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Structured Linearizations for Palindromic Matrix Polynomials of Odd Degree *

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Abstract

The standard way to solve polynomial eigenvalue problems $P(\lambda)x = 0$ is to convert the matrix polynomial $P(\lambda)$ into a matrix pencil that preserves its spectral information– a process known as linearization. When $P(\lambda)$ is palindromic, the eigenvalues, elementary divisors, and minimal indices of $P(\lambda)$ have certain symmetries that can be lost when using the classical first and second companion linearizations for numerical computations, since these linearizations do not preserve the palindromic structure. Recently new families of linearizations have been introduced with the goal of finding linearizations that retain whatever structure that the original $P(\lambda)$ might possess, with particular attention paid to the preservation of palindromic structure. However, no general construction of palindromic linearizations valid for all palindromic polynomials has as yet been achieved. In this paper we present a family of linearizations for odd degree polynomials $P(\lambda)$ which are palindromic whenever $P(\lambda)$ is, and which are valid for all palindromic polynomials of odd degree. We illustrate our construction with several examples. In addition, we establish a simple way to recover the minimal indices of the polynomial from those of the linearizations in the new family.

Key words. matrix polynomial, matrix pencil, minimal indices, palindromic, structured linearization

AMS subject classification. 65F15, 15A18, 15A21, 15A22

1 Introduction

Consider an $n \times n$ matrix polynomial with degree $k \geq 2$ over an arbitrary field $\mathbb{F}$, i.e.,

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n \times n}, \quad A_k \neq 0\,. \tag{1.1}$$

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Then $P(\lambda)$ is said to be *T-palindromic* [21] if $A_i^T = A_{k-i}$ for $i = 0, 1, \ldots, k$, or in other words if

$$\operatorname{rev} P(\lambda) = P(\lambda)^T ,$$

where $\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i}$ denotes the *reversal* polynomial of $P(\lambda)$. For polynomials over the particular field $\mathbb{F} = \mathbb{C}$, one can also consider $P(\lambda)$ that are $*$-*palindromic* [17, 21], i.e., polynomials that satisfy $A_i^* = A_{k-i}$ for $i = 0, 1, \ldots, k$, or equivalently $\operatorname{rev} P(\lambda) = P(\lambda)^*$, where $*$ denotes conjugate transpose. Since everything that we do in this paper for T-palindromic polynomials works equally well for $*$-palindromic polynomials, from now on we will just refer to "palindromic" polynomials for the sake of simplicity, except in those few situations in this introduction where the distinction is significant. Polynomials $P(\lambda)$ satisfying $\operatorname{rev} P(\lambda) = -P(\lambda)^T$ or $\operatorname{rev} P(\lambda) = -P(\lambda)^*$, sometimes referred to as *anti-palindromic* polynomials [21], are also of some interest, and can be handled in a similar manner.

Palindromic polynomials arise in a number of application areas. For example, the mathematical modelling and numerical simulation of the behavior of periodic surface acoustic wave filters [13, 24], as well as the analysis of rail track vibrations produced by high speed trains [12, 14, 15, 21], each lead to a quadratic T-palindromic polynomial eigenvalue problem. Also, discrete-time optimal control problems can be formulated as $*$-palindromic eigenproblems of degree 2 and higher [3].

The spectral structure of palindromic matrix polynomials enjoys certain symmetries. For example, the elementary divisors of T-palindromic polynomials corresponding to eigenvalues $\lambda_0 \neq \pm 1$ always come in pairs $(\lambda - \lambda_0)^s, (\lambda - 1/\lambda_0)^s$ [19, 21, 23]. For palindromic polynomials $P(\lambda)$ that are *singular*[1], the minimal indices also are paired; if $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_\ell$ and $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_m$ are respectively the left and right minimal indices of $P(\lambda)$, then $\ell = m$ and $\eta_j = \varepsilon_j$ for $j = 1, \ldots, \ell$ [5, Thm. 3.6].

The usual way to numerically solve polynomial eigenproblems for *regular* polynomials $P(\lambda)$ is to first linearize $P(\lambda)$ into a matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk \times nk}$, then compute the eigenvalues and eigenvectors of $L(\lambda)$ using well-known algorithms for general matrix pencils. When $P(\lambda)$ is singular, linearizations can also be used to compute the minimal indices and bases of $P(\lambda)$ [5, 6]. The classical approach uses the *first* or *second companion forms* of $P(\lambda)$ as linearizations [11]. However, these companion forms are never palindromic, even when $P(\lambda)$ is. Consequently, the rounding errors inherent in numerical computations may destroy the symmetry of elementary divisors and minimal indices of palindromic polynomials if such unstructured linearizations are employed. A numerical procedure that preserves palindromic structure throughout the computation would thus be more appropriate than using standard methods for general polynomials. In order to gain more accuracy and reliability in the numerical solution of palindromic eigenvalue and minimal index problems by linearization, then, two steps should be addressed:

(1) Design linearizations that share the palindromic structure of $P(\lambda)$.

(2) Develop specific numerical methods for computing eigenvalues and minimal indices of palindromic pencils, methods that preserve and exploit the palindromic structure throughout the computation.

[1] Recall that an $n \times n$ polynomial $P(\lambda)$ is *singular* if $\det P(\lambda) \equiv 0$, i.e., if all the coefficients of the scalar polynomial $\det P(\lambda)$ are zero, and it is *regular* otherwise.

Step (2) has been addressed for the regular case in [22], where a structured Schur-like form for T-palindromic pencils and an algorithm to compute it are presented. Step (1) has been addressed in [21], but again only for regular palindromic polynomials $P(\lambda)$. In [21], necessary and sufficient conditions are given for the existence of palindromic linearizations within certain families of matrix pencils associated with P that were introduced in [20]. A procedure to construct these structured linearizations, when they exist, is also given in [21]. However, the problem of finding palindromic linearizations that are valid for *all* palindromic polynomials, regular and singular, remained open.

In this paper we will construct a family of strong linearizations for all *odd degree* matrix polynomials P, regular and singular, that are palindromic whenever P is, thus completely solving the open problem for the odd degree case. Note that the even degree case is more challenging because in this case there exist palindromic polynomials that do not have any palindromic linearization [21]. The even degree case will be addressed separately in [7] and [23]. Our construction for the odd degree case is based on the Fiedler pencils, a family of linearizations introduced in [1] for regular polynomials, and extended and further analyzed in [6] for both regular and singular matrix polynomials. An important advantage of the Fiedler linearizations is that they allow the recovery of the minimal indices of P from those of the linearization by means of a very simple formula [6]. We will see that this property is inherited by the palindromic linearizations constructed in this paper. We want to stress that minimal indices are intrinsic quantities associated with any singular matrix polynomial, and are relevant in many control problems [8, 16]. *We emphasize that linearizations in the new family introduced in this work are very easy to construct from the coefficients of the polynomial, see in Theorem* 4.7 *our main result, but it requires considerable work to prove that they are palindromic whenever P is.*

The paper is organized as follows. In Section 2 we introduce the basic definitions, background facts, and notations used throughout the paper, including the Fiedler pencils and their basic properties. Then in Section 3 certain block matrices closely related to the Fiedler pencils, but with some factors deleted, are introduced and algorithmically constructed. These block matrices will be the basis of our construction of palindromic linearizations. These linearizations are then introduced and their basic properties established in Section 4. We also prove some useful structural properties of these linearizations and provide a number of concrete examples. Section 5 then shows how any anti-palindromic polynomial of odd degree can be linearized by an anti-palindromic pencil, using an appropriately modified version of any of the palindromic linearizations constructed in Section 4. In Section 6 we show how the minimal indices of an odd degree palindromic polynomial can be recovered in an extremely simple way from the minimal indices of any one of our palindromic linearizations. Finally, conclusions and future work are discussed in Section 7.

2 Basic definitions and background

In this paper we follow the notation and definitions from [6]. In particular, $\mathbb{F}(\lambda)$ will denote the field of rational functions with coefficients in $\mathbb{F}$, and I_ℓ is the $\ell \times \ell$ identity matrix. Since the $n \times n$ identity appears frequently throughout the paper (n being the size of $P(\lambda)$ in (1.1)), for this particular size we will drop the subscript and denote it simply by I.

The spectral structure of a regular matrix polynomial $P(\lambda)$ is comprised of its *finite* and *infinite elementary divisors* (see definition in [9]). For singular matrix polynomials $P(\lambda)$, there is an additional structure comprised of the *minimal indices*. Minimal indices are only considered in Section 6, so we will introduce its formal definition in that section. We just mention here that minimal indices are related to the existence of *right* and *left null vectors* of $P(\lambda)$, that is, nonzero vectors $x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1}$ and $y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n}$ such that $P(\lambda)x(\lambda) \equiv 0$ and $y(\lambda)^T P(\lambda) \equiv 0$. The existence of these null vectors leads us to introduce the notion of *right* and *left* nullspaces of $P(\lambda)$. These are the following vector subspaces over $\mathbb{F}(\lambda)$:

$$\begin{aligned} \mathcal{N}_r(P) &:= \left\{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0\right\}, \\ \mathcal{N}_\ell(P) &:= \left\{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T\right\}. \end{aligned}$$

Notice that, since $P(\lambda)$ is square, we have $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_\ell(P)$.

Two matrix pencils $L_1(\lambda)$ and $L_2(\lambda)$ are *strictly equivalent* if there exist two constant invertible matrices E, F such that

$$E \cdot L_1(\lambda) \cdot F = L_2(\lambda).$$

The notion of strict equivalence is also valid for matrix polynomials with degree greater than one, but in this paper we will only need it for matrix pencils. It is well known [9] that two pencils of the same size are strictly equivalent if and only if they have the same elementary divisors and minimal indices.

Now we recall the notion of linearization as introduced in [11], and also the related notion of strong linearization introduced in [10] and named in [18]. Note that a *unimodular matrix* is a square matrix polynomial whose determinant is a nonzero constant in $\mathbb{F}$.

Definition 2.1. *A matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk\times nk}$ is a* linearization *of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k if there exist two unimodular $nk \times nk$ matrices $U(\lambda)$ and $V(\lambda)$ such that*

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2.1}$$

or, in other words, if $L(\lambda)$ is unimodularly equivalent *to* $\operatorname{diag}(I_{(k-1)n}, P(\lambda))$. *A linearization $L(\lambda)$ is called a* strong linearization *if* $\operatorname{rev} L(\lambda)$ *is also a linearization of* $\operatorname{rev} P(\lambda)$.

