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Locally Finitely Presented Categories of Sheaves of Modules

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1 Introduction

We know what is meant by an element of a module. What should we mean by an *element* of a sheaf of modules? One answer is that it is simply a section (over an open set). If, however, one takes the algebraic view that an element should be of finitary character and hence should belong to a sum of subobjects iff it belongs to some finite subsum, then this is not a good answer: sections are, in general, of infinitary character.

One of our original motivations was to develop some model theory for sheaves of modules, at least to see under what conditions on a ringed space a reasonable model theory for sheaves of modules may be developed (for this see [**?**], also [12]). Here we mean the usual model theory which is based on elements (of finitary character). Through this we were lead to the problem of determining when a sheaf of modules is finitely generated or finitely presented in the usual algebraic sense and to the problem of determining when the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a ringed space, is locally finitely presented. This condition on a category is equivalent to its objects being determined by their elements.

So far as we could determine, these questions in full generality had not hitherto been addressed. Presheaves are genuinely algebraic objects and categories of presheaves are always locally finitely presented. But the sheaf property is not an algebraic one and does not fit well with notions like "finitely presented" and "finitely generated" unless the base space has strong compactness properties (such as is usually the case for those spaces considered in algebraic geometry and analysis, where finiteness conditions on sheaves are of central importance).

Our other initial motivation was to investigate the representation of R-modules, where R is any ring, as sheaves over a certain ringed space which originally arose in the model theory of modules. This ringed space is the sheaf of locally definable scalars over the rep-Zariski (=dual-Ziegler) spectrum of R (see, e.g. [12] [13]). This spectrum has bad (separation and compactness) properties compared with the spaces usually considered in algebraic geometry and analysis. But our interest in this space explains why we consider ringed spaces in full generality (arbitrary spaces and arbitrary rings with 1).

Our main result (3.5) is that if X has a basis of compact open sets and if $\mathcal{O}_X$ is any sheaf of rings over X then the category Mod-$\mathcal{O}_X$ of sheaves of $\mathcal{O}_X$-modules is locally finitely presented, with the $j_!\mathcal{O}_U$, where U ranges over any basis of compact open sets, forming a generating set of finitely presented objects. Here $j_!\mathcal{O}_U$ is the extension by 0 of the restriction, $\mathcal{O}_U$, of $\mathcal{O}_X$ to U. Indeed, over any ringed space, $j_!\mathcal{O}_U$ is finitely presented iff the open set U is compact (3.7). The dependence of this result on X but not on the sheaf $\mathcal{O}_X$ prompted us to ask (see the earlier versions, [14], of this paper) whether the answer to the question also depends only on X; that is, does Mod-$\mathcal{O}_X$ being locally finitely presented depend only on the underlying space X? This question is still open but the independence, given X, of 3.5 from the choice of sheaf $\mathcal{O}_X$ is explained in a paper [3] by Bridge: there it is shown that if $(\mathcal{C}, J)$ is a Grothendieck site such that the topos of sheaves of sets over $(\mathcal{C}, J)$ is locally finitely presented and, if R is any ring object in that topos, then the category of R-modules will be locally finitely presented. Taking the site to be the poset of open subsets of X, regarded as a category in the usual way, with the usual notion of covering, gives 3.5. In the original version of this paper we also commented that it seemed our results would generalise to locales; that also is covered by Bridge's result.

We also investigate the weaker condition that Mod-$\mathcal{O}_X$ be locally finitely generated and we begin by showing that if F is a finitely generated sheaf then the support of F is compact (4.5). We also prove that if K is a locally closed subset of X then $j_!\mathcal{O}_K$ is finitely generated iff K is compact (4.6). Using this we obtain a necessary, but not sufficient, condition for Mod-$\mathcal{O}_X$ to be locally finitely generated. Namely, if Mod-$\mathcal{O}_X$ is locally finitely generated then for every $x \in X$ and every open neighbourhood, U, of x there is a compact locally closed set K with $x \in K \subseteq U$ (4.8). We give examples which show that the property of local finite generation depends on the structure sheaf, not just on the space: if X is the (closed) unit interval in $\mathbb{R}$ with the usual topology and if $\mathcal{O}_X$ is the sheaf of continuous functions on X then Mod-$\mathcal{O}_X$ is locally finitely generated whereas, if we let $\mathcal{O}'_X$ be the constant sheaf on the same space then Mod-$\mathcal{O}_X$ is not locally finitely generated (4.9, 4.10). We also show that, in the first case, although Mod-$\mathcal{O}_X$ is locally finitely generated, it is not locally finitely presented (5.5); for that we develop a criterion (5.3, 5.4) for $j_!\mathcal{O}_K$ to be finitely presented over Hausdorff locally compact spaces.

