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1 Introduction

It is shown in [5] that for a ringed space with a basis of compact open sets, the category of modules on the space is locally finitely presented. In this paper we generalise this result to sheaves of modules over a ring object in an arbitrary locally finitely presentable topos. This is a consequence of the monadicity of the module category over the base topos, which we present as a consequence of Beck's theorem.[1]

Throughout, we take λ to be a regular cardinal. We will think mainly of the case when $\lambda = \aleph_0$, but all of the results carry over for arbitrary regular cardinals. By a λ-small set, we will mean a set with cardinality less than λ.

In section 2, we recall basic facts about monads, and state Beck's Theorem.

In section 3, we provide necessary and sufficient conditions for the category of sheaves on a space to be locally finitely presented.

In section 4, we show that the associated sheaf functor for presheaves of modules commutes with the forgetful functor.

In section 5, we use this fact to prove that the category of modules over the sheaf of rings is monadic over the base topos, and show that our result is a consequence of this fact.

2 Monads

Recall that a monad on a category $\mathcal{C}$ is given by a triple (T, ϵ, μ) where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\epsilon : 1_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ are natural transformations, such that the diagrams below commute:

$$\begin{tikzcd} T \arrow[r, "\epsilon_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\epsilon"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

[1] The use of monadic functors to approach this problem was suggested to us by Tibor Beke. The work reported in this paper will form part of the doctoral thesis of the author.

Given a monad $\mathbb{T} = (T, \epsilon, \mu)$ on a category $\mathcal{C}$, an algebra on $\mathbb{T}$ consists of a pair (A, α), where A is an object of $\mathcal{C}$, and $\alpha : TA \to A$ is a map such that the diagrams below commute.

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd}$$

A morphism of algebras $f : (A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ in $\mathcal{C}$ such that $f\alpha = \beta T f$.

For a monad $\mathbb{T}$, the category of algebras and their morphisms is called the Eilenberg-Moore category, denoted $\mathcal{C}^{\mathbb{T}}$.

There is an obvious forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, mapping each algebra (A, α) to A. Also, there is a 'free algebra' functor $F : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$, left adjoint to U, sending each object C of $\mathcal{C}$ to the algebra (TC, μ_C). We say a functor $R : \mathcal{X} \to \mathcal{C}$ is monadic if there is some monad $\mathbb{T}$ on $\mathcal{C}$ such that $\mathcal{X}$ is equivalent to $\mathcal{C}^{\mathbb{T}}$ and R corresponds to the forgetful functor U under the equivalence.

The following result characterizes such functors, see eg [2, vol.2, 4.4.4] or [4, IV.4.2], for a proof.

Theorem 1 (Beck's Theorem)**.** *A functor $R : \mathcal{X} \to \mathcal{C}$ is monadic iff it has a left adjoint L, it reflects isomorphisms, and it creates split coequalisers.*

Here a split coequaliser diagram is a diagram

$$\begin{tikzcd} C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & D \arrow[l, bend left, "r"] \arrow[r, "q"] & Q \arrow[l, bend left, "s"] \end{tikzcd}$$

where $qu = qv$, $qs = 1_Q$, $ur = 1_D$ and $vr = sq$ (this of course makes q a coequaliser for u and v). R is said to create such coequalisers if given $u, v : X \to Y$ in $\mathcal{X}$ such that Ru and Rv have a split coequaliser in $\mathcal{C}$, then there is a coequaliser of u and v in $\mathcal{X}$, and this is preserved by R.

We will also use the following result, which is derived from the reflection theorem for accessible categories on p.124 of [1]. We provide a direct proof here for completeness.

Lemma 2. *If $\mathbb{T} = (T, \epsilon, \mu)$ is a monad on a locally λ-presentable category $\mathcal{C}$, and T preserves λ-directed colimits, then the category of algebras $\mathcal{C}^{\mathbb{T}}$ is also locally λ-presented.*

Proof. A direct proof of the cocompleteness of $\mathcal{C}^{\mathbb{T}}$ is given at [2, vol.2, 4.3.6].

We will show that the free $\mathbb{T}$-algebras on the λ-presented objects in $\mathcal{C}$ are λ-presented in $\mathcal{C}^{\mathbb{T}}$, and that these form a strong generating set in $\mathcal{C}^{\mathbb{T}}$.

Let C be a λ-presented object in $\mathcal{C}$. The free $\mathbb{T}$-algebra generated by this is the object (TC, μ_C) in $\mathcal{C}^{\mathbb{T}}$.

Suppose we have a λ-directed colimit cocone in $\mathcal{C}^{\mathbb{T}}$, say

$$\{ (D_i, \gamma_i) \xrightarrow{d_i} (D, \gamma) \}$$

and some $\mathbb{T}$-algebra map $f : (TC, \mu_C) \to (D, \gamma)$.