The relevance of linearizations (resp. strong linearizations) in the study of both regular and singular matrix polynomials lies in the fact that these are the only matrix pencils preserving the dimension of the left and right null spaces and the finite (resp. finite and infinite) elementary divisors of the polynomial [5, Lemma 2.3].

Note that the size of linearizations in Definition 2.1 is assumed to be exactly $nk \times nk$. Linearizations with smaller size have been considered recently in [2], and their minimal possible size has been determined in [4]. In particular, it is shown in [4] that all strong linearizations of a regular $n \times n$ matrix polynomial with degree k must have size exactly $nk \times nk$. Since we are interested in finding strong linearizations valid for all matrix polynomials of degree k (including regular ones), in this paper we only consider linearizations of size $nk \times nk$.

Our construction of palindromic linearizations is based on the *Fiedler pencils*, introduced in [1] for regular matrix polynomials, and later extended in [6] to the singular case. To construct these pencils for the polynomial $P(\lambda)$ in (1.1) we need the following block-partitioned matrices:

$$M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{2.2}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1\,. \tag{2.3}$$

These matrices are partitioned into $k \times k$ blocks of size $n \times n$, and are the basic factors used to build the *Fiedler pencils* [1, 6] of $P(\lambda)$:

$$\lambda M_k - M_{i_0} M_{i_1} \cdots M_{i_{k-1}}\,, \tag{2.4}$$

where $(i_0, i_1, \ldots, i_{k-1})$ is any possible permutation of the n-tuple $(0, 1, \ldots, k-1)$. The following fact is fundamental is the rest of the paper.

Theorem 2.2 ([1, 6]). *Let $P(\lambda)$ be an $n \times n$ matrix polynomial (regular or singular). Then any Fiedler pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

This result was shown to hold for regular $P(\lambda)$ over $\mathbb{F} = \mathbb{C}$ in [1], while a proof valid for arbitrary regular and singular polynomials over an arbitrary field $\mathbb{F}$ was given in [6]. As background for the work in this paper, this fact is crucial in guaranteeing that our construction produces strong linearizations of $P(\lambda)$.

We recall the commutativity relations

$$M_i M_j = M_j M_i \qquad \text{for } |i - j| \neq 1\,, \tag{2.5}$$

that will be used later. Unless otherwise stated, the matrices M_i for $i = 0, \ldots, k$ refer to the matrix polynomial $P(\lambda)$ in (1.1). When necessary, we will explicitly indicate the dependence on a certain matrix polynomial $Q(\lambda)$ with the notation $M_i(Q)$. This convention will also be applied to other matrices appearing in this paper.

In the following example we exhibit a Fiedler pencil for polynomials of degree $k = 5$.

Example 2.3. Let $k = 5$ and $(i_0, i_1, i_2, i_3, i_4) = (3, 4, 0, 1, 2)$. Then the Fiedler pencil associated with this permutation is

$$\lambda M_5 - M_3 M_4 M_0 M_1 M_2 = \begin{bmatrix} \lambda A_5 + A_4 & -I & 0 & 0 & 0 \\ A_3 & \lambda I & A_2 & -I & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & 0 & A_1 & \lambda I & -I \\ 0 & 0 & A_0 & 0 & \lambda I \end{bmatrix}.$$

Example 2.3 illustrates the general structure of the Fiedler pencils. The zero-degree term contains all the coefficients of $P(\lambda)$ except the leading one, i.e. A_k, together with $k-1$ identity blocks (with a minus sign). The remaining blocks of this term are null blocks. The first-degree coefficient contains the leading coefficient of $P(\lambda)$ in the $(1,1)$ position together with $k-1$ identities in the remaining diagonal positions. Again, the other blocks are zero.

3 Fiedler-like block matrices with deleted factors

For further developments, we construct matrices analogous to the ones in the zero-degree term of (2.4), but with some of the factors missing. For working effectively with this type of matrices we introduce the following notation: let $s \leq k$ be a positive integer and $C_s := \{j_1, \ldots, j_s\} \subseteq \{0, 1, \ldots, k-1\}$ be a set of s distinct numbers. Let also $\tau : C_s \to \{1, 2, \ldots, s\}$ be a bijection. Then we consider the matrix

$$\mathbb{M}_\tau := M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(s)} \,. \tag{3.1}$$

Notice that $\tau(j)$ for $j \in C_s$ describes the position of the matrix M_j in the product defining $\mathbb{M}_\tau$. Observe that $\mathbb{M}_\tau$ can be obtained from the zero-degree term of one of the Fiedler pencils of $P(\lambda)$ (2.4) by removing $k-s$ of the M_j factors.

Definition 3.1. *Let $\tau : C_s \to \{1, 2, \ldots, s\}$ be a bijection. For $j \in C_s$ we say that τ has a* consecution *at j if $j+1 \in C_s$ and $\tau(j) < \tau(j+1)$, and that τ has an* inversion *at j if $j+1 \in C_s$ and $\tau(j) > \tau(j+1)$.*

The following theorem provides an algorithm to construct the matrix $\mathbb{M}_\tau$ without performing multiplications. `Algorithm 1` in Theorem 3.2 will be used to establish certain properties of the matrix $\mathbb{M}_\tau$ that are needed in Section 4. We assume that all matrices appearing in `Algorithm 1` are block partitioned matrices with $n \times n$ blocks and that *MATLAB notation for submatrices is used on block indices. We will follow this convention in the rest of the paper.*

Theorem 3.2. *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *with degree $k \geq 2$, let $C_s = \{j_1, j_2, \ldots, j_s\} \subseteq \{0, 1, \ldots, k-1\}$ be a set of s distinct numbers such that $0 \in C_s$, let $\tau : C_s \to \{1, 2, \ldots, s\}$ be a bijection and let $\mathbb{M}_\tau$ be the matrix defined in* (3.1). *Then* `Algorithm 1` *below computes $\mathbb{M}_\tau$.*

`Algorithm 1`: Given $P(\lambda)$, C_s and τ computes $\mathbb{M}_\tau$

if τ has a consecution at 0

$$W_0 = \begin{bmatrix} -A_1 & I \\ -A_0 & 0 \end{bmatrix}$$

elseif τ has an inversion at 0

$$W_0 = \begin{bmatrix} -A_1 & -A_0 \\ I & 0 \end{bmatrix}$$

else `%` this happens if $1 \notin C_s$

$$W_0 = \begin{bmatrix} I & 0 \\ 0 & -A_0 \end{bmatrix}$$

endif

for $i = 1 : k-2$

if τ has a consecution at i

$$W_i = \begin{bmatrix} -A_{i+1} & I & 0 \\ W_{i-1}(:,1) & 0 & W_{i-1}(:,2:i+1) \end{bmatrix}$$

elseif τ has a inversion at i

$$W_i = \begin{bmatrix} -A_{i+1} & W_{i-1}(1,:) \\ I & 0 \\ 0 & W_{i-1}(2:i+1,:) \end{bmatrix}$$

elseif ($i \notin C_s$ and $i+1 \in C_s$)

$$W_i = \begin{bmatrix} -A_{i+1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & W_{i-1}(2:i+1, 2:i+1) \end{bmatrix}$$

else $\quad$ % this happens if $i+1 \notin C_s$

$$W_i = \begin{bmatrix} I & 0 \\ 0 & W_{i-1} \end{bmatrix}$$

endif

endfor

$I\!M_\tau = W_{k-2}$

Proof. The proof proceeds by induction on the degree k. The result is obvious for $k = 2$ because in this case there are only three possibilities for $I\!M_\tau$, namely: $I\!M_\tau = M_0 M_1$ if τ has a consecution at 0, $I\!M_\tau = M_1 M_0$ if τ has an inversion at 0 and $I\!M_\tau = M_0$ if $1 \notin C_s$. A direct computation shows that these three matrices correspond to the matrices computed by `Algorithm 1` for $k = 2$.

Assume now that the result is true for matrix polynomials of degree $k - 1 \geq 2$ and let us prove it for the polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ of degree k and the bijection $\tau : C_s \to \{1, 2, \ldots, s\}$, where $C_s = \{j_1, j_2, \ldots, j_s\}$ is as in the statement. Notice first that the matrices $M_i(P)$ defined in (2.2) and (2.3) for $P(\lambda)$ satisfy

$$M_i(P) = \operatorname{diag}(I, M_i(Q)), \qquad \text{for } i = 0, \ldots, k-2, \tag{3.2}$$

where $M_i(Q)$ are the $n(k-1) \times n(k-1)$ matrices corresponding to the polynomial $Q(\lambda) = \sum_{i=0}^{k-1} \lambda^i A_i$. In the proof, we distinguish four cases that correspond to the four possibilities in the "if" statement inside the "for loop" of `Algorithm 1` for $i = k - 2$.

Case 1. If τ has a consecution at $k-2$ then the commutativity relations (2.5) imply

$$I\!M_\tau(P) = M_{i_1}(P) \cdots M_{i_{s-1}}(P)\, M_{k-1}(P),$$

where $(i_1, \ldots, i_{s-1})$ is a permutation of $C_s \setminus \{k-1\}$. Notice that for $i = 0, 1, \ldots, k-2$, $i \in C_s \setminus \{k-1\}$ if and only if $i \in C_s$. Now, by using (3.2) we can write

$$I\!M_\tau(P) = \operatorname{diag}(I, I\!M_{\widetilde{\tau}}(Q))\, M_{k-1}(P), \tag{3.3}$$

where $\widetilde{\tau} : C_s \setminus \{k-1\} \to \{1, 2, \ldots, s-1\}$ is a bijection such that for $i = 0, 1, \ldots, k-3$, $\widetilde{\tau}$ has a consecution (resp. inversion) at i if and only if τ has a consecution (resp. inversion) at i. So, `Algorithm 1` applied on $Q(\lambda)$, $C_s \setminus \{k-1\}$ and $\widetilde{\tau}$ produces the same W_{k-3} that `Algorithm 1` applied on $P(\lambda)$, C_s and τ. Therefore, the induction hypothesis guarantees that $I\!M_{\widetilde{\tau}}(Q) = W_{k-3}$. Finally, we perform the simple block product in (3.3) as follows

$$\begin{aligned} I\!M_\tau(P) &= \begin{bmatrix} I & 0 & 0 \\ 0 & W_{k-3}(:,1) & W_{k-3}(:,2:k-1) \end{bmatrix} \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I_{(k-2)n} \end{bmatrix} \\ &= \begin{bmatrix} -A_{k-1} & I & 0 \\ W_{k-3}(:,1) & 0 & W_{k-3}(:,2:k-1) \end{bmatrix}, \end{aligned}$$

which is precisely the matrix obtained for $i = k-2$ in the "for loop" in `Algorithm 1` when τ has a consecution at $k - 2$.