A direction that we have not pursued here is that of replacing Mod-$\mathcal{O}_X$ with the full subcategory of quasicoherent sheaves or of sheaves with quasicoherent cohomology.

Although we have not been able to answer in full the question with which we started, Bridge's result, as well as recent interest (see [16]) in this topic, has prompted the preparation of this (somewhat shorter) version of [14] for publication.

2 Some general constructions and results on sheaves

First, some basic definitions and notation.

Let $\mathcal{O}_X$ be a sheaf of rings (all our rings will be associative with an identity $1 \neq 0$). We denote by PreMod-$\mathcal{O}_X$ the category of presheaves over X which are $\mathcal{O}_X$-pre-modules. That is, $M \in \text{PreMod-}\mathcal{O}_X$ means that M is a presheaf of abelian groups such that, for each open set $U \subseteq X$, $M(U)$ is a right R_U-module, where we set $R_U = \mathcal{O}_X(U)$, and such that, for every inclusion $V \subseteq U \subseteq X$ of open subsets of X, the restriction map, $\text{res}^M_{U,V} : M(U) \longrightarrow M(V)$, is a homomorphism of R_U-modules, where we regard $M(V)$ as an R_U-module *via* $\text{res}^{\mathcal{O}_X}_{U,V} : R_U \longrightarrow R_V$. The full subcategory of sheaves, that is, of $\mathcal{O}_X$-modules is denoted $\text{Mod-}\mathcal{O}_X$. Then (see [2, Sections I.3, II.4]) both $\text{PreMod-}\mathcal{O}_X$ and $\text{Mod-}\mathcal{O}_X$ are Grothendieck abelian categories.

An object C of a category $\mathcal{C}$ is **finitely presented** (**fp**) if the representable functor $(C, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ commutes with direct limits in $\mathcal{C}$. If $\mathcal{C}$ is Grothendieck abelian, then it is sufficient to check that for every directed system $((D_\lambda)_\lambda, (g_{\lambda\mu} : D_\lambda \longrightarrow D_\mu)_{\lambda<\mu})$ in $\mathcal{C}$, with limit $(D, (g_{\lambda\infty} : D_\lambda \longrightarrow D)_\lambda)$, every $f \in (C, D)$ factors through some $g_{\lambda\infty}$. A category $\mathcal{C}$ is **finitely accessible** if the full subcategory, $\mathcal{C}^{\text{fp}}$, of finitely presented objects is skeletally small and if every object of $\mathcal{C}$ is a direct limit of finitely presented objects; if $\mathcal{C}$ also is complete (equivalently, [1, 2.47], cocomplete) then $\mathcal{C}$ is said to be **locally finitely presented** (**lfp**). Abelian categories which are finitely accessible hence, [6, 2.4], Grothendieck and lfp are in many ways as well-behaved as categories of modules over rings. In particular, objects of $\mathcal{C}$ are determined by their "elements" (morphisms from finitely presented objects) and these "elements" have finitary character (as opposed to what one has for merely presentable categories). Such categories have a good model theory and they admit a useful embedding into a related functor category (see e.g., [8], [10], [11], [12]). The category $\text{PreMod-}\mathcal{O}_X$ is locally finitely presented (see [5, p. 7] for example), indeed it is a variety of finitary many-sorted algebras in the sense of [1, Section 3A], but $\text{Mod-}\mathcal{O}_X$ need not be (see 4.10, 5.5).

