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Dynamics of poles with position-dependent strengths and its optical analogues

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Abstract

Dynamics of point vortices is generalized in two ways. Firstly by allowing complex strengths which allows for sources and sinks in combination with the usual vorticity, and secondly by allowing the strengths to be functions of position. We describe several exact solutions with optical analogues, notably Snell's law and law of reflection off a mirror.

Keywords: Vortex dynamics, Snell's law, geometric optics, hybrid systems

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1. Introduction

Dynamics of point vortices with fixed strengths in a 2-dimensional ideal fluid has a classical pedigree (e.g. Lamb [4], Art. 154–160). We extend it in two ways, firstly allowing, besides vortices, sources/sinks as well as their superpositions ('poles'), and secondly allowing the strengths of these poles to vary as functions of position in the plane.

The first extension goes back to a 1928 paper by Friedman and Polubarinova-Kochina (the former is the same Friedman as in the eponymous cosmological model); the rather more recent paper by Borisov & Mamaev [2] contains references as well as a good theoretical analysis. Here we present a couple of new exact solutions and alternative derivations of some old ones.

The second extension seems less explored, and leads to a rich dynamics, which we illustrate with a variety of exact solutions with analogues in geometric optics, the position-dependent 'strength' of a pole replacing the medium-dependent index of refraction. As typical examples we shall detail the analogues of Snell's law and law of reflection off a mirror, in generalized forms. Optical analogy is not so obvious: indeed, even though it was suggested by Kimura [3] that in the dipole limit a classical vortex pair travels along a geodesic, the principle governing light rays in geometric optics is one of least time, not of least distance. Nevertheless, dynamics of poles with position-dependent strengths turns out to be quite versatile in mimicking the ray representation of phenomena of wave propagation. Take for instance the work by Berry [1] on focusing and defocusing of surface waves by underwater landscape. It will become clear that such phenomena can be realized by our dynamics, too.

2. Equation of motion, conservation laws

We begin with the generic background of 2-dimensional ideal fluid flow in terms of complex potential, but shall specialize soon. Consider N interacting points $z_1, \dots, z_N \in \mathbb{C}$ called *poles*, each pole z_i carrying

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with it a family of complex-valued functions $\{\mu_n^i(z)\}_{n \in \mathbb{Z}}$ called *strengths*, only finitely many of which are nonzero. The poles move according to

$$\frac{d}{dt}z_i(t) = \sum_{j: j \neq i} \sum_{n \in \mathbb{Z}} \overline{\mu_n^j(z_j(t))} (z_i(t) - z_j(t))^n \quad (i = 1, \dots, N). \quad (1)$$

The dynamics of (1), being 1st-order in time t , has no inertia, in the sense that the poles' instantaneous positions determine their velocities: the phase space is a product of N copies of \mathbb{C} (minus diagonals if we wish a priori to exclude collisions), not a (co)tangent bundle. We can set up something like this dynamics on any domain of any Riemann surface, provided $\sum_i \mu_{-1}^i$ is chosen pure imaginary and equal to $1/2\pi\sqrt{-1}$ times the total circulation along the boundary (else Stokes's formula is violated). In simple domains that arise in practice, solutions can be found by the method of images.

A term of the form $\mu(z - z_j)^n$ on the right side of (1) represents a flow velocity induced by z_j at z . The pictures for $n = -1$ have rotational symmetry: source or sink of flux $2\pi\mu$ for μ real, vortex of circulation $2\pi\sqrt{-1}\mu$ for μ pure imaginary, in general a superposition of these, i.e. a spiral node. Other n exhibit other symmetries: multipolar flows for $n < -1$, uniform flow for $n = 0$, and corner flows for $n > 0$.

