class of infinitely divisible measures can be characterised by the set of triplets (h, s, ν) where $h \in H$, $s : H \rightarrow H$ is a trace class operator and ν is a σ -finite measure satisfying

$$\int_H \left(\|u\|^2 \wedge 1 \right) \nu(du) < \infty, \quad (1.1)$$

see for example Parthasarathy [14]. However, in Banach spaces such an explicit description of infinitely divisible measures is not known in general. But it turns out in this work, that the class of infinitely divisible cylindrical measures on an arbitrary Banach space can be described by a set of triplets with some explicit conditions on the entries of the triplets.

This result enables us to solve the following problem: even in the finite dimensional case, a probability measure on \mathbb{R}^2 which satisfies that all image measures under linear projections to \mathbb{R} are infinitely divisible might be not infinitely divisible, see Giné and Hahn [6] and Marcus [12]. However, the question left open if a probability measure on an infinite dimensional space is infinitely divisible if all linear projections to \mathbb{R}^n for all finite dimensions $n \in \mathbb{N}$ are infinitely divisible. By the characterisation of the set of infinitely divisible cylindrical measures mentioned above we are able to answer this question affirmative.

As already indicated, the condition (1.1) might be neither sufficient nor necessary for a σ -finite measure on an arbitrary Banach space U to guarantee that there exists an infinitely divisible measure with characteristics $(0, 0, \nu)$, see for example $U = C[0, 1]$ in Araujo [3]. However, we show in the last part of this work that a σ -finite measure ν satisfying the weaker condition

$$\int_U \left(|\langle u, a \rangle|^2 \wedge 1 \right) \nu(du) < \infty \quad \text{for all } a \in U^*, \quad (1.2)$$

always generates an infinitely divisible cylindrical measure μ . This result reduces the question whether a σ -finite measure generates an infinitely divisible measure to the question whether the infinitely divisible cylindrical measure μ extends to a Radon measure. In fact, but without relating it to cylindrical measures, this can be considered as the starting point in some work, e.g. Araujo [4] and Paulauskas [15], where for specific examples of a Banach space the set or subset of all Lévy measures is described explicitly.

2 Preliminaries

For a measure space (S, \mathcal{S}, μ) we denote by $L^p_\nu(S, \mathcal{S})$, $p \geq 0$ the space of equivalence classes of measurable functions $f : S \rightarrow \mathbb{R}$ with $\int |f(s)|^p \mu(ds) < \infty$.

Let U be a Banach space with dual U^* . The dual pairing is denoted by $\langle u, a \rangle$ for $u \in U$ and $a \in U^*$. The Borel σ -algebra in U is denoted by $\mathcal{B}(U)$ and the ball by $B_U := \{u \in U : \|u\| \leq 1\}$.

For every $a_1, \dots, a_n \in U^*$ and $n \in \mathbb{N}$ we define a linear map

$$\pi_{a_1, \dots, a_n} : U \rightarrow \mathbb{R}^n, \quad \pi_{a_1, \dots, a_n}(u) = (\langle u, a_1 \rangle, \dots, \langle u, a_n \rangle).$$

Let Γ be a subset of U^* . Sets of the form

$$\begin{aligned} Z(a_1, \dots, a_n; B) &:= \{u \in U : (\langle u, a_1 \rangle, \dots, \langle u, a_n \rangle) \in B\} \\ &= \pi_{a_1, \dots, a_n}^{-1}(B), \end{aligned}$$

where $a_1, \dots, a_n \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$ are called *cylindrical sets*. The set of all cylindrical sets is denoted by $\mathcal{Z}(U, \Gamma)$ and it is an algebra. The generated σ -algebra is denoted by $\mathcal{C}(U, \Gamma)$ and it is called the *cylindrical σ -algebra with respect to (U, Γ)* . If $\Gamma = U^*$ we write $\mathcal{Z}(U) := \mathcal{Z}(U, \Gamma)$ and $\mathcal{C}(U) := \mathcal{C}(U, \Gamma)$.

A function $\mu : \mathcal{Z}(U) \rightarrow [0, \infty]$ is called a *cylindrical measure on $\mathcal{Z}(U)$* , if for each finite subset $\Gamma \subseteq U^*$ the restriction of μ to the σ -algebra $\mathcal{C}(U, \Gamma)$ is a measure. A cylindrical measure is called finite if $\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$.

For every function $f : U \rightarrow \mathbb{C}$ which is measurable with respect to $\mathcal{C}(U, \Gamma)$ for a finite subset $\Gamma \subseteq U^*$ the integral $\int f(u) \mu(du)$ is well defined as a complex valued Lebesgue integral if it exists. In particular, the characteristic function $\varphi_\mu : U^* \rightarrow \mathbb{C}$ of a finite cylindrical measure μ is defined by

$$\varphi_\mu(a) := \int_U e^{i\langle u, a \rangle} \mu(du) \quad \text{for all } a \in U^*.$$

In contrary to probability measures on $\mathcal{B}(U)$ there exists an analogue of Bochner's theorem for cylindrical probability measures, see [20, Prop.VI.3.2]: a function $\varphi : U^* \rightarrow \mathbb{C}$ with $\varphi(0) = 1$ is the characteristic function of a cylindrical probability measure if and only if it is positive-definite and continuous on every finite-dimensional subspace.

For every $a_1, \dots, a_n \in U^*$ we obtain an image measure $\mu \circ \pi_{a_1, \dots, a_n}^{-1}$ on $\mathcal{B}(\mathbb{R}^n)$. Its characteristic function $\varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}$ is determined by that of μ :

$$\varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}(t) = \varphi_\mu(t_1 a_1 + \dots + t_n a_n) \quad (2.3)$$

for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$.

If μ_1 and μ_2 are cylindrical probability measures on $\mathcal{Z}(U)$ their convolution is the cylindrical probability measure defined by

$$(\mu_1 * \mu_2)(Z) = \int_U \mu_1(Z - u) \mu_2(du),$$

for each $Z \in \mathcal{Z}(U)$. Indeed if $Z = \pi_{a_1, \dots, a_n}^{-1}(B)$ for some $a_1, \dots, a_n \in U^*$, $B \in \mathcal{B}(\mathbb{R}^n)$, then it is easily verified that

$$(\mu_1 * \mu_2)(Z) = (\mu_1 \circ \pi_{a_1, \dots, a_n}^{-1}) * (\mu_2 \circ \pi_{a_1, \dots, a_n}^{-1})(B).$$

A standard calculation yields $\varphi_{\mu_1 * \mu_2} = \varphi_{\mu_1} \varphi_{\mu_2}$. For more information about convolution of cylindrical probability measures, see [17]. The k -times convolution of a cylindrical probability measure μ with itself is denoted by μ^{*k} .

3 Infinitely divisible cylindrical measures

A probability measure ζ on $\mathcal{B}(\mathbb{R})$ is called *infinitely divisible* if for every $k \in \mathbb{N}$ there exists a probability measure ζ_k such that $\zeta = (\zeta_k)^{*k}$. It is well known that infinitely divisible probability measures on $\mathcal{B}(\mathbb{R})$ are characterised by their characteristic function. The characteristic function is unique but its specific representation depends on the chosen truncation function.

Definition 3.1. *A truncation function is any measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$ which is bounded and satisfies $h = \text{Id}$ in a neighborhood $D(h)$ of 0.*

In general the set of admissible truncation functions is much larger but this is not our concern here. Given a truncation function h a probability measure ζ on $\mathcal{B}(\mathbb{R})$ is infinitely divisible if and only if its characteristic function is of the form

$$\varphi_\zeta : \mathbb{R} \rightarrow \mathbb{C}, \quad \varphi_\zeta(t) = \exp \left(imt - \frac{1}{2} r^2 t^2 + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) \eta(ds) \right) \quad (3.4)$$

for some constants $m \in \mathbb{R}$, $r \geq 0$ and a Lévy measure η , which is a σ -finite measure η on $\mathcal{B}(\mathbb{R})$ with $\eta(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (|s|^2 \wedge 1) \eta(ds) < \infty.$$

The function $\tilde{\psi}_h$ is defined by

$$\tilde{\psi}_h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}, \quad \tilde{\psi}_h(s, t) := e^{ist} - 1 - ith(s).$$

In this situation we call the triplet $(m, r, \eta)_h$ the *characteristics* of ζ . If h' is another truncation function then $(m', r, \eta)_{h'}$ is the characteristics of ζ with respect to h' , where

$$m' := m + \int_{\mathbb{R}} (h'(s) - h(s)) \eta(ds).$$

The integral exists because $h'(s) - h(s) = 0$ for $s \in D(h') \cap D(h)$ and h is bounded. From Bochner's theorem and the Schoenberg correspondence it follows that the function

$$t \mapsto - \int_{\mathbb{R}} \tilde{\psi}_h(s, t) \eta(s)$$

is negative-definite for all Lévy measures η . By choosing $\eta = \delta_{s_0}$, where δ_{s_0} denotes the Dirac measure in s_0 for a constant $s_0 \in \mathbb{R}$, we conclude that

$$t \mapsto -\tilde{\psi}_h(s_0, t) \quad \text{is negative-definite for all } s_0 \in \mathbb{R}. \quad (3.5)$$