Next, we recall, following [9] (also see [7], [17]) some standard functors on categories of sheaves. Let $Y \subseteq X$ and let $F \in \text{Mod-}\mathcal{O}_X$. Let $j : Y \longrightarrow X$ denote the inclusion. The sheaf j^*F is defined by: for any open subset U of Y we set $j^*F.U$ to be the set of those s such that s is a set-theoretic section of the stalk space of F over the set U such that for all $y \in U$ there exists $V \subseteq X$ open with $y \in V \cap Y \subseteq U$ and there exists $t \in FV$ such that for all $z \in V \cap Y$ we have $s_z = t_z$. That is, sections of j^*F locally look like sections of F. If Y is open in X then $j^*F.U = FU$ for $U \subseteq Y$ open.

One may check that j^*F is, indeed, a sheaf (exercise in [9, p. 65] or [17, p. 58]). It is also denoted $F \mid_Y$ or $F \upharpoonright Y$ and called the **restriction** of F to Y. If $F \in \text{Mod-}\mathcal{O}_X$ then $j^*F \in \text{Mod-}\mathcal{O}_Y$ where we let $\mathcal{O}_Y$ denote $\mathcal{O}_X \mid_Y$ (cf. p. 110 of [7] where the notation j^{-1} is used for what we have denoted j^*: here the structure sheaf over a subspace is always that induced by the structure sheaf of the whole space, so the j^*/j^{-1} distinction does not arise).

Fact 2.1. *(e.g. [9, p. 97]) $j^* : \text{Mod-}\mathcal{O}_X \longrightarrow \text{Mod-}\mathcal{O}_Y$ is exact and is left adjoint to the left exact functor $j_* : \text{Mod-}\mathcal{O}_Y \longrightarrow \text{Mod-}\mathcal{O}_X$ which is given by $j_*G.U = G(U \cap Y)$ for $G \in \text{Mod-}\mathcal{O}_Y$ and U open in X (the* **direct image** *functor): $(j^*F, G) \simeq (F, j_*G)$ for $F \in \text{Mod-}\mathcal{O}_X$, $G \in \text{Mod-}\mathcal{O}_Y$.*

Let $K \subseteq X$ be **locally closed** (the intersection of an open set with a closed set); denote the inclusion by $j : K \longrightarrow X$, and let $G \in \text{Mod-}\mathcal{O}_K$. Define the sheaf $j_!G$ on X, the **extension** of G **by zero** by: $j_!G.U = \{s \in G(U \cap K) : \text{supp}(s) \text{ is closed in } U\}$. This is a sheaf and $j_!G \in \text{Mod-}\mathcal{O}_X$ (for an alternative description, see [17, p. 63/4]). Recall that the **support** of a section $s \in FU$ is $\mathrm{supp}(s) = \{x \in X : s_x \neq 0\}$ where s_x is the germ of s at x; this is a closed subset of U.

Fact 2.2. *(e.g. [9, p. 106/7])* $j_! : \text{Mod-}\mathcal{O}_K \longrightarrow \text{Mod-}\mathcal{O}_X$ *is an exact functor which is an equivalence between* $\text{Mod-}\mathcal{O}_K$ *and the category of* $\mathcal{O}_X$*-modules which have all stalks over* $X \setminus K$ *equal to 0.*

Now, given $F \in \text{Mod-}\mathcal{O}_X$ and $K \subseteq X$ locally closed, let F^K be the sheaf (clearly it is a sheaf) given by $F^K U = \{s \in FU : \mathrm{supp}(s) \subseteq K\}$. So F^K is a subsheaf of F. Set $j^! F = j^* F^K \in \text{Mod-}\mathcal{O}_K$; sections of $j^! F$ are locally given by sections of F with support contained in K.

Fact 2.3. *(e.g. [9, p. 108/9]) The functor* $j^! : \text{Mod-}\mathcal{O}_X \longrightarrow \text{Mod-}\mathcal{O}_K$ *is left exact and is right adjoint to the functor* $j_! : (j_! G, F) \simeq (G, j^! F)$.

Fact 2.4. *(e.g. [9, p. 109])*
If $K = C$ *is closed then* $j_* = j_!$.
If $K = U$ *is open then* $j^* = j^!$.
Always $j^! F \leq j^* F$.