Case 2. If τ has an inversion at $k-2$ then the proof is similar to that of *Case 1*, but with $M_{k-1}(P)$ placed on the left, that is,

$$\mathbb{M}_{\tau}(P) = M_{k-1}(P)\, M_{i_1}(P) \cdots M_{i_{s-1}}(P) = M_{k-1}(P)\,\mathrm{diag}(I, \mathbb{M}_{\widetilde{\tau}}(Q))\,.$$

Case 3. If $k-2 \notin C_s$ and $k-1 \in C_s$, we can argue as in *Case 1* and write again

$$\mathbb{M}_{\tau}(P) = M_{i_1}(P) \cdots M_{i_{s-1}}(P) M_{k-1}(P) = \mathrm{diag}(I, \mathbb{M}_{\widetilde{\tau}}(Q))\, M_{k-1}(P)\,, \qquad (3.4)$$

and, by the induction hypothesis,

$$\mathbb{M}_{\widetilde{\tau}}(Q) = W_{k-3} = \begin{bmatrix} I & \\ & W_{k-4} \end{bmatrix},$$

where W_{k-3} and W_{k-4} are the matrices obtained for $i = k-3, k-4$ in the "for loop" of `Algorithm 1` applied on $Q(\lambda)$, $C_s \setminus \{k-1\}$ and $\widetilde{\tau}$, that are the same that W_{k-3} and W_{k-4} obtained when `Algorithm 1` is applied on $P(\lambda)$, C_s and τ. Finally, performing the block product in (3.4) we get

$$\begin{aligned}\mathbb{M}_{\tau}(P) &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & W_{k-3}(2:k-1, 2:k-1) \end{bmatrix} \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I_{(k-2)n} \end{bmatrix} \\ &= \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & W_{k-3}(2:k-1, 2:k-1) \end{bmatrix},\end{aligned}$$

which is precisely the matrix obtained for $i = k-2$ in the "for loop" of `Algorithm 1` when $k-2 \notin C_s$ and $k-1 \in C_s$.

Case 4. If $k-1 \notin C_s$, we have that $C_s \setminus \{k-1\} = C_s$ and we can simply write

$$\mathbb{M}_{\tau}(P) = \mathrm{diag}(I, \mathbb{M}_{\tau}(Q)).$$

By the induction hypothesis, $\mathbb{M}_{\tau}(Q) = W_{k-3}$ with W_{k-3} the matrix obtained for $i = k-3$ in the "for loop" in `Algorithm 1`. Therefore $\mathbb{M}_{\tau}(P) = W_{k-2}$ and the proof is complete. $\square$

We finish this section by establishing a simple corollary of Theorem 3.2 that will be used in Section 4.

Corollary 3.3. *With the same notation and assumptions as in Theorem* 3.2*, let* $W_0, W_1, \ldots, W_{k-2} = \mathbb{M}_{\tau}$ *be the sequence of matrices computed by* `Algorithm 1`*, assume* $k \geq 3$ *and recall that MATLAB notation refers to block indices. Then for* $i = 0, \ldots, k-3$*,*

(a) $\mathbb{M}_{\tau}(k-i:k, k-i:k) = W_i\,(2:i+2, 2:i+2)$;

(b) $\mathbb{M}_{\tau}(j,:)$ *(respectively,* $\mathbb{M}_{\tau}(:,j)$*), for* $j = k-i, \ldots, k$*, is obtained from* $W_i(j+2-k+i,:)$ *(respectively,* $W_i(:,j+2-k+i)$*) by adding* $k-i-2$ *zero blocks of size* $n \times n$ *in certain positions.*

Proof. Part (a). The result for $i = k-3$ follows directly from the way W_{k-2} is obtained from W_{k-3} in `Algorithm 1`, that implies

$$\mathbb{M}_\tau(3:k, 3:k) = W_{k-2}\,(3:k, 3:k) = W_{k-3}\,(2:k-1, 2:k-1).$$

Now, we proceed by induction: We assume that the result is true for an index $i+1$ such that $1 \le i+1 \le k-3$ and we will prove it for i. By the induction assumption $\mathbb{M}_\tau(k-i-1 : k, k-i-1 : k) = W_{i+1}\,(2 : i+3, 2 : i+3)$. On the other hand, by the way W_{i+1} is obtained from W_i in `Algorithm 1`, it is clear that $W_{i+1}\,(3 : i+3, 3 : i+3) = W_i\,(2 : i+2, 2 : i+2)$. Combining the two identities above we get $\mathbb{M}_\tau(k-i : k, k-i : k) = W_i\,(2 : i+2, 2 : i+2)$, which is the result in part (a).

Part (b). We prove the result only for block rows, since for block columns the proof is analogous. The result for $i = k-3$ follows directly from `Algorithm 1`. Now, we proceed by induction: We assume that the result is true for an index $i+1$ such that $1 \le i+1 \le k-3$ and we will prove it for i. This induction assumption implies that

$$\mathbb{M}_\tau(j,:), \quad \text{for } j = k-i-1, \ldots, k, \text{ is obtained from} \quad W_{i+1}(j+3-k+i,:) \tag{3.5}$$

by adding zero blocks. On the other hand, by the way W_{i+1} is obtained from W_i in `Algorithm 1`, it is clear that

$$W_{i+1}(j,:), \quad \text{for } j = 3, \ldots, i+3, \text{ is obtained from} \quad W_i(j-1,:) \tag{3.6}$$

by adding one zero block. Finally, combine (3.5) and (3.6) to get the result. □

4 Palindromic linearizations for odd degree

The technical results presented in Section 3 allow us to deal in this section with the main result of this paper: the construction of strong linearizations $\lambda X + Y$ for any odd degree matrix polynomial $P(\lambda)$, with coefficients as in (1.1), such that are palindromic whenever $P(\lambda)$ is palindromic. For this purpose, we will *construct pencils strictly equivalent to certain Fiedler pencils* and that satisfy

$$X^T = Y \qquad \text{whenever} \qquad A_{k-i} = A_i^T \quad \text{for all } i = 0, 1, \ldots, k. \tag{4.1}$$

Observe that Theorem 2.2 guarantees immediately that the pencils we construct are strong linearizations for any odd degree $P(\lambda)$. The starting step in our strategy will consist in multiplying a carefully selected Fiedler pencil by the inverses of some of the M_i matrices in (2.3), that are always invertible for $i = 1, \ldots, k-1$ and whose inverses are given by

$$M_i^{-1} = \begin{bmatrix} I_{(k-i-1)n} & & & \\ & 0 & I & \\ & I & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}. \tag{4.2}$$

So, starting with (2.4), we will get a pencil $\lambda X' + Y'$ where X' is a product of M_k times some inverses M_i^{-1} and $-Y'$ is a product of those M_j matrices that have not been inverted (this product always involves M_0). An analogous strategy was introduced in [1, Corollary 2.4, Theorem 3.1] to build self-adjoint linearizations of self-adjoint regular matrix polynomials.

However for preserving the palindromic structure we must do two additional tasks: first reversing the order of the block rows of $\lambda X' + Y'$ and, then, changing the signs of a selected subset of block rows. Both tasks can be performed through strict equivalences on $\lambda X' + Y'$.

Two important points in the strategy described in the previous paragraph are how to select the initial Fiedler pencil and the inverses M_i^{-1} for getting (4.1). Note that a simple necessary condition follows trivially from (4.1): if for some $i = 1, \ldots, k-1$ the factor M_i is in Y, then X must contain the factor M_{k-i}^{-1} with "complementary" index. This key fact forces the degree to be odd and plays a central role in our construction of palindromic linearizations based on Fiedler pencils. Before we address the general construction leading to our main Theorem 4.7, we will illustrate this initial discussion with Example 4.1.

Example 4.1. Let $k = 5$ and set

$$\mathbb{M}_0 = M_0 M_1 M_2, \qquad \mathbb{M}_1 = M_3^{-1} M_4^{-1} M_5 \,.$$

Then

$$\lambda \mathbb{M}_1 - \mathbb{M}_0 = \begin{bmatrix} -I & \lambda I & 0 & 0 & 0 \\ 0 & -I & \lambda I & 0 & 0 \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & 0 & A_1 & \lambda I & -I \\ 0 & 0 & A_0 & 0 & \lambda I \end{bmatrix}. \tag{4.3}$$

Note that (4.3) is strictly equivalent to the Fiedler pencil $\lambda M_5 - M_4 M_3 M_0 M_1 M_2$. Reversing in (4.3) the order of the block rows, we get the strictly equivalent pencil

$$\begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & -I & \lambda I & 0 & 0 \\ -I & \lambda I & 0 & 0 & 0 \end{bmatrix}.$$

If we finally change the sign of the fourth and fifth block rows, then we achieve the pencil

$$\lambda X + Y = \begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix},$$

which satisfies (4.1). Observe that the block rows whose signs have been changed have only $\pm I$, $\pm \lambda I$ and 0 blocks.

Example 4.1 and the paragraphs before it sketch a procedure to construct palindromic linearizations for odd degree matrix polynomials from Fiedler pencils in three main steps:

(S1) Build up a pencil $\lambda \mathbb{M}_1 - \mathbb{M}_0$ strictly equivalent to a Fiedler pencil, where $\mathbb{M}_0$ is a product of M_0 and some of the M_i matrices for $i = 1, \ldots, k-1$, and $\mathbb{M}_1$ is a product of M_k and the matrices M_{k-i}^{-1} with "complementary indices" to those in $\mathbb{M}_0$.