For an arbitrary Banach space U a Radon probability measure μ on $\mathcal{B}(U)$ is called *infinitely divisible* if for each $k \in \mathbb{N}$ there exists a Radon probability measure μ_k such that $\mu = (\mu_k)^{*k}$. We generalise this definition to cylindrical measures:

Definition 3.2. *A cylindrical probability measure μ on $\mathcal{Z}(U)$ is called infinitely divisible if there exists for each $k \in \mathbb{N}$ a cylindrical probability measure μ_k such that $\mu = (\mu_k)^{*k}$.*

Bochner's theorem for cylindrical probability measures, [20, Prop.VI.3.2], implies that a cylindrical probability measure μ on $\mathcal{Z}(U)$ is infinitely divisible if and only if for every $k \in \mathbb{N}$ there exists a characteristic function φ_{μ_k} of a cylindrical probability measure μ_k such that

$$\varphi_\mu(a) = (\varphi_{\mu_k}(a))^k \quad \text{for all } a \in U^*.$$

One might conjecture that a cylindrical probability measure μ is infinitely divisible if every image measure $\mu \circ a^{-1}$ is infinitely divisible for all $a \in U^*$. But this is wrong already in the case $U = \mathbb{R}^2$ which is shown by Giné and Hahn [6] and Marcus [12]. They constructed a probability measure μ on $\mathcal{B}(\mathbb{R}^2)$ such that all projections $\mu \circ a^{-1}$ are infinitely divisible for all linear functions $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ but μ is not infinitely divisible. However, we will show in this work that a cylindrical probability measure or a Radon probability measure μ is infinitely divisible if and only if

$$\mu \circ \pi_{a_1, \dots, a_n}^{-1} \text{ is infinitely divisible for all } a_1, \dots, a_n \in U^* \text{ and } n \in \mathbb{N}. \quad (3.6)$$

A cylindrical probability measure μ satisfies (3.6) if and only if for each $k \in \mathbb{N}$ and all $a_1, \dots, a_n \in U^*$, $n \in \mathbb{N}$ there exists a characteristic function $\varphi_{\xi_{k, a_1, \dots, a_n}}$ of a probability measure ξ_{k, a_1, \dots, a_n} on $\mathcal{B}(\mathbb{R}^n)$ such that

$$\varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}(t) = (\varphi_{\xi_{k, a_1, \dots, a_n}}(t))^k \quad \text{for all } t \in \mathbb{R}^n. \quad (3.7)$$

Every infinitely divisible cylindrical probability measure μ satisfies (3.6) because for each $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ we have

$$\begin{aligned} \varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}(t) &= \varphi_\mu(t_1 a_1 + \dots + t_n a_n) \\ &= (\varphi_{\mu_k}(t_1 a_1 + \dots + t_n a_n))^k \\ &= \left(\varphi_{\mu_k \circ \pi_{a_1, \dots, a_n}^{-1}}(t) \right)^k \end{aligned}$$

for all $a_1, \dots, a_n \in U^*$ and all $k \in \mathbb{N}$. In particular, we obtain for $n = 1$ that $\mu \circ a^{-1}$ is infinitely divisible and thus,

$$\begin{aligned}\varphi_\mu(a) &= \varphi_{\mu \circ a^{-1}}(1) \\ &= \exp \left(im_a - \frac{1}{2} r_a^2 + \int_{\mathbb{R}} (e^{is} - 1 - is \mathbb{1}_{B_{\mathbb{R}}}(s)) \eta_a(ds) \right)\end{aligned}\quad (3.8)$$

for some constants $m_a \in \mathbb{R}$, $r_a \geq 0$ and a Lévy measure η_a on $\mathcal{B}(\mathbb{R})$. This representation can be significantly improved as we have shown in Applebaum and Riedle [1]. Not only for completeness of this work but also because we formulate the result under the weaker condition (3.6) we give the details of the proof here. Despite the weaker condition (3.6) the proof remains the same.

Theorem 3.3. *Let μ be a cylindrical probability measure on $\mathcal{Z}(U)$ which satisfies (3.6). Then its characteristic function $\varphi_\mu : U^* \rightarrow \mathbb{C}$ is given by*

$$\varphi_\mu(a) = \exp \left(iw(a) - \frac{1}{2} q(a) + \int_U \left(e^{i\langle u, a \rangle} - 1 - i\langle u, a \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, a \rangle) \right) \nu(du) \right), \quad (3.9)$$

where $w : U^* \rightarrow \mathbb{R}$ is a mapping, $q : U^* \rightarrow \mathbb{R}$ is a quadratic form and ν is a cylindrical measure on $\mathcal{Z}(U)$ such that $\nu \circ \pi_{a_1, \dots, a_n}^{-1}$ is the Lévy measure on $\mathcal{B}(\mathbb{R}^n)$ of $\mu \circ \pi_{a_1, \dots, a_n}^{-1}$ for all $a_1, \dots, a_n \in U^*$, $n \in \mathbb{N}$.

Proof. For fixed $a_1, \dots, a_n \in U^*$ let ν_{a_1, \dots, a_n} denote the Lévy measure on $\mathcal{B}(\mathbb{R}^n)$ of the infinitely divisible probability measure $\mu \circ \pi_{a_1, \dots, a_n}^{-1}$. Define the family of cylindrical sets

$$\mathcal{G} := \{Z(a_1, \dots, a_n; B) : a_1, \dots, a_n \in U^*, n \in \mathbb{N}, B \in \mathcal{F}_{a_1, \dots, a_n}\},$$

where

$$\mathcal{F}_{a_1, \dots, a_n} := \{(\alpha, \beta] \subseteq \mathbb{R}^n : \nu_{a_1, \dots, a_n}(\partial(\alpha, \beta]) = 0, 0 \notin [\alpha, \beta]\}$$

and $\partial(\alpha, \beta]$ denotes the boundary of the n -dimensional interval

$$(\alpha, \beta] := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \alpha_i < x_i \leq \beta_i, i = 1, \dots, n\}$$

for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$.

Our proof relies on the relation

$$\lim_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx) = \int_{\mathbb{R}^n} \mathbb{1}_B(x) \nu_{a_1, \dots, a_n}(dx) \quad (3.10)$$

for all sets $B \in \mathcal{F}_{a_1, \dots, a_n}$. This can be deduced from Corollary 2.8.9. in [18] which states that

$$\lim_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} f(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx) = \int_{\mathbb{R}^n} f(x) \nu_{a_1, \dots, a_n}(dx) \quad (3.11)$$

for all bounded and continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which vanish on a neighborhood of 0. The relation (3.10) can be seen in the following way: let $B = (\alpha, \beta]$ be a set in $\mathcal{F}_{a_1, \dots, a_n}$ for $\alpha, \beta \in \mathbb{R}^n$. Because $0 \notin \bar{B}$ there exists $\varepsilon > 0$ such that $0 \notin [\alpha - \varepsilon, \beta + \varepsilon]$ where $\alpha - \varepsilon := (\alpha_1 - \varepsilon, \dots, \alpha_n - \varepsilon)$ and $\beta + \varepsilon := (\beta_1 + \varepsilon, \dots, \beta_n + \varepsilon)$. Define for $i = 1, \dots, n$ the functions $g_i : \mathbb{R} \rightarrow [0, 1]$ by

$$g_i(t) = \left(1 - \frac{(\alpha_i - t)}{\varepsilon}\right) \mathbb{1}_{(\alpha_i - \varepsilon, \alpha_i]}(t) + \mathbb{1}_{(\alpha_i, \beta_i]}(t) + \left(1 - \frac{(t - \beta_i)}{\varepsilon}\right) \mathbb{1}_{(\beta_i, \beta_i + \varepsilon]}(t),$$

and interpolate the function $x \mapsto \mathbb{1}_{(\alpha, \beta]}(x)$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by

$$f((x_1, \dots, x_n)) := g_1(x_1) \cdot \dots \cdot g_n(x_n).$$

Because $\mathbb{1}_B \leq f \leq \mathbb{1}_{(\alpha - \varepsilon, \beta + \varepsilon]}$ we have

$$\frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx) \leq \frac{1}{t_k} \int_{\mathbb{R}^n} f(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx)$$

and

$$\int_{\mathbb{R}^n} f(x) \nu_{a_1, \dots, a_n}(dx) \leq \int_{\mathbb{R}^n} \mathbb{1}_{(\alpha - \varepsilon, \beta + \varepsilon]}(x) \nu_{a_1, \dots, a_n}(dx) = \nu_{a_1, \dots, a_n}((\alpha - \varepsilon, \beta + \varepsilon]).$$

Since f is bounded, continuous and vanishes on a neighborhood of 0, it follows from (3.11) that

$$\limsup_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx) \leq \nu_{a_1, \dots, a_n}((\alpha - \varepsilon, \beta + \varepsilon]). \quad (3.12)$$

By considering $(\alpha + \varepsilon, \beta - \varepsilon] \subseteq (\alpha, \beta]$ we obtain similarly that

$$\nu_{a_1, \dots, a_n}((\alpha + \varepsilon, \beta - \varepsilon]) \leq \liminf_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx). \quad (3.13)$$

Because $\nu_{a_1, \dots, a_n}(\partial B) = 0$ the inequalities (3.12) and (3.13) imply (3.10).