By [2, 4.3.2], the forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ preserves λ-directed colimits, so

$$\{D_i \xrightarrow{Ud_i} D\}$$

is a λ-directed diagram in $\mathcal{C}$. Thus the map $f\epsilon_C : C \to D$ must factor through one of the cocone maps Ud_i, say $f\epsilon_C = Ud_i\tilde{f}$, where $\tilde{f} : C \to D_i$ is some map in $\mathcal{C}$. But now by the universal property of the free functor, there must be some $\mathbb{T}$-algebra map $\bar{f} : (TC, \mu_C) \to (D_i, \gamma_i)$ such that $\tilde{f} = \bar{f}\epsilon_C$. Furthermore, we must have $Ud_iU\bar{f} = Uf$ since both are factorisations of $Uf\epsilon_C$ through the free $\mathbb{T}$-algebra (TC, μ_C). But this is just the statement that $d_i\bar{f} = f$. Similarly, the essential uniqueness of this factorisation follows from the essential uniqueness of the factorisation of $f\epsilon_C$ in the category $\mathcal{C}$. This proves that (TC, μ_C) is λ-presented in $\mathcal{C}^{\mathbb{T}}$.

To show that the objects (TC, μ_C) form a generating set for $\mathcal{C}^{\mathbb{T}}$, suppose we have a pair of arrows $f \neq g : (A, \alpha) \to (B, \beta)$ in $\mathcal{C}^{\mathbb{T}}$. These are represented by arrows $Uf, Ug : A \to B$ in $\mathcal{C}$. Now since the objects C form a generating set in $\mathcal{C}$, there is some map $x : C \to A$ with C λ-presented such that $Ufx \neq Ugx$. Letting $\bar{x} : (TC, \mu_C) \to (A, \alpha)$ be the factorisation of x through ϵ_C given by the universal property of the free functor, we get that $f\bar{x} \neq g\bar{x}$. This shows that the objects (TC, μ_C) form a generating set.

To show this generating set is strong, suppose we are given an object (A, α) in $\mathcal{C}^{\mathbb{T}}$, and a proper subobject $(S, \sigma) \xrightarrow{s} (A, \alpha)$. It is easy to show that the map $Us : S \to A$ in $\mathcal{C}$ is a proper subobject in $\mathcal{C}$, and so there is some map $x : C \to A$ that does not factor through it. The map $\bar{x} : (TC, \mu_C) \to (A, \alpha)$ in $\mathcal{C}^{\mathbb{T}}$ cannot factor through s, since if it did this would give a factorisation of x through s. This shows that the objects (TC, μ_C) form a strong generating set for $\mathcal{C}^{\mathbb{T}}$, completing the proof. $\square$

3 Sheaves on a space

Our main result concerns categories of module objects in locally λ-presentable toposes. It is therefore natural to ask when a topos is locally λ-presentable. In this section we give necessary and sufficient conditions for the category of sheaves on a space to be locally λ-presentable, for any regular cardinal λ.

Let $\mathcal{C}$ be a small category with pullbacks, and let J be a Grothendieck topology on $\mathcal{C}$, with a basis K as in [4, III.2]. Then a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ is a sheaf with respect to this topology iff for any cover $\{f_i : C_i \to C \mid i \in I\}$ in K, the diagram

$$PC \xrightarrow{e} \prod_{i\in I} PC_i \overset{p_2}{\underset{p_1}{\rightrightarrows}} \prod_{i,j\in I} P(C_i \times_C C_j)$$

is an equaliser, where e is given by the collection of maps $Pf : PC \to PC_i$, and the maps p_1, p_2 are given by the first and second projections of the pullbacks, see [4, III.4.1].

We say a Grothendieck topology is of λ-type if there exists a basis K in which all the covering families have size less than λ. In particular, for such a topology, the diagram above is λ-small.

Since in a locally λ-presentable category, we have that λ-small limits commute with λ-directed colimits [1, 1.59], we have the following result:

Lemma 3. *Let $\mathcal{C}$ be a small category with pullbacks, and let J be a topology on $\mathcal{C}$ with a basis K consisting entirely of λ-small families. Then $\mathbf{Sh}(\mathcal{C}, J)$ is a locally λ-presentable category.*

In the case where $\lambda = \aleph_0$, this is the well-known fact that coherent toposes are locally finitely presented, see eg [3, D.3.3.12].

Remark. Alternatively, we can write a λ-ary essentially algebraic theory as in [1, 3.34], whose models are precisely the sheaves for the topology J.

The sorts for our language will be the objects of $\mathcal{C}$.

For each arrow $f : C \to C'$ in $\mathcal{C}$, we include in our language a (total) function symbol $\mathbf{f} : C' \to C$, and include equations in our theory expressing that the composition of two function symbols is the function symbol corresponding to the composition of the two arrows.

In addition, for each covering family $S = \{f_i : C_i \to C\}$ in K, we add a partial morphism $\sigma : \prod_{i \in I} C_i \to C$.