Now suppose (1) is homogeneous so that $\mu_n^i = 0$ except for a certain exponent $n = n_0$, and moreover all $\mu^i := \mu_{n_0}^i$ are fixed. Then (1) may be recast in the 'canonical' form

$$\frac{d}{dt}z_i = \frac{1}{\mu^i} \frac{\partial}{\partial \bar{z}_i} H, \quad H(z_1, \dots, z_N) = \operatorname{Re} \sum_{i, j: i \neq j} \mu^i \mu^j G(z_i - z_j)$$

with $G(z_i - z_j) = (z_i - z_j)^{n_0+1}/(n_0+1)$ when $n_0 \neq -1$, $\log(z_i - z_j)$ when $n_0 = -1$ (G as in 'Green'). From

$$\frac{d}{dt}H(z_1(t), \dots, z_N(t)) = \sum_i \left(\frac{\partial}{\partial z_i} H \cdot \frac{d}{dt}z_i + \frac{\partial}{\partial \bar{z}_i} H \cdot \frac{d}{dt}\bar{z}_i \right) = \sum_i (\mu^i + \bar{\mu}^i) \left| \frac{d}{dt}z_i(t) \right|^2$$

we see

Theorem 2.1. *If all μ^i are pure imaginary, then H is conserved.*

Next let the homogeneity degree n_0 be odd, with μ^i still fixed. Pairwise cancellation in (1) yields $\sum_i \bar{\mu}^i dz_i/dt = 0$, whence

Theorem 2.2. *If $\mu := \sum_i \mu^i \neq 0$, then the 'center of strength' $c = \sum_i \bar{\mu}^i z_i / \bar{\mu}$ is conserved. If $\mu = 0$, then for every partition $I \sqcup I' = \{1, \dots, N\}$ such that $\mu_I := \sum_{i \in I} \mu^i \neq 0$ and $\mu_{I'} := \sum_{i' \in I'} \mu^{i'} \neq 0$, the difference between the 'subcenters' $\sum_{i \in I} \bar{\mu}^i z_i / \bar{\mu}_I - \sum_{i' \in I'} \bar{\mu}^{i'} z_{i'} / \bar{\mu}_{I'}$ is conserved.*

The 'partition' part of this result is elementary but does not appear to have been used prior to the paper by Montaldi, Soulière, Tokieda [5].

If instead the homogeneity degree n_0 is even, and there are just 2 poles z, z' with strengths μ, μ' , then the quantity $\bar{\mu}z - \bar{\mu}'z'$ is a conserved quantity. However, this does not appear to extend to more than 2 poles.

Finally, how can we extend the affine symmetry of the phase space to that of the phase space-time so as to conserve the invariance of (1)? The requirement that time t be real gives the answer.

Theorem 2.3. *The system (1) is $\mathbb{C}^* \times \mathbb{C}$ -invariant if and only if it is homogeneous of degree $n_0 = -1$: here $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ sends (t, z) to $(|a|^2 t, az + b)$.*

3. Exact solutions

We shall, in what follows, focus on the case where (1) involves only the exponent $n_0 = -1$ and all the strengths share a common dependence on the position: $\mu^i(z) = \mu^i \cdot S(z)$ for some constants μ^i ($i = 1, \dots, N$) and some real-valued function S ('seabed' or 'step'). Thus the equations of motion are

$$\frac{d}{dt} z_i(t) = \sum_{j:j \neq i} \frac{\overline{\mu^j S(z_j(t))}}{z_i(t) - z_j(t)} \quad (i = 1, \dots, N). \quad (2)$$

In most of our examples S will be piecewise constant. Theorems 2.1, 2.2, 2.3 hold piecewise until one of the poles crosses a discontinuity of S , at which instant H and the centers of strengths jump to new values.

When $S(z) = 1$ and all μ^i are pure imaginary, we are back to classical point vortices and recover H as their Hamiltonian.