Now we define a set function

$$\nu : \mathcal{Z}(U) \rightarrow [0, \infty], \quad \nu(Z(a_1, \dots, a_n; B)) := \nu_{a_1, \dots, a_n}(B).$$

First, we show that ν is well defined. For $Z(a_1, \dots, a_n; B) \in \mathcal{G}$ equation (3.10) allows us to conclude that

$$\begin{aligned} \nu(Z(a_1, \dots, a_n; B)) &= \lim_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu \circ \pi_{a_1, \dots, a_n}^{-1})^{*t_k}(dx) \\ &= \lim_{t_k \rightarrow 0} \frac{1}{t_k} \int_{\mathbb{R}^n} \mathbb{1}_B(x) (\mu^{*t_k} \circ \pi_{a_1, \dots, a_n}^{-1})(dx) \\ &= \lim_{t_k \rightarrow 0} \frac{1}{t_k} \int_U \mathbb{1}_B(\pi_{a_1, \dots, a_n}(u)) \mu^{*t_k}(du) \\ &= \lim_{t_k \rightarrow 0} \frac{1}{t_k} \mu^{*t_k}(Z(a_1, \dots, a_n; B)). \end{aligned}$$

It follows that for two sets in \mathcal{G} with $Z(a_1, \dots, a_n; B) = Z(b_1, \dots, b_m; C)$ that

$$\nu(Z(a_1, \dots, a_n; B)) = \nu(Z(b_1, \dots, b_m; C)),$$

which verifies that ν is well defined on \mathcal{G} .

Having shown that ν is well-defined on \mathcal{G} for fixed $a_1, \dots, a_n \in U$ we now demonstrate that its restriction to the σ -algebra $\mathcal{Z}(U, \{a_1, \dots, a_n\})$ is a measure so that it yields a cylindrical measure on $\mathcal{Z}(U)$.

Define a set of n -dimensional intervals by

$$\mathcal{H} := \{(\alpha, \beta] \subseteq \mathbb{R}^n : 0 \notin [\alpha, \beta]\}.$$

Because ν_{a_1, \dots, a_n} is a σ -finite measure the set

$$\mathcal{H} \setminus \mathcal{F}_{a_1, \dots, a_n} = \{(\alpha, \beta] \in \mathcal{H} : \nu_{a_1, \dots, a_n}(\partial(\alpha, \beta]) \neq 0\}$$

is countable. Thus, the set $\mathcal{F}_{a_1, \dots, a_n}$ generates the same σ -algebra as \mathcal{H} because the countably missing sets in $\mathcal{F}_{a_1, \dots, a_n}$ can easily be approximated by sets in $\mathcal{F}_{a_1, \dots, a_n}$. But \mathcal{H} is known to be a generator of the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$ which implies that

$$\mathcal{G}_{a_1, \dots, a_n} := \{Z(a_1, \dots, a_n; B) : B \in \mathcal{F}_{a_1, \dots, a_n}\}$$

generates $\mathcal{Z}(U, \{a_1, \dots, a_n\})$.

Furthermore, $\mathcal{G}_{a_1, \dots, a_n}$ is a semi-ring because $\mathcal{F}_{a_1, \dots, a_n}$ is a semi-ring. Secondly, ν restricted to $\mathcal{G}_{a_1, \dots, a_n}$ is well defined and is a pre-measure. For, if $\{Z_k := Z_k(a_1, \dots, a_n; B_k) : k \in \mathbb{N}\}$ are a countable collection of disjoint sets in $\mathcal{G}_{a_1, \dots, a_n}$ with $\cup Z_k \in \mathcal{G}_{a_1, \dots, a_n}$ then the Borel sets B_k are disjoint and it follows that

$$\begin{aligned} \nu \left(\bigcup_{k \geq 1} Z_k \right) &= \nu \left(\bigcup_{k \geq 1} \pi_{a_1, \dots, a_n}^{-1}(B_k) \right) = \nu \left(\pi_{a_1, \dots, a_n}^{-1} \left(\bigcup_{k \geq 1} B_k \right) \right) \\ &= \nu_{a_1, \dots, a_n} \left(\bigcup_{k \geq 1} B_k \right) = \sum_{k=1}^{\infty} \nu_{a_1, \dots, a_n}(B_k) = \sum_{k=1}^{\infty} \nu(Z_k). \end{aligned}$$

Thus, ν restricted to $\mathcal{G}_{a_1, \dots, a_n}$ is a pre-measure and because it is σ -finite it can be extended uniquely to a measure on $\mathcal{Z}(U, \{a_1, \dots, a_n\})$ by Carathéodory's extension theorem, which verifies that ν is a cylindrical measure on $\mathcal{Z}(U)$.

By using (3.8) we define $w(a) := m_a$ and the function

$$q : U^* \rightarrow \mathbb{R}_+, \quad q(a) := r_a^2$$

and it remains to show that q is a quadratic form.

According to Proposition IV.4.2 in [20] there exists a probability space (Ω, \mathcal{A}, P) and a cylindrical random variable X with cylindrical distribution μ . By the Lévy-Itô decomposition in \mathbb{R} it follows that

$$Xa = w(a) + r_a W_a + \int_{0 < |s| < 1} s \tilde{N}_a(ds) + \int_{|s| \geq 1} s N_a(ds) \quad P\text{-a.s.}, \quad (3.14)$$

where W_a is a real valued centred Gaussian random variable with $EW_a^2 = 1$, N_a is an independent Poisson random measure on $\mathbb{R} \setminus \{0\}$ and \tilde{N}_a is the compensated Poisson random measure.

By applying (3.14) to Xa , Xb and $X(a+b)$ for arbitrary $a, b \in U^*$ we obtain

$$r_{a+b}W_{a+b} = r_aW_a + r_bW_b \quad P\text{-a.s.} \quad (3.15)$$

Similarly, for $t \in \mathbb{R}$ we have

$$r_{ta}W_{ta} = tr_aW_a \quad P\text{-a.s.} \quad (3.16)$$

By squaring both sides of (3.16) and then taking expectations it follows that the function q satisfies $q(ta) = t^2q(a)$. Similarly, one derives from (3.15) that $r_{a+b}^2 = r_a^2 + r_b^2 + 2\rho(a, b)$, where $\rho(a, b) := \text{Cov}(r_aW_a, r_bW_b)$. Equation (3.15) yields for $c \in U^*$

$$\begin{aligned} \rho(a+c, b) &= \text{Cov}(r_{a+c}W_{a+c}, r_bW_b) \\ &= \text{Cov}(r_aW_a + r_cW_c, r_bW_b) \\ &= \rho(a, b) + \rho(c, b), \end{aligned}$$

which implies together with properties of the covariance that ρ is a bilinear form. Thus, the function

$$F : U^* \times U^* \rightarrow \mathbb{R}, \quad F(a, b) := q(a+b) - q(a) - q(b) = 2\rho(a, b)$$

is a bilinear form and q is verified as a quadratic form. \square

Definition 3.4. A cylindrical measure $\nu : \mathcal{Z}(U) \rightarrow [0, \infty]$ is called a cylindrical Lévy measure if $\nu \circ a^{-1}$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ for all $a \in U^*$.

From (3.9) we can easily derive a representation of the characteristic function φ_μ of an infinitely divisible cylindrical probability measure μ for an arbitrary truncation function h . Since $h = \text{Id}$ on $D(h)$ one can define

$$p : U^* \rightarrow \mathbb{R}, \quad p(a) := w(a) + \int_U (h(\langle u, a \rangle) - \langle u, a \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, a \rangle)) \nu(du).$$

It follows from (3.9) that

$$\varphi_\mu(a) = \exp \left(ip(a) - \frac{1}{2}q(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right), \quad (3.17)$$

where the kernel function ψ_h is defined by

$$\psi_h : \mathbb{R} \rightarrow \mathbb{C}, \quad \psi_h(t) := e^{it} - 1 - ih(t)$$

for an arbitrary truncation function h .

Definition 3.5. Let h be an truncation function and let μ be an infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with characteristic function (3.17). Then we call the triplet $(p, q, \nu)_h$ the cylindrical characteristics of μ .

Analogously to the one-dimensional situation after Definition 3.1 one can convert the cylindrical characteristics $(p, s, \nu)_h$ into $(p', s, \nu)_{h'}$ if h' is another truncation function. It follows from (3.17) that the characteristic function $\varphi_{\mu \circ a^{-1}}$ of the probability measure $\mu \circ a^{-1}$ on $\mathcal{B}(\mathbb{R})$ is for all $t \in \mathbb{R}$ given by

$$\begin{aligned} \varphi_{\mu \circ a^{-1}}(t) &= \varphi_{\mu}(at) \\ &= \exp \left(ip(at) - \frac{1}{2}q(a)t^2 + \int_{\mathbb{R}} \psi_h(st) (\nu \circ a^{-1})(ds) \right). \end{aligned} \quad (3.18)$$

This representation of $\varphi_{\mu \circ a^{-1}}$ does not coincide with the representation (3.4) because the functions $\widetilde{\psi}_h$ and ψ_h do not coincide. Thus, we can not read out directly the characteristics of $\mu \circ a^{-1}$ from (3.18).

