

*Finite Elements and Anisotropic EIT
reconstruction*

Lionheart, William R. B. and Paridis, Kyriakos

2010

MIMS EPrint: **2010.35**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

Finite Elements and Anisotropic EIT reconstruction

William R. B. Lionheart¹, Kyriakos Paridis¹

¹School of Mathematics, The University of Manchester, Manchester, UK.

E-mail: `bill.lionheart@manchester.ac.uk`

Abstract. We study the indeterminacy of the inverse problem for the finite element approximation to the anisotropic inverse problem of EIT

1. Introduction

It is well known that many body tissues such as muscle have an anisotropic conductivity. It is also known that the inverse problem for anisotropic EIT does not have a unique solution even with complete data (arbitrarily many, arbitrarily small electrodes) of arbitrary precision [1]. The usual numerical treatment of EIT uses a finite element (FE) model to solve the forward problem and the conductivity in this model is adjusted to fit the measured data. Extending the usual approach to the isotropic problem we approximate the potential as piecewise linear on tetrahedra elements and it seems natural to represent the anisotropic conductivity as a constant symmetric matrix on each tetrahedron.

There are two pit-falls with this approach. The finite element system matrix can have non zero elements only for pairs of vertex indices corresponding to an edge in the FE mesh. The diagonal elements are simply the negative of the sum of the off diagonal elements for that row or column so the maximum number of degrees of freedom in our FE model is the number of edges. Representing the anisotropic conductivity as a symmetric matrix on each element gives $6n_t$ degrees of freedom (n_t is number of tetrahedra) – more than n_e the number of edges. This means that even if we knew the FE system matrix (let alone just the boundary data) we still could not uniquely determine the conductivity matrix on each tetrahedron uniquely.

However this is not the non-uniqueness that arises in the continuum problem. Let $\sigma(x) = (\sigma_{ij}(x))$ be an anisotropic conductivity on a domain $\Omega \subset \mathbb{R}^3$, and let $F : \Omega \rightarrow \Omega$ be a smooth invertible map (diffeomorphism) with $F(x) = x$ for $x \in \partial\Omega$ then the known non-uniqueness in EIT arises as the Neumann-Dirichlet map (transfer impedance) defined by

$$R_\sigma : \sigma \nabla u \cdot \mathbf{n} \mapsto u|_{\partial\Omega} \quad (1)$$

(where $\nabla \cdot \sigma \nabla u = 0$) satisfies

$$R_\sigma = R_{\tilde{\sigma}} \quad (2)$$

for another conductivity

$$\tilde{\sigma}(F(x)) = \frac{DF(x)^T \sigma(x) DF(x)}{\det DF(x)} \quad (3)$$

where $DF(x)$ is the Jacobian matrix of F . Using a fixed finite element mesh does not reveal this non-uniqueness in the inverse problem as a smooth map F does not preserve the mesh. So

in this obvious approach we have chosen one of the infinite family of anisotropic conductivities consistent with the data simply by our choice of mesh.

In this paper we look at the singular value decomposition of the linearised forward problem (Jacobian) for anisotropic EIT, and we show for a variety of meshes the problem of determining edge conductances exhibits the same ill-conditioning as the isotropic problem, with no further drop in rank of the Jacobian. This indicates that we are as likely to be as successful in recovering edge conductances from EIT data as we are isotropic conductivities on elements. This shifts the problem of anisotropic EIT to finding a mesh, and an assignment of anisotropic conductivity matrices on elements, consistent with *a priori* data.

2. Finite element formulation

The finite element mesh consists of vertices $x_i \in \Omega$ $i = 1, \dots, n_v$ and tetrahedra T_k , $k = 1, \dots, n_t$. The tetrahedra are convex hull of the sets of four distinct vertices, and they intersect at most in faces (that is the convex hull of the three vertices they share). The union $\bigcup_k T_k$ is a polyhedron approximating Ω . The nodal basis functions $\phi_i(x)$ are the piecewise linear functions such that $\phi(x_i) = 1$ and $\phi(x_j) = 0$, $j \neq i$. We approximate the potential as $u(x) = \sum_i u_i \phi_i(x)$, and assign a positive definite matrix σ^k to each tetrahedron. In this context the finite element system matrix $K \in \mathbb{R}^{n_v \times n_v}$ is given by

$$K_{ij} = \sum_{k:\{x_i, x_j\} \subset T_k} \nabla \phi_i \cdot \sigma^k \nabla \phi_j |T_k| \quad (4)$$

where $|T_k|$ is the volume of the tetrahedron and we note that on each tetrahedron $\nabla \phi_i$ is constant. For a boundary current density $J = \sigma \nabla u \cdot \mathbf{n}$ we define the current vector $I \in \mathbb{R}^{n_v}$ by