For each pair $i, j \in I$, let $p_1^{i,j}$ and $p_2^{i,j}$ be the function symbols corresponding to the arrows in the pullback diagram

$$\begin{tikzcd} C_i \times_C C_j \arrow[r, "p_1^{i,j}"] \arrow[d, "p_2^{i,j}"'] & C_i \arrow[d, "f_i"] \\ C_j \arrow[r, "f_j"] & C \end{tikzcd}$$

We now define

The collection of models for this theory are precisely the sheaves over the site $(\mathcal{C}, J)$; since each operation symbol is λ-ary, each equation uses less than λ variables, and each $\mathrm{Def}(\sigma)$ uses less than λ-equations. It follows from [1, 3.36] that the category of models is locally λ-presentable. $\mathrm{Def}(\sigma)$ to be the collection of equations $p_1^{i,j}(\bar{x}) = p_2^{i,j}(\bar{x})$, as i and j range over all possible pairs from I.

The partial map σ thus maps each 'matching family' to a unique 'amalgamation'; we add equations to our theory $\sigma(\{f_i(x)\}_{i \in I}) = x$ and $f_j(\sigma(\{x_i\}_{i \in I}) = x_j$ to express this.

Definition 4. *Suppose X is a topological space; we define an open subset of X to be λ-compact if every open cover of it has a λ-small subcover.*

That is, an open set is λ-compact if it is λ-presented in $\mathcal{O}(X)$, the lattice of open sets in X.

Example. 1. If $\lambda = \aleph_0$, this is just the usual notion of compact.

2. The open intervals (p, q) form a basis for the topology on $\mathbb{R}$, where $p, q \in \mathbb{Q}$. Every open subset of $\mathbb{R}$ is therefore $\aleph_1$-compact, since every cover can be broken down into elements of this basis, and countably many of these sets must suffice to cover a given open set in the real line.

3. In general, if a space X has a λ-small basis, then all the open sets in it are λ-compact.

Now suppose X is a topological space with a basis $\mathcal{B}$ of λ-compact sets. Then the category of sheaves on X is equivalent to the category of sheaves on the basis, see for example [4, II.1.3]. Moreover, the covers in $\mathcal{B}$ are precisely those containing λ-small covers, by λ-compactness. Thus the λ-small covers form a basis for the Grothendieck topology on $\mathcal{B}$, and by the previous result, the category of sheaves is locally λ-presentable.

The rest of this section is dedicated to showing that the converse is true.

To prove the next result, we will need to look at slice categories, see eg [2, vol.1, p.92], [4, p.12]. Given an object C in a category $\mathcal{C}$, the category of objects over C, denoted $\mathcal{C}/C$, has as its objects arrows $B \xrightarrow{f} C$ in $\mathcal{C}$, and as its morphisms, commutative triangles

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f"'] & B' \arrow[d, "f'"] \\ & C \end{tikzcd} .$$

Proposition 5. *Let $\mathcal{C}$ be a locally λ-presentable category; let C be an object of $\mathcal{C}$. Then Sub(C) is locally λ-presentable.*

Proof. We start by showing that $\mathcal{C}/C$ is locally λ-presentable.

That $\mathcal{C}/C$ is cocomplete is proved in [2, vol.1, 2.16.3]; we describe the colimits here. Suppose $\mathcal{I}$ is a small category and we have some functor $D : \mathcal{I} \to \mathcal{C}/C$, sending each object i in I to the object $D_i \xrightarrow{d_i} C$ in $\mathcal{C}/C$, and each map $\mu : i \to j$ in $\mathcal{I}$ to $D_\mu : D_i \to D_j$. Then we can compose with the forgetful functor $U : \mathcal{C}/C \to \mathcal{C}$ and form the colimit $\{D_i \xrightarrow{l_i} L\}$ of UD in $\mathcal{C}$. The maps $\{d_i\}$ now form a compatible cocone over the diagram, so by the colimit property we get a unique map $L \xrightarrow{d} C$. One can easily verify that this map is a colimit for the diagram D in $\mathcal{C}/C$.

We now claim that $\mathcal{C}/C$ has a strong generating set of λ-presented objects. Suppose B is a λ-presented object in $\mathcal{C}$; then we claim that any map $B \xrightarrow{b} C$ is a λ-presented object in $\mathcal{C}/C$. This is clear, since by the construction of the colimits in $\mathcal{C}/C$, they are preserved by the forgetful functor U; thus given a map f from $B \xrightarrow{b} C$ to a λ-directed colimit in $\mathcal{C}/C$, we apply the forgetful functor and can factorize Uf through the colimit cocone. One may easily check that this is also a factorization in $\mathcal{C}/C$, and so the object $B \xrightarrow{b} C$ is λ-presented in $\mathcal{C}/C$.

The collection of all maps $B \to C$ with B λ-presented is a strong generating set of objects for $\mathcal{C}/C$: this follows immediately from the fact that the λ-presented objects form a strong generating set in $\mathcal{C}$.

This proves that $\mathcal{C}/C$ is locally λ-presentable.

Now if $D \xrightarrow{f} C$ is any object in $\mathcal{C}/C$, then defining rf to be the image of f, we get a reflection functor $\mathcal{C}/C \to \mathrm{Sub}(C)$; the image exists because $\mathcal{C}$ has strong epi - mono factorisations.

This shows that Sub(C) is reflective in $\mathcal{C}/C$. It now suffices to show that Sub(C) is closed in $\mathcal{C}/C$ under λ-directed colimits; the result will then follow by [1, 1.39].