(S2) Reverse the order of the block rows of $\lambda \mathbb{M}_1 - \mathbb{M}_0$.

(S3) Change the sign of appropriate block rows in the pencil obtained in (**S2**).

In subsequent developments we adopt the following notation for simplicity. From the matrices M_i introduced in (2.2) and (2.3), we define for $j = 0, 1, \ldots, k-1$

$$\widetilde{M}_{k-j} = \begin{cases} M_k & \text{if } j = 0 \\ M_{k-j}^{-1} & \text{otherwise} \end{cases} . \tag{4.4}$$

The ordering and the selection of the factors in the pencil $\lambda I\!\!M_1 - I\!\!M_0$ in step (**S1**) above will be crucial in our construction. This is established in Definition 4.2.

Definition 4.2 (Admissible index set and associated pencils)**.** *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let the degree k be odd and $h := (k+1)/2$. A subset $C \subset \{0, 1, \ldots, k-1\}$ is said to be an* admissible index set *if*

- $0 \in C$,
- $C = \{j_1, \ldots, j_h\}$ *has cardinality h, and*
- $C \cap \{k - j_1, \ldots, k - j_h\} = \emptyset$.

In addition, given any bijection $\tau : C \to \{1, 2, \ldots, h\}$, the pencil of $P(\lambda)$ associated with C and τ is the $nk \times nk$ matrix pencil

$$L_\tau(\lambda) := \lambda \widetilde{M}_{k-\tau^{-1}(h)} \cdots \widetilde{M}_{k-\tau^{-1}(2)} \widetilde{M}_{k-\tau^{-1}(1)} - M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(h)}. \tag{4.5}$$

For brevity, we denote the coefficients of this pencil by

$$I\!\!M_0 := M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(h)}, \quad I\!\!M_1 := \widetilde{M}_{k-\tau^{-1}(h)} \cdots \widetilde{M}_{k-\tau^{-1}(2)} \widetilde{M}_{k-\tau^{-1}(1)}. \tag{4.6}$$

The construction of admissible index sets is a simple task because $\{1, 2, \ldots, k-1\} = \cup_{j=1}^{(k-1)/2} \{j, k-j\}$, so any admissible index set is formed by taking for each $j = 1, 2, \ldots, (k-1)/2$ exactly one element from $\{j, k-j\}$, and then adding 0. Given an admissible index set C there are many possible bijections τ and, as a consequence, there may be more than one different pencils $L_\tau(\lambda)$ of $P(\lambda)$ associated with C. Nevertheless, every pencil $L_\tau(\lambda)$ can be obtained by multiplying a certain Fiedler pencil of $P(\lambda)$ on the left and/or on the right by the inverses of the matrices $M_{k-j_1}, \ldots, M_{k-j_h}$ with $j_\ell \neq 0$. Therefore, every pencil $L_\tau(\lambda)$ is strictly equivalent to a Fiedler pencil and it is always a strong linearization of $P(\lambda)$ by Theorem 2.2. Finally, observe that C is a particular case of set C_s introduced in Section 3, with $s = h$, and that the matrix $I\!\!M_0$ is a particular case of matrix $I\!\!M_\tau$ in (3.1). In the arguments of Subsection 4.1, it is convenient to bear in mind that $\tau(j)$ for $j \in C$ is the position of the factor M_j in the product defining $I\!\!M_0$.

4.1 Technical lemmas

We gather in this subsection four technical lemmas that are used in the proof of the main result of the paper, i.e., Theorem 4.7. These lemmas study the block structure of the matrix $I\!\!M_0 \in \mathbb{F}^{nk\times nk}$, introduced in Definition 4.2, viewed as a $k \times k$ block matrix with $n \times n$ blocks.

Lemma 4.3. *Let $I\!\!M_0$ be as in Definition 4.2. Then the following statements hold.*

(a) *If τ has a consecution at i, $0 \leq i \leq k-2$, then the $(k-i)$th block-column of $\mathbb{M}_0$ contains exactly one identity block and its remaining blocks are zero.*

(b) *If τ has an inversion at i, $0 \leq i \leq k-2$, then the $(k-i)$th block-row of $\mathbb{M}_0$ contains exactly one identity block and its remaining blocks are zero.*

Proof. The result is an immediate consequence of Corollary 3.3 and `Algorithm 1` in Theorem 3.2. We only prove (a) because the proof of (b) is analogous. For $i = k-2$, the result follows from `Algorithm 1` and the fact that $\mathbb{M}_0 = W_{k-2}$. For other i recall that, from Corollary 3.3 (b), we know that $\mathbb{M}_0(:, k-i)$ is obtained from $W_i(:,2)$ by adding zero blocks. But, if τ has a consecution at i, then $W_i(:,2) = [I \;\; 0]^T$ by `Algorithm 1`. □

Lemma 4.4. *Let $\mathbb{M}_0$ be as in Definition 4.2 and $0 \leq i \leq k-1$. If $i \notin C$, then the $(k-i)$th block-row of $\mathbb{M}_0$ contains exactly one identity block and its remaining blocks are zero. The same is true for the $(k-i)$th block-column of $\mathbb{M}_0$.*

Proof. Recall that $k \geq 3$ and that $i > 0$ since $i \notin C$. We prove the result for block-rows, because for block-columns is analogous. If $i = k-1$, then `Algorithm 1` gives $\mathbb{M}_0 = W_{k-2} = \operatorname{diag}(I, W_{k-3})$ and the result is proven. If $0 < i \leq k-2$, then $\mathbb{M}_0(k-i,:)$ has the same nonzero blocks as $W_i(2,:)$. This follows from Corollary 3.3 (b) for $i < k-2$ and from $\mathbb{M}_0 = W_{k-2}$ for $i = k-2$. Therefore in the rest of the proof we focus in proving that $W_i(2,:)$ has only one nonzero block equal to I. `Algorithm 1` provides two possibilities for W_i when $i \notin C$:

$$W_i = \begin{bmatrix} -A_{i+1} & I & \\ I & 0 & \\ & & W_{i-1}(2:i+1, 2:i+1) \end{bmatrix} \quad \text{if } i+1 \in C,$$

or $W_i = \operatorname{diag}(I, W_{i-1})$ if $i+1 \notin C$. But $i \notin C$ in `Algorithm 1` implies that $W_{i-1}(1,:) = [I \;\; 0]$. So in any case $W_i(2,:)$ contains exactly one identity block and its remaining blocks are zero. □

Lemma 4.5. *Let $\mathbb{M}_0$ be as in Definition 4.2. Then the following statements hold.*

(a) *The matrix $\mathbb{M}_0$ contains exactly $k-1$ identity blocks.*

(b) *If the (i,j) block-entry of $\mathbb{M}_0$, with $i \neq j$, is equal to I, then a block $-A_d$, for some $0 \leq d \leq k-1$, is in the ith block-row or in the jth block-column of $\mathbb{M}_0$.*

Proof. Part (a) follows from `Algorithm 1`, that constructs $\mathbb{M}_0$ in $k-1$ steps. Observe that in each step exactly one identity block is added. This is evident in all cases except when $i \notin C$ and $i+1 \in C$, for $i \geq 1$. In this case W_i is obtained by adding as nonzero blocks two I and $-A_{i+1}$, and by removing the first block-row and the first block-column of W_{i-1}. But, $i \notin C$ implies $W_{i-1}(1,:) = [I \;\; 0]$ and $W_{i-1}(:,1) = [I \;\; 0]^T$, so in turn only one I is added.

Part (b). We will prove by induction that the result is true for every matrix $W_0, W_1, \ldots,$ $W_{k-2} = \mathbb{M}_0$ computed by `Algorithm 1`. The result is obviously true for W_0. Assume that it is true for W_{i-1}, with $i-1 \geq 0$, and let us prove it for W_i. Get W_i from W_{i-1} according to `Algorithm 1`, then a simple inspection shows that those off-diagonal identity blocks of W_i that are not in W_{i-1} satisfy the condition of the statement. For those off-diagonal identity blocks of W_i that are in W_{i-1}, note that: (1) off-diagonal blocks of W_{i-1} remain as off-diagonal blocks of W_i; and, (2) the block-rows and block-columns of W_{i-1} corresponding to off-diagonal identity blocks are contained in W_i. The result follows from (1) and (2). □

Lemma 4.6. *Let $\mathbb{M}_0$ be as in Definition 4.2, $0 \leq i \leq k-1$ and recall $h = (k+1)/2$. The $(k-i, k-i)$ block-entry of $\mathbb{M}_0$ is equal to I if and only if $i \notin C$ and $i+1 \notin C$. As a consequence, the (h, h) block-entry of $\mathbb{M}_0$ is not equal to I.*

Proof. Recall that $k \geq 3$. Consider first the case $i = k-1$, i.e., $k - i = 1$. Then, according to `Algorithm 1`, $\mathbb{M}_0(1,1) = W_{k-2}(1,1) = I$ if and only if $k-1 \notin C$. This proves the result for $i = k-1$ because $k \notin C$ by definition.

Now, we consider $i = k-2$, i.e., $k-i = 2$. Then, according to `Algorithm 1`, $\mathbb{M}_0(2,2) = W_{k-2}(2,2) = I$ if and only if $k-1 \notin C$ and $W_{k-3}(1,1) = I$. Use again `Algorithm 1` to see that $W_{k-3}(1,1) = I$ if and only if $k-2 \notin C$. This proves the result.

Finally consider $i \leq k-3$ and use Corollary 3.3 (a) to establish that $\mathbb{M}_0(k-i, k-i) = I$ if and only if $W_i(2,2) = I$. This never happens for $i = 0 \in C$ because $W_0(2,2) \neq I$. For $i \geq 1$, `Algorithm 1` says that $W_i(2,2) = I$ if and only if $i+1 \notin C$ and $W_{i-1}(1,1) = I$, which is equivalent to $i+1 \notin C$ and $i \notin C$.

