Witten-Hodge theory for manifolds with boundary

Al-Zamil, Qusay and Montaldi, James

2010

MIMS EPrint: **2010.31**

ISSN 1749-9097

Witten-Hodge theory for manifolds with boundary

Qusay S.A. Al-Zamil & James Montaldi

April 11, 2010

Abstract

We consider a compact, oriented, smooth Riemannian manifold M (with or without boundary) and we suppose G is a torus acting by isometries on M. Given X in the Lie algebra and corresponding vector field X_M on M, one defines Witten's inhomogeneous operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M} : \Omega_G^{\pm} \to \Omega_G^{\mp}$ (even/odd invariant forms on M). Witten [10] showed that the resulting cohomology classes have X_M-harmonic representatives (forms in the null space of $\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2$), and the cohomology groups are isomorphic to the ordinary de Rham cohomology groups of the fixed point set. Our principal purpose is to extend these results to manifolds with boundary. In particular, we define relative (to the boundary) and absolute versions of the X_M-cohomology and show the classes have representative X_M-harmonic fields with appropriate boundary conditions. To do this we present the relevant version of the Hodge-Morrey-Friedrichs decomposition theorem for invariant forms in terms of the operator d_{X_M} and its adjoint δ_{X_M}; the proof involves showing that certain boundary value problems are elliptic. We also elucidate the connection between the X_M-cohomology groups and the relative and absolute equivariant cohomology, following work of Atiyah and Bott [2]. This connection is then exploited to show that every harmonic field with appropriate boundary conditions on F has a uniqe extension to an X_M-harmonic field on M, with corresponding boundary conditions.

Keywords: Hodge theory, manifolds with boundary, equivariant cohomology, Killing vector fields

1 Introduction

Let M be a compact oriented Riemannian manifold of dimension n without boundary, and for each k denote by $\Omega^k = \Omega^k(M)$ the space of smooth differential k-forms on M. The de Rham cohomology of M is defined to be $H^k(M) = \ker \mathsf{d}_k / \operatorname{im} \mathsf{d}_{k-1}$, where d_k is the restriction of the exterior differential d to Ω^k. Based on the Riemannian structure, there is a natural inner product on each Ω^k defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta), \tag{1.1}$$

where $\star : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator [1, 9]. One defines $\delta : \Omega^k \to \Omega^{k-1}$ by

$$\delta\omega = (-1)^{n(k+1)+1}(\star \mathsf{d} \star)\omega. \tag{1.2}$$

This is seen to be the formal adjoint of d relative to the inner product (1.1): $\langle \mathsf{d}\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. The Hodge Laplacian is defined by $\Delta = (\mathsf{d}+\delta)^2 = d\delta + \delta d$, and a form ω is said to be *harmonic* if $\Delta\omega = 0$.

In the 1930s, Hodge [5] proved the fundamental result that each cohomology class contains a unique harmonic form. A more precise statement is that, for each k,

$$\Omega^k(M) = \mathcal{H}^k \oplus \mathsf{d}\Omega^{k-1} \oplus \delta\Omega^{k+1}. \tag{1.3}$$

The direct sums are orthogonal with respect to the inner product (1.1), and the direct sum of the first two subspaces is equal to the subspace of all closed k-forms (that is, $\ker \mathsf{d}_k$).

Furthermore, on a manifold without boundary, any harmonic form $\omega \in \ker\Delta$ is both closed ($\mathsf{d}\omega = 0$) and co-closed ($\delta\omega = 0$), as

$$0 = \langle\Delta\omega, \omega\rangle = \langle \mathsf{d}\delta\omega, \omega\rangle + \langle\delta\mathsf{d}\omega, \omega\rangle = \langle\delta\omega, \delta\omega\rangle + \langle\mathsf{d}\omega, \mathsf{d}\omega\rangle = \|\delta\omega\|^2 + \|\mathsf{d}\omega\|^2. \tag{1.4}$$

For manifolds with boundary this is no longer true, and in general we write

$$\mathcal{H}^k = \mathcal{H}^k(M) = \ker d \cap \ker\delta.$$

Thus for manifolds without boundary $\mathcal{H}(M) = \ker\Delta$, the space of harmonic forms, and it follows that the Hodge star operator realizes Poincaré duality at the level of harmonic forms.

An interesting observation which follows from the theorem of Hodge is the following. If a group G acts on M then there is an induced action on each $H^k(M)$, and if this action is trivial (for example, if G is connected) and the action on M is by isometries, then each harmonic form is invariant under the action of G.

Now suppose K is a Killing vector field on M (meaning that the Lie derivative of the metric vanishes). Witten [10] defines an operator on differential forms

$$\mathsf{d}_s := \mathsf{d} + s\iota_K = \mathsf{d} + \iota_{X_M},$$

for $X_M = sK$ where ι_K is interior multiplication of a form with K (we write X_M as we will think of X as an element of a Lie algebra acting on M and X_M its associated vector field). This operator is no longer homogeneous in the degree of the form: if $\omega \in \Omega^k(M)$ then $\mathsf{d}_s\omega \in \Omega^{k+1} \oplus \Omega^{k-1}$. Note then that $\mathsf{d}_s : \Omega^{\pm} \to \Omega^{\mp}$, where $\Omega^{\pm}$ is the space of forms of even $(+)$ or odd $(-)$ degree. Let us write $\delta_s = \mathsf{d}_s^*$ for the formal adjoint of d_s (so given by $\delta_s = \delta + s(-1)^{n(k+1)+1}(\star\iota_K\star)$ on each homogenous form of degree k). By Cartan's formula, $\mathsf{d}_s^2 = \mathcal{L}_K = s\mathcal{L}_{X_M}$ (the Lie derivative along K). On the space $\Omega^{\pm}_{X_M} = \Omega^{\pm} \cap \ker\mathcal{L}_{X_M}$ of invariant forms, $\mathsf{d}_s^2 = 0$ so one can define two cohomology groups $H_s^{\pm} := \ker\mathsf{d}_s^{\pm}/\operatorname{im}\mathsf{d}_s^{\mp}$. Witten then defines

$$\Delta_s := (\mathsf{d}_s + \delta_s)^2 : \Omega^{\pm}_{X_M}(M) \to \Omega^{\pm}_{X_M}(M),$$

(which he denotes H_s as it represents a Hamiltonian operator, but for us this would cause confusion), and he observes that using standard Hodge theory arguments, there is an isomorphism

$$\mathcal{H}_s^{\pm} := (\ker\Delta_s)^{\pm} \simeq H_s^{\pm}(M), \tag{1.5}$$

although no details of the proof are given (nor are they to be found elsewhere in the literature). We call $H_s^{\pm}(M)$ the X_M- *cohomology* of M and denote it by $H^{\pm}_{X_M}(M)$. Witten also shows, among other things, that for $s \neq 0$, the dimensions of $\mathcal{H}_s^{\pm}$ are respectively equal to the total even and odd Betti numbers of the subset N of zeros of X_M, which in particular implies the finiteness of $\dim\mathcal{H}_s$. Atiyah and Bott [2] relate this result of Witten's to their localization theorem in equivariant cohomology.

The principal purpose of this paper is to extend Witten's results to manifolds with boundary. In order to do this, in Section 2 we outline a proof of Witten's results using classical Hodge theory arguments, which in Section 3 we extend to deal with the case of manifolds with boundary. Finally in Section 4 we describe Atiyah and Bott's localization and its conclusions in the case of manifolds with boundary. Section 5 provides a few conclusions.

In the remainder of this introduction we recall the standard extension of Hodge theory to manifolds with boundary, leading to the Hodge-Morrey-Friedrichs decompositions. So now we let M be a compact orientable Riemannian manifold with boundary ∂M, and let $i : \partial M \hookrightarrow M$ be the inclusion. In this setting, there are two types of de Rham cohomology, the absolute cohomology $H^k(M)$ and the relative cohomology $H^k(M,\partial M)$. The first is the cohomology of the de Rham complex $(\Omega^k(M), \mathsf{d})$, while the second is the cohomology of the subcomplex $(\Omega^k_D(M), \mathsf{d})$, where $\omega \in \Omega^k_D$ if it satisfies $i^*\omega = 0$ (the D is for Dirichlet boundary condition). One also defines $\Omega^k_N(M) = \left\{\alpha \in \Omega^k(M) \mid i^*(\star\alpha) = 0\right\}$ (Neumann boundary condition). Here i^* is the pullback by the inclusion map. Clearly, the Hodge star provides an isomorphism

$$\star : \Omega^k_D \xrightarrow{\sim} \Omega^{n-k}_N.$$

Furthermore, because d and i^* commute, it follows that d preserves Dirichlet boundary conditions while δ preserves Neumann boundary conditions.