Now suppose we have a λ-directed system in Sub(C), that is, a λ-directed collection of maps over C

$$\{(S_i \xrightarrow{f_i} C)\}$$

such that each f_i is monic. Then the colimit in $\mathcal{C}/C$ is given by taking the colimit

$$\{(S_i \xrightarrow{\mu_i} S)\}$$

of the system in C, and taking the map $S \xrightarrow{f} C$ induced by the f_i to get a colimit in $\mathcal{C}/C$. To show that this colimit is in $\mathrm{Sub}(C)$, it suffices to show that f is monic (in $\mathcal{C}$).

So suppose we have distinct arrows $B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} S$ such that $f\alpha = f\beta$. Then since C has a generating set of λ-presentable object, we can choose an arrow $G \xrightarrow{x} B$ with G λ-presentable, such that $\alpha x \neq \beta x$. Now since the S_i form a λ-directed system, αx and βx both factor through one of the S_i's, and taking an upper bound we can choose S_i with maps

$$G \underset{b}{\overset{a}{\rightrightarrows}} S_i \xrightarrow{\mu_i} S$$

such that $\mu_i a = \alpha x$, $\mu_i b = \beta x$.

But now $f\mu_i a = f\alpha x = f\beta x = f\mu_i b$. Since $f\mu_i = f_i$ which is monic, this gives us that $a = b$ and therefore $\alpha x = \beta x$. This is a contradiction. $\square$

For any topological space X, the lattice of open sets $\mathcal{O}(X)$ is equivalent to the lattice of subobjects of the terminal object in $\mathbf{Sh}(X)$ [4, III.8(17)], so we have the following result:

Corollary 6. *Let X be a topological space such that $\mathbf{Sh}(X)$ is locally λ-presented. Then $\mathcal{O}(X)$ is locally λ-presented.*

Since the sets λ-presented in this lattice are precisely the λ-compact opens, each open set can therefore be presented as the λ-directed union of its λ-compact subsets. In particular, the λ-compact open subsets form a basis.

We summarize:

Corollary 7. *The category $\mathbf{Sh}(X)$ is locally λ-presented iff X has a basis of λ-compact open sets.*

Remark. This corollary shows that for spatial toposes, being coherent is equivalent to being locally finitely presentable. Examples are given in [3, D.3.3.12] of locally finitely presentable toposes that are not coherent. It is shown that if the category of sheaves on a space is a coherent topos, then the space has a basis of compact open sets in [3, D.3.3.14].

4 The Associated Sheaf Functor for a Presheaf of Modules

The results in this section will be used to prove our main theorem. We could not find a full account of this material in the literature, so we have proved the results we need.

Given a ring object in a category of sheaves, we want to show that the associated sheaf functor for presheaves of modules over the ring commutes with the forgetful functor. We begin by summarizing the construction of the associated sheaf functor for sheaves of sets, which is given in (for example) [4, III.5].

Let $(\mathcal{C}, J)$ be a site and let P be a presheaf of sets on $\mathcal{C}$. For any object C in $\mathcal{C}$ and any cover $S \in JC$, a matching family of elements of P is a map of presheaves $S \to P$, where we consider S to be a subpresheaf of $\mathrm{Hom}(-, C)$. Equivalently, we can think of it as a set $\{x_f\}_{f \in S}$ where for each $f : D \to C$ in R, $x_f \in PD$, and such that $x_{fg} = Pf(x_g)$. We write $\mathrm{Match}(S, P)$ for the collection of all matching families of elements of P for the cover R. An amalgamation for a matching family $\{x_f\}_{f \in S}$ is some $x \in PC$ such that $x_f = Pf(x)$ for each $f \in S$.

The associated sheaf functor is defined by two applications of the 'plus-functor' $P \mapsto P^+$. For an arbitrary presheaf P, this is given by

$$P^+C = \varinjlim_{S \in JC} \mathrm{Match}(S, P),$$

that is, each element of P^+C is an equivalence class of matching families for covers of C where two matching families $\{x_f\}_{f \in S}$ and $\{y_g\}_{g \in T}$ are equivalent if there is a common refinement of S and T on which they agree, that is, if there exists $U \subseteq S \cap T$ such that for all $f \in U$, $x_f = y_f$.

For each map $h : C' \to C$, and each subpresheaf S of $\mathrm{Hom}(-, C)$, we write $h^*S = \{g : D \to C' \mid hg \in S\}$.

We define $P^+h : P^+C \to P^+C'$ by

$$\{x_f \mid f \in S\} \mapsto \{x_{hf'} \mid f' \in h^*S\}.$$

Thus P^+ is a presheaf. Each map $\phi : P \to Q$ of presheaves induces a map $\phi^+ : P^+ \to Q^+$ of presheaves by taking a matching family $S \longrightarrow P$ to the composite $S \longrightarrow P \xrightarrow{\phi} Q$.

There is a map of presheaves $\eta : P \to P^+$ defined by sending each $x \in PC$ to the equivalence class of $\{Pf(x) \mid f \in t_C\}$, where t_C is the maximal sieve on C.