Lemma 3.6. Let μ be a cylindrical probability measure on $\mathcal{Z}(U)$ which satisfies (3.6) and has therefore the characteristic function $\varphi_{\mu} : U^* \rightarrow \mathbb{C}$,

$$\varphi_{\mu}(a) = \exp \left(ip(a) - \frac{1}{2}q(a) + \int_U \left(e^{i\langle u, a \rangle} - 1 - i\langle u, a \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle u, a \rangle) \right) \nu(du) \right).$$

Then $\mu \circ a^{-1}$ has the characteristics $(p(a), q(a), \nu \circ a^{-1})_h$ for all $a \in U^*$.

Proof. As above we can rewrite the characteristic function $\varphi_{\mu \circ a^{-1}}$ in the form (3.18) for the given truncation function h . In order to write $\varphi_{\mu \circ a^{-1}}$ in the standard form (3.4), we introduce the function $\widetilde{p} : U^* \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\widetilde{p}(a, t) := \begin{cases} p(at) + \int_{\mathbb{R}} (t h(s) - h(st)) (\nu \circ a^{-1})(ds), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Note, that the integral is well defined because for each $t \neq 0$ we have

$$t h(s) - h(st) = 0 \quad \text{for all } s \in D(h) \cap \frac{1}{t}D(h)$$

and because h is bounded. By defining the function

$$\widetilde{\psi}_h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}, \quad \widetilde{\psi}_h(s, t) = e^{ist} - 1 - it h(s),$$

we can rewrite the characteristic function (3.18) of $\mu \circ a^{-1}$ for all $t \in \mathbb{R}$:

$$\varphi_{\mu \circ a^{-1}}(t) = \exp \left(i\widetilde{p}(a, t) - \frac{1}{2}q(a)t^2 + \int_{\mathbb{R}} \widetilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds) \right).$$

By Theorem 3.3 the Lévy measure of the infinitely divisible probability measure $\mu \circ a^{-1}$ is given by $\nu \circ a^{-1}$ for each $a \in U^*$. Thus, there exist some constants

$m_a \in \mathbb{R}$ and $r_a \geq 0$ such that $(m_a, r_a, \nu \circ a^{-1})_h$ is the characteristics of $\mu \circ a^{-1}$. For all $t \in \mathbb{R}$ it follows that

$$\begin{aligned}\varphi_{\mu \circ a^{-1}}(t) &= \exp \left(i\tilde{p}(a, t) - \frac{1}{2}q(a)t^2 + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds) \right) \\ &= \exp \left(im_a t - \frac{1}{2}r_a^2 t^2 + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds) \right),\end{aligned}$$

which results in $\tilde{p}(a, t) = m_a t = \tilde{p}(a, 1)t = p(a)t$. Consequently, we have

$$\varphi_{\mu \circ a^{-1}}(t) = \exp \left(ip(a)t - \frac{1}{2}q(a)t^2 + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds) \right),$$

which completes the proof. \square

Recalling the Lévy-Khintchine decomposition for infinitely divisible measures on Banach spaces we could expect from (3.17) that

$$a \mapsto \exp(ip(a)), \quad a \mapsto \exp \left(\int_U \psi_h(\langle u, a \rangle) \nu(du) \right)$$

are characteristic functions of cylindrical measures on $\mathcal{Z}(U)$, respectively. But the following example shows that we can not separate the drift part p and the cylindrical Lévy measure ν in order to obtain cylindrical measures.

Example 3.7. Let $\ell : U^* \rightarrow \mathbb{R}$ be a linear but not necessarily a continuous functional and $\lambda > 0$ a constant. We will see later in Example 3.10 that

$$\varphi : U^* \rightarrow \mathbb{C}, \quad \varphi(a) := \exp \left(\lambda \left(e^{i\ell(a)} - 1 \right) \right)$$

is the characteristic function of an infinitely divisible cylindrical probability measure. In order to write φ in the form (3.17) let ν be the cylindrical measure on $\mathcal{Z}(U)$ defined by

$$\nu(Z(a_1, \dots, a_n; B)) := \begin{cases} \lambda, & \text{if } (\ell(a_1), \dots, \ell(a_n)) \in B, \\ 0, & \text{else,} \end{cases}$$

for every $a_1, \dots, a_n \in U^*$, $B \in \mathcal{B}(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Then we can represent φ by

$$\varphi(a) = \exp \left(ip(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right),$$

where $p(a) := \lambda h(\ell(a))$. Since $a \mapsto \exp(ip(a))$ is not positive-definite in general there does not exist a cylindrical measure with this function as its characteristic function.

The Example 3.7 leads us to the insight that some necessary conditions guaranteeing the existence of an infinitely divisible cylindrical probability measure with cylindrical characteristics $(p, 0, \nu)$ rely on the interplay of the entries p and ν . The following result gives some properties of the entries p , q and ν of the cylindrical characteristics, respectively, but also the interplay of p and ν . For later purpose, we formulate the result under the weaker condition (3.6).

Lemma 3.8. *Let μ be a cylindrical probability measure on $\mathcal{Z}(U)$ which satisfies (3.6) and has therefore the characteristic function $\varphi_\mu : U^* \rightarrow \mathbb{C}$,*

$$\varphi_\mu : U^* \rightarrow \mathbb{C}, \quad \varphi_\mu(a) = \exp \left(ip(a) - \frac{1}{2}q(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right).$$

If h is a continuous truncation function it follows that:

- (a) $a \mapsto \kappa(a) := - \left(ip(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right)$ is negative-definite.
- (b) for every sequence $a_n \rightarrow a$ in a finite dimensional subspace $V \subseteq U^*$ equipped with $\|\cdot\|_{U^*}$ we have:
 - (i) $p(a_n) \rightarrow p(a)$;
 - (ii) $q(a_n) \rightarrow q(a)$.
 - (iii) $\left(|s|^2 \wedge 1 \right) (\nu \circ a_n^{-1})(ds) \rightarrow \left(|s|^2 \wedge 1 \right) (\nu \circ a^{-1})(ds)$ weakly;

Proof. (a): We define the functions $\varphi_i : U^* \rightarrow \mathbb{C}$, $i = 1, 2$ by

$$\varphi_1(a) := \exp(-\frac{1}{2}q(a)), \quad \varphi_2(a) := \exp(-\kappa(a)).$$

Since q is a quadratic form the function φ_1 is the characteristic function of a cylindrical probability measure μ_1 , see [20, p.393]. Since $\varphi_\mu(\cdot) = \varphi_1(\cdot) \cdot \varphi_2(\cdot)$ and both φ_μ and φ_1 are characteristic functions of cylindrical probability measures, it follows that φ_2 is also the characteristic function of a cylindrical probability measure, say μ_2 . For fixed $a_1, \dots, a_n \in U^*$ and $k \in \mathbb{N}$ define the functions

$$\begin{aligned} \varphi_{a_1, \dots, a_n} : \mathbb{R}^n &\rightarrow \mathbb{C}, & \varphi_{a_1, \dots, a_n}(t) &:= \left(\varphi_{\mu \circ \pi_{a_1^{-1}, \dots, a_n}^{-1}}(t) \right)^{1/k} \\ \varphi_{a_1, \dots, a_n}^{(1)} : \mathbb{R}^n &\rightarrow \mathbb{C}, & \varphi_{a_1, \dots, a_n}^{(1)}(t) &:= \left(\varphi_{\mu_1 \circ \pi_{a_1^{-1}, \dots, a_n}^{-1}}(t) \right)^{1/k} \\ \varphi_{a_1, \dots, a_n}^{(2)} : \mathbb{R}^n &\rightarrow \mathbb{C}, & \varphi_{a_1, \dots, a_n}^{(2)}(t) &:= \left(\varphi_{\mu_2 \circ \pi_{a_1^{-1}, \dots, a_n}^{-1}}(t) \right)^{1/k}. \end{aligned}$$

It is easy to see that $\mu_1 \circ \pi_{a_1^{-1}, \dots, a_n}^{-1}$ is a Gaussian measure on $\mathcal{B}(\mathbb{R}^n)$ and thus, it follows that $\varphi_{a_1, \dots, a_n}^{(1)}$ is the characteristic function of a probability measure on $\mathcal{B}(\mathbb{R}^n)$. By condition (3.6) the function $\varphi_{a_1, \dots, a_n}$ is the characteristic function of a probability measure on $\mathcal{B}(\mathbb{R}^n)$ and because

$$\varphi_{a_1, \dots, a_n}(\cdot) = \varphi_{a_1, \dots, a_n}^{(1)}(\cdot) \varphi_{a_1, \dots, a_n}^{(2)}(\cdot)$$

it follows that also $\varphi_{a_1, \dots, a_n}^{(2)}$ is the characteristic function of a probability measure on $\mathcal{B}(\mathbb{R}^n)$. Thus, the cylindrical measure μ_2 satisfies (3.6) and has the characteristic function $\varphi_2(a) = \exp(-\kappa(a))$. Consequently, we can assume in the following that $q = 0$.