Fact 2.5. *(e.g. [9, p. 110], [17, 3.8.11]) If* $U \subseteq X$ *is open,* $C = X \setminus U$ *is its complement and* $F \in \text{Mod-}\mathcal{O}_X$ *then there is an exact sequence*

$$0 \longrightarrow j_!(F\,|_U) \longrightarrow F \longrightarrow i_!(i^* F) = i_*(i^* F) \longrightarrow 0$$

where $j : U \longrightarrow X$ *and* $i : C \longrightarrow X$ *are the inclusions, where the first map is the natural inclusion and where* $F_x \longrightarrow (i_! i^* F)_x$ *is the identity if* $x \in C$ *and is* 0 *otherwise (see [9, p. 97, 4.3]).*

In particular we have an exact sequence

$$0 \longrightarrow j_! \mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C \longrightarrow 0.$$

It follows that if C is closed, K is locally closed and $C \subseteq K \subseteq X$ then there is an exact sequence

$$0 \longrightarrow i'_! \mathcal{O}_{K \setminus C} \longrightarrow \mathcal{O}_K \longrightarrow i_! \mathcal{O}_C \longrightarrow 0$$

where $i : C \longrightarrow K$ and $i' : K \setminus C \longrightarrow K$ are the inclusions. Then, since $j_!$ is exact, where $j : K \longrightarrow X$ is the inclusion, we have

$$0 \longrightarrow j_! i'_! \mathcal{O}_{K \setminus C} \longrightarrow j_! \mathcal{O}_K \longrightarrow j_! i_! \mathcal{O}_C \longrightarrow 0$$

that is,

$$0 \longrightarrow (ji')_! \mathcal{O}_{K \setminus C} \longrightarrow j_! \mathcal{O}_K \longrightarrow (ji)_! \mathcal{O}_C \longrightarrow 0$$

where $ji' : K \setminus C \longrightarrow X$, $j : K \longrightarrow X$ and $ji : C \longrightarrow X$ are the inclusions.

Lemma 2.6. *Let* $C \subseteq X$ *be closed and let* $U = X \setminus C$*. The canonical sequence* $0 \longrightarrow j_! \mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C \longrightarrow 0$ *is split iff* C *is open. (Here and elsewhere letters such as* j, i *will denote the obvious inclusions.)*

Proof. Suppose that we have $g : i_! \mathcal{O}_C \longrightarrow \mathcal{O}_X$ splitting the canonical surjection $f : \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C$. Setting $s = g1_C$, we have $s_x = \begin{cases} 1_x \in \mathcal{O}_{X,x} & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$. Then $(1_X - s)_x = \begin{cases} 0 & \text{if } x \in C \\ 1_x & \text{if } x \notin C \end{cases}$, so $\mathrm{supp}(1_X - s) = X \setminus C$ and, since the support of any section must be closed, we deduce that C is open in X.

Conversely, if C is open in X then the inclusion $i_! \mathcal{O}_C \longrightarrow \mathcal{O}_X$ clearly splits f, as required. □

Since, if $K \subseteq X$ is locally closed, the functor $j_!$ is exact, where $j : K \longrightarrow X$ is the inclusion, we have the more general statement.

Lemma 2.7. *Let $C \subseteq K \subseteq X$ with C closed and K locally closed, $i : C \longrightarrow K$ and $j : K \longrightarrow X$ the inclusions. Then the canonical epimorphism $j_!\mathcal{O}_K \longrightarrow (ji)_!\mathcal{O}_C$ (from the exact sequence before 2.6) is split iff C is open in K.*

Proof. If C is open in K then we have a split exact sequence as in 2.6 with K replacing X so then apply $j_!$.

Conversely, if $j_!\mathcal{O}_K \longrightarrow j_!i_!\mathcal{O}_C$ is split then apply $(j)^*$, noting (see [9, II.6.4]) that $(j)^*j_! = \mathrm{Id}$, and then apply 2.6. □

The dual statement follows immediately.