$$I_i = \int_{\partial \Omega} J \phi_i dx \quad (5)$$

and the FE system is

$$K \underline{u} = I \quad (6)$$

where \underline{u} is the vector of u_i . One additional condition is required for a unique solution as the voltage is only determined up to an additive constant, one way to do this is to choose one (“grounded”) vertex i_g and enforce $u_{i_g} = 0$ by deleting the i_g row and column from the system (6). It is clear from (4) that for a pair of vertices indexed by i, j that are not both in any tetrahedron, $K_{i,j} = 0$. For an isotropic conductivity, that is $\sigma^k = \gamma^k \mathbf{I}$ for scalars γ^k and \mathbf{I} the identity matrix, the system (6) is equivalent to Ohm’s and Kirchoff’s law for a resistor network with distinct vertices labeled by i and j connected by a resistor with conductance

$$K_{ij} = 6 \sum_{k:\{x_i, x_j\} \subset T_k} \gamma_k \cot \theta_{ij}^k L_{ij} \quad (7)$$

(see Appendix) where θ_{ij}^k is the angle between the faces of T_k where they meet the edge *not* containing the vertices indexed by i or j and L_{ij} the length of that edge. Obviously some restrictions on the angles are necessary to ensure the conductances are nonnegative, if they are all acute that is certainly sufficient. If we allow negative conductances we can interpret K_{ij} as an “edge conductance” even for anisotropic conductivity and possible non-acute angles. The total Ohmic power for the network will still be nonnegative for any solution. The mapping $(\sigma^k) \mapsto K$ is linear, with a domain of dimension $6n_t$ and range of dimension at most n_e . In a typical mesh $6n_t > n_e$.

3. Rank analysis of Jacobian

In practical EIT a system of electrodes is used that does not typically cover the whole of $\partial\Omega$ which means some conductivity information at the boundary could be inaccessible in between electrodes. We therefore consider the idealisation that any electrodes can be used. In the finite element context it means that any current vector I can be specified with zero sum and supported on boundary vertices, and all measurements of voltage made at boundary vertices. Without loss of generality we will set a current of -1 at i_g and then apply currents of 1 at each other boundary i vertex in turn. The voltage $V_{ij} = u(x_j)$ is measured at x_j . The Jacobian matrix gives the rate of change of each of these voltages when σ_{lm}^k is changed in tetrahedron T_k . This is

$$\frac{\partial V_{ij}}{\partial \sigma_{lm}^k} = - \int_{T_k} \frac{\partial u^i}{\partial x_l} \frac{\partial u^j}{\partial x_m} dx \tag{8}$$

where $u^i = \sum_q u_q^i \phi_q(x)$ is the finite element approximation to the potential for the current of 1 at vertex x_i .

Using Netgen [2] to generate meshes of geometric objects and a modification of code derived by Abascal [3] from Polydorides' EIDORS-3D [4] to calculate the Jacobian we studied the singular values of the anisotropic Jacobian, see table 1.

n_t	n_e	rank(J)	n_t	n_e	rank(J)	n_t	n_e	rank(J)
6	19	19	40	82	81	89	162	162
28	66	66	70	142	141	251	418	418
48	98	98	122	246	245	361	656	656
168	289	289	320	504	504	712	1049	1049
224	382	382	336	590	589	(c) Cylinder		
384	604	604	560	879	879			
1344	1922	1922	976	1529	1529			

Table 1: Meshes with n_t tetrahedra and n_e edges against the numerical rank of the Jacobian.

We note in figure 1 that sudden fall in the singular values after n_e , and the linear fall on a log scale before that, confirms that the edge conductances can be determined by the transfer impedance data, with a similar degree of illconditioning as the isotropic inverse problem. We note that this rank of the Jacobian is less than the $6n_t$ degrees of freedom in the anisotropic conductivity and as expected we cannot hope to recover these uniquely.

4. Discussion and Conclusion

The relation between finite element meshes and resistor network only applies to linear tetrahedral elements. It is also not necessarily appropriate to treat the conductivities on elements as each individually variable. In isotropic EIT it is quite usual to use a coarser mesh for the conductivity than for the potential. In this case with fewer degrees of freedom for the conductivity, and perhaps basis functions reflecting *a priori* information, the loss of information we observed going from conductivity matrices to edge conductances is not so important. More important perhaps is the necessity in our numerical representation to reintroduce the non-uniqueness resulting from the undetermined diffeomorphism. One way to do this is to reconstruct edge conductances on a mesh, then use *a priori* information, for example about the orientation of muscle fibres, to constrain the mapping of the abstract resistor mesh, that has no geometric information, to the