As alluded to before, because of boundary terms, the null space of Δ no longer coincides with the closed and co-closed forms. Elements of $\ker\Delta$ are called *harmonic forms*, while ω satisfying $\mathsf{d}\omega = \delta\omega = 0$ are called *harmonic fields* (following Kodaira); it is clear that every harmonic field is a harmonic form, but the converse is false. The space of harmonic *k-fields* is denoted $\mathcal{H}^k(M)$ (so $\mathcal{H}^*(M) \subset \ker\Delta$). In fact, the space $\mathcal{H}^k(M)$ is infinite dimensional and so is much too big to represent the cohomology, and to recover the Hodge isomorphism one has to impose boundary conditions. One restricts $\mathcal{H}^k(M)$ into each of two finite dimensional subspaces, namely $\mathcal{H}^k_D(M)$ and $\mathcal{H}^k_N(M)$ with the obvious meanings (Dirichlet and Neumann harmonic k-fields, respectively). There are therefore two different candidates for harmonic representatives when the boundary is present.

The Hodge-Morrey decomposition [8] states that

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N.$$

(We will make a more precise functional analytic statement below.) This decomposition is again orthogonal with respect to the inner product given above. Friedrichs [3] subsequently showed that

$$\mathcal{H}^k = \mathcal{H}^k_D \oplus \mathcal{H}^k_{\mathrm{co}}; \qquad \mathcal{H}^k = \mathcal{H}^k_N \oplus \mathcal{H}^k_{\mathrm{ex}}$$

where $\mathcal{H}^k_{\mathrm{ex}}$ are the exact harmonic fields and $\mathcal{H}^k_{\mathrm{co}}$ the coexact ones. These give the orthogonal *Hodge-Morrey-Friedrichs* [9] decompositions,

$$\begin{aligned}\Omega^k(M) &= \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_D \oplus \mathcal{H}^k_{\mathrm{co}} \\ &= \mathsf{d}\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_N \oplus \mathcal{H}^k_{\mathrm{ex}}.\end{aligned}$$

The two decompositions are related by the Hodge star operator. The consequence for cohomology is that each class in $H^k(M)$ is represented by a unique harmonic field in $\mathcal{H}^k_N(M)$, and each relative class in $H^k(M,\partial M)$ is represented by a unique harmonic field in $\mathcal{H}^k_D(M)$. Again, the Hodge star operator acts as Poincaré duality (or rather Poincaré-Lefschetz duality) on the harmonic fields, sending Dirichlet fields to Neumann fields. And again, if a group acts by isometries on $(M,\partial M)$ in a manner that is trivial on the cohomology, then the harmonic fields are invariant.

In this paper, we suppose G is a compact connected Abelian Lie group (a torus) acting by isometries on M, with Lie algebra $\mathfrak{g}$, and we let $X \in \mathfrak{g}$. If M has a boundary then the G-action necessarily restricts to an action on the boundary and X_M must therefore be tangent to the boundary. We denote by $\Omega_G = \Omega_G(M)$ the set of invariant forms on M: $\omega \in \Omega_G$ if $g^*\omega = \omega$ for all $g \in G$; in particular if ω is invariant then $\mathcal{L}_{X_M}\omega = 0$. Note that because the action preserves the metric and the orientation it follows that, for each $g \in G$, $\star(g^*\omega) = g^*(\star\omega)$, so if $\omega \in \Omega_G$ then $\star\omega \in \Omega_G$.

Remark on typesetting: Since the letter H plays three roles in this paper, we use three different typefaces: a script $\mathcal{H}$ for harmonic fields, a sans-serif H for Sobolev spaces and a normal (italic) H for cohomology. We hope that will prevent any confusion.

Acknowledgment The first named author would like to express his gratitude to the Ministry of Higher Education and Scientific Research of Iraq for the financial support for his PhD studies in Mathematics at the University of Manchester. This work will form part of the thesis for that PhD.

2 Witten-Hodge theory for manifolds without boundary

In this section we prove some of the results of Witten [10], providing details we will need in the next section for manifolds with boundary. We will use the notation from the introduction.

We have an oriented boundaryless compact Riemannian manifold M with an action of a torus G, and we fix an element $X \in \mathfrak{g}$. The associated vector field on M is X_M, and using this one defines Witten's inhomogeneous operator $\mathsf{d}_{X_M} : \Omega^{\pm}_G \to \Omega^{\mp}_G$, $\mathsf{d}_{X_M}\omega = \mathsf{d}\omega + \iota_{X_M}\omega$, and the corresponding operator (cf. eq. (1.2))

$$\delta_{X_M} = (-1)^{n(k+1)+1} \star \mathsf{d}_{X_M} \star = \delta + (-1)^{n(k+1)+1} \star \iota_{X_M} \star$$

(which is the adjoint operator to d_{X_M} by Proposition 2.2 below). The resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M} : \Omega^{\pm}_G \to \Omega^{\pm}_G$ defined by $\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2 = \mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M}$. We write the space of X_M-harmonic fields

$$\mathcal{H}_{X_M} = \ker \mathsf{d}_{X_M} \cap \ker \delta_{X_M},$$

which (for manifolds without boundary) satisfies $\mathcal{H}_{X_M} = \ker \Delta_{X_M}$. The last equality follows for the same reason as for ordinary Hodge theory, namely the argument in (1.4), with Δ replaced by Δ_{X_M} etc.

We recast Stokes' theorem and Green's formula in terms of the operators d_{X_M} and δ_{X_M} by defining $\int_M \omega = 0$ if $\omega \in \Omega^k(M)$ with $k \neq n$. For any form $\omega \in \Omega(M)$ one has $\int_M \iota_{X_M}\omega = 0$ as $\iota_{X_M}\omega$ has no term of degree n, and the following version of Stokes' theorem follows from the ordinary Stokes' theorem. For future use, we allow M to have a boundary.

Theorem 2.1 (Stokes' theorem for d_{X_M}) *Let M be a compact manifold with boundary ∂M (possibly empty) for all differential forms $\omega \in \Omega(M)$ then*

$$\int_M \mathsf{d}_{X_M}\omega = \int_{\partial M} i^*\omega,$$

where $i : \partial M \hookrightarrow M$ is the inclusion, and where the right-hand-side is taken to be zero if M has no boundary.

Proposition 2.2 (Green's formula for d_{X_M} and δ_{X_M}) *Let $\alpha, \beta \in \mathsf{H}^1\Omega_G$ be invariant differential forms on the compact manifold M with boundary ∂M (possibly empty) , then*

$$\langle \mathsf{d}_{X_M}\alpha, \beta\rangle = \langle \alpha, \delta_{X_M}\beta\rangle + \int_{\partial M} i^*(\alpha \wedge \star\beta) \tag{2.1}$$

PROOF: For technical reasons we write α and β as:

$$\alpha = \alpha^+ + \alpha^-, \quad \beta = \beta^+ + \beta^- \in \mathsf{H}^1\Omega_G$$

then

$$\begin{aligned} \mathsf{d}_{X_M}(\alpha\wedge(\star\beta)) &= \mathsf{d}_{X_M}(\alpha^+ + \alpha^-)\wedge\star(\beta^+ + \beta^-) + \\ & \alpha^+ \wedge \mathsf{d}_{X_M}(\star(\beta^+ + \beta^-)) - \alpha^- \wedge \mathsf{d}_{X_M}(\star(\beta^+ + \beta^-)) \end{aligned}$$

integrating both sides over M, applying Theorem 2.1 and using $\star\delta_{X_M} = \pm \mathsf{d}_{X_M}\star$ on $\Omega_G^{\pm}(M)$ and then by using the linearity and orthogonality of $\Omega_G(M) = \Omega_G^+(M) \oplus \Omega_G^-(M)$ then we obtain eq. (2.1). ❐

Returning now to the case of a manifold without boundary, we obtain the following.

Theorem 2.3 *The Witten-Hodge-Laplacian Δ_{X_M} is a self-adjoint elliptic operator.*

PROOF: The self-adjoint property follows from the same argument as for the classical Hodge Laplacian, namely that δ_{X_M} is the adjoint of d_{X_M}. For the ellipticity, we can expand Δ_{X_M} from its definition as,

$$\Delta_{X_M} = \Delta + (-1)^{n(k+1)+1}(\mathsf{d}\star\iota_{X_M}\star + \star\iota_{X_M}\star\mathsf{d} + \star\iota_{X_M}\star\iota_{X_M} + \iota_{X_M}\star\iota_{X_M}\star) + \iota_{X_M}\delta + \delta\iota_{X_M}. \tag{2.2}$$

It follows that Δ_{X_M} and Δ have the same principal symbol (indeed $\Delta_{X_M} - \Delta$ is a first order differential operator). Since Δ is elliptic, it follows that so too is Δ_{X_M}. ❐

Every elliptic operator is Fredholm, in the following sense. For each space $\Omega_G^{\pm}$, let $\mathsf{H}^s\Omega_G^{\pm}$ be the corresponding Sobolev space (the completion of $\Omega_G^{\pm}$ under an appropriate norm). Then for each $s \in \mathbb{R}$,

$$\Delta_{X_M} : \mathsf{H}^s\Omega_G^{\pm} \to \mathsf{H}^{s-2}\Omega_G^{\pm}$$

is a Fredholm operator, so has finite dimensional kernel and cokernel, and closed range.

The regularity and Fredholm properties of elliptic operators imply the following.