3.1. Self-similar solutions

If a collection of poles happens to move in a self-similar manner, then Theorem 2.3 reduces the (complex) degree of freedom from N to 1, down to a single equation

$$\frac{d}{dt} Z = \frac{M}{\overline{Z}}, \quad (3)$$

or in polars $\frac{d}{dt} |Z|^2/2 = \text{Re } M$, $|Z|^2 \frac{d}{dt} \arg Z = \text{Im } M$. The solution is

$$Z(t) = T \exp\left(\sqrt{-1} \frac{\text{Im } M}{\text{Re } M} \log T\right) Z(0), \quad \text{where } T = \sqrt{1 + \frac{\text{Re } M}{|Z(0)|^2/2} t} \quad \text{if } \text{Re } M \neq 0, \quad (4)$$

and

$$Z(t) = \exp\left(\sqrt{-1} \frac{\text{Im } M}{|Z(0)|^2} t\right) Z(0) \quad \text{if } \text{Re } M = 0. \quad (5)$$

We remark however that there is really no need to treat the case (5) apart from (4), since (4) converges to (5) in the limit $\text{Re } M \rightarrow 0$.

The poles spiral in and collapse to their center of strength after time $-|Z(0)|^2/2 \text{Re } M$ if $\text{Re } M \neq 0$ (in the future if $\text{Re } M < 0$, in the past if > 0). If $\text{Re } M = 0$, then the poles merely spin with the pair remaining congruent to itself.

3.2. Pair

A pole pair z, z' of fixed strengths μ, μ' moves self-similarly around its center of strength c (defined in Theorem 2.2). With a little manipulation (2) takes the form of (3):

$$\frac{d}{dt}(z - c) = \frac{|\mu'|^2}{\mu + \mu'} \frac{1}{z - c}$$

and the same equation for $z' - c$ with prime and unprime exchanged. If $\mu + \mu' = 0$, then Theorem 2.2 implies that the pair moves along parallel trajectories ($c \rightarrow \infty$). In particular if μ, μ' are real and $\mu + \mu' = 0$, then the source chases the sink and the sink runs away from the source, while their mutual separation remains constant.

3.3. Regular polygons

Place N poles of equal fixed strengths μ at the vertices of a regular N -gon, plus 1 pole of strength μ' at the center c . The poles spiral self-similarly out or in, clockwise or anticlockwise, depending on N, μ, μ' . In terms of any one of the vertices z , (2) takes the form of (3):

$$\frac{d}{dt}(z - c) = \left(\frac{N-1}{2} \overline{\mu} + \overline{\mu'}\right) \frac{1}{z - c}.$$

If $\mu' = -(N-1)\mu/2$, then all the poles are immobilized.

4. Position-dependent strengths, optical analogy

4.1. Analogue of Snell's law

Let $S(z) = s_1$ in the lower half-plane $\text{Im } z < 0$ and $S(z) = s_2$ in the upper half-plane $\text{Im } z \geq 0$, where $s_1, s_2 \in \mathbb{R}_+$. A pair z, z' with strengths $-\mu S(z), \mu S(z')$, moving along parallel trajectories, arrives from the lower half-plane (Figure 1).

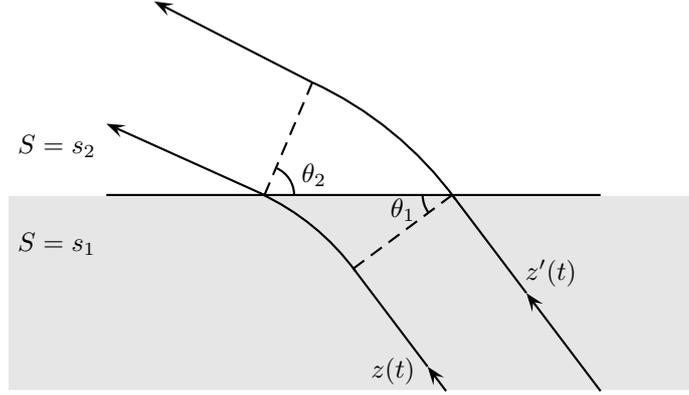


Figure 1: Snell's law for vortices

When z' crosses the real axis, the pair starts swerving self-similarly around its center of strength $c = (s_2 z' - s_1 z)/(s_2 - s_1)$. When, after time t say, z crosses the real axis, the pair resumes moving along parallel trajectories inside the upper half-plane. The angles of inclination θ_1, θ_2 of these trajectories pre- and post-crossing satisfy