Observe that $\mathbb{M}_0(h,h) \neq I$ because $h = k - (h-1)$ and Definition 4.2 for C does not allow $h-1 \notin C$ and $h \notin C$. Note that $2 \leq h \leq k-1$. $\square$

4.2 Main result, consequences and examples

Now we can state and prove the most important result in this work: Theorem 4.7, that presents a simple procedure to construct a family of palindromic strong linearizations of palindromic matrix polynomials with odd degree. From now on, $R \in \mathbb{F}^{nk \times nk}$ denotes the $k \times k$ block reverse identity matrix with $n \times n$ blocks, that is

$$R := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix}.$$

Theorem 4.7 (Palindromic linearizations for odd degree polynomials). *Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, with $A_i \in \mathbb{F}^{n\times n}$ and $A_k \neq 0$, be a (regular or singular) matrix polynomial of odd degree $k \geq 3$, let $h = (k+1)/2$ and let $C \subset \{0, 1, \ldots, k-1\}$ be an admissible index set. Let $\tau : C \to \{1, 2, \ldots, h\}$ be a bijection and let $L_\tau(\lambda)$ be the pencil of $P(\lambda)$ associated with C and τ, as it was defined in (4.5). Define $S \in \mathbb{F}^{nk\times nk}$ as the $k \times k$ block-diagonal matrix whose $n \times n$ diagonal block $S(i,i)$ is given for $i = 1, \ldots, k$ by*

$$S(i,i) := \begin{cases} -I & \text{if} \quad \begin{cases} \tau \text{ has an inversion at } i-1, \text{ or} \\ \tau \text{ has a consecution at } k-i, \text{ or} \\ i \in C \text{ and } i-1 \notin C \end{cases} \\ I & \text{otherwise} \end{cases} . \tag{4.7}$$

Then the pencil $S \cdot R \cdot L_\tau(\lambda)$ is a strong linearization of $P(\lambda)$ which is palindromic whenever $P(\lambda)$ is palindromic.

Remark 4.8. The reader is invited to check that the number of $-I$ blocks in S is $(k-1)/2$. For this purpose, prove first that the three conditions "τ has an inversion at $i-1$", "τ has a consecution at $k-i$" and "$i \in C$ and $i-1 \notin C$" satisfy that if any of them holds, then the other two do not hold. After this, note that the number of inversions of τ plus the number of consecutions of τ plus the number of indices i such that "$i \in C$ and $i-1 \notin C$" is $h-1 = (k-1)/2$.

Proof of Theorem 4.7. Since $S \cdot R \cdot L_\tau(\lambda)$ is strictly equivalent to a Fiedler pencil for $P(\lambda)$, we know from Theorem 2.2 that it is a strong linearization for $P(\lambda)$. The remaining task is to prove that $S \cdot R \cdot L_\tau(\lambda)$ is palindromic whenever $P(\lambda)$ is. For this, it is convenient to define the matrices:

$$\widehat{M}_{k-i} := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & A_{k-i} & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \qquad \text{for } i = 1, \ldots, k-1,$$

and

$$\widehat{M}_k = \begin{bmatrix} I_{(k-1)n} & \\ & A_k \end{bmatrix}.$$

We will use the following notation: $\mathbb{M}_0(P)$ and $\mathbb{M}_1(P)$ are the matrices defined in (4.6) for $P(\lambda)$, and $\mathbb{M}_0(-P)$ and $\mathbb{M}_1(-P)$ are the corresponding matrices for $-P(\lambda)$.

The proof will be carried out in two steps:

Step 1. We will prove that, if $P(\lambda)$ is palindromic, then

$$(R \cdot \mathbb{M}_1(P))^T = R \cdot \mathbb{M}_0(-P). \tag{4.8}$$

Step 2. We will prove that

$$R \cdot \mathbb{M}_0(-P) \cdot S = -S \cdot R \cdot \mathbb{M}_0(P). \tag{4.9}$$

Observe that (4.8) and (4.9) imply easily that $S \cdot R \cdot L_\tau(\lambda)$ is palindromic if $P(\lambda)$ is, because $S \cdot R \cdot L_\tau(\lambda) = \lambda S \cdot R \cdot \mathbb{M}_1(P) - S \cdot R \cdot \mathbb{M}_0(P)$ from (4.5). So,

$$(S \cdot R \cdot \mathbb{M}_1(P))^T = (R \cdot \mathbb{M}_1(P))^T \cdot S^T = R \cdot \mathbb{M}_0(-P) \cdot S = -S \cdot R \cdot \mathbb{M}_0(P),$$

which means that $S \cdot R \cdot L_\tau(\lambda)$ is palindromic. Note that in (4.8) we are assuming that P is palindromic, whereas (4.9) is true for an arbitrary $n \times n$ polynomial, P, of odd degree k.

Step 1. Let us begin by proving (4.8). From (4.2),

$$\widehat{M}_{k-i} = R M_{k-i}^{-1} R, \quad i = 1, \ldots, k-1, \quad \text{and} \quad \widehat{M}_k = R M_k R,$$

and taking into account that $R^2 = I$, we get

$$R \cdot \mathbb{M}_1(P) = \widehat{M}_{k-\tau^{-1}(h)} \cdots \widehat{M}_{k-\tau^{-1}(2)} \widehat{M}_{k-\tau^{-1}(1)} \cdot R.$$

In addition, if P is palindromic, i.e., $A_{k-i} = A_i^T$, then (2.2) and (2.3) imply

$$\widehat{M}_{k-i} = (M_i(-P))^T \quad \text{for } i = 0, 1, \ldots, k-1.$$

From the previous two equations we get (4.8):

$$\begin{aligned} (R \cdot \mathbb{M}_1(P))^T &= R^T \cdot \widehat{M}^T_{k-\tau^{-1}(1)} \widehat{M}^T_{k-\tau^{-1}(2)} \cdots \widehat{M}^T_{k-\tau^{-1}(h)} \\ &= R \cdot M_{\tau^{-1}(1)}(-P) \cdot M_{\tau^{-1}(2)}(-P) \cdots M_{\tau^{-1}(h)}(-P) = R \cdot \mathbb{M}_0(-P). \end{aligned}$$

Step 2. Now we address the proof of (4.9). We will use the matrix $\widetilde{S} := RSR \in \mathbb{F}^{nk \times nk}$. Viewed as a $k \times k$ block matrix with $n \times n$ blocks, $\widetilde{S}$ is block diagonal with diagonal blocks

$$\widetilde{S}(i,i) = S(k+1-i, k+1-i) \quad \text{for } i = 1, \ldots, k.$$

Observe that if $S(i,i) = -I$ and $H \in \mathbb{F}^{nk \times nk}$ is an arbitrary matrix viewed as a $k \times k$ block matrix with $n \times n$ blocks, then the ith block-column of HS is minus the ith block-column of H, whereas the ith block-row of SH is minus the ith block-row of H.

The identities $SR = R\widetilde{S}$ and $S^2 = I$ allow us to show that (4.9) is equivalent to

$$\widetilde{S} \cdot I\!\!M_0(-P) \cdot S = -I\!\!M_0(P)\,. \tag{4.10}$$

Therefore we focus in proving (4.10) in the rest of the proof, which relies in Lemmas 4.3, 4.4, 4.5 and 4.6 and is somewhat messy although elementary. In what follows all matrices are viewed as $k \times k$ block matrices with $n \times n$ blocks and we often use MATLAB notation on block indices. For brevity we use expressions like $H(i,:) = [0 \cdots 0\ I\ 0 \cdots 0]$ to indicate that the ith block-row of H has only one nonzero block equal to I that can be in any block-entry, including the first and the last ones.

From `Algorithm 1` for constructing $I\!\!M_0(P)$ and $I\!\!M_0(-P)$ and Lemma 4.5, it is easy to see that: (1) $I\!\!M_0(P)$ has $k-1$ blocks equal to I, h blocks $-A_{j_1}, \ldots, -A_{j_h}$, where $C = \{j_1, \ldots, j_h\}$, and the remaining blocks are zero; and, (2) if the blocks $-A_{j_1}, \ldots, -A_{j_h}$ in $I\!\!M_0(P)$ are replaced by $A_{j_1}, \ldots, A_{j_h}$, then $I\!\!M_0(-P)$ is obtained. Therefore, as S and $\widetilde{S}$ are block diagonal with diagonal blocks $\pm I$, proving (4.10) is equivalent to proving that the *only* effect of $\widetilde{S}$ and S in the product $\widetilde{S} \cdot I\!\!M_0(-P) \cdot S$ is transforming all $k-1$ identity blocks of $I\!\!M_0(-P)$ into minus identities or, equivalently in terms of block-entries, that

$$I\!\!M_0(-P)\,(i,j) = -\left(\widetilde{S} \cdot I\!\!M_0(-P) \cdot S\right)(i,j), \quad \text{whenever } I\!\!M_0(-P)(i,j) = I, \tag{4.11}$$

$$I\!\!M_0(-P)\,(i,j) = \left(\widetilde{S} \cdot I\!\!M_0(-P) \cdot S\right)(i,j), \qquad \text{otherwise,} \tag{4.12}$$

for $1 \leq i, j \leq k$. We will prove (4.11)-(4.12) through the following three steps:

(a) We will prove that if $S(j,j) = -I$, $1 \leq j \leq k$, then $I\!\!M_0(-P)\,(:,j) = [0 \cdots 0\ I\ 0 \cdots 0]^T$.

(b) We will prove that if $\widetilde{S}(i,i) = -I$, $1 \leq i \leq k$, then $I\!\!M_0(-P)\,(i,:) = [0 \cdots 0\ I\ 0 \cdots 0]$.

(c) We will prove that if $I\!\!M_0(-P)\,(i,j) = I$, then $\widetilde{S}(i,i) \neq -I$ or $S(j,j) \neq -I$.

Observe that **(a)**, **(b)** and **(c)** imply that each $-I$ block in S and $\widetilde{S}$ has the only effect of transforming one identity block of $I\!\!M_0(-P)$ into a minus identity block of $\widetilde{S} \cdot I\!\!M_0(-P) \cdot S$. But, this means that all identity blocks of $I\!\!M_0(-P)$ are transformed into minus identity blocks of $\widetilde{S} \cdot I\!\!M_0(-P) \cdot S$, because the total number of $-I$ blocks in S and $\widetilde{S}$ is $k-1$.