Corollary 2.4 *The set of X_M-harmonic (even/odd) forms $\mathcal{H}_{X_M}^{\pm}$ is finite dimensional and consists of smooth C^∞ forms.*

The following result is the analogue of the Hodge decomposition theorem, and is a standard consequence of the fact that Δ_{X_M} is self-adjoint.

Theorem 2.5 *The following is an orthogonal decomposition*

$$\Omega_G^{\pm} = \mathcal{H}_{X_M}^{\pm} \oplus \mathsf{d}_{X_M}\Omega_G^{\mp} \oplus \delta_{X_M}\Omega_G^{\mp},$$

and in terms of Sobolev spaces ($\forall s \in \mathbb{R}$)

$$\mathsf{H}^s\Omega_G^{\pm} = \mathcal{H}_{X_M}^{\pm} \oplus \mathsf{d}_{X_M}\mathsf{H}^{s+1}\Omega_G^{\mp} \oplus \delta_{X_M}\mathsf{H}^{s+1}\Omega_G^{\mp}.$$

The orthogonality is with respect to the L^2 inner product, given in (1.1).

As consequences for our decomposition above to the invariant differential forms $\Omega_G^{\pm}$, we have the following topological properties for X_M-cohomology.

Proposition 2.6 *Every X_M-cohomology class has a unique X_M-harmonic form representative.*

Corollary 2.7 *The X_M-cohomology groups $H^{\pm}_{X_M}(M)$ for a compact, oriented differentiable Riemannian manifold M with an action of a torus G are all finite dimensional.*

We infer the following form of Poincaré duality but in terms of X_M-cohomology. Here and elsewhere we write $n-\pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n.

Theorem 2.8 (Poincaré duality for $H^{\pm}_{X_M}$) *Let M be a compact, oriented smooth Riemannian manifold of dimension n and with an action of a torus G. The bilinear function*

$$(\, , \,) : H^{\pm}_{X_M} \times H^{n-\pm}_{X_M} \longrightarrow \mathbb{R}$$

defined by setting

$$([\alpha],[\beta]) = \int_M \alpha \wedge \beta \tag{2.3}$$

is well-defined, non-singular pairing and consequently gives isomorphisms of $H^{n-\pm}_{X_M}$ with the dual space of $H^{\pm}_{X_M}$. i.e.

$$H^{n-\pm}_{X_M} \cong (H^{\pm}_{X_M})^*.$$

PROOF: Clearly, the bilinear map (2.3) is well-defined and the non-singularity follows from Proposition 2.6 as follows: given a non-zero X_M-cohomology class $[\omega] \in H^{\pm}_{X_M}$, we must find a non-zero X_M-cohomology class $[\psi] \in H^{n-\pm}_{X_M}$ such that $([\omega],[\psi]) \neq 0$. According to Proposition 2.6, that ω is the harmonic representative of the non zero X_M-cohomology class $[\omega]$, it follows that ω is not identically zero. Applying the fact that $\star\Delta_{X_M} = \Delta_{X_M}\star$, it gives that $\star\omega$ is also harmonic and represents a X_M-cohomology class $[\star\omega] \in H^{n-\pm}_{X_M}$. Thus the pairing (2.3)

$$([\omega],[\star\omega]) = \int_M \omega \wedge \star\omega = \|\omega\|^2 \neq 0$$

is non-singular while the isomorphisms $H^{n-\pm}_{X_M} \cong (H^{\pm}_{X_M})^*$ follow from the finite dimensionality of X_M-cohomology (cf. Corollary 2.4 and Proposition 2.6) and the non-singularity above. ❒

Let $N = N(X_M)$ be the set of zeros of X_M. Witten observed that if ω is d_{X_M}-closed then its pullback to N is closed in the usual (de Rham) sense. And exact forms pull back to exact forms. Consequently, pullback defines a natural map $H^{\pm}_{X_M} \to H^{\pm}(N)$, where $H^{+}(N)$ is the direct sum of the even cohomology groups of N, and $H^{-}_{X_M}$ of the odd ones.

Theorem 2.9 (Witten [10]) *The pullback to N induces an isomorphism between the X_M-cohomology groups $H^{\pm}_{X_M}(M)$ and the cohomology groups $H^{\pm}(N)$.*

Witten gave a fairly explicit proof of this theorem by extending closed forms on N to X_M-closed forms on M. Atiyah and Bott [2] give a proof using their localization theorem in equivariant cohomology which we discuss, and adapt to the case of manifolds with boundary, in Section 4.

Furthermore, the restriction to N of an X_M-harmonic form on M is harmonic in the usual sense, so it follows from the theorem that every harmonic form on N has a unique extension to an X_M-harmonic form on M.

Remark 2.10 Suppose X generates the torus $G(X)$, and G is a larger torus containing $G(X)$ and acting on M by isometries. Then the action of G preserves X_M. It follows that G acts trivially on the de Rham cohomology of N, and hence on the X_M-cohomology of M, and consequently on the space of X_M-harmonic forms. In other words, $\mathcal{H}^{\pm}_{X_M} \subset \Omega^{\pm}_G$. There is therefore no loss in considering just forms invariant under the action of the larger torus in that the X_M-cohomology, or the space of X_M-harmonic forms, is independent of the choice of torus.

Example 2.11 Consider $M = S^2$ (the unit 2-sphere in $\mathbb{R}^3$), and use cylindrical polar coordinates $z \in [-1,1]$ and $\phi \in [0,2\pi]$. Let the group $G = S^1$ act on S^2 by rotations about the z-axis, with infinitesimal generator $\partial/\partial\phi$. Let $X \in \mathfrak{g}$, so $X_M = s\,\partial/\partial\phi$, for some $s \in \mathbb{R}$. Invariant even and odd forms are of the form

$$\omega_+ = f_0(z) + f_2(z)\,\mathrm{d}\phi \wedge \mathrm{d}z \in \Omega^+_G, \quad \omega_- = f_1(z)\,\mathrm{d}z + g_1(z)\,\mathrm{d}\phi \in \Omega^-_G.$$

In order that ω_- is smooth, g_1 must vanish at the poles $z = \pm 1$. The invariant volume form is $\mathrm{d}\phi \wedge \mathrm{d}z$, with total volume 4π, and the metric is $\mathrm{d}s^2 = (1-z^2)^{-1}\mathrm{d}z^2 + (1-z^2)\,\mathrm{d}\phi^2$. Consequently, $\star(\mathrm{d}z) = -(1-z^2)\,\mathrm{d}\phi$ and $\star(\mathrm{d}\phi) = (1-z^2)^{-1}\,\mathrm{d}z$, so

$$\mathrm{d}_{X_M}\omega_+ = (f_0'(z) + s f_2(z))\,\mathrm{d}z, \quad \delta_{X_M}\omega_+ = -(1-z)^2(f_2'(z) + s f_0(z))\,\mathrm{d}\phi.$$

One finds ω_+ is X_M-harmonic if and only if

$$\omega_+ = A\mathrm{e}^{sz}(1 - \mathrm{d}\phi \wedge \mathrm{d}z) + B\mathrm{e}^{-sz}(1 + \mathrm{d}\phi \wedge \mathrm{d}z),$$

for $A, B \in \mathbb{R}$, and one finds that there are no non-zero odd X_M-harmonic forms. Furthermore, the pullback of ω_+ to N (which here is the two poles at $z = \pm 1$) is $A(\mathrm{e}^s, \mathrm{e}^{-s}) + B(\mathrm{e}^{-s}, \mathrm{e}^s)$ which for $s \neq 0$ are linearly independent, as predicted by Theorem 2.9. It is notable that the two fundamental solutions for ω_+ (those with $A = 1, B = 0$ and vice versa) depend analytically on s.

3 Witten-Hodge theory for manifolds with boundary

In this section we adapt the results and methods of Hodge theory for manifolds with boundary to study the X_M-cohomology and the space of X_M-harmonic forms and fields for manifolds with boundary. As for ordinary (singular) cohomology, there are both absolute and relative X_M-cohomology groups. So from now on our manifold will be with boundary and with torus action which acts by isometry on this manifold unless otherwise indicated, and as before $i : \partial M \hookrightarrow M$ denotes the inclusion of the boundary.

3.1 The difficulties if the boundary is present

Firstly, d_{X_M} and δ_{X_M} are no longer adjoint because the boundary terms arise when we integrate by parts and then Δ_{X_M} will not be self-adjoint. In addition, the space of all harmonic fields is infinite dimensional and there is no reason to expect the X_M-harmonic fields $\mathcal{H}_{X_M}(M)$ to be any different. To overcome these problems, we follows the method which is used to solve this problem in classical case, i.e. with d and δ [1, 9], and impose certain boundary conditions on our invariant forms $\Omega_G(M)$. Hence we make the following definitions.