$$s_1 \sin \theta_1 \exp\left(-\frac{\text{Re } \mu}{\text{Im } \mu} \theta_1\right) = s_2 \sin \theta_2 \exp\left(-\frac{\text{Re } \mu}{\text{Im } \mu} \theta_2\right) \quad \text{if } \text{Im } \mu \neq 0, \quad (6)$$

and

$$\theta_1 = \theta_2 \quad \text{if } \text{Im } \mu = 0. \quad (7)$$

If μ is pure imaginary, then θ_1, θ_2 become the angles of incidence and (6) reduces to

$$s_1 \sin \theta_1 = s_2 \sin \theta_2 \quad (8)$$

which is the analogue of Snell's law in optics; s is then the analogue of the index of refraction. At one extreme $\theta_1 = \theta_2 = 0$ there is no refraction. At the other extreme $\theta_2 = \pi/2$ the pair 'skids' along the real axis, hence reflection into the lower half-plane occurs for $\theta_1 > \arcsin(s_2/s_1)$.

One proof of (6) and (7) goes as follows. During the crossing, i.e. while z' is already in the upper half-plane but z is still in the lower half-plane, $z' - z$ evolves according to

$$\frac{d}{dt}(z' - z) = \frac{\overline{\mu(s_2 - s_1)}}{z' - z}$$

which is in the form (3) with $Z = z' - z$ and $M = \overline{\mu(s_2 - s_1)} = \bar{\mu}(s_2 - s_1)$. As remarked after (5), the special case of $\text{Re } \mu = 0$ can be treated as the limit $\text{Re } \mu \rightarrow 0$, so we shall proceed with the proof assuming $\text{Re } \mu \neq 0$. With this in mind, we have $\text{Im } M/\text{Re } M = -\text{Im } \mu/\text{Re } \mu$, so by (4)

$$z'(t) - z(t) = T \exp\left(-\sqrt{-1} \frac{\text{Im } \mu}{\text{Re } \mu} \log T\right) (z'(0) - z(0)), \quad (9)$$

where

$$T = \sqrt{1 + \frac{(s_2 - s_1)\operatorname{Re} \mu}{|Z(0)|^2/2} t}.$$

Since

$$z - c = \frac{s_2}{s_1 - s_2}(z' - z) \quad (10)$$

(of course: the point is that during the crossing the solution is self-similar), we also have

$$z(t) - c = T \exp\left(-\sqrt{-1} \frac{\operatorname{Im} \mu}{\operatorname{Re} \mu} \log T\right)(z(0) - c). \quad (11)$$

The equations (9) and (11) convey all the information we need to compare the positions of the pair at time 0 (or more precisely 0+) when the crossing starts and at time t (or $t-$) when it ends.

First, taking the argument of (9) we find

$$\theta_2 = -\frac{\operatorname{Im} \mu}{\operatorname{Re} \mu} \log T + \theta_1. \quad (12)$$

Next, taking the imaginary part of (11) we find

$$\text{left side} = 0 - \operatorname{Im} c = \frac{s_1}{s_2 - s_1} \operatorname{Im} z(0)$$

because $\operatorname{Im} z(t) = \operatorname{Im} z'(0) = 0$, and

$$\text{right side} = T \frac{s_2}{s_1 - s_2} |z'(0) - z(0)| \sin\left(-\frac{\operatorname{Im} \mu}{\operatorname{Re} \mu} \log T + \theta_1\right)$$

because (10) shows that $z(0) - c$ may be written as $\frac{s_2}{s_1 - s_2} |z'(0) - z(0)| \exp(\sqrt{-1} \theta_1)$. Equating the two sides and using (12),

$$T s_2 \sin \theta_2 = s_1 \frac{-\operatorname{Im} z(0)}{|z'(0) - z(0)|} = s_1 \sin \theta_1.$$

But again by (12)

$$\text{if } \operatorname{Im} \mu = 0, \text{ then } \theta_2 = \theta_1,$$

and

$$\text{if } \operatorname{Im} \mu \neq 0, \text{ then } T = \exp\left(\frac{\operatorname{Re} \mu}{\operatorname{Im} \mu} (\theta_1 - \theta_2)\right).$$

The formulae (6) and (7) are proved.