With this assumption we show (a) by applying Schoenberg's correspondence, see [20, Property(h), p.192], for which we have to show that $a \mapsto \exp(-\frac{1}{k}\kappa(a))$ is positive-definite for all $k \in \mathbb{N}$ and that κ is Hermitian, i.e. $\overline{\kappa(-a)} = \kappa(a)$ for all $a \in U^*$. To prove positive-definiteness, fix $k \in \mathbb{N}$, $a_1, \dots, a_n \in U^*$ and $z_1, \dots, z_n \in \mathbb{C}$ and let e_i denote the i -th unit vector in \mathbb{R}^n . By (3.7) there exists a characteristic function $\varphi_{\xi_{k, a_1, \dots, a_n}}$ of a probability measure ξ_{k, a_1, \dots, a_n} on $\mathcal{B}(\mathbb{R}^n)$ such that

$$\varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}(t) = \left(\varphi_{\xi_{k, a_1, \dots, a_n}}(t) \right)^k \quad \text{for all } t \in \mathbb{R}^n.$$

Consequently, we have

$$\begin{aligned} \sum_{i,j=1}^n z_i \bar{z}_j \exp\left(-\frac{1}{k}\kappa(a_i - a_j)\right) &= \sum_{i,j=1}^n z_i \bar{z}_j (\varphi_{\mu}(a_i - a_j))^{1/k} \\ &= \sum_{i,j=1}^n z_i \bar{z}_j \left(\varphi_{\mu \circ \pi_{a_1, \dots, a_n}^{-1}}(e_i - e_j) \right)^{1/k} \\ &= \sum_{i,j=1}^n z_i \bar{z}_j \varphi_{\xi_{k, a_1, \dots, a_n}}(e_i - e_j) \\ &\geq 0, \end{aligned}$$

where the last line follows from the fact that $\varphi_{\xi_{k, a_1, \dots, a_n}}$ is a characteristic function on \mathbb{R}^n .

Next, we want show that κ is Hermitian. Since rewriting the characteristic function for different truncation functions does not effect the function κ we can fix $h(s) = s \mathbb{1}_{B_{\mathbb{R}}}(s)$ for $s \in \mathbb{R}$ which yields $\tilde{\psi}_h(-s, t) = \tilde{\psi}_h(s, -t)$ for all $s, t \in \mathbb{R}$. By Lemma 3.6 we obtain for all $t \in \mathbb{R}$ that

$$\begin{aligned} \varphi_{\mu \circ a^{-1}}(t) &= \varphi_{\mu \circ (-a)^{-1}}(-t) \\ &= \exp\left(ip(-a)(-t) + \int_{\mathbb{R}} \tilde{\psi}_h(s, -t) (\nu \circ (-a)^{-1})(ds)\right) \\ &= \exp\left(ip(-a)(-t) + \int_{\mathbb{R}} \tilde{\psi}_h(-s, -t) (\nu \circ a^{-1})(ds)\right) \\ &= \exp\left(ip(-a)(-t) + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds)\right), \end{aligned}$$

which implies $p(-a) = -p(a)$. It follows that

$$\begin{aligned}\overline{\kappa(-a)} &= \overline{-ip(-a)} - \int_U \overline{\psi_h(\langle u, -a \rangle)} \nu(du) \\ &= -ip(a) - \int_U \psi_h(\langle u, a \rangle) \nu(du) \\ &= \kappa(a)\end{aligned}$$

for all $a \in U^*$, which completes the proof of (a).

To see (b) let $a_n \rightarrow a$ in a finite-dimensional subspace $V \subseteq U^*$ and let the truncation function h be continuous. Then Bochner's theorem implies that

$$\lim_{n \rightarrow \infty} \varphi_{\mu \circ a_n^{-1}}(t) = \lim_{n \rightarrow \infty} \varphi_\mu(ta_n) = \varphi_\mu(ta) = \varphi_{\mu \circ a^{-1}}(t) \quad (3.19)$$

for all $t \in \mathbb{R}$. By Lemma 3.6 the measures $\mu \circ a_n^{-1}$ are infinitely divisible with characteristics $(p(a_n), q(a_n), \nu \circ a_n^{-1})$. It follows from (3.19) that the infinitely divisible measures with characteristics $(p(a_n), q(a_n), \nu \circ a_n^{-1})$ converge weakly to $\mu \circ a^{-1}$ which has the characteristics $(p(a), q(a), \nu \circ a^{-1})$. Applying Theorem VII.2.9 and Remark VII.2.10 (p.396) in Jacod and Shiryaev which characterise the weak convergence of infinitely divisible measures in terms of their characteristics implies $p(a_n) \rightarrow p(a)$ and

$$\begin{aligned}q(a_n) \delta_0(ds) + (|s|^2 \wedge 1) (\nu \circ a_n^{-1})(ds) \\ \rightarrow q(a) \delta_0(ds) + (|s|^2 \wedge 1) (\nu \circ a^{-1})(ds) \quad \text{weakly.}\end{aligned}$$

But since q is a quadratic form and V is a finite-dimensional space we have $q(a_n) \rightarrow q(a)$ which is property (ii) and which yields (iii). □

Theorem 3.9. *Let $\nu : \mathcal{Z}(U) \rightarrow [0, \infty]$ be a set function and $p, q : U^* \rightarrow \mathbb{R}$ some functions and let h be a continuous truncation function. Then the following are equivalent:*

- (a) *there exists an infinitely divisible cylindrical probability measure μ with cylindrical characteristics $(p, q, \nu)_h$;*
- (b) *the following is satisfied:*
 - (1) $p(0) = 0$ and $p(a_n) \rightarrow p(a)$ for every sequence $a_n \rightarrow a$ in a finite dimensional subspace $V \subseteq U^*$ equipped with $\|\cdot\|_{U^*}$;
 - (2) $q : U^* \rightarrow \mathbb{R}$ is a quadratic form;
 - (3) ν is a cylindrical Lévy measure;
 - (4) $a \mapsto \kappa(a) := -\left(ip(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du)\right)$ is negative-definite.

In this situation, the characteristic function of μ is given by

$$\varphi_\mu : U^* \rightarrow \mathbb{C}, \quad \varphi_\mu(a) = \exp \left(ip(a) - \frac{1}{2}q(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right)$$

and $\mu = \mu_1 * \mu_2$ where μ_1 and μ_2 are cylindrical probability measures with characteristic functions $\varphi_{\mu_1}(a) = \exp(-\frac{1}{2}q(a))$ and $\varphi_{\mu_2}(a) = \exp(-\kappa(a))$.

Proof. (a) \Rightarrow (b): The properties (2) and (3) are stated in Theorem 3.3 and the properties (1) and (4) are derived in Lemma 3.8. The property $p(0) = 0$ is an immediate consequence of Bochner's theorem and the fact that $q(0) = 0$.

(b) \Rightarrow (a): Property (2) implies that

$$\varphi_q : U^* \rightarrow \mathbb{C}, \quad \varphi_q(a) := e^{-\frac{1}{2}q(a)}$$

is the characteristic function of a Gaussian cylindrical probability measure μ_1 , see [16] or [20, p.393]. Since also $\frac{1}{k}q$ is a quadratic form for every $k \in \mathbb{N}$ it follows that $(\varphi_q)^{1/k}$ is the characteristic function of a cylindrical measure which verifies μ_1 as infinitely divisible. Thus, we can assume $q = 0$ in the sequel.

We have to show that the functions

$$\varphi_k : U^* \rightarrow \mathbb{C}, \quad \varphi_k(a) := \exp \left(\frac{1}{k} \left(ip(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du) \right) \right)$$

are the characteristic function of a cylindrical probability measure for each $k \in \mathbb{N}$. The case $k = 1$ shows that there exists a cylindrical measure μ with characteristic function φ_1 and the cases $k \geq 1$ show that μ is infinitely divisible. Note firstly, that the integral in the definition of φ_k exists and is finite because of condition (3).