Definition 3.1 (1) We define the following two sets of smooth invariant forms on the manifold M with boundary and with torus action

$$\Omega_{G,D} \quad = \quad \Omega_G \cap \Omega_D \; = \{\omega \in \Omega_G \mid i^*\omega = 0\} \tag{3.1}$$

$$\Omega_{G,N} \quad = \quad \Omega_G \cap \Omega_N \; = \{\omega \in \Omega_G \mid i^*(\star\omega) = 0\} \tag{3.2}$$

and the spaces $\mathsf{H}^s\Omega_{G,D}$ and $\mathsf{H}^s\Omega_{G,N}$ are the corresponding closures with respect to suitable Sobolev norms, for $s > \frac{1}{2}$. This can be refined to take into account the parity of the forms, so defining $\Omega^{\pm}_{G,D}$ etc. Since $\omega \in \Omega^k$ implies $\star\omega \in \Omega^{n-k}$ we write that for $\omega \in \Omega^{\pm}_G$ we have $\star\omega \in \Omega^{n-\pm}_G$.
(2) We define the two subspaces of $\mathcal{H}_{X_M}(M)$

$$\mathcal{H}_{X_M,D}(M) = \{\omega \in \mathsf{H}^1\Omega_{G,D} \mid \mathsf{d}_{X_M}\omega = 0, \delta_{X_M}\omega = 0\} \tag{3.3}$$
$$\mathcal{H}_{X_M,N}(M) = \{\omega \in \mathsf{H}^1\Omega_{G,N} \mid \mathsf{d}_{X_M}\omega = 0, \delta_{X_M}\omega = 0\} \tag{3.4}$$

which we call Dirichlet and Neumann X_M-harmonic fields, respectively. We will show below that these forms are smooth. Clearly, the Hodge star operator $\star$ defines an isomorphism $\mathcal{H}_{X_M,D}(M) \cong \mathcal{H}_{X_M,N}(M)$. Again, these can be refined to take the parity into account, defining $\mathcal{H}^{\pm}_{X_M,D}(M)$ etc.

As for ordinary Hodge theory, on a manifold with boundary one has to distinguish between X_M-harmonic *forms* (i.e. $\ker\Delta_{X_M}$) and X_M-harmonic *fields* (i.e. $\mathcal{H}_{X_M}(M)$) because they are not equal: one has $\mathcal{H}_{X_M}(M) \subseteq \ker\Delta_{X_M}$ but not conversely. The following proposition shows the conditions on ω to be fulfilled in order to ensure $\omega \in \ker\Delta_{X_M} \Longrightarrow \omega \in \mathcal{H}_{X_M}(M)$ when $\partial M \neq \emptyset$.

Proposition 3.2 *If $\omega \in \Omega_G(M)$ is an X_M-harmonic form (i.e. $\Delta_{X_M}\omega = 0$) and in addition any one of the following four pairs of boundary conditions is satisfied then $\omega \in \mathcal{H}_{X_M}(M)$.*

$$(1)\quad i^*\omega = 0,\ i^*(\star\omega) = 0; \qquad (2)\quad i^*\omega = 0,\ i^*(\delta_{X_M}\omega) = 0;$$
$$(3)\quad i^*(\star\omega) = 0,\ i^*(\star\mathsf{d}_{X_M}\omega) = 0; \qquad (4)\quad i^*(\delta_{X_M}\omega) = 0,\ i^*(\star\mathsf{d}_{X_M}\omega) = 0.$$

PROOF: Because $\Delta_{X_M}\omega = 0$, one has $\langle\Delta_{X_M}\omega, \omega\rangle = 0$. Now applying Proposition 2.2 to this and using any of these conditions (1)–(4) ensures ω is an X_M-harmonic field. ❐

Remark 3.3 An averaging argument shows that $\mathsf{H}^1\Omega_{G,D}$ and $\mathsf{H}^1\Omega_{G,N}$ are dense in $L^2\Omega_G$, because the corresponding statements hold for the spaces of all (not invariant) forms.

3.2 Elliptic boundary value problem

We prove the ellipticity of certain boundary value problem (BVP) which is given in Theorem 3.4. This theorem represents the keystone to extending Witten's results to manifolds with boundary, via our extension of the Hodge-Morrey decomposition theorem in terms of d_{X_M} and δ_{X_M}. We then to relate our results to the equivariant cohomology ring. The proofs in this section rely heavily on the corresponding statements for the usual Laplacian Δ on a manifold with boundary, as described in the book of Schwarz [9].

We consider the BVP

$$\begin{cases} \Delta_{X_M}\omega = \eta & \text{on } M \\ i^*\omega = 0 & \text{on } \partial M \\ i^*(\delta_{X_M}\omega) = 0 & \text{on } \partial M. \end{cases} \tag{3.5}$$

Theorem 3.4

1. *The BVP (3.5) is elliptic in the sense of Lopatinskiĭ-Šapiro, where $\Delta_{X_M} : \Omega_G(M) \longrightarrow \Omega_G(M)$.*

2. *The BVP (3.5) is Fredholm of index 0.*

3. *All $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$ are smooth.*

PROOF:
(1) Firstly, as in the proof of Theorem 2.3, we can see that Δ and Δ_{X_M} have the same principal symbol. Similarly, expanding the second boundary condition gives

$$\delta_{X_M} = \delta + (-1)^{n(k+1)+1} \star \iota_{X_M} \star$$

so δ_{X_M} and δ have the same first-order part. Hence our BVP (3.5) has the same principal symbol as the following BVP

$$\begin{cases} \Delta\varepsilon &= \xi \quad \text{on} \quad M \\ i^*\varepsilon &= 0 \quad \text{on} \quad \partial M \\ i^*(\delta\varepsilon) &= 0 \quad \text{on} \quad \partial M \end{cases} \tag{3.6}$$

for $\varepsilon, \xi \in \Omega(M)$, because the principal symbol does not change when terms of lower order are added to the operator. However the BVP (3.6) is elliptic in the sense of Lopatinskiĭ-Šapiro conditions [6, 9], and thus so is (3.5).

(2) From part (1), since the BVP (3.5) is elliptic, by using Theorem 1.6.2 in [9] or Theorem 20.1.2 in [6] we conclude that the BVP (3.5) is a Fredholm operator and the regularity theorem holds. In addition, we observe that the only differences between BVP (3.6) and our BVP (3.5) are all lower order operators and it is proved in [9] that the index of BVP (3.6) is zero but Theorem 20.1.8 in [6] asserts generally that if the difference between two BVP'S are just lower order operators then they must have the same index. Hence, the index of the BVP (3.5) must be zero.

(3) Let $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$. If $\omega \in \mathcal{H}_{X_M,D}$ then it satisfies the BVP (3.5) with $\eta = 0$, so by the regularity properties of elliptic BVPs, the smoothness of ω follows. If on the other hand $\omega \in \mathcal{H}_{X_M,N}$ then $\star\omega \in \mathcal{H}_{X_M,D}$ which is therefore smooth and consequently $\omega = \pm\star(\star\omega)$ is smooth as well. ❐

We consider the resulting operator obtained by restricting Δ_{X_M} to the subspace of smooth invariant forms satisfying the boundary conditions

$$\overline{\Omega}_G(M) = \{\omega \in \Omega_G(M) \mid i^*\omega = 0, i^*(\delta_{X_M}\omega) = 0\} \tag{3.7}$$

Since the trace map i^* is well-defined on $\mathsf{H}^s\Omega_G$ for $s > 1/2$ it follows that it makes sense to consider $\mathsf{H}^2\overline{\Omega}_G(M)$, which is a closed subspace of $\mathsf{H}^2\Omega_G(M)$ and hence a Hilbert space. For simplicity, we rewrite our BVP (3.5) as follows: consider the restriction/extension of Δ_{X_M} to this space:

$$A = \Delta_{X_M}|_{\mathsf{H}^2\overline{\Omega}_G(M)} : \mathsf{H}^2\overline{\Omega}_G(M) \longrightarrow L^2\Omega_G(M).$$

and consider the BVP,

$$A\omega = \eta \tag{3.8}$$

for $\omega \in \mathsf{H}^2\overline{\Omega}_G(M)$ and $\eta \in L^2\Omega_G(M)$ instead of BVP (3.5) which are in fact compatible. In addition, from Theorem 3.4 we deduce that A is an elliptic and Fredholm operator and

$$\operatorname{index}(A) = \dim(\ker A) - \dim(\ker A^*) = 0 \tag{3.9}$$

where A^* is the adjoint operator of A.

From Green's formula (Proposition 2.2) we deduce the following property.

Lemma 3.5 *A is L^2-self-adjoint on $\mathsf{H}^2\overline{\Omega}_G(M)$, meaning that for all $\alpha, \beta \in \mathsf{H}^2\overline{\Omega}_G(M)$ we have*

$$\langle A\alpha, \beta\rangle = \langle \alpha, A\beta\rangle\,,$$

where $\langle -,-\rangle$ is the L^2-pairing.

Theorem 3.6 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G, the space $\mathcal{H}_{X_M,D}(M)$ is finite dimensional and*

$$L^2\Omega_G(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,D}(M)^{\perp}. \tag{3.10}$$

PROOF: We begin by showing that $\ker A = \mathcal{H}_{X_M,D}(M)$. It is clear that $\mathcal{H}_{X_M,D}(M) \subseteq \ker A$, so we need only prove that $\ker A \subseteq \mathcal{H}_{X_M,D}(M)$.