Proof of **(a)**. $S(j,j) = -I$ implies that "τ has an inversion at $j-1$" or "τ has a consecution at $k-j$" or "$j \in C$ and $j-1 \notin C$". Let us analyze separately these three possibilities. If "τ has an inversion at $j-1$", then $j \in C$, which is equivalent to $k-j \notin C$ and Lemma 4.4 implies the result. If "τ has a consecution at $k-j$", then Lemma 4.3 (a) implies the result. Finally, if "$j \in C$ and $j-1 \notin C$", then $k-j \notin C$ and Lemma 4.4 implies the result.

Proof of **(b)**. $\widetilde{S}(i,i) = S(k+1-i, k+1-i) = -I$ implies that "τ has an inversion at $k-i$" or "τ has a consecution at $i-1$" or "$k+1-i \in C$ and $k-i \notin C$". Let us analyze separately these three possibilities. If "τ has an inversion at $k-i$", then Lemma 4.3 (b) implies the result. If "τ has a consecution at $i-1$", then $i \in C$, which is equivalent to $k-i \notin C$ and Lemma 4.4 implies the result. Finally, if "$k+1-i \in C$ and $k-i \notin C$", then Lemma 4.4 implies again the result.

Proof of **(c)**. For $i \neq j$ proceed by contradiction: assume $\widetilde{S}(i,i) = -I$ and $S(j,j) = -I$. Therefore, from **(b)** and **(a)**, $I\!M_0(-P)\,(i,:) = [0 \cdots 0\ I\ 0 \cdots 0]$ and $I\!M_0(-P)\,(:,j) = [0 \cdots 0\ I\ 0 \cdots 0]^T$. This implies $I\!M_0(-P)\,(i,j) \neq I$ by Lemma 4.5 (b).

For $i = j$, we give a direct argument. $I\!M_0(-P)\,(i,i) = I$ implies $k-i \notin C$ and $k-i+1 \notin C$ by Lemma 4.6. This is equivalent to $i \in C$ and $i-1 \in C$, by Definition 4.2. So, in this situation the definition of S implies that $S(i,i) = -I$ holds only if "τ has an inversion at $i-1$" and that $\widetilde{S}(i,i) = S(k+1-i, k+1-i) = -I$ holds only if "τ has a consecution at $i-1$". Therefore $S(i,i) \neq -I$ or $\widetilde{S}(i,i) \neq -I$. □

Theorem 4.7 provides many strong linearizations $S \cdot R \cdot L_\tau(\lambda)$ for $P(\lambda)$ that are palindromic whenever $P(\lambda)$ is. Note in the first place that there exist $2^{(k-1)/2}$ different[2] admissible index sets C, and that for each of these sets exists many different bijections $\tau : C \to \{1, 2, \ldots, h\}$. In this context, it is important to note that different bijections of the same C may produce the same linearization due to the commutativity relations (2.5) and, as a consequence, that different index sets may have associated different numbers of linearizations. This can be observed in Table 4.1. However, if $C_1 \neq C_2$ are two admissible index sets, then linearizations associated with C_1 are never equal to linearizations associated with C_2, because the coefficients of $P(\lambda)$ in the zero-degree terms of the linearizations associated with C_1 are different from those in the zero-degree terms of the linearizations associated with C_2.

Next, we present some examples to emphasize that the palindromic linearizations in Theorem 4.7 are easily constructible from the coefficients of the polynomial, and that certain selections of the set C and the bijection τ produce particularly simple patterns.

Example 4.9. Let $k \geq 3$ be an odd integer. Consider the following admissible set

$$C = \{2j : j = 0, 1, \ldots, (k-1)/2\} = \{0, 2, 4, \ldots, k-1\},$$

and the bijection $\tau : C \to \{1, 2, \ldots, h\}$ defined by $\tau(2j) = j+1$, for $j = 0, 1, \ldots, (k-1)/2$. Then, the pencil (4.5) associated with C and τ is

$$L_\tau(\lambda) = \lambda M_1^{-1} M_3^{-1} \cdots M_{k-2}^{-1} M_k - M_0 M_2 \cdots M_{k-3} M_{k-1}, \tag{4.13}$$

and the matrix S in (4.7) satisfies $S(i,i) = I$ for odd i, and $S(i,i) = -I$ for even i. For $L_\tau(\lambda)$ in (4.13), we denote by $L_k(\lambda) := S \cdot R \cdot L_\tau(\lambda)$ the pencil in the statement of Theorem 4.7 associated with $P(\lambda)$, then we have

$$L_3(\lambda) = \lambda \begin{bmatrix} & I & A_1 \\ & 0 & -I \\ A_3 & & \end{bmatrix} + \begin{bmatrix} & & A_0 \\ I & 0 & \\ A_2 & -I & \end{bmatrix},$$

$$L_5(\lambda) = \lambda \begin{bmatrix} & & & I & A_1 \\ & & & 0 & -I \\ & I & A_3 & & \\ & 0 & -I & & \\ A_5 & & & & \end{bmatrix} + \begin{bmatrix} & & & & A_0 \\ & & I & 0 & \\ & & A_2 & -I & \\ I & 0 & & & \\ A_4 & -I & & & \end{bmatrix},$$

[2]Recall that $j \in C$, $1 \leq j \leq k-1$, if and only if $k-j \notin C$ and that $\{1, \ldots, k-1\} = \bigcup_{j=1}^{(k-1)/2} \{j, k-j\}$, so there are $2^{(k-1)/2}$ ways of selecting the elements of C.

and a direct inductive argument gives

$$L_k(\lambda) = \begin{bmatrix} & & & & & & \lambda I & \lambda A_1 + A_0 \\ & & & & & I & 0 & -\lambda I \\ & & & & \lambda I & \lambda A_3 + A_2 & -I & \\ & & & \iddots & \iddots & \iddots & & \\ & & I & 0 & -\lambda I & & & \\ & \lambda I & \lambda A_{k-2} + A_{k-3} & -I & & & & \\ I & 0 & -\lambda I & & & & & \\ \lambda A_k + A_{k-1} & -I & & & & & & \end{bmatrix},$$

which has a reverse tridiagonal pattern. Note that, by the commutativity relations (2.5) and the analogous ones for the inverses of the matrices M_i's, each bijection $\tau : \{0, 2, 4, \ldots, k-1\} \to \{1, 2, \ldots, h\}$ gives the same pencil $L_k(\lambda)$.

Example 4.10. In this example we show for polynomials of degree $k = 5$ several of the palindromic linearizations in Theorem 4.7 having special patterns. The reader can generalize these patterns to arbitrary odd degrees. First, we illustrate that there exist other "reverse" tridiagonal patterns apart from the one in Example 4.9. For this purpose, choose the admissible index set $C_1 = \{0, 1, 3\}$ and the bijection $\tau_1 : C_1 \to \{1, 2, 3\}$ defined by $\tau_1(0) = 1$, $\tau_1(1) = 2$, $\tau_1(3) = 3$. Then

$$L_{\tau_1}(\lambda) = \lambda M_2^{-1} M_4^{-1} M_5 - M_0 M_1 M_3 .$$

The blocks of S in (4.7) are $S(i,i) = I$ for $i = 1, 2, 4$ and $S(i,i) = -I$ for $i = 3, 5$. Then

$$S \cdot R \cdot L_{\tau_1}(\lambda) = \lambda \begin{bmatrix} & & & & I \\ & & I & A_2 & \\ & & 0 & -I & \\ A_5 & A_4 & & & \\ 0 & -I & & & \end{bmatrix} + \begin{bmatrix} & & & A_0 & 0 \\ & & & A_1 & -I \\ & I & 0 & & \\ & A_3 & -I & & \\ I & & & & \end{bmatrix}.$$

Next, for the admissible index set $C_2 = \{0, 1, 2\}$, we present two palindromic linearizations with maximum block-bandwith. Let $\tau_2 : C_2 \to \{1, 2, 3\}$ be defined by $\tau(0) = 3\,, \tau(1) = 2\,, \tau(2) = 1$, then

$$L_{\tau_2}(\lambda) = \lambda M_5 M_4^{-1} M_3^{-1} - M_2 M_1 M_0 .$$

In this case $S(i,i) = I$ for $i = 3, 4, 5$ and $S(i,i) = -I$ for $i = 1, 2$, so

$$S \cdot R \cdot L_{\tau_2}(\lambda) = \lambda \begin{bmatrix} & & & & -I \\ & & & -I & \\ 0 & I & A_3 & & \\ I & 0 & A_4 & & \\ 0 & 0 & A_5 & & \end{bmatrix} + \begin{bmatrix} & & 0 & I & 0 \\ & & I & 0 & 0 \\ & & A_2 & A_1 & A_0 \\ & -I & & & \\ -I & & & & \end{bmatrix}.$$

Observe that the zero and the first degree terms consist of three factors with consecutive indices, what causes the structure of the 3×3 block matrix in the upper-right (resp. lower-left) corner in the zero (resp. first) degree term. Other 3×3 structures of this type can be produced by taking different orders of the M_0, M_1 and M_2 factors in the zero degree term. For instance, take $\tau_3 : C_2 \to \{1,2,3\}$ defined by $\tau(0)=1\,,\tau(1)=3\,,\tau(2)=2$, then

$$L_{\tau_3}(\lambda) = \lambda M_4^{-1} M_3^{-1} M_5 - M_0 M_2 M_1\,.$$

Now $S(i,i)=I$ for $i=1,3,4$ and $S(i,i)=-I$ for $i=2,5$, so

$$S \cdot R \cdot L_{\tau_3}(\lambda) = \lambda \begin{bmatrix} & & & & I \\ & & & -I & \\ 0 & I & A_3 & & \\ A_5 & 0 & A_4 & & \\ 0 & 0 & -I & & \end{bmatrix} + \begin{bmatrix} & & 0 & A_0 & 0 \\ & & I & 0 & 0 \\ & & A_2 & A_1 & -I \\ & -I & & & \\ I & & & & \end{bmatrix}.$$

Table 4.1 displays all pencils that may be constructed for degree $k=5$ with the procedure of Theorem 4.7, including the examples above.