If F is a sheaf and P is a presheaf, then any map $\phi : P \to F$ of presheaves factors uniquely through η; we write $\phi = \tilde{\phi}\eta$, as in the diagram

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[dr, "\phi"'] & P^+ \arrow[d, "\tilde{\phi}"] \\ & F \end{tikzcd}$$

Moreover, for every presheaf P, $(P^+)^+$ is a sheaf. Thus defining $aP = P^{++}$, we get that $a : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sh}(\mathcal{C}, J)$ is a left adjoint to the inclusion functor $i : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

We now turn to the question of sheaves of modules for a given sheaf of rings R.

Lemma 8. *Let R be a sheaf of rings on a site $(\mathcal{C}, J)$, and let M be a presheaf of R-modules. Then there is a canonical R-module structure on UM^+, for which $\eta : UM \to UM^+$ is a homomorphism.*

Proof. For each object C in $\mathcal{C}$, the elements of UM^+C are given by equivalence classes of families

$$\{x_f | f : D \to C, f \in S\}$$

where $S \in JC$, and two such families $\mathbf{x} = \{x_f \mid f \in S\}$ and $\mathbf{y} = \{y_g \mid g \in T\}$ are equivalent if there is a common refinement $U \subseteq S \cap T$ with $U \in JC$ such that $x_f = y_f$ for all $f \in W$.

So if $\mathbf{x}$ and $\mathbf{y}$ are two such classes, choose representations $\{x_f | f \in S\}$ and $\{y_g | g \in T\}$; then choose the sum $\mathbf{x} + \mathbf{y}$ to be (the equivalence class of) the point-wise sum of x_f and y_g on $S \cap T$.

To show this is well-defined, suppose we choose another representative $\{x'_f | f \in S'\}$ of $\mathbf{x}$. Then there is a refinement $W \in S \cap S'$ such that $x_f = x'_f$ for all $f \in W$. So for $f \in W \cap T$, we have $x_f + y_f = x'_f + y_f$. Now

$$W \cap T \subseteq (S \cap T) \cap (S' \cap T)$$

and $W \cap T \in JC$. So this shows that the two sums above are equivalent, that is, that the addition operation is well-defined.

If $r \in RC$, and $\mathbf{x}$ is an equivalence class represented by $\{x_f | f \in S\}$ as above, then we can define the scalar product rx to be the equivalence class represented by $\{Rf(r).x_f | f \in S\}$. It is easy to show that this definition is independent of the choice of matching family to represent x.

We have now shown that $UM^+(C)$ has a canonical RC-module structure.

Now suppose $h : D \to C$ is a morphism in $\mathcal{C}$. Then $UM^+(h)$ is defined by

$$UM^+(h)(\{x_f | f \in S\}) = \{x_{hf'} | f' \in h^*S\}$$

and this is well-defined on equivalence classes.

To verify this commutes with addition, suppose $\mathbf{x}$ and $\mathbf{y}$ are equivalence classes with representatives $\{x_f | f \in S\}$, $\{y_g | g \in T\}$. Adding their images under UM^+, we get:

$$\{x_{hf'} | f' \in h^*S\} + \{y_{hg'} | g \in h^*T\} = \{x_{hf'} + y_{hf'} | f' \in h^*S \cap h^*T\} \qquad (1)$$

Now we note that $h^*S \cap h^*T = h^*(S \cap T)$ (from the definition of h^*) and that

$$x_{hf'} + y_{hf'} = UM(h)(x_{f'}) + UM(h)(y_{f'}) = UM(h)(x_{f'} + y_{f'})$$

since $UM(h)$ is a module map. It follows that the right hand side of (2) is just the image of the sum $\mathbf{x} + \mathbf{y}$ under $UM^+(h)$, which is precisely what we set out to prove.

To show that $UM^+(h)$ commutes with scalar multiplication is similar.

This shows that the plus-operator maps presheaves of modules to presheaves of modules. It remains to show that it maps morphisms to morphisms. Let $M \xrightarrow{\phi} N$ be a morphism of presheaves of R-modules. Then $U\phi^+ : UM^+ \to UN^+$ is defined by

$$\{x_f | f \in R\} \mapsto \{\phi_{dom(f)}(x_f) | f \in R\}$$

This clearly commutes with the module operations since ϕ does.

To show ϕ is a natural transformation, let $h : C \to C'$ be a morphism in $\mathcal{C}$. We need to show the diagram:

$$\begin{CD} UM^+C @>{UM^+h}>> UM^+C' \\ @V{U\phi^+_C}VV @VV{U\phi^+_{C'}}V \\ UN^+C @>{UN^+h}>> UN^+C' \end{CD}$$

commutes. But this can be immediately verified by inspection.