Let $\omega \in \ker A$. Then ω satisfies the BVP (3.5). Therefore, by condition (2) of Proposition 3.2, it follows that $\omega \in \mathcal{H}_{X_M,D}(M)$, as required.

Now, $\ker A = \mathcal{H}_{X_M,D}(M)$ but $\dim \ker A$ is finite, it follows that so too is $\dim \mathcal{H}_{X_M,D}(M)$. This implies that $\mathcal{H}_{X_M,D}(M)$ is a closed subspace of the Hilbert space $L^2\Omega_G(M)$, hence eq. (3.10) holds. ❐

Theorem 3.7

$$\text{Range}(A) = \mathcal{H}_{X_M,D}(M)^{\perp} \tag{3.11}$$

where $\perp$ denotes the orthogonal complement in $L^2\Omega_G(M)$.

PROOF: Firstly, we should observe that eq. (3.9) asserts that $\ker A \cong \ker A^*$ but Theorem 3.6 shows that $\ker A = \mathcal{H}_{X_M,D}(M)$, thus

$$\ker A^* \cong \mathcal{H}_{X_M,D}(M) \tag{3.12}$$

Since $\text{Range}(A)$ is closed in $L^2\Omega_G(M)$ because A is Fredholm operator, it follows from the closed range theorem in Hilbert spaces that

$$\text{Range}(A) = (\ker A^*)^{\perp} \quad \equiv \quad \text{Range}(A)^{\perp} = \ker A^* \tag{3.13}$$

Hence, we just need to prove that $\ker A^* = \mathcal{H}_{X_M,D}(M)$, and to show that we need first to prove

$$\text{Range}(A) \subseteq \mathcal{H}_{X_M,D}(M)^{\perp}. \tag{3.14}$$

So, if $\alpha \in \mathrm{H}^2\overline{\Omega}_G(M)$ and $\beta \in \mathcal{H}_{X_M,D}(M)$ then applying Lemma 3.5 gives

$$\langle A\alpha, \beta \rangle = 0$$

hence, eq. (3.14) holds. Moreover, equations (3.13) and (3.14) and the closedness of $\mathcal{H}_{X_M,D}(M)$ imply

$$\mathcal{H}_{X_M,D}(M) \subseteq \ker A^* \tag{3.15}$$

but eq. (3.12) and eq. (3.15) force $\ker A^* = \mathcal{H}_{X_M,D}(M)$. Hence, $\text{Range}(A) = \mathcal{H}_{X_M,D}(M)^{\perp}$. ❐

Following [9], we denote the L^2-orthogonal complement of $\mathcal{H}_{X_M,D}(M)$ in the space $\mathrm{H}^2\Omega_{G,D}$ by

$$\mathcal{H}_{X_M,D}(M)^{\oplus} = \mathrm{H}^2\Omega_{G,D} \cap \mathcal{H}_{X_M,D}(M)^{\perp} \tag{3.16}$$

(although in [9] it denotes H^1-forms rather than H^2).

Proposition 3.8 *For each $\eta \in \mathcal{H}_{X_M,D}(M)^{\perp}$ there is a unique differential form $\omega \in \mathcal{H}_{X_M,D}(M)^{\oplus}$ satisfying the BVP (3.5).*

PROOF: Let $\eta \in \mathcal{H}_{X_M,D}(M)^\perp$. Because of Theorem (3.7) there is a differential form $\gamma \in \mathsf{H}^2\overline{\Omega}_G(M)$ such that γ satisfies the BVP (3.5). Since $\gamma \in \mathsf{H}^2\overline{\Omega}_G(M) \subseteq L^2\Omega_G(M)$ then there are unique differential forms $\alpha \in \mathcal{H}_{X_M,D}(M)$ and $\omega \in \mathcal{H}_{X_M,D}(M)^\perp$ such that $\gamma = \alpha + \omega$ because of eq. (3.10).

Since γ satisfies the BVP (3.5) it follows that ω satisfies the BVP (3.5) as well because $\alpha \in \mathcal{H}_{X_M,D}(M) = \ker(\Delta_{X_M}|_{\mathsf{H}^2\overline{\Omega}_G(M)})$. Since $\omega = \gamma - \alpha$, it follows that $\omega \in \mathsf{H}^2\Omega_{G,D}$, hence $\omega \in \mathcal{H}_{X_M,D}(M)^\oplus$ and it is unique ❒

Remark 3.9

(1) ω satisfying the BVP (3.5) in Proposition 3.8 can be recast to the condition

$$\langle \mathsf{d}_{X_M}\omega, \mathsf{d}_{X_M}\xi\rangle + \langle \delta_{X_M}\omega, \delta_{X_M}\xi\rangle = \langle \eta, \xi\rangle, \quad \forall \xi \in \mathsf{H}^1\Omega_{G,D} \tag{3.17}$$

(2) All the results above can be recovered but in terms of $\mathcal{H}_{X_M,N}(M)$ because the Hodge star operator defines an isomorphism $L^2\Omega_G \cong L^2\Omega_G$ which restricts to $\mathcal{H}_{X_M,D}(M) \cong \mathcal{H}_{X_M,N}(M)$.

3.3 Decomposition theorems

We adapt the Hodge-Morrey and Freidrichs decompositions arising for Hodge theory on manifolds with boundary, to the present setting with d_{X_M} and δ_{X_M}.

Definition 3.10 Define the following two sets of exact and coexact forms on the manifold M with boundary and with an action of the torus G:

$$\mathcal{E}_G(M) = \{\mathsf{d}_{X_M}\alpha \mid \alpha \in \mathsf{H}^1\Omega_{G,D}\} \subseteq L^2\Omega_G(M), \tag{3.18}$$
$$\mathcal{C}_G(M) = \{\delta_{X_M}\beta \mid \beta \in \mathsf{H}^1\Omega_{G,N}\} \subseteq L^2\Omega_G(M). \tag{3.19}$$

Clearly, $\mathcal{E}_G(M) \perp \mathcal{C}_G(M)$ because of Proposition 2.2. We denote by $L^2\mathcal{H}_{X_M}(M) = \overline{\mathcal{H}_{X_M}(M)}$ the L^2-closure of the space $\mathcal{H}_{X_M}(M)$.

Proposition 3.11 (Algebraic decomposition and L^2-closedness)

(a) Each $\omega \in L^2\Omega_G(M)$ can be split uniquely into

$$\omega = \mathsf{d}_{X_M}\alpha_\omega + \delta_{X_M}\beta_\omega + \kappa_\omega$$

where $\mathsf{d}_{X_M}\alpha_\omega \in \mathcal{E}_G(M)$, $\delta_{X_M}\beta_\omega \in \mathcal{C}_G(M)$ and $\kappa_\omega \in (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$.

(b) The spaces $\mathcal{E}_G(M)$and $\mathcal{C}_G(M)$ are closed subspaces of $L^2\Omega_G(M)$.

(a) and (b) mean that there is the following orthogonal decomposition

$$L^2\Omega_G(M) = \mathcal{E}_G(M) \oplus \mathcal{C}_G(M) \oplus (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp \tag{3.20}$$

PROOF: (a) We have shown that

$$L^2\Omega_G(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,D}(M)^\perp = \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,N}(M)^\perp.$$

Let $\omega \in L^2\Omega_G(M)$ then corresponding to these decompositions we can split it uniquely into

$$\omega = \lambda_D + (\omega - \lambda_D), \quad \omega = \lambda_N + (\omega - \lambda_N)$$

where $(\omega - \lambda_D) \in \mathcal{H}_{X_M,D}(M)^\perp$ and $(\omega - \lambda_N) \in \mathcal{H}_{X_M,N}(M)^\perp$. By Proposition 3.8 there are unique elements $\theta_D \in \mathcal{H}_{X_M,D}(M)^\oplus$ and $\theta_N \in \mathcal{H}_{X_M,N}(M)^\oplus$ satisfying the BVP (3.5) with η replaced by $(\omega-\lambda_D)$ and $(\omega - \lambda_N)$ respectively.

From Theorem (3.7) we infer that θ_D and θ_N are of Sobolev class H^2, so define

$$\alpha_\omega = \delta_{X_M}\theta_D \in \mathsf{H}^1\Omega_{G,D} \quad \text{and} \quad \beta_\omega = \mathsf{d}_{X_M}\theta_N \in \mathsf{H}^1\Omega_{G,N} \tag{3.21}$$

Now let

$$\kappa_\omega = \omega - \mathsf{d}_{X_M}\alpha_\omega - \delta_{X_M}\beta_\omega \in L^2\Omega_G(M)$$

The next step is to show that κ_ω is orthogonal to $\mathcal{E}_G(M)$ but form proposition 2.2 we can prove that $\lambda_D, \delta_{X_M}\beta \in \mathcal{E}_G(M)^\perp$, in addition, $(\omega - \lambda_D) = \Delta_{X_M}\theta_D$ then

$$\langle \kappa_\omega, \mathsf{d}_{X_M}\alpha\rangle = \langle \Delta_{X_M}\theta_D, \mathsf{d}_{X_M}\alpha\rangle - \langle \mathsf{d}_{X_M}\delta_{X_M}\theta_D + \delta_{X_M}\mathsf{d}_{X_M}\theta_D, \mathsf{d}_{X_M}\alpha\rangle = 0, \quad \forall \mathsf{d}_{X_M}\alpha \in \mathcal{E}_G(M)$$

Analogously we can show that $\langle \kappa_\omega, \delta_{X_M}\beta\rangle = 0$, $\forall\, \delta_{X_M}\beta \in \mathcal{C}_G(M)$. Therefore $\kappa_\omega \in (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$.