If we look carefully at the patterns of the pencils $S \cdot R \cdot L_\tau(\lambda)$ in Table 4.1, we find that, up to the signs of the identity blocks, these pencils are paired up by block symmetry through the main block anti-diagonal. More precisely, the first one is paired with the third one, the second one with the fourth one, the fifth one with the sixth one and the seventh one with the eighth one. The ninth one is self-paired, because, up to signs, it is block symmetric through the main block antidiagonal. Lemma 4.11 below shows that this is not by chance. Before stating this lemma, we need to introduce the concept of *reversal bijection.* Let C be an admissible index set and $\tau : C \to \{1,2,\ldots,h\}$ be a bijection, then the *reversal bijection* of τ is $\operatorname{rev}\tau : C \to \{1,2,\ldots,h\}$ where $\operatorname{rev}\tau(j) = h+1-\tau(j)$. In plain words, the M_j factors of $\mathbb{M}_{\operatorname{rev}\tau}$ are the ones of $\mathbb{M}_\tau$ in (3.1) but placed in reverse order. Then Lemma 4.11 proves that each pencil $S \cdot R \cdot L_\tau(\lambda)$ constructed in Theorem 4.7 is paired up with the pencil $S \cdot R \cdot L_{\operatorname{rev}\tau}(\lambda)$. We will also need the block-transpose operation: Let $A=(A_{ij})$ be a block $r \times s$ matrix with $m \times n$ blocks A_{ij}. The *block transpose* of A is the block $s \times r$ matrix $A^{\mathcal{B}}$ with $m \times n$ blocks defined by $\left(A^{\mathcal{B}}\right)_{ij} = A_{ji}$.

Lemma 4.11. *Let τ and $L_\tau(\lambda)$ be as in the statement of Theorem* 4.7 *and let* $\operatorname{rev}\tau$ *be the reversal bijection of τ. Then*

$$R\,(R \cdot L_\tau(\lambda))^{\mathcal{B}}\,R = R \cdot L_{\operatorname{rev}\tau}(\lambda)\,.$$

Proof. We first recall [19, Chapter 3] that if A and C are block partitioned matrices with $n \times n$ blocks A_{ij} and C_{ij} such that $A_{ij}C_{jp} = C_{jp}A_{ij}$, for all i,j,p, then $(AC)^{\mathcal{B}} = C^{\mathcal{B}}A^{\mathcal{B}}$. This property implies that

$$R(R \cdot L_\tau(\lambda))^{\mathcal{B}} R = R \cdot (L_\tau(\lambda))^{\mathcal{B}} \cdot R^{\mathcal{B}} R = R \cdot (L_\tau(\lambda))^{\mathcal{B}}.$$

Next, it can be proved that $(L_\tau(\lambda))^{\mathcal{B}} = L_{\operatorname{rev}\tau}(\lambda)$ with some care. We only sketch the proof. For the zero degree terms, note that $\operatorname{rev}\tau$ has a consecution (resp. inversion) at j if and only if τ has an inversion (resp. consecution) at j and use `Algorithm 1` for proving the result through induction on the matrices $W_0, W_1, \ldots, W_{k-2}$ of `Algorithm 1`. For the first degree terms, one has to develop and use an analogous algorithm to construct the matrices $\mathbb{M}_1$ in (4.6). $\square$

C	τ	S	$S \cdot R \cdot L_\tau(\lambda)$
$\{0,1,2\}$	$(1,2,3)$	$\mathrm{diag}(I,I,I,-I,-I)$	$\begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(3,1,2)$	$\mathrm{diag}(-I,I,I,-I,I)$	$\begin{bmatrix} 0 & 0 & I & 0 & -\lambda I \\ 0 & 0 & A_1 & \lambda I & A_0 \\ \lambda I & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ -I & \lambda A_5 & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(3,2,1)$	$\mathrm{diag}(-I,-I,I,I,I)$	$\begin{bmatrix} 0 & 0 & 0 & I & -\lambda I \\ 0 & 0 & I & -\lambda I & 0 \\ 0 & \lambda I & \lambda A_3 + A_2 & A_1 & A_0 \\ \lambda I & -I & \lambda A_4 & 0 & 0 \\ -I & 0 & \lambda A_5 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(1,3,2)$	$\mathrm{diag}(I,-I,I,I,-I)$	$\begin{bmatrix} 0 & 0 & 0 & A_0 & \lambda I \\ 0 & 0 & I & -\lambda I & 0 \\ 0 & \lambda I & \lambda A_3 + A_2 & A_1 & -I \\ \lambda A_5 & -I & \lambda A_4 & 0 & 0 \\ I & 0 & -\lambda I & 0 & 0 \end{bmatrix}$
$\{0,1,3\}$	$(1,2,3)$	$\mathrm{diag}(I,I,-I,I,-I)$	$\begin{bmatrix} 0 & 0 & 0 & A_0 & \lambda I \\ 0 & 0 & \lambda I & \lambda A_2 + A_1 & -I \\ 0 & I & 0 & -\lambda I & 0 \\ \lambda A_5 & \lambda A_4 + A_3 & -I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,3\}$	$(3,2,1)$	$\mathrm{diag}(-I,I,-I,I,I)$	$\begin{bmatrix} 0 & 0 & 0 & I & -\lambda I \\ 0 & 0 & \lambda I & \lambda A_2 + A_1 & A_0 \\ 0 & I & 0 & -\lambda I & 0 \\ \lambda I & \lambda A_4 + A_3 & -I & 0 & 0 \\ -I & \lambda A_5 & 0 & 0 & 0 \end{bmatrix}$
$\{0,3,4\}$	$(1,2,3)$	$\mathrm{diag}(I,-I,-I,I,I)$	$\begin{bmatrix} 0 & 0 & \lambda I & \lambda A_2 & \lambda A_1 + A_0 \\ 0 & 0 & 0 & I & -\lambda I \\ I & 0 & 0 & -\lambda I & 0 \\ A_3 & \lambda I & -I & 0 & 0 \\ \lambda A_5 + A_4 & -I & 0 & 0 & 0 \end{bmatrix}$
$\{0,3,4\}$	$(3,2,1)$	$\mathrm{diag}(I,I,-I,-I,I)$	$\begin{bmatrix} 0 & 0 & 0 & \lambda I & \lambda A_1 + A_0 \\ 0 & 0 & \lambda I & -I & \lambda A_2 \\ 0 & I & 0 & 0 & -\lambda I \\ I & -\lambda I & 0 & 0 & 0 \\ \lambda A_5 + A_4 & A_3 & -I & 0 & 0 \end{bmatrix}$
$\{0,2,4\}$	$(1,2,3)$	$\mathrm{diag}(I,-I,I,-I,I)$	$\begin{bmatrix} 0 & 0 & 0 & \lambda I & \lambda A_1 + A_0 \\ 0 & 0 & I & 0 & -\lambda I \\ 0 & \lambda I & \lambda A_3 + A_2 & -I & 0 \\ I & 0 & -\lambda I & 0 & 0 \\ \lambda A_5 + A_4 & -I & 0 & 0 & 0 \end{bmatrix}$

Table 4.1: This table shows all possible palindromic linearizations for polynomials with degree 5 constructed by using Theorem 4.7. For each admissible index set $C = \{j_1, j_2, j_3\}$, the bijections $\tau : C \to \{1,2,3\}$ are described as $(\tau(j_1), \tau(j_2), \tau(j_3))$.

Lemma 4.11 tells us that, up to the change of signs given by the matrix S, the pencils constructed in Theorem 4.7 are paired up by the operation $R\,(\cdot)^{\mathcal{B}}\,R$. This operation is precisely a block symmetry through the main block anti-diagonal. Notice that when $C = \{2j : j = 0, 1, \ldots, (k-1)/2\}$, due to the commutativity relations (2.5), the pencils $L_\tau(\lambda)$ are equal for all bijections $\tau : C \to \{1, 2, \ldots, h\}$. This unique pencil satisfies the identity $R\,(R \cdot L_\tau(\lambda))^{\mathcal{B}} \cdot R = R \cdot L_\tau(\lambda)$, that is, $R \cdot L_\tau(\lambda)$ is block symmetric through the main block anti-diagonal, and then it is self-paired by the operation $R\,(\cdot)^{\mathcal{B}}\,R$. This corresponds to the pencil at the bottom of Table 4.1. As a consequence, given any k odd, the number of different pencils constructed in Theorem 4.7 is odd.

Theorem 4.7 asserts, in particular, that each palindromic polynomial with odd degree has a palindromic strong linearization. It is worthwhile stating this as a separate fact.

Corollary 4.12. *Let k be an odd number and $P(\lambda)$ be an $n \times n$ palindromic matrix polynomial of degree k. Then there exists an $nk \times nk$ palindromic strong linearization of $P(\lambda)$.*

We want to stress that Corollary 4.12 is not true for even degree polynomials. For instance, the palindromic scalar polynomial $p(\lambda) = \lambda^2 + 2\lambda + 1$ does not have any palindromic linearization with size 2×2 [21, Example 3.10].

5 Anti-palindromic linearizations for odd degree

We have already commented in Section 1 that anti-palindromic matrix polynomials, i.e. those satisfying $\operatorname{rev} P(\lambda) = -P(\lambda)^T$, have also some interest in applications. Therefore, it is natural to look for anti-palindromic linearizations of anti-palindromic polynomials. We show in this section that for polynomials with *odd degree*, any method for constructing palindromic linearizations of palindromic matrix polynomials can be very easily adapted to construct anti-palindromic linearizations of anti-palindromic polynomials. This is proved in Theorem 5.3, that can be applied, in particular, to the linearizations introduced in Theorem 4.7. We need to prove first a couple of simple lemmas.