Another, only slightly different, formulation of this generalized Snell's law is to say that during the crossing $s \sin \theta \exp\left(-\frac{\operatorname{Re} \mu}{\operatorname{Im} \mu} \theta\right)$ remains constant if $\operatorname{Im} \mu \neq 0$, and θ remains constant if $\operatorname{Im} \mu = 0$. In this formulation, the law clearly holds in the more general set-up where a real-valued 'seabed' function $S(z)$ depends only on $\operatorname{Im} z$ but otherwise varies arbitrarily: just divide the plane into thin strips parallel to the real axis and approximate $S(z)$ by a function constant on each strip, like a sloping beach.

When μ is pure imaginary, an alternative short proof of (8) is available using Noether's theorem (a refinement of Theorem 2.2). The invariance under translations along the real direction gives rise to the conserved quantity $\operatorname{Im}(\mu S(z')z' - \mu S(z)z)$. From the fact already observed that the mutual separation between the poles remains constant, (8) follows immediately by geometry.

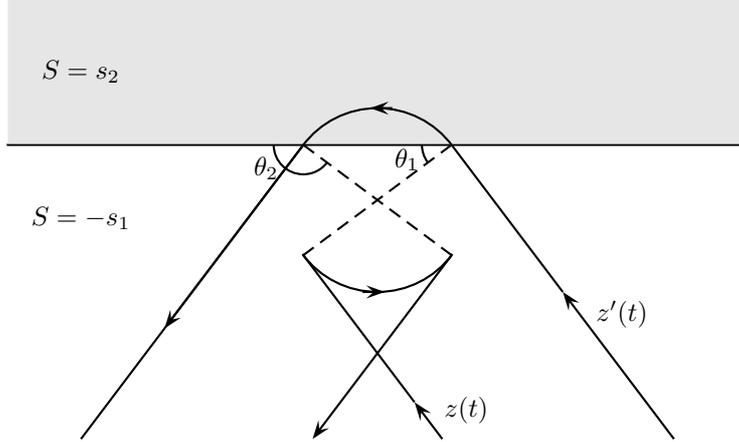


Figure 2: Law of reflection for vortices

4.2. Analogue of the law of reflection

Let $S(z) = -s_1$ in the lower half-plane $\text{Im } z < 0$ and $S(z) = s_2$ in the upper half-plane $\text{Im } z \geq 0$, where $s_1, s_2 \in \mathbb{R}_+$. The difference from the set-up for Snell's law is that S changes sign between the half-planes. To fix ideas, let us think of μ such that $\text{Re } \mu \geq 0$, $\text{Im } \mu \leq 0$. A pair z, z' with strengths $-\mu S(z)$, $\mu S(z')$, moving along parallel trajectories, arrives from the lower half-plane (Figure 2).

When z' crosses the real axis, the pair starts spiraling out self-similarly around its center of strength $c = (s_2 z' + s_1 z)/(s_2 + s_1)$. When, after time t say, z' again crosses the real axis (it does so necessarily before z does), the pair resumes moving along parallel trajectories inside the lower half-plane. Exactly the same line of calculation as for Snell's law proves that the angles of inclination θ_1, θ_2 of these trajectories pre- and post-reflection satisfy

$$\sin \theta_1 \exp\left(-\frac{\text{Re } \mu}{\text{Im } \mu} \theta_1\right) = \sin \theta_2 \exp\left(-\frac{\text{Re } \mu}{\text{Im } \mu} \theta_2\right) \quad \text{if } \text{Im } \mu \neq 0, \quad (13)$$

If μ is pure imaginary, then (13) reduces to $\sin \theta_1 = \sin \theta_2$, from which we extract

$$\theta_1 = \pi - \theta_2 \quad \text{if } \text{Im } \mu \neq 0, \text{Re } \mu = 0. \quad (14)$$