(b) Let $\{\mathsf{d}_{X_M}\alpha_j\}_{j\in\mathbb{N}}$ be an L^2-Cauchy sequence in $\mathcal{E}_G(M)$ then $\mathsf{d}_{X_M}\alpha_j \longrightarrow \gamma \in L^2\Omega_G(M)$. Hence we get from part (a) above that

$$\gamma = \mathsf{d}_{X_M}\alpha_\gamma + \delta_{X_M}\beta_\gamma + \kappa_\gamma$$

where $\mathsf{d}_{X_M}\alpha_\gamma \in \mathcal{E}_G(M)$, $\delta_{X_M}\beta_\gamma \in \mathcal{C}_G(M)$ and $\kappa_\gamma \in (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$. Because $\mathcal{E}_G(M) \perp \mathcal{C}_G(M) \perp (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$ and $\langle \gamma - \mathsf{d}_{X_M}\alpha_j, \gamma - \mathsf{d}_{X_M}\alpha_j\rangle \longrightarrow 0$ it follows that $\delta_{X_M}\beta\gamma = 0$ and $\kappa_\gamma = 0$, thus $\gamma = \mathsf{d}_{X_M}\alpha_\gamma \in \mathcal{E}_G(M)$. Hence $\mathcal{E}_G(M)$ is closed. The corresponding argument applies to $\mathcal{C}_G(M)$. ❒

Now we can present the main theorems for this section.

Theorem 3.12 (X_M-Hodge-Morrey decomposition theorem) *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G. Then*

$$L^2\Omega_G(M) = \mathcal{E}_G(M) \oplus \mathcal{C}_G(M) \oplus L^2\mathcal{H}_{X_M}(M) \tag{3.22}$$

PROOF: From Proposition 3.11 we infer eq. (3.20) and then we first observe that the spaces $\mathcal{E}_G(M)$, $\mathcal{C}_G(M)$ and $L^2\mathcal{H}_{X_M}(M)$ are mutually orthogonal with respect to the L^2-inner product which is an immediate consequence of Green's formulae (Proposition 2.2), and hence

$$L^2\mathcal{H}_{X_M}(M) \subseteq (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$$

So we need only to prove the converse and then using eq. (3.20) we will get the decomposition (3.22). So, let $\omega \in (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$, so

$$\begin{aligned} \langle \omega, \mathsf{d}_{X_M}\alpha\rangle &= \langle \delta_{X_M}\omega, \alpha\rangle = 0 \quad \forall \alpha \in \mathsf{H}^1\Omega_{G,D} \\ \langle \omega, \delta_{X_M}\beta\rangle &= \langle \mathsf{d}_{X_M}\omega, \beta\rangle = 0 \quad \forall \beta \in \mathsf{H}^1\Omega_{G,N}. \end{aligned} \tag{3.23}$$

From Remark 3.3 we know that $\mathsf{H}^1\Omega_{G,D}$ and $\mathsf{H}^1\Omega_{G,N}$ are dense in $L^2\Omega_G(M)$, hence eq. (3.23) implies that $\mathsf{d}_{X_M}\omega = 0$ and $\delta_{X_M}\omega = 0$ which shows that $\omega \in L^2\mathcal{H}_{X_M}(M)$. Hence $L^2\mathcal{H}_{X_M}(M) = (\mathcal{E}_G(M) \oplus \mathcal{C}_G(M))^\perp$. ❒

Theorem 3.13 (X_M-Friedrichs Decomposition Theorem) *Let M be a compact, oriented smooth Riemannian manifold with boundary of dimension n and with an action of a torus G. Then the space $\mathcal{H}_{X_M}(M) \subseteq \mathrm{H}^1\Omega_G(M)$ of X_M- harmonic fields can respectively be decomposed into*

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,\mathrm{co}}(M) \quad (3.24)$$

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,\mathrm{ex}}(M) \quad (3.25)$$

where the right hand sides are coexact and exact harmonic forms respectively:

$$\mathcal{H}_{X_M,\mathrm{co}}(M) = \{\eta \in \mathcal{H}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\} \quad (3.26)$$

$$\mathcal{H}_{X_M,\mathrm{ex}}(M) = \{\xi \in \mathcal{H}_{X_M}(M) \mid \xi = \mathrm{d}_{X_M}\sigma\} \quad (3.27)$$

For $L^2\mathcal{H}_{X_M}(M)$ these decompositions are valid accordingly.

PROOF: We prove eq. (3.24); the argument for the dual eq. (3.25) is analogous. Proposition 2.2 shows the orthogonality of the decomposition (3.24), i.e.

$$\langle \delta_{X_M}\alpha, \lambda_D \rangle = 0 \quad \forall \lambda_D \in \mathcal{H}_{X_M,D}(M). \quad (3.28)$$

The space $\mathcal{H}_{X_M}(M) \subseteq L^2\Omega_G(M)$, hence equation (3.10) asserts that $\mathcal{H}_{X_M}(M)$ can be decomposed into:

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,D}(M)^{\perp} \cap \mathcal{H}_{X_M}(M) \quad (3.29)$$

where $\mathcal{H}_{X_M,D}(M)^{\perp} \cap \mathcal{H}_{X_M}(M)$ is the orthogonal complement of $\mathcal{H}_{X_M,D}(M)$ inside the space $\mathcal{H}_{X_M}(M)$. So, we need only prove that

$$\mathcal{H}_{X_M,\mathrm{co}}(M) = \mathcal{H}_{X_M,D}(M)^{\perp} \cap \mathcal{H}_{X_M}(M).$$

But, it is clear that $\mathcal{H}_{X_M,\mathrm{co}}(M) \subseteq \mathcal{H}_{X_M,D}(M)^{\perp} \cap \mathcal{H}_{X_M}(M)$ so, we just need to prove that

$$\mathcal{H}_{X_M,D}(M)^{\perp} \cap \mathcal{H}_{X_M}(M) \subseteq \mathcal{H}_{X_M,\mathrm{co}}(M).$$

Now, let $\omega \in \mathcal{H}_{X_M}(M) \cap \mathcal{H}_{X_M,D}(M)^{\perp}$ then Proposition 3.8 asserts that there is a unique element $\theta_D \in \mathcal{H}_{X_M,D}(M)^{\oplus}$ such that θ_D satisfies the BVP (3.5). One can infer from eq. (3.28) that also $\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D \in \mathcal{H}_{X_M,D}(M)^{\perp}$. Hence,

$$\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D = \Delta_{X_M}\theta_D - \delta_{X_M}\mathrm{d}_{X_M}\theta_D = \mathrm{d}_{X_M}\delta_{X_M}\theta_D.$$

The above equation gives that

$$i^*(\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D) = 0, \quad \mathrm{d}_{X_M}(\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D) = 0, \quad \text{and} \quad \delta_{X_M}(\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D) = 0$$

which mean that $\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D \in \mathcal{H}_{X_M,D}(M)$ but $\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D \in \mathcal{H}_{X_M,D}(M)^{\perp}$. Hence $\omega = \delta_{X_M}\mathrm{d}_{X_M}\theta_D \in \mathcal{H}_{X_M,\mathrm{co}}(M)$ as required. Thus, equation (3.24) holds.

For $\omega \in L^2\mathcal{H}_{X_M}(M)$ all the arguments up to $\omega - \delta_{X_M}\mathrm{d}_{X_M}\theta_D$ apply similarly. ❐

Combining Theorems 3.12 and 3.13 gives the following.

Corollary 3.14 (The X_M-Hodge-Morrey-Friedrichs decompositions) *The space $L^2\Omega_G(M)$ can be decomposed into L^2-orthogonal direct sums as follows:*

$$L^2\Omega_G(M) = \mathcal{E}_G(M) \oplus \mathcal{C}_G(M) \oplus \mathcal{H}_{X_M,D}(M) \oplus L^2\mathcal{H}_{X_M,\mathrm{co}}(M) \quad (3.30)$$

$$L^2\Omega_G(M) = \mathcal{E}_G(M) \oplus \mathcal{C}_G(M) \oplus \mathcal{H}_{X_M,N}(M) \oplus L^2\mathcal{H}_{X_M,\mathrm{ex}}(M) \quad (3.31)$$

Remark 3.15 All the results above can be recovered but in terms of $\pm$-spaces, for instance,

$$\mathcal{H}^{\pm}_{X_M,D}(M) \cong \mathcal{H}^{n-\pm}_{X_M,N}(M), \quad L^2\Omega^{\pm}_G(M) = \mathcal{E}^{\pm}_G(M) \oplus \mathcal{C}^{\pm}_G(M) \oplus \mathcal{H}^{\pm}_{X_M,D}(M) \oplus L^2\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)$$

. . . etc.