We also need to show that $\eta : UM \to UM^+$ is an R-module homomorphism. To show it's a homomorphism of (sheaves of) abelian groups, consider an object C in $\mathcal{C}$, and take two elements $x, y \in MC$. Then since Pf is a module morphism for every morphism f in $\mathcal{C}$, we have

$$\eta_C(x+y) = \{Pf(x+y)|f \in t_C\} = \{Pf(x) + Pf(y)|f \in t_C\} = \eta_C(x) + \eta_C(y)$$

To show η commutes with scalar multiplication is similar. ☐

Lemma 9. *Let P be a presheaf of modules, F a sheaf of modules, and $\phi : P \to F$ be a map of presheaves of modules. Then $\tilde{\phi} : P^+ \to F$ is a map of presheaves of modules.*

Proof. Let $\mathbf{x}$and $\mathbf{y}$ be elements of P^+C, represented by $\{x_f \mid f \in S\}$ and $\{y_g \mid g \in T\}$ respectively, for some $C \in \mathcal{C}$.

The image $\tilde{\phi}(x)$ is the unique amalgamation of $\{\phi(x_f) \mid f \in S\}$ in FC, $\tilde{\phi}(y)$ is the unique amalgamation of $\{\phi(y_g) \mid g \in T\}$. So for all $f \in R \cap S$, we have

$$Ff(\tilde{\phi}(x) + \tilde{\phi}(y)) = \phi(x_f) + \phi(y_f)$$

But then $\tilde{\phi}(x) + \tilde{\phi}(y)$ is an amalgamation of

$$\{\phi(x_f) + \phi(y_f) \mid f \in R \cap S\} = \{\phi(x_f + y_f) \mid f \in R \cap S\} = \tilde{\phi}(x + y)$$

The proof that $\tilde{\phi}$ respects scalar multiplication is similar. ☐

Let M be a presheaf of modules over R. Then let M^+ be the presheaf of modules whose underlying presheaf of sets is UM^+, and whose R-module structure is that just given.

Corollary 10. *Let P be a presheaf of modules. Then for any morphism $\phi : P \to F$ where F is a sheaf of modules, there is a unique morphism (of presheaves of modules) $\tilde{\phi} : P^+ \to F$ such that the diagram below commutes:*

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[dr, "\phi"'] & P^+ \arrow[d, "\tilde{\phi}"] \\ & F. \end{tikzcd} \tag{2}$$

Now we define $\tilde{a} : \mathbf{PreMod}\text{-}R \to \mathbf{PreMod}\text{-}R$ by $\tilde{a}M = M^{++}$. Since the underlying set of aM is just UM^{++}, we have proved the following.

Corollary 11. *The functor $\tilde{a}$: **PreMod**-R $\to$ **Mod**-R is left adjoint to the inclusion functor $\tilde{i}$: **Mod**-R $\to$ **PreMod**-R.*

Furthermore, sheafification commutes with the forgetful functor, in the sense that the diagram below commutes:

$$\begin{tikzcd} \mathbf{PreMod}\text{-}R \arrow[r, "\tilde{a}"] \arrow[d, "U"] & \mathbf{Mod}\text{-}R \arrow[d, "U'"'] \\ \mathbf{Sets}^{\mathcal{C}^{op}} \arrow[r, "a"] & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

We also need the following lemma.

Lemma 12. *The forgetful functor U : **PreMod**-R $\to$ $\mathbf{Sets}^{\mathcal{C}^{op}}$ preserves directed colimits.*

Proof. We begin by showing that directed colimits in **PreMod-**R can be taken pointwise.

Let $\{M_i\}_{i \in I}$ be a directed system in **PreMod-**R for some directed poset $(I, \leq)$. Then define $L \in$ **PreMod-**R by taking LC to be the colimit of the directed system $\{M_iC\}$, for each object C in $\mathcal{C}$. Write l^i_C for the colimit maps $M_iC \to LC$.

Given a map $C \xrightarrow{f} D$ in $\mathcal{C}$, we get maps $\{\, M_iC \xrightarrow{M_if} M_iD \xrightarrow{l^i_D} LD \,\}$ forming a compatible cocone over the directed system $\{M_iC\}$; the map $Lf : LC \to LD$ is now given by the universal property of colimits. It follows from this that the families of maps $l^i = \{l^i_C\}_{C \in \mathcal{C}}$ are natural transformations, as implied by the notation.

To see this construction forms a colimit in **PreMod**-R, suppose we have some compatible cocone $\{\, M_i \xrightarrow{\alpha^i} P \,\}_{i \in I}$ in **PreMod**-R. Then in each component C, we get a unique factorisation map $LC \xrightarrow{\tilde{\alpha}_C} PC$. We need to show that the $\tilde{\alpha}$ so defined is a natural transformation, that is, given any map $f : C \to D$ in $\mathcal{C}$, that the following square commutes:

$$\begin{tikzcd} LC \arrow[r, "\tilde{\alpha}_C"] \arrow[d, "Lf"'] & PC \arrow[d, "Pf"] \\ LD \arrow[r, "\tilde{\alpha}_D"] & PD. \end{tikzcd}$$

This is true because for each $i \in I$, we have that

$$Pf\tilde{\alpha}_C l^i_C = Pf.\alpha^i_C = \alpha^i_D M_i f = \tilde{\alpha}_D l^i_D M_i f = \tilde{\alpha}_D Lf l^i_C$$

and so $Pf\tilde{\alpha}_C = \tilde{\alpha}_D Lf$ by uniqueness of the factorisation through the colimit.