This is the analogue of the law of reflection in optics, θ_1 being the angle of incidence, $\pi - \theta_2$ the angle of reflection. Note that s does not feature in the results (13) and (14). The reason is most readily appreciated in the case $\text{Re } \mu = 0$: then reflection results from 'pivoting' of $z'(0) - z(0)$ to $z'(t) - z(t)$, and the relative magnitudes of s_1, s_2 determine where the pivot is (at the intersection of the dashed lines in Figure 2), but the net pivoting angle is the same regardless of where the pivot is.

If $\text{Im } \mu = 0$, then no reflection occurs: z' and z repel each other along a straight line, z' going away in the upper half-plane and z going away in the lower half-plane.

There arises a curious degenerate case when $\text{Re } \mu = 0$ and the pair hits the mirror head-on, $\theta_1 = 0$. In this case the intuitive picture is that, barely into the upper half-plane, both poles switch the sign of their strengths simultaneously, step back barely into the lower half-plane, change the sign, etc., oscillating upper, lower, upper, lower, ... The pair gets trapped in the mirror.

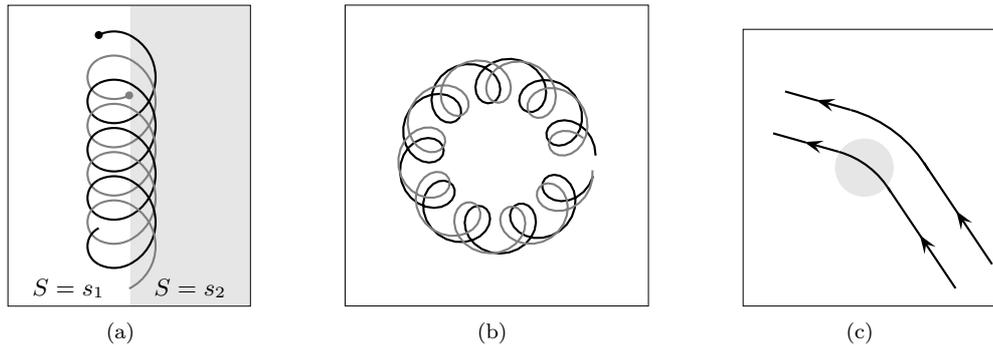


Figure 3: Leapfrogging and rainbow

4.3. Leapfrogging and rainbow

In vortex dynamics, *two* vortex pairs perform a motion called leapfrogging (e.g. [5] and references therein). With position-dependent strengths *one* pole pair can leapfrog all by itself. Let $S(z) = s_1$ in the left half-plane $\text{Re } z < 0$ and $S(z) = s_2$ in the right half-plane $\text{Re } z \geq 0$, where $s_1, s_2 \in \mathbb{R}_+$. A pair z, z' with strengths $\sqrt{-1} S(z), \sqrt{-1} S(z')$ is initially at positions $\Delta z \in \mathbb{C}$ and 0. It is easy to check that this pair leapfrogs along piecewise circular paths, as in Figure 3(a), advancing by $\frac{s_2 - s_1}{s_2 + s_1} \sqrt{-1} |\text{Im } \Delta z|$ per half-period. By distributing S in a circular bump, like a plateau or a crater, we can also persuade a pair to leapfrog (quasi-)periodically as in Figure 3(b).

A pole pair of opposite $\mu = \pm\sqrt{-1}$ and of separation $d > 0$ approaches a disk of radius $r > 0$. Let $S(z) = 1/r$ inside the disk and $S(z) = 1$ outside. Figure 3(c) shows how the pair gets refracted if just one of the pair passes through the disk; the angle of refraction is $2 \arctan(1 - r)/d$. If both enter the disk, then the pair gets internally reflected a certain number of times before it re-emerges, as in a rainbow.

As a final remark, note that the dynamical systems in this section may be viewed as examples of hybrid systems, with different equations of motion in different regions of the phase space.

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