3.4 Relative and absolute X_M-cohomology

Using d_{X_M} and δ_{X_M} we can form a number of $\mathbb{Z}_2$-graded complexes. A $\mathbb{Z}_2$-graded complex is a pair of Abelian groups $C^{\pm}$ with homomorphisms between them:

$$C^+ \underset{\mathsf{d}_-}{\overset{\mathsf{d}_+}{\rightleftarrows}} C^-$$

satisfying $\mathsf{d}_+ \circ \mathsf{d}_- = 0 = \mathsf{d}_- \circ \mathsf{d}_+$. The two (co)homology groups of such a complex are defined in the obvious way: $H^{\pm} = \ker \mathsf{d}_{\pm} / \operatorname{im} \mathsf{d}_{\mp}$.

The complexes we have in mind are,

$$\begin{array}{cc} (\Omega_G^{\pm}, \mathsf{d}_{X_M}) & (\Omega_G^{\pm}, \delta_{X_M}) \\ (\Omega_{G,D}^{\pm}, \mathsf{d}_{X_M}) & (\Omega_{G,N}^{\pm}, \delta_{X_M}). \end{array}$$

The two on the lower line are subcomplexes of the corresponding upper ones, and are defined by

$$\begin{aligned} \Omega_{G,D}^{\pm} &= \{\omega \in \Omega_G^{\pm} \mid i^*\omega = 0\} \subseteq \Omega_G^{\pm} \\ \Omega_{G,N}^{\pm} &= \{\omega \in \Omega_G^{\pm} \mid i^* \star \omega = 0\} \subseteq \Omega_G^{\pm}. \end{aligned}$$

These are subcomplexes because i^* commutes with d_{X_M}. By analogy with the de Rham groups, we denote

$$\begin{aligned} H_{X_M}^{\pm}(M) &:= H^{\pm}(\Omega_G, \mathsf{d}_{X_M}), \\ H_{X_M}^{\pm}(M, \partial M) &:= H^{\pm}(\Omega_{G,D}, \mathsf{d}_{X_M}). \end{aligned}$$

Theorem 3.16 (X_M-Hodge Isomorphism) *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G. Let $X \in \mathfrak{g}$. There are the following isomorphisms of vector spaces:*

(a) $H_{X_M}^{\pm}(M, \partial M) \cong \mathcal{H}_{X_M,D}^{\pm}(M) \cong H^{\pm}(\Omega_G^{\pm}, \delta_{X_M})$*;*

(b) $H_{X_M}^{\pm}(M) \cong \mathcal{H}_{X_M,N}^{\pm}(M) \cong H^{\pm}(\Omega_{G,N}^{\pm}, \delta_{X_M})$*;*

(c) (X_M-Poincaré-Lefschetz duality): The Hodge star operator $\star$ on $\Omega_G(M)$ induces an isomorphism

$$H_{X_M}^{\pm}(M) \cong H_{X_M}^{n-\pm}(M, \partial M).$$

PROOF: We use the various decomposition theorems to prove (a). Part (b) is proved similarly, and part (c) follows from (a), (b) and the fact that the Hodge star operator defines an isomorphism $\mathcal{H}_{X_M,D}^{\pm}(M) \cong \mathcal{H}_{X_M,N}^{n-\pm}(M)$.

For the first isomorphism in (a), Theorem 3.12 (the X_M-Hodge-Morrey decomposition theorem) implies a unique splitting of any $\gamma \in \Omega_{G,D}^{\pm}$ into,

$$\gamma = \mathsf{d}_{X_M}\alpha_\gamma + \delta_{X_M}\beta_\gamma + \kappa_\gamma$$

where $\mathsf{d}_{X_M}\alpha_\gamma \in \mathcal{E}_G^{\pm}(M)$, $\delta_{X_M}\beta_\gamma \in \mathcal{C}_G^{\pm}(M)$ and $\kappa_\gamma \in L^2\mathcal{H}_{X_M}^{\pm}(M)$. If $\mathsf{d}_{X_M}\gamma = 0$ then $\delta_{X_M}\beta_\gamma = 0$, but $i^*\gamma = 0$ implies $i^*(\kappa_\gamma) = 0$ so that $\kappa_\gamma \in \mathcal{H}_{X_M,D}^{\pm}(M)$. Thus,

$$\gamma \in \ker \mathsf{d}_{X_M}|_{\Omega_{G,D}} \Longleftrightarrow \gamma = \mathsf{d}_{X_M}\alpha_\gamma + \kappa_\gamma.$$

This establishes the isomorphism $H^{\pm}_{X_M}(M,\partial M) \cong \mathcal{H}^{\pm}_{X_M,D}(M)$.

For the second isomorphism in (a), the X_M-Hodge-Morrey-Friedrichs decomposition (Corollary 3.14) eq. (3.31) implies as well a unique splitting of any $\gamma \in \Omega^{\pm}_G(M)$ into,

$$\gamma = \mathsf{d}_{X_M}\xi_\gamma + \delta_{X_M}\eta_\gamma + \delta_{X_M}\zeta_\gamma + \lambda_\gamma$$

where $\mathsf{d}_{X_M}\xi_\gamma \in \mathcal{E}^{\pm}_G(M)$, $\delta_{X_M}\eta_\gamma \in \mathcal{C}^{\pm}_G(M)$, $\delta_{X_M}\zeta_\gamma \in L^2\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)$ and $\lambda_\gamma \in \mathcal{H}^{\pm}_{X_M,D}(M)$.

If $\delta_{X_M}\gamma = 0$, then $\mathsf{d}_{X_M}\xi_\gamma = 0$, and hence

$$\gamma \in \ker \delta_{X_M} \Longleftrightarrow \gamma = \delta_{X_M}(\eta_\gamma + \zeta_\gamma) + \lambda_\gamma.$$

This establishes the isomorphism $\mathcal{H}^{\pm}_{X_M,D}(M) \cong H^{\pm}_{X_M}(\Omega^{\pm}_G, \delta_{X_M})$. ❒

4 Relation with equivariant cohomology

When the manifold in question has no boundary, Atiyah and Bott [2] discuss the relationship between equivariant cohomology and X_M-cohomology by using their localization theorem. In this section we will relate our relative and absolute X_M-cohomology with the relative and absolute equivariant cohomology $H^{\pm}_G(M,\partial M)$ and $H^{\pm}_G(M)$; the arguments are no different to the ones in [2]. First we recall briefly the basic definitions of equivariant cohomology, and the relevant localization theorem, and then state the conclusions for X_M-cohomology.

If a torus G acts on a manifold M (with or without boundary), the Cartan model for the equivariant cohomology is defined as follows. Let $\{X_1,\dots,X_\ell\}$ be a basis of $\mathfrak{g}$ and $\{u_1,\dots,u_\ell\}$ the corresponding coordinates. The "Cartan complex" consists of poynomial[1] maps from $\mathfrak{g}$ to the space of invariant differential forms, so $\Omega^*_G(M)\otimes R$ where $R = \mathbb{R}[u_1,\dots,u_\ell]$, with differential

$$\mathsf{d}_{\mathrm{eq}}(\omega) = \mathsf{d}\omega + \sum_{j=1}^{\ell} u_j\, \iota_{X_j}\omega.$$

The equivariant cohomology $H^*_G(M)$ is the cohomology of this complex. The relative equivariant cohomology $H^*_G(M,\partial M)$ (if M has non-empty boundary) is formed by taking the subcomplex with forms that vanish on the boundary $i^*\omega = 0$, with the same differential.

The cohomology groups are graded by giving the u_i weight 2 and a k-form weight k, so the differential d_{eq} is of degree 1. Futhermore, as the cochain groups are R-modules, and d_{eq} is a homomorphism of R-modules, it follows that the equivariant cohomology is an R-module. The localization theorem of Atiyah and Bott [2] gives information on the module structure (there it is only stated for absolute cohomology, but it is equally true in the relative setting, with the same proof; see also Appendix C of [4]).

First we define the following subset of $\mathfrak{g}$,

$$Z := \bigcup_{K \subsetneq G} \mathfrak{k}$$

where the union is over proper isotropy subgroups of the action on M. If M is compact, then Z is a finite union of proper subspaces of $\mathfrak{g}$. Let $F = \mathrm{Fix}(G,M) = \{x \in M \mid G\cdot x = x\}$ be the set of fixed points in M. It follows from the local structure of group actions that F is a submanifold of M, with boundary $\partial F = F \cap \partial M$.

[1] we use real valued polynomials, though complex valued ones works just as well, and all tensor products are thus over $\mathbb{R}$, unless stated otherwise

Theorem 4.1 (Atiyah-Bott [2, Theorem 3.5]) *The inclusion $j: F \hookrightarrow M$ induces homomorphisms of R-modules*

$$H^*_G(M) \xrightarrow{j^*} H^*_G(F)$$

$$H^*_G(M,\partial M) \xrightarrow{j^*} H^*_G(F,\partial F)$$

whose kernel and cokernel have support in Z.