For each C, the theory of RC modules is an algebraic theory and by [1, 3.4(4), 3.6(6)], the forgetful functor U preserves directed colimits. The same argument as above now shows that the maps $UM_i \to UL$ form a colimit cocone for the directed system UM_i in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. $\square$

5 Monadicity of the module category

Theorem 13. *Let $\mathcal{E}$ be a Grothendieck topos, and let R be a ring object in $\mathcal{E}$. Then* **Mod**-R*, the category of right R-modules in $\mathcal{E}$ is monadic over $\mathcal{E}$.*

Proof. Let $(\mathcal{C}, J)$ be an underlying site for $\mathcal{E}$.

Let $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ be the category of presheaves of sets over $\mathcal{C}$. Now R is a ring object in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, (since the inclusion functor $i : \mathcal{E} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ preserves the finite product $R \times R$) so we can define a category **PreMod**-R of (presheaves of) modules over R, that is, the R-module objects in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$.

We have the forgetful functor

$$U : \mathbf{PreMod}\text{-}R \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}.$$

This restricts to a forgetful functor between the sheaf categories (recall that a sheaf of modules is a presheaf such that the composition with the forgetful functor is a sheaf).

$$U' : \mathbf{Mod}\text{-}R \to \mathbf{Sh}(\mathcal{C}, J)$$

We seek to show that this functor is monadic.

We can construct a 'free module' functor, left adjoint to U, point-wise. Let

$$F : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{PreMod}\text{-}R$$

be the functor defined on each presheaf of sets S by taking $(FS)(C)$ to be the free RC-module generated by SC, for each object $C \in \mathcal{C}$. This induces a unique definition on arrows. So FS is a presheaf of modules.

We now claim that F is left adjoint to U, that is, that it has the universal property of a free module. So let S be a presheaf of sets, and let M be a presheaf of modules, as in the diagram

$$\begin{tikzcd} S \arrow[r, "\epsilon"] \arrow[dr, "\alpha"'] & UFS \\ & UM \end{tikzcd}$$

with ϵ being the canonical map $S \to UFS$ and α being an arbitrary natural transformation. Then for each $C \in \mathcal{C}$, we can define $\tilde{\alpha}_C : (FS)(C) \to MC$ by the free module property of $(FS)(C)$. It remains to show that $\tilde{\alpha}$ defined in this way is a natural transformation.

So let $C \xrightarrow{f} D$ be an arrow in $\mathcal{C}$. We need to show that the diagram below commutes:

$$\begin{tikzcd} FS(C) \arrow[r, "\tilde{\alpha}_C"] \arrow[d, "FSf"'] & MC \arrow[d, "Mf"] \\ FS(D) \arrow[r, "\tilde{\alpha}_D"] & MD \end{tikzcd}$$

Now we can expand this diagram in $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ to

$$\begin{tikzcd}
SC \arrow[r, "\epsilon_C"] \arrow[d, "Sf"'] & UFS(C) \arrow[r, "U\tilde{\alpha}_C"] \arrow[d, "UFSf"'] & UMC \arrow[d, "UMf"] \\
SD \arrow[r, "\epsilon_D"] & UFS(D) \arrow[r, "U\tilde{\alpha}_D"] & UMD
\end{tikzcd}$$

with $(U\tilde{\alpha}_C)\epsilon_C = \alpha_C$ and $(U\tilde{\alpha}_D)\epsilon_D = \alpha_D$.

Since α is natural, the outer rectangle commutes and the square on the left commutes by the definitions of F and ϵ. So we have a map of sets $SC \to UMD$, so there is a unique factorization through $SC \xrightarrow{\epsilon_C} UFS(C)$; and since both arms of our original diagram are such factorizations, they must be equal. Thus $F \dashv U$.

We now have the diagram:

$$\begin{tikzcd}
\mathbf{Mod}\text{-}R \arrow[r, shift left, "\tilde{i}"] \arrow[d, "U'"'] & \mathbf{PreMod}\text{-}R \arrow[l, shift left, "\tilde{a}"] \arrow[d, shift left, "U"] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[r, shift left, "i"] & \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \arrow[l, shift left, "a"] \arrow[u, shift left, "F"]
\end{tikzcd}$$

where by composition of adjunctions, $\tilde{a}.F \dashv U.\tilde{i} = iU'$. So in particular, for a sheaf of modules M and a sheaf of sets S, we have a natural isomorphism

$$(\tilde{a}F(iS), M) \cong (iS, iU'M) \cong (S, U'M)$$

and so the above adjunction restricts to an adjunction on the sheaf categories $\mathbf{Mod}\text{-}R \underset{\tilde{a}Fi}{\overset{U'}{\rightleftarrows}} \mathbf{Sh}(\mathcal{C}, J)$.

By Beck's Theorem, U' is monadic if it reflects isomorphisms and creates split coequalisers.