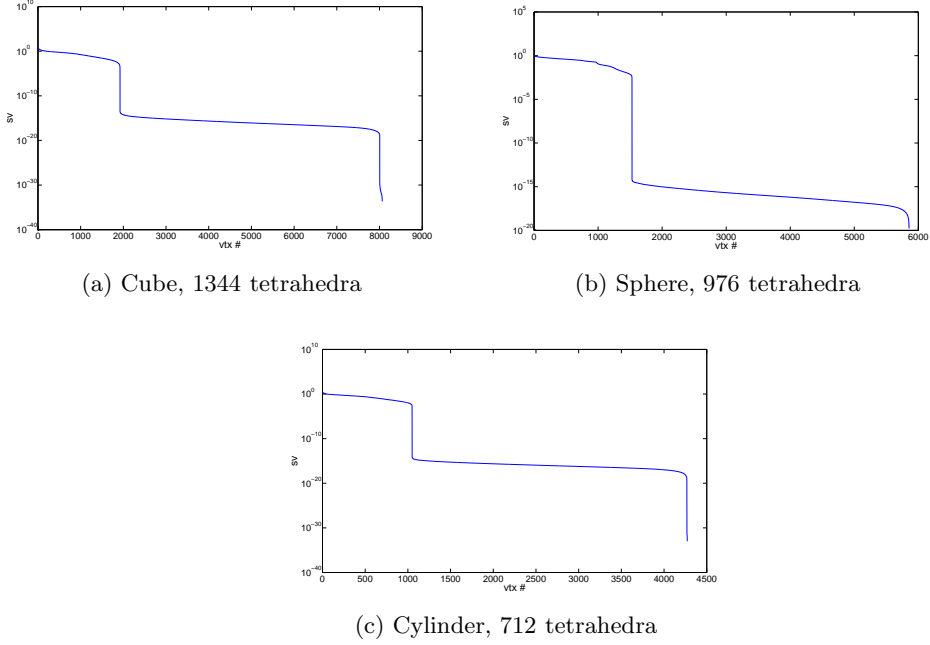


Figure 1: Singular values of the Jacobian for selected meshes

vertex positions and hence conductivities. That is our intention for future work. This work is supported by EPSRC grant EP/F033974/1.

References

- [1] J. M. Lee and G. Uhlmann, “Determining anisotropic real-analytic conductivities by boundary measurements,” *Comm. Pure Appl. Math.*, vol. 42, no. 8, pp. 1097–1112, 1989.
- [2] J. Schöberl, “Netgen an advancing front 2D/3D-mesh generator based on abstract rules,” *Comput. Vis. Sci.*, vol. 1, no. 1, pp. 41–52, 1997.
- [3] J.-F. P. J. Abascal, S. R. Arridge, D. Atkinson, R. Horesh, L. Fabrizi, M. De Lucia, L. Horesh, R. H. Bayford, and D. S. Holder, “Use of anisotropic modelling in electrical impedance tomography; Description of method and preliminary assessment of utility in imaging brain function in the adult human head,” *Neuroimage*, vol. 43, pp. 258–268, Nov 1 2008.
- [4] N. Polydorides and W. R. B. Lionheart, “A matlab toolkit for three-dimensional electrical impedance tomography: a contribution to the electrical impedance and diffuse optical reconstruction software project,” *Meas. Sci. Technol.*, vol. 13, no. 12, pp. 1871–1883, 2002.
- [5] J. R. Lee, “The law of cosines in a tetrahedron,” *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.*, vol. 4, no. 1, pp. 1–6, 1997.

Appendix

The derivation of (7) is fairly elementary apart from one step. Consider one tetrahedron T with vertices x_1, \dots, x_4 . The unit normal to plane of vertices 2, 3, 4 is

$$\mathbf{N}_1 = \frac{(x_2 - x_3) \times (x_2 - x_4)}{|(x_2 - x_3) \times (x_2 - x_4)|} \quad (9)$$

Hence the nodal basis function on this tetrahedron is

$$\phi_1(x) = \frac{(x - x_2) \cdot (x_2 - x_3) \times (x_2 - x_4)}{(x_1 - x_2) \cdot (x_2 - x_3) \times (x_2 - x_4)} \quad (10)$$

and

$$\nabla \phi_1 \cdot \nabla \phi_2|_T = \mathbf{N}_1 \cdot \mathbf{N}_2 \frac{4\text{Area}(234)\text{Area}(134)}{|T|} \quad (11)$$

Clearly $\mathbf{N}_1 \cdot \mathbf{N}_2 = \cos \theta_{12}$ where θ_{12} is the dihedral angle at the edge 34 and from [5] we see that

$$|T| = \frac{2}{3L_{12}} \text{Area}(234) \text{Area}(134) \sin \theta_{12}$$

where L_{12} is the length of the *opposite* edge 34. The result follows.