Obviously, $\varphi_k(0) = 1$ by (1). Property (4) implies by the Schoenberg correspondence for functions on Banach spaces (property (h), p.192 in Vakhania et al) that φ_k is positive-definite. In order to show the last condition of Bochner's theorem let $V \subseteq U^*$ be a finite-dimensional subspace, say $V = \text{span}\{b_1, \dots, b_d\}$ for $b_1, \dots, b_d \in U^*$ and $a_n \rightarrow a_0$ in V . Then $(U, \mathcal{Z}(U, \{b_1, \dots, b_d\}), \nu)$ is a standard measure space. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function and define

$$g_n : U \rightarrow \mathbb{R}, \quad g_n(u) := f(\langle u, a_n \rangle) \left(|\langle u, a_n \rangle|^2 \wedge 1 \right)$$

for $n \in \mathbb{N} \cup \{0\}$. It follows by (3) that each $g_n \in L^1_\nu(U, \mathcal{Z}(U, \{b_1, \dots, b_d\}))$ and

$$|g_n(u)| \leq \|f\|_\infty (1 + c) \left(|\langle u, a_0 \rangle|^2 \wedge 1 \right)$$

for a constant $c > 0$. Lebesgue's theorem of dominated convergence implies that

$$\lim_{n \rightarrow \infty} \int_U g_n(u) \nu(du) = \int_U g_0(u) \nu(du),$$

which shows that

$$\left(|s|^2 \wedge 1\right) (\nu \circ a_n^{-1})(ds) \rightarrow \left(|s|^2 \wedge 1\right) (\nu \circ a^{-1})(ds) \text{ weakly.} \quad (3.20)$$

Condition (3) guarantees for each $a \in U^*$, that

$$\varphi_{\mu_a} : \mathbb{R} \rightarrow \mathbb{C}, \quad \varphi_{\mu_a}(t) = \exp \left(ip(a)t + \int_{\mathbb{R}} \tilde{\psi}_h(s, t) (\nu \circ a^{-1})(ds) \right)$$

is the characteristic function of an infinitely divisible probability measure, say μ_a on $\mathcal{B}(\mathbb{R})$ with characteristics $(p(a), 0, \nu \circ a^{-1})_h$. Then condition (1) together with the weak convergence in (3.20) imply by Theorem VII.2.9 and Remark VII.2.10 (p.396) in [8] that $\varphi_{\mu_{a_n}}(t) \rightarrow \varphi_{\mu_{a_0}}(t)$ for all $t \in \mathbb{R}$. Because $\varphi_k(a) = (\varphi_{\mu_a}(1))^k$ for all $a \in U^*$ and $k \in \mathbb{N}$ the functions φ_k are verified as continuous on every finite-dimensional subspace which is the last condition in Bochner's theorem.

The remaining part on the form of the characteristic function is derived in Theorem 3.3. \square

Example 3.10. Now we can show that the function φ in Example 3.7 is in fact the characteristic function of an infinitely divisible cylindrical measure. The linearity of ℓ yields that the mapping $a \mapsto p(a) = \lambda h(\ell(a))$ is continuous on each finite dimensional subspace of U^* if the truncation function h is continuous. The measure ν satisfies $\nu \circ a^{-1} = \delta_{\ell(a)}$ for each $a \in U^*$ and is therefore a cylindrical Lévy measure. Since (3.5) yields that

$$f : \mathbb{R} \rightarrow \mathbb{C}, \quad f(t) := -\lambda (e^{it} - 1)$$

is a negative-definite function it follows for $z_1, \dots, z_n \in \mathbb{C}$, $a_1, \dots, a_n \in U^*$ that

$$\sum_{i,j=1}^n z_i \bar{z}_j \kappa(a_i - a_j) = \sum_{i,j=1}^n z_i \bar{z}_j f(\ell(a_i) - \ell(a_j)) \leq 0.$$

Thus, the map κ is negative-definite which proves the claim due to Theorem 3.9.

For a given cylindrical Lévy measure ν there does not exist in general an infinitely divisible cylindrical probability measure with cylindrical characteristics $(0, 0, \nu)$. But one might be able to construct a function $p : U^* \rightarrow \mathbb{R}$ such that there exists a cylindrical probability measure with cylindrical characteristics $(p, 0, \nu)$.

The following example shows the construction of the function p for a given cylindrical Lévy measure ν with weak second moments. In Section 5 we consider the case if the cylindrical Lévy measure extends to a σ -finite measure on $\mathcal{B}(U)$.

Example 3.11. Let ν be a cylindrical Lévy measure which satisfies

$$\int_U |\langle u, a \rangle|^2 \nu(du) < \infty \quad \text{for all } a \in U^*.$$

The existence of the weak second moments enables us to define

$$p : U^* \rightarrow \mathbb{R}, \quad p(a) := \int_U (h(\langle u, a \rangle) - \langle u, a \rangle) \nu(du).$$

for a continuous truncation function h . With a careful analysis similar to the one in the proof of Theorem 5.1 it can be shown that p is continuous on every finite-dimensional subspace of U^* . From (3.5) it follows that

$$f : \mathbb{R} \rightarrow \mathbb{C} \quad f(t) := -(e^{it} - 1 - it)$$

is negative-definite. For $z_1, \dots, z_n \in \mathbb{C}$, $a_1, \dots, a_n \in U^*$ we have:

$$\begin{aligned} & \sum_{i,j=1}^n -z_i \bar{z}_j \left(ip(a_i - a_j) + \int_U \psi_h(\langle u, a_i - a_j \rangle) \nu(du) \right) \\ &= \int_U \sum_{i,j=1}^n z_i \bar{z}_j f(\langle u, a_i \rangle - \langle u, a_j \rangle) \nu(du) \leq 0. \end{aligned}$$

Theorem 3.9 shows that there exists an infinitely divisible cylindrical measure with cylindrical characteristics $(p, 0, \nu)_h$.

We finish this section with characterising infinitely divisible cylindrical probability measures by their finite-dimensional projections. This result enables us to derive the analogue result for infinitely divisible Radon probability measures.

Theorem 3.12.

- (a) *A set function $\mu : \mathcal{Z}(U) \rightarrow [0, \infty]$ is an infinitely divisible cylindrical probability measure if and only if*

$$\begin{aligned} & \mu \circ \pi_{a_1, \dots, a_n}^{-1} \text{ is an infinitely divisible probability measure for all} \\ & a_1, \dots, a_n \in U^*, n \in \mathbb{N}. \end{aligned}$$

- (b) *A Radon probability measure μ on $\mathcal{B}(U)$ is infinitely divisible if and only if*

$$\begin{aligned} & \mu \circ \pi_{a_1, \dots, a_n}^{-1} \text{ is an infinitely divisible probability measure for all} \\ & a_1, \dots, a_n \in U^*, n \in \mathbb{N}. \end{aligned}$$

Proof. (a) Since all image measures $\mu \circ \pi_{a_1, \dots, a_n}^{-1}$ are probability measures on $\mathcal{B}(\mathbb{R}^n)$ it follows that μ is a cylindrical probability measure. Theorem 3.3 and Lemma 3.8 guarantee that the characteristic function of μ satisfies the conditions in Theorem 3.9.

(b) For the restriction of μ to $\mathcal{Z}(U)$ it follows from (a) that for each $k \in \mathbb{N}$ there exists a cylindrical probability measure μ_k such that $\mu = \mu_k^{*k}$. Theorem 1 in [17]

implies that there exists ℓ in the algebraic dual $U^{*'} of U^* such that $\mu_k * \delta_\ell$ is a Radon probability measure where$

$$\delta_\ell(Z) := \begin{cases} 1, & \text{if } (\ell(a_1), \dots, \ell(a_n)) \in B, \\ 0, & \text{otherwise} \end{cases}$$

for every $Z := Z(a_1, \dots, a_n; B) \in \mathcal{Z}(U)$. Since

$$\mu * \delta_\ell^{*k} = \mu_k^{*k} * \delta_\ell^{*k} = (\mu_k * \delta_\ell)^{*k}$$

and the right hand side is Radon it follows from [2, Prop.7.14.50] that δ_ℓ^{*k} is a Radon probability measure which yields $\ell \in U$. Since $\mu_k * \delta_\ell$ is Radon a further application of [2, Prop.7.14.50] shows that μ_k is a Radon probability measure and μ is verified as an infinitely divisible Radon measure. \square

4 Continuous infinitely divisible cylindrical measures

Continuity of cylindrical measures is defined with respect to an arbitrary vector topology \mathcal{O} in U^* . We assume here that the topological space (U^*, \mathcal{O}) satisfies the first countability axiom, that is that every neighborhood system of every point in U^* has a countable local base. In such spaces, convergence is equivalent to sequentially convergent. In particular, U^* equipped with the norm topology satisfies the first countability axiom.

Definition 4.1. *A cylindrical probability measure μ on $\mathcal{Z}(U)$ is called \mathcal{O} -continuous if for each $\varepsilon > 0$ there exists a neighborhood N of 0 such that*

$$\mu(\{u \in U : |\langle u, a \rangle| \geq 1\}) \leq \varepsilon$$

for all $a \in N$. If \mathcal{O} is the norm topology we say μ is continuous.

A cylindrical probability measure μ is \mathcal{O} -continuous if and only if its characteristic function $\varphi_\mu : U^* \rightarrow \mathbb{C}$ is continuous in the topology \mathcal{O} , see [19, Th.II.3.1]. This enables us to derive the following criteria:

Lemma 4.2. *Let μ be an infinitely divisible cylindrical probability measure on $\mathcal{Z}(U)$ with cylindrical characteristics $(p, q, \nu)_h$ for a continuous truncation function h . Then the following are equivalent:*

- (a) μ is \mathcal{O} -continuous;
- (b) for every sequence $a_n \rightarrow a$ in (U^*, \mathcal{O}) we have:
 - (i) $p(a_n) \rightarrow p(a)$;
 - (ii) $q(a_n)\delta_0(ds) + \left(|s|^2 \wedge 1\right) (\nu \circ a_n^{-1})(ds) \rightarrow q(a)\delta_0(ds) + \left(|s|^2 \wedge 1\right) (\nu \circ a^{-1})(ds)$ weakly.