Lemma 5.1. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial and define $Q(\lambda) := P(-\lambda)$. If $L(\lambda)$ is a linearization (resp. strong linearization) of $P(\lambda)$, then $\widetilde{L}(\lambda) := L(-\lambda)$ is a linearization (resp. strong linearization) of $Q(\lambda)$.*

Proof. Assume that the degree of $P(\lambda)$ is k. If $L(\lambda)$ is a linearization of $P(\lambda)$, then, by definition, there exist two unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $U(\lambda)\,L(\lambda)\,V(\lambda) = \operatorname{diag}(I_{(k-1)n}, P(\lambda))$. So, $U(-\lambda)\,L(-\lambda)\,V(-\lambda) = \operatorname{diag}(I_{(k-1)n}, P(-\lambda))$, which proves that $\widetilde{L}(\lambda)$ is a linearization of $Q(\lambda)$ because $U(-\lambda)$ and $V(-\lambda)$ are unimodular. The result for strong linearizations requires more attention. Observe that

$$
\begin{aligned}
(\operatorname{rev} Q)\,(\lambda) = \lambda^k Q(1/\lambda) = \lambda^k P(-1/\lambda) &= (-1)^k \, \Big((-\lambda)^k P(-1/\lambda)\Big) \\
&= (-1)^k \, (\operatorname{rev} P)\,(-\lambda). \qquad (5.1)
\end{aligned}
$$

If $L(\lambda)$ is a strong linearization of $P(\lambda)$, then we also have that $Y(\lambda)\,(\operatorname{rev} L)(\lambda)\,Z(\lambda) = \operatorname{diag}(I_{(k-1)n}, (\operatorname{rev} P)(\lambda))$ for some unimodular matrices $Y(\lambda)$ and $Z(\lambda)$. As a consequence, $Y(-\lambda)\,(\operatorname{rev} L)(-\lambda)\,Z(-\lambda) = \operatorname{diag}(I_{(k-1)n}, (\operatorname{rev} P)(-\lambda))$, and from (5.1)

$$
Y(-\lambda)\,\Big(-(\operatorname{rev}\widetilde{L})(\lambda)\Big)\,Z(-\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & (-1)^k\,(\operatorname{rev} Q)\,(\lambda) \end{bmatrix}.
$$

This implies

$$E(\lambda)\,(\operatorname{rev}\widetilde{L})(\lambda)\,Z(-\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & (\operatorname{rev}Q)\,(\lambda) \end{bmatrix},$$

with $E(\lambda) = -\operatorname{diag}(I_{(k-1)n}, (-1)^k I)\,Y(-\lambda)$. Note that $E(\lambda)$ and $Z(-\lambda)$ are unimodular matrices and, therefore, $(\operatorname{rev}\widetilde{L})(\lambda)$ is a linearization of $(\operatorname{rev}Q)\,(\lambda)$. $\square$

Lemma 5.2. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial with* odd degree *and define $Q(\lambda) := P(-\lambda)$. Then $P(\lambda)$ is anti-palindromic if and only if $Q(\lambda)$ is palindromic. Also, $P(\lambda)$ is palindromic if and only if $Q(\lambda)$ is anti-palindromic.*

Proof. It follows directly from (5.1). $\square$

Next, we state Theorem 5.3, the main result in this section. It is a trivial consequence of Lemmas 5.1 and 5.2 and so its proof is omitted. Simply note that Lemma 5.2 has to be applied also on linearizations.

Theorem 5.3. *Let $P(\lambda)$ be any $n \times n$ anti-palindromic matrix polynomial with* odd degree *and define $Q(\lambda) := P(-\lambda)$. Let $\widetilde{L}(\lambda)$ be any strong palindromic linearization of the palindromic polynomial $Q(\lambda)$. Then $L(\lambda) := \widetilde{L}(-\lambda)$ is a strong anti-palindromic linearization of $P(\lambda)$.*

6 The recovery of minimal indices

We have seen in Sections 1 and 2 that the minimal indices are intrinsic quantities associated with singular matrix polynomials that are relevant in many control problems [8, 16]. In this section we show how to recover very easily the minimal indices of a polynomial from those of the linearizations introduced in Theorem 4.7. The results in this section are consequence of results in [6].

Let us recall very briefly the concept of minimal indices (see [5, Section 2] or [6, Section 2] for more complete summaries). A *vector polynomial* is a vector whose entries are polynomials in the variable λ, and its *degree* is the greatest degree of its components. For any subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$ it is always possible to find a basis consisting entirely of vector polynomials. Then, the *order* of a polynomial basis of $\mathcal{V}$ is the sum of the degrees of its vectors [8, p. 494] and a minimal basis of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$. It can be shown [8] that for any subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. The left (resp. right) minimal indices of a singular matrix polynomial $P(\lambda)$ are the minimal indices of its left (resp. right) null spaces (see Section 2). Observe that any square matrix polynomial $P(\lambda)$ has the same number of left and right minimal indices, and recall that if $P(\lambda)$ is palindromic, then its left minimal indices are equal to its right minimal indices [5, Theorem 3.6].

As mentioned in Section 2, strong linearizations preserve the elementary divisors of a polynomial $P(\lambda)$ and also the number of left and right minimal indices, but they do not preserve in general the values of the minimal indices. Therefore, the recovery of the minimal indices of $P(\lambda)$ from those of one of its linearization is, in general, a non trivial task [5, 6]. However, Theorem 6.1 shows that this recovery is very simple from the linearizations in Theorem 4.7.

Theorem 6.1. *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with odd degree $k \geq 3$ and let $S \cdot R \cdot L_\tau(\lambda)$ be one of the strong linearizations of $P(\lambda)$ introduced in Theorem* 4.7. *Let $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ and $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ be, respectively, the left and right minimal indices of $P(\lambda)$. Then the following statements hold.*

(a) *The left and right minimal indices of $S \cdot R \cdot L_\tau(\lambda)$ are, respectively,*

$$\begin{aligned} &\eta_1 + \frac{k-1}{2} \leq \eta_2 + \frac{k-1}{2} \leq \cdots \leq \eta_p + \frac{k-1}{2} \qquad \textit{and} \\ &\varepsilon_1 + \frac{k-1}{2} \leq \varepsilon_2 + \frac{k-1}{2} \leq \cdots \leq \varepsilon_p + \frac{k-1}{2}. \end{aligned}$$

(b) *If $P(\lambda)$ is palindromic, then $\eta_i = \varepsilon_i$ for $i = 1, \ldots, p$ and the left and right minimal indices of $S \cdot R \cdot L_\tau(\lambda)$ are both equal to $\varepsilon_1 + \frac{k-1}{2} \leq \varepsilon_2 + \frac{k-1}{2} \leq \cdots \leq \varepsilon_p + \frac{k-1}{2}$.*

Proof. The minimal indices are preserved under strict equivalence and we have already discussed in Section 4 that every pencil $S \cdot R \cdot L_\tau(\lambda)$ is strictly equivalent to a certain Fiedler pencil of $P(\lambda)$. We denote by $F_\sigma(P)$ this Fiedler pencil, following the notation in [6]. In this proof, we also denote $S \cdot R \cdot L_\tau(\lambda)$ by $S \cdot R \cdot L_\tau(P)$ to make explicit the dependence on $P(\lambda)$ and dropping the dependence on λ for brevity. Therefore the minimal indices of $S \cdot R \cdot L_\tau(P)$ are equal to those of $F_\sigma(P)$.

The pencil $F_\sigma(P)$ is a function of $P(\lambda)$ and we can consider it for any other $n \times n$ matrix polynomial $Q(\lambda)$ with degree k. In this case, it is denoted by $F_\sigma(Q)$. If $0 \leq \eta_1' \leq \eta_2' \leq \cdots \leq \eta_q'$ and $0 \leq \varepsilon_1' \leq \varepsilon_2' \leq \cdots \leq \varepsilon_q'$ are, respectively, the left and right minimal indices of $Q(\lambda)$, then the left and right minimal indices of $F_\sigma(Q)$ are given, respectively, by [6, Corollaries 5.8 and 5.11]

$$\eta_1' + \mathfrak{c}(\sigma) \leq \eta_2' + \mathfrak{c}(\sigma) \leq \cdots \leq \eta_q' + \mathfrak{c}(\sigma) \qquad \text{and} \tag{6.1}$$

$$\varepsilon_1' + \mathfrak{i}(\sigma) \leq \varepsilon_2' + \mathfrak{i}(\sigma) \leq \cdots \leq \varepsilon_q' + \mathfrak{i}(\sigma). \tag{6.2}$$

The quantities $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$ are defined in [6], but only two properties of them are of interest to us: (1) $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = k - 1$; and, (2) they are the same for any $n \times n$ singular polynomial $Q(\lambda)$ of degree k. Therefore, we can determine $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$ by applying (6.1) and (6.2) to a particular matrix polynomial. Let us assume that $Q(\lambda)$ is singular and palindromic, then $\eta_i' = \varepsilon_i'$ for $i = 1, \ldots, q$ [5, Theorem 3.6]. Moreover, the minimal indices of $F_\sigma(Q)$ are equal to those of $S \cdot R \cdot L_\tau(Q)$, but this linearization of $Q(\lambda)$ is palindromic by Theorem 4.7 and then $\eta_i' + \mathfrak{c}(\sigma) = \varepsilon_i' + \mathfrak{i}(\sigma)$ for $i = 1, \ldots, q$. So, $\mathfrak{c}(\sigma) = \mathfrak{i}(\sigma) = (k-1)/2$ and Theorem 6.1 follows from applying (6.1) and (6.2) to $P(\lambda)$. $\square$

Our last result is a corollary of Theorem 6.1 that asserts that all pencils constructed in Theorem 4.7 are strictly equivalent.

Corollary 6.2. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with odd degree $k \geq 3$. Then all pencils constructed in Theorem* 4.7 *for $P(\lambda)$ are strictly equivalent.*

Proof. By Theorem 6.1 all pencils constructed in Theorem 4.7 for $P(\lambda)$ have the same minimal indices. On the other hand, all these pencils are strong linearizations of $P(\lambda)$, so they all have the same finite and infinite elementary divisors. Since two matrix pencils are strictly equivalent if and only if they have the same elementary divisors and minimal indices [9], the result follows. $\square$

7 Conclusions and future work

We have presented a symbolic procedure to construct many strong linearizations of square matrix polynomials which are valid for all polynomials of odd degree $k \geq 3$ over an arbitrary field, and are palindromic whenever the polynomial is palindromic. These linearizations are easily constructible from the coefficients of the polynomial and have been simply adapted to construct anti-palindromic linearizations of anti-palindromic polynomials of odd degree. Finally, we have also shown that the minimal indices of these linearizations are very easily related to the minimal indices of the polynomial.

The results in this paper are in sharp contrast with the situation for even degree palindromic polynomials, since it is known that there are palindromic matrix polynomials of even degree that have no palindromic linearizations of any kind. Thus the natural continuation of the present paper is to address the even degree case in more detail, and obtain necessary and sufficient conditions for the existence of palindromic linearizations. These conditions will be presented in [7].

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