U' reflects isomorphisms: let $\alpha : M \to N$ be a natural transformation between sheaves of modules (that is, a morphism in $\mathbf{Mod}\text{-}R$) such that $U'\alpha$ is an isomorphism. Then $(U'\alpha)_C$ is a bijection for each $C \in \mathcal{C}$. Therefore, α_C is a bijection, and thus an isomorphism of modules. Since α_C is an isomorphism for every object C, it follows that it is an isomorphism of functors.

U' creates split coequalisers: let $\alpha, \beta : M \to N$ be maps in $\mathbf{Mod}\text{-}R$ such that there is a split coequaliser diagram

$$\begin{tikzcd}
U'M \arrow[r, shift left, "U'\alpha"] \arrow[r, shift right, "U'\beta"'] & U'N \arrow[r, shift left, "\gamma"] \arrow[l, bend left=40, "i"] & P \arrow[l, shift left, "j"]
\end{tikzcd}$$

with $U'\alpha.i = 1_{U'N},\ \gamma j = 1_P,\ U'\beta.i = j\gamma,\ \gamma.U'\alpha = \gamma.U'\beta$.

To show that U' creates this coequaliser, it suffices to show that P underlies a sheaf of R-modules, with γ being a module map. Since P is already a sheaf, we just need to show it has R-module structure, that is, for each $C \in \mathcal{C}$, PC has an RC-module structure, that γ_C is an RC linear map, and that the restriction maps in P are linear. For each such C we have

$$U'MC \underset{\underset{i_C}{\longleftarrow}}{\overset{U'\alpha_C}{\underset{U'\beta_C}{\rightrightarrows}}} U'NC \underset{j_C}{\overset{\gamma_C}{\rightleftarrows}} PC$$

Since $U'\alpha_C.i_C = 1_{U'NC}$ and $\gamma_C j_C = 1_{PC}$, i_C and j_C must both be injective, and so $PC \subseteq U'NC \subseteq U'MC$, with maps

$$U'MC \overset{U'\alpha_C}{\underset{U'\beta_C}{\rightrightarrows}} U'NC \xrightarrow{\gamma_C} PC$$

Here $U'\alpha_C$ is the identity on $U'NC$, $U'\beta_C$ agrees with γ_C on $U'NC$.

To show that PC with the module structure inherited from NC is a submodule of NC, let $a, b \in PC$; then since γ is epi, there exist $x, y \in U'NC$ such that $\gamma_C(x) = a, \gamma_C(y) = b$; then

$$\gamma_C(x + y) = U'\beta_C(x + y) = U'\beta_C(x) + U'\beta_C(y) = a + b$$

so $a + b \in PC$. Likewise if $r \in RC$, then $\gamma_C(rx) = ra$. And since γ agrees with $U'\beta$ on $U'NC$, it follows that it is linear.

It remains to prove that if $f : C \to D$ is a morphism in $\mathcal{C}$, then Pf is linear. However this follows immediately from the fact that Pf is just the restriction to PC of Nf.

We have now shown that $U' : \mathbf{Mod}\text{-}R \to \mathbf{Sh}(\mathcal{C}, J)$ is a monadic functor. $\square$

Corollary 14. *Let $\mathcal{E}$ be a locally λ-presentable topos (and so in particular a Grothendieck topos) and let R be a ring object in $\mathcal{E}$. Then the category of modules over R is locally λ-presented also.*

Proof. By Lemma 2, it suffices to prove that U' preserves λ-directed colimits. If this is the case, then since left adjoints preserve all colimits, the monad functor $U'\tilde{a}Fi$ will preserve them, and the result will follow.

Suppose $(I, \leq)$ is a λ-directed poset, viewed as a category, and let

$$D : (I, \leq) \to \mathbf{Mod}\text{-}R$$

$$i \mapsto D_i$$

be a functor. **PreMod**-R is cocomplete, so there is a colimit cocone

$$\{\, \tilde{i}D_i \xrightarrow{d_i} E \,\}$$

over the diagram $\tilde{i}D : (I, \leq) \to \mathbf{PreMod}\text{-}R$. Since **Mod**-$R$ is a reflective subcategory of **PreMod**-R, the cocone

$$\{\, D_i \xrightarrow{\eta_E d_i} \tilde{a}E \,\} \tag{3}$$

is a colimit cocone in **Mod**-R.

Now by Lemma 12, U preserves directed colimits, and so preserves λ-directed colimits also; since a is a left adjoint it preserves all colimits. Thus the cocone

$$\{\, aU(\tilde{i}D_i) \xrightarrow{aU(d_i)} aU(E) \,\}$$

is a colimit. But now by Corollary 11, $aU = U'\tilde{a}$, so the above cocone is the collection of maps

$$\{U'\tilde{a}\tilde{i}D_i \xrightarrow{U'\tilde{a}(d_i)} U'\tilde{a}E\}$$

which is just the cocone

$$\{U'D_i) \xrightarrow{U'\eta_E d_i)} U'\tilde{a}E\}.$$

This is just U' applied to the colimit cocone (3).

This shows that U' preserves λ-directed colimits, completing our proof. $\square$

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