In particular, this means that if $f \in I(Z)$ (the ideal in R of polynomials vanishing on Z) then the localizations[2] $H^*_G(M)_f$ and $H^*_G(F)_f$ are isomorphic R_f-modules. Notice that the act of localization destroys the integer grading of the cohomology, but since the u_i have weight 2, it preserves the parity of the grading, so that the separate even and odd parts are maintained: $H^{\pm}_G(M)_f \simeq H^{\pm}_G(F)_f$. The same reasoning applies to the cohomology relative to the boundary, so $H^{\pm}_G(M,\partial M)_f \simeq H^{\pm}_G(F,\partial F)_f$

Since the action on F is trivial, it is immediate from the definition that there is an isomorphism of R-modules, $H^*_G(F) \simeq H^*(F) \otimes R$ so that the localization theorem shows j^* induces an isomorphism of R_f-modules,

$$H^{\pm}_G(M)_f \xrightarrow{j^*} H^{\pm}(F) \otimes R_f. \tag{4.1}$$

It follows that $H^{\pm}_G(M)_f$ is a free R_f module whenever $f \in I(Z)$. Of course, analogous statements hold for the relative versions. Since localization does not alter the rank of a module (it just removes torsion elements), we have that

$$\operatorname{rank} H^{\pm}_G(M) = \dim H^{\pm}(F), \qquad \operatorname{rank} H^{\pm}_G(M,\partial M) = \dim H^{\pm}(F,\partial F).$$

For $X \in \mathfrak{g}$, define $N(X_M) = \{x \in M \mid X_M(x) = 0\}$, the set of zeros of the vector field X_M. Clearly $N(X_M) \supset F$, and $N(X_M) = F$ if and only if $X \notin Z$.

Theorem 4.2 *Let $X = \sum_j s_j X_j \in \mathfrak{g}$. If the set of zeros of the corresponding vector field X_M is equal to the fixed point set for the G-action (i.e. $N(X_M) = F$) then*

$$H^{\pm}_{X_M}(M,\partial M) \cong H^{\pm}_G(M,\partial M)/\mathfrak{m}_X H^{\pm}_G(M,\partial M), \tag{4.2}$$

and

$$H^{\pm}_{X_M}(M) \cong H^{\pm}_G(M)/\mathfrak{m}_X H^{\pm}_G(M) \tag{4.3}$$

where $\mathfrak{m}_X = \langle u_1 - s_1, \dots, u_l - s_l \rangle$ is the ideal of polynomials vanishing at X.

PROOF: Our assumption $N(X_M) = F$ is equivalent to $X \in \mathfrak{g} \setminus Z$. Therefore there is a polynomial $f \in I(Z)$ such that $f(X) \neq 0$. In addition, we can use f and replace the ring R by R_f and then localize $H^{\pm}_G(M)$ and $H^{\pm}_G(M,\partial M)$ to make $H^{\pm}_G(M)_f$ and $H^{\pm}_G(M,\partial M)_f$ which are free R_f-modules.

We now apply the lemma stated below, in which the left-hand side is obtained by putting $u_i = s_i$ before taking cohomology, so results in $H^{\pm}_{X_M}(M)$ (or similar for the relative case), while the right-hand side is the right-hand side of (4.2) and (4.3), so proving the theorem. ❒

[2]The localized ring R_f consists of elements of R divided by a power of f and if K is an R-module, its localization is $K_f := K \otimes_R R_f$; they correspond to restricting to the open set where f is non-zero. See the notes by Libine [7] for a good discussion of localization in this context.

Lemma 4.3 (Atiyah-Bott [2, Lemma 5.6]) *Let (C^*,d) be a cochain complex of free R-modules and assume that, for some polynomial f, $H(C^*,d)_f$ is a free module over the localized ring R_f. Then, if $s \in \mathbb{R}^l$ with $f(s) \neq 0$,*

$$H^{\pm}(C_s^*,d_s) \cong H^{\pm}(C^*,d) \bmod \mathfrak{m}_s$$

where $\mathfrak{m}_s$ is the ideal $\langle u_1 - s_1, \ldots, u_l - s_l \rangle$.

Corollary 4.4 *With the hypotheses of the theorem, the pullback j^* to the fixed point set induces isomorphisms:*

1- $H^{\pm}_{X_M}(M) \simeq H^{\pm}(F)$,

2- $H^{\pm}_{X_M}(M,\partial M) \simeq H^{\pm}(F,\partial F)$.

PROOF: Reduce equation (4.1) modulo $\mathfrak{m}_X$ and apply Theorem 4.2. ❐

5 Conclusions

In previous sections, we began with the action of a torus G; here we state results for a given Killing vector field K on a compact Riemannian manifold M (with or without boundary), more in keeping with Witten's original work [10]. Recall that the group $\mathrm{Isom}(M)$ of isometries of M is a compact Lie group, and the smallest closed subgroup $G(K)$ containing K in its Lie algebra is Abelian, so a torus. Furthermore, the submanifold $N = N(K)$ of zeros of K coincides with $\mathrm{Fix}(G(K),M)$

The equivariant cohomology constructions of Section 4 give us the proof of the following result, which extends the theorem of Witten (our Theorem 2.9) to manifolds with boundary.

Given a K-harmonic form on M, its pullpack to N is harmonic in the ordinary sense.

Theorem 5.1 *Let K be a Killing vector field on the compact Riemannian manifold M (with or without boundary), and let $N = N(K)$ be the submanifold of zeros of K. Then pullback to N induces isomorphisms*

$$H^{\pm}_K(M) \cong H^{\pm}(N), \quad \textit{and} \quad H^{\pm}_K(M,\partial M) \cong H^{\pm}(N,\partial N).$$

PROOF: Apply Corollary 4.4 to the equivariant cohomology for the action of the torus $G(K)$. ❐

Furthermore, using the Hodge star operator, the Poincaré-Lefschetz duality of Theorem 3.16(c) corresponds under the isomorphisms in the theorem above, to Poincaré-Lefschetz duality on the fixed point space.

Translating this theorem into the language of harmonic fields, shows

$$\mathcal{H}^{\pm}_{K,N}(M) \cong \mathcal{H}^{\pm}_N(N) \quad \text{and} \quad \mathcal{H}^{\pm}_{K,D}(M) \cong \mathcal{H}^{\pm}_D(N)$$

where $\mathcal{H}^{\pm}_N(N)$ and $\mathcal{H}^{\pm}_D(N)$ are the ordinary Neumann and Dirichlet harmonic fields on N respectively. More explicitly, since the isomorphisms are induced by the inclusion $j: N \hookrightarrow M$, one has the following.

Corollary 5.2 *Given any harmonic field on N with either Dirichlet or Neumann boundary conditions, there is a unique K-harmonic field on M with the corresponding boundary conditions.*

Note that if $\partial N = \emptyset$ then the boundary condition on N is non-existent, and so every harmonic form (=field) on N can be uniquely extended to each of a Dirichlet and a Neumann harmonic form on M.

Euler characteristics As is well known, given a complex of $\mathbb{R}[s]$ (or $\mathbb{C}[s]$) modules whose cohomology is finitely generated, the Euler characterstic of the complex is independent of s. This remains true for a $\mathbb{Z}_2$-graded complex, for the same reasons (briefly, the cohomology is the direct sum of a torsion module and a free module, and the torsion cancels in the Euler characterstic).

Consequently, $\chi(M) = \chi(N)$ and $\chi(M, \partial M) = \chi(N, \partial N)$, and furthermore applying the same arguments to the manifold ∂M, one has $\chi(\partial M) = \chi(\partial N)$ (since $\mathrm{Fix}(G, \partial M) = \partial N$). This is of course compatible with the usual relation between Euler characteristics of manifolds with boundary,

$$\chi(M) = \chi(\partial M) + \chi(M, \partial M) = \chi(\partial N) + \chi(N, \partial N) = \chi(N).$$

These facts about Euler characteristics can of course be obtained in a more elementary manner using basic algebraic topology and Mayer-Vietoris sequences.

References

[1] R. Abraham, J.E. Marsden, and T.S. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[2] M.F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[3] K.O. Friedrichs. Differential forms on Riemannian manifolds. *Comm. Pure Appl. Math.*, 8:551–590, 1955.

[4] V. Guillemin, V. Ginzburg, and Y. Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[5] W.V.D. Hodge. A Dirichlet problem for harmonic functionals, with applications to analytic varieties. *Proc. London Math. Soc.*, s2-36(1):257303, 1934.

[6] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[7] M. Libine. Lecture notes on equivariant cohomology. *arXiv:0709.3615v1*, 2007.

[8] C.B. Morrey, Jr. A variational method in the theory of harmonic integrals. II. *Amer. J. Math.*, 78:137–170, 1956.

[9] G. Schwarz. *Hodge decomposition—a method for solving boundary value problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[10] E. Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692, 1982.

School of Mathematics,
University of Manchester,
Oxford Road,
Manchester M13 9PL,
UK.

`Qusay.Abdul-Aziz@postgrad.manchester.ac.uk`
`j.montaldi@manchester.ac.uk`