Proof. The cylindrical measure μ is \mathcal{O} -continuous if and only if its characteristic function $\varphi_\mu : U^* \rightarrow \mathbb{C}$ is continuous in (U^*, \mathcal{O}) , or equivalently, that $\varphi_\mu : U^* \rightarrow \mathbb{C}$ is sequentially continuous. It follows as in the proof of Theorem 3.9 that:

$$\begin{aligned} \varphi_\mu(a_n) &\rightarrow \varphi_\mu(a) \quad \text{for all sequences } a_n \rightarrow a \text{ in } (U^*, \mathcal{O}) \\ \iff \varphi_{\mu \circ a_n^{-1}}(t) &\rightarrow \varphi_{\mu \circ a^{-1}}(t) \quad \text{for all sequences } a_n \rightarrow a \text{ in } (U^*, \mathcal{O}) \text{ and } t \in \mathbb{R}. \end{aligned}$$

By applying Theorem VII.2.9 and Remark VII.2.10 (p.396) in [8] and Lemma 3.6 the right hand side is equivalent to the conditions (i) and (ii) in (b) which completes the proof. \square

In Corollary 4.2 it does not follow from (a) that we can consider separately the quadratic form q and the cylindrical Lévy measure ν in condition (b). This is due to the well known fact, that a sequence of infinitely divisible measures on $\mathcal{B}(\mathbb{R})$ can converge weakly such that the small jumps contribute to the Gaussian part in the limit. But because of the representation of an infinitely divisible cylindrical measure as the convolution of two cylindrical probability measures according to Theorem 3.9 this is an important property which motivates the following definition:

Definition 4.3. *An infinitely divisible cylindrical probability measure with cylindrical characteristics $(p, q, \nu)_h$ is called regularly \mathcal{O} -continuous if the infinitely divisible cylindrical measures with cylindrical characteristics $(0, q, 0)_h$ and $(p, 0, \nu)_h$ are \mathcal{O} -continuous.*

Lemma 4.4. *Let h be a continuous truncation function. For an \mathcal{O} -continuous infinitely divisible cylindrical probability measure μ with cylindrical characteristics $(p, q, \nu)_h$ the following are equivalent:*

- (a) μ is regularly \mathcal{O} -continuous;
- (b) $q : U^* \rightarrow \mathbb{R}$ is continuous in (U^*, \mathcal{O}) ;
- (c) for every sequence $a_n \rightarrow a$ in (U^*, \mathcal{O}) we have:

- (i) $p(a_n) \rightarrow p(a)$;
- (ii) $\left(|s|^2 \wedge 1\right) (\nu \circ a_n^{-1})(ds) \rightarrow \left(|s|^2 \wedge 1\right) (\nu \circ a^{-1})(ds)$ weakly.

Proof. Let φ_μ be the characteristic function of μ . Then $\varphi_\mu = \varphi_1 \cdot \varphi_2$ where φ_1 is the characteristic function of the cylindrical measure μ_1 with cylindrical characteristics $(0, q, 0)_h$ and φ_2 is the characteristic function of the cylindrical measure μ_2 with cylindrical characteristics $(p, 0, \nu)_h$. Since the characteristic function of an infinitely divisible measure does not vanish in any point it follows that continuity of φ_1 and φ_μ results in the continuity of φ_2 and analogously if φ_2 is continuous. Thus, μ is regularly \mathcal{O} -continuous if and only if either μ_1 or μ_2 is \mathcal{O} -continuous.

Applying Lemma 4.2 to μ_1 shows the equivalence (a) \Leftrightarrow (b) and applying Lemma 4.2 to μ_2 shows the equivalence (a) \Leftrightarrow (c). \square

Remark 4.5. If U^* is equipped with the norm topology then (b) in Lemma 4.4 can be replaced by

(b') there exists a positive, symmetric operator $Q : U^* \rightarrow U^{**}$ such that $q(a) = \langle a, Qa \rangle$ for all $a \in U^*$;

Proof. According to Proposition IV.4.2 in [20] there exist a probability space (Ω, \mathcal{A}, P) and a cylindrical random variable $X : U^* \rightarrow L_P^0(\Omega, \mathcal{A})$ with cylindrical distribution $(0, q, 0)$ and with characteristic function $a \mapsto \varphi(a) = \exp(-\frac{1}{2}q(a))$. If q is continuous Proposition VI.5.1 in [20] implies that the mapping $X : U^* \rightarrow L_P^0(\Omega, \mathcal{A})$ is continuous. It follows from Theorem 4.7 in [1] that $(Qa)b := E[(Xa)(Xb)]$ for $a, b \in U^*$ defines a positive, symmetric operator $Q : U^* \rightarrow U^{**}$. Obviously, it satisfies $q(a) = (Qa)a$ for each $a \in U^*$. \square

Example 4.6. Let (Ω, \mathcal{A}, P) be a probability space and let $L := (L(t) : t \geq 0)$ be a cylindrical process, that is the mappings $L(t) : U^* \rightarrow L_P^0(\Omega, \mathcal{A})$ are linear. In Applebaum and Riedle [1] we call L a *cylindrical Lévy process* if

$$((L(t)a_1, \dots, L(t)a_n) : t \geq 0)$$

is a Lévy process in \mathbb{R}^n for all $a_1, \dots, a_n \in U^*$, $n \in \mathbb{N}$. If L is a cylindrical Lévy process we derive in [1] that it can be decomposed according to

$$L(t) = W(t) + Y(t) \quad \text{for all } t \geq 0 \text{ } P\text{-a.s.},$$

where $(W(t) : t \geq 0)$ and $(Y(t) : t \geq 0)$ are cylindrical processes. Their characteristic functions are for all $a \in U^*$ given by

$$\varphi_{W(t)}(a) := E[\exp(iW(t)a)] = \exp\left(-\frac{1}{2}q(a)t\right)$$

for a quadratic form $q : U^* \rightarrow \mathbb{R}$ and

$$\varphi_{Y(t)}(a) := E[\exp(iY(t)a)] = \exp\left(t\left(ip(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du)\right)\right)$$

for a mapping $p : U^* \rightarrow \mathbb{R}$ and a cylindrical Lévy measure ν . Obviously, (p, q, ν) is a cylindrical characteristics. If the corresponding cylindrical measure is regularly continuous, i.e. the cylindrical measures with the characteristic functions $\varphi_{W(1)}$ and $\varphi_{L(1)}$ are continuous, it follows that also the mappings

$$W(t) : U^* \rightarrow L_P^0(\Omega, \mathcal{A}), \quad Y(t) : U^* \rightarrow L_P^0(\Omega, \mathcal{A}),$$

are continuous, see [20, Prop.VI.5.1]. Moreover, according to Remark 4.5 the quadratic form q is of the form $q(a) = \langle a, Qa \rangle$ for all $a \in U^*$ and for a symmetric, positive operator $Q : U^* \rightarrow U^{**}$. If $Q(U^*) \subseteq U$ then W is a cylindrical Wiener process in a strong sense, which is usually referred to in the literature, see Riedle [16].

5 Lévy measures on Banach spaces

In this section we consider the situation that the cylindrical Lévy measure ν extends to a σ -finite measure on $\mathcal{B}(U)$ which is also denoted by ν .

Theorem 5.1. *Let ν be a cylindrical Lévy measure which extends to a σ -finite measure on $\mathcal{B}(U)$ with $\nu(B_U^c) < \infty$. Then there exists a regularly continuous infinitely divisible cylindrical probability measure μ with cylindrical characteristics $(d_\nu, 0, \nu)_h$, where*

$$d_\nu : U^* \rightarrow \mathbb{R}, \quad d_\nu(a) := \int_U (h(\langle u, a \rangle) - \langle u, a \rangle \mathbb{1}_{B_U}(u)) \nu(du).$$

Proof. First we show that the integral in the definition of the function d_ν is well defined for every truncation function h . Choose a constant $c > 0$ such that

$$\{t \in \mathbb{R} : |t| \leq c\} \subseteq D(h)$$

and define for every $a \in U^*$ the set

$$D(a) := \{v \in U : |\langle v, a \rangle| \leq c\}.$$

For the integrand $f_a(u) := h(\langle u, a \rangle) - \langle u, a \rangle \mathbb{1}_{B_U}(u)$ it follows for every $u \in U$ that

$$f_a(u) \neq 0 \Rightarrow u \in (D(a) \cap B_U^c) \cup (D^c(a) \cap B_U) \cup (D^c(a) \cap B_U^c).$$

But on these three domains we obtain

$$\int_{D(a) \cap B_U^c} |f_a(u)| \nu(du) \leq \int_{B_U^c} c \nu(du) = c \nu(B_U^c) < \infty,$$

and

$$\begin{aligned} \int_{D^c(a) \cap B_U} |f_a(u)| \nu(du) &\leq \int_{c < |\langle u, a \rangle| \leq \|a\|} |h(\langle u, a \rangle) - \langle u, a \rangle| \nu(du) \\ &= \int_{c < |s| \leq \|a\|} |h(s) - s| (\nu \circ a^{-1})(ds) \\ &\leq (\|h\|_\infty + \|a\|) (\nu \circ a^{-1})(\{s \in \mathbb{R} : |s| > c\}) < \infty, \end{aligned}$$

because $\nu \circ a^{-1}$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ and

$$\int_{D^c(a) \cap B_U^c} |f_a(u)| \nu(du) \leq \|h\|_\infty \int_{B_U^c} \nu(du) = \|h\|_\infty \nu(B_U^c) < \infty.$$

Now we choose the truncation function h continuous and show by a similar decomposition that d_ν is continuous. Let $a_n \rightarrow a$ in U^* and choose a constant $c > 0$ such that

$$\{t \in \mathbb{R} : |t| \leq c + \varepsilon\} \subseteq D(h).$$

for some $\varepsilon > 0$. Let $D(a) = \{v \in U : |\langle v, a \rangle| \leq c\}$. Since for every $u \in B_U$ we have

$$|\langle u, a_n \rangle - \langle u, a \rangle| \leq \|a_n - a\|,$$

we can conclude that there exists $n_0 \in \mathbb{N}$ such that $u \in D(a) \cap B_U$ implies that $\langle u, a \rangle, \langle u, a_n \rangle \in D(h)$ for every $n \geq n_0$. Consequently, we have for $f_{a,n}(u) := h(\langle u, a_n \rangle) - h(\langle u, a \rangle) - (\langle u, a_n \rangle - \langle u, a \rangle) \mathbb{1}_B(u)$ and $n \geq n_0$ the implication:

$$f_{a,n}(u) \neq 0 \Rightarrow u \in (D(a) \cap B_U^c) \cup (D^c(a) \cap B_U) \cup (D^c(a) \cap B_U^c).$$

As above it can be shown that $f_{a,n}$ is dominated by an integrable function on all three sets and therefore, Lebesgue's theorem on dominated convergence shows that d_ν is continuous.

It follows for $h'(s) := s \mathbb{1}_{B_{\mathbb{R}}}(s)$ from (3.5) that

$$f : \mathbb{R} \rightarrow \mathbb{C} \quad f(s_0, t) := -\tilde{\psi}_{h'}(s_0, t)$$

is negative-definite for each $s_0 \in \mathbb{R}$. For $z_1, \dots, z_n \in \mathbb{C}$, $a_1, \dots, a_n \in U^*$ we have:

$$\begin{aligned} & \sum_{i,j=1}^n -z_i \bar{z}_j \left(i d_\nu(a_i - a_j) + \int_U \psi_h(\langle u, a_i - a_j \rangle) \nu(du) \right) \\ &= \sum_{i,j=1}^n -z_i \bar{z}_j \int_U \left(e^{i\langle u, a_i - a_j \rangle} - 1 - i\langle u, a_i - a_j \rangle \mathbb{1}_{B_U}(u) \right) \nu(du) \\ &= \sum_{i,j=1}^n -z_i \bar{z}_j \int_U \left(e^{i \frac{\langle u, a_i - a_j \rangle}{\|u\|} \|u\|} - 1 - i \frac{\langle u, a_i - a_j \rangle}{\|u\|} \|u\| \mathbb{1}_{B_{\mathbb{R}}}(\|u\|) \right) \nu(du) \\ &= \int_U \sum_{i,j=1}^n z_i \bar{z}_j f(\|u\|, \frac{1}{\|u\|} (\langle u, a_i \rangle - \langle u, a_j \rangle)) \nu(du) \leq 0. \end{aligned}$$

Theorem 3.9 implies that there exists an infinitely divisible cylindrical probability measure μ with cylindrical characteristics $(d_\nu, 0, \nu)_h$. In order to show that μ is continuous let $a_n \rightarrow a_0$ in U^* . For a bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ define

$$g_n : U \rightarrow \mathbb{R}, \quad g_n(u) := f(\langle u, a_n \rangle) \left(|\langle u, a_n \rangle|^2 \wedge 1 \right)$$

for $n \in \mathbb{N} \cup \{0\}$. It follows that each $g_n \in L_\nu^1(U, \mathcal{B}(U))$ and

$$|g_n(u)| \leq \|f\|_\infty (1 + c) \left(|\langle u, a_0 \rangle|^2 \wedge 1 \right)$$

for a constant $c > 0$. Lebesgue's theorem on dominated convergence implies that

$$\lim_{n \rightarrow \infty} \int_U g_n(u) \nu(du) = \int_U g_0(u) \nu(du),$$

which shows that

$$\left(|s|^2 \wedge 1\right) (\nu \circ a_n^{-1})(ds) \rightarrow \left(|s|^2 \wedge 1\right) (\nu \circ a_0^{-1})(ds) \text{ weakly.} \quad (5.21)$$

Lemma 4.2 implies that μ is continuous and thus μ is regular continuous by Lemma 4.4. \square

A cylindrical Lévy measure which extends to a σ -finite measure on $\mathcal{B}(U)$ is an obvious candidate to be a Lévy measure in the usual sense. We recall the definition from Linde [11]: a σ -finite measure ν on $\mathcal{B}(U)$ is called a *Lévy measure* if

- (a) $\int_U (\langle u, a \rangle^2 \wedge 1) \nu(du) < \infty$ for all $a \in U^*$;
- (b) there exists a Radon probability measure μ on $\mathcal{B}(U)$ with characteristic function

$$\varphi_\mu : U^* \rightarrow \mathbb{C}, \quad \varphi_\mu(a) = \exp \left(\int_U \left(e^{i\langle u, a \rangle} - 1 - i\langle u, a \rangle \mathbb{1}_{B_U}(u) \right) \nu(du) \right). \quad (5.22)$$

In fact, this is rather a result (Theorem 5.4.8) in Linde [11] than his definition. Note furthermore, that this definition includes already the requirement that a Radon probability measure on $\mathcal{B}(U)$ exists with the corresponding characteristic function. In general, there are no conditions known on a measure ν which guarantee that ν is a Lévy measure. In particular, the condition

$$\int_U (\|u\|^2 \wedge 1) \nu(du) < \infty$$

is sufficient and necessary in Hilbert spaces, but in general spaces it is even neither sufficient nor necessary, for example in the space of continuous functions on $[0, 1]$, see Araujo [3].

Corollary 5.2. *Let ν be a σ -finite measure on $\mathcal{B}(U)$ and h be a truncation function. Then the following are equivalent:*

- (a) ν is a Lévy measure;
- (b) there exists an infinitely divisible cylindrical probability measure μ with cylindrical characteristics $(d_\nu, 0, \nu)_h$ and it extends to a Radon measure on $\mathcal{B}(U)$.

In this situation, the Radon probability measure with characteristic function (5.22) corresponding to the Lévy measure ν coincides with the Radon extension of μ .

Proof. It is easily seen that the characteristic function of the cylindrical measure with cylindrical characteristics $(d_\nu, 0, \nu)_h$ is of the form (5.22). Consequently, (b) implies (a). If ν is a Lévy measure Proposition 5.4.5 in [11] guarantees that $\nu(B_U^c) < \infty$. Theorem 5.1 implies that there exists a cylindrical probability measure with cylindrical characteristics $(d_\nu, 0, \nu)_h$ which extends to a Radon probability measure because its characteristic function is of the form (5.22). \square

Remark 5.3. If ν is a Lévy measure and μ the infinitely divisible Radon probability measure with characteristic function (5.22) one might call the triplet $(0, 0, \nu)$ the characteristics of μ . However, according to Corollary 5.2 the measure μ considered as an infinitely divisible cylindrical probability measure has the cylindrical characteristics $(d_\nu, 0, \nu)_h$. This asymmetry illustrates the interaction of the components p and ν of the cylindrical characteristics $(p, 0, \nu)$ of an infinitely divisible cylindrical measure. Even if ν is a Lévy measure and $p = d_\nu$ then the function

$$a \mapsto \kappa(a) := - \left(\int_U \left(e^{-i\langle u, a \rangle} - 1 - i\langle u, a \rangle \mathbb{1}_{B_U}(u) \right) \nu(du) \right)$$

is negative-definite by Bochner's theorem and the Schoenmacher correspondence. But although

$$\kappa(a) = -(id_\nu(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du))$$

none of the both summands in this representation respectively are negative-definite in general.

In general, the condition (b) in Corollary 5.2 might be verified by applying Prohorov's theorem, [20, Th.VI.3.2], and proving that the cylindrical measure μ is tight. In Sazanov spaces this is simplified:

Remark 5.4. If U is a Sazanov space then condition (b) in Corollary 5.2 can be replaced by:

- (b') (i) there exists an infinitely divisible cylindrical probability measure μ with characteristics $(d_\nu, 0, \nu)$;
- (ii) $a \mapsto \kappa(a) = -(id_\nu(a) + \int_U \psi_h(\langle u, a \rangle) \nu(du))$ is continuous in an admissible topology.

Example 5.5. If U is a Hilbert space then the Sazanov topology is admissible. If $(d_\nu, 0, \nu)_h$ is a cylindrical characteristics the function κ is necessarily negative-definite and if it is also continuous in the Sazanov topology one obtains the well known Lévy-Khintchine decomposition in Hilbert spaces, see [14, Th.6.4.10].

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