Lemma 2.8. *Let $U \subseteq K \subseteq X$ with U open and K locally closed, $i : U \longrightarrow X$ and $j : K \longrightarrow X$ the inclusions. Then the canonical monomorphism $i_!\mathcal{O}_U \longrightarrow j_!\mathcal{O}_K$ is split iff U is closed in K.*

Lemma 2.9. *Let $C = \bigcap_\lambda C_\lambda$ be closed sets, where the intersection is directed (i.e. $\forall \lambda, \mu\, \exists \nu\ C_\nu \subseteq C_\lambda \cap C_\mu$). Let $((j_!\mathcal{O}_{C_\lambda})_\lambda, (g_{\lambda\mu} : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_{C_\mu})_{C_\lambda \supseteq C_\mu})$ be the corresponding directed system of epimorphisms (we use j generically to denote inclusions). Then $\varinjlim_\lambda (j_!\mathcal{O}_{C_\lambda}) = j_!\mathcal{O}_C$ and each limit map $g_{\lambda\infty} : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_C$ is an epimorphism.*

Proof. Let $G = \varinjlim j_!\mathcal{O}_{C_\lambda}$ with limit maps $g_{\lambda\infty} : j_!\mathcal{O}_{C_\lambda} \longrightarrow G$.

For each λ the inclusion $C \longrightarrow C_\lambda$ gives rise, by the canonical exact sequence (2.5), to an epimorphism $h_\lambda : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_C$ and these are compatible, so we have a unique induced map $h : G \longrightarrow j_!\mathcal{O}_C$ with $hg_{\lambda\infty} = h_\lambda$ for all λ. We show that h is an isomorphism - so it is sufficient to show that h is an isomorphism at each stalk.

Let $x \in X$. Then $G_x = \varinjlim_{x\in U} GU = \varinjlim_{x\in U} \varinjlim_\lambda G_\lambda U$ (the presheaf and sheaf limits agree on stalks) $= \varinjlim_\lambda \varinjlim_{x\in U} G_\lambda U = \varinjlim_\lambda (G_\lambda)_x = \begin{cases} \mathcal{O}_x & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$.

So G has the same stalks as $j_!\mathcal{O}_C$ and since all maps in the system are, at the level of stalks, either the identity or zero, we can check that $h_x = \begin{cases} \mathrm{id} : \mathcal{O}_x \longrightarrow \mathcal{O}_x & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$. So $G \longrightarrow j_!\mathcal{O}_C$ is an isomorphism. Also, we have seen that each $g_{\lambda\infty}$ is stalkwise surjective and hence is an epimorphism. □

3 The category Mod-$\mathcal{O}_X$: local finite presentation

Every category Mod-$\mathcal{O}_X$ is locally presentable because it is a Grothendieck category (e.g. [4, 3.4.2, 3.4.16]); we would like to determine when Mod-$\mathcal{O}_X$ is locally *finitely* presented.

Remark 3.1. *(cf. [2, p. 260]) The sheaves $j_!\mathcal{O}_U$, with $U \subseteq X$ open, together generate* Mod-$\mathcal{O}_X$*. This follows, for instance, from the corresponding result (see [2, Section I.3]) for presheaves by localising/sheafifying (which preserves generating sets) to the category of sheaves.*

Proposition 3.2. *If $\mathcal{U}$ is a basis of open sets for X then the $j_!\mathcal{O}_U$ for $U \in \mathcal{U}$ together generate* Mod-$\mathcal{O}_X$.

Proof. Let $F \in$ Mod-$\mathcal{O}_X$, let $x \in X$ and take $a \in F_x$. Since $F_x = \varinjlim_{x\in U} FU$ there is $U = U(x,a)$ open and $b = b_{U,a} \in FU$ such that the canonical map from FU to F_x takes b to a. Without loss of generality $U \in \mathcal{U}$. Define $f' : \mathcal{O}_U \longrightarrow F \upharpoonright U$ by $1 \in \mathcal{O}_U U \mapsto b_{U,a} \in FU$ (by linearity and restriction this defines a presheaf morphism, which is enough).

Since $j_!$ is left adjoint to $(-)_U$ there is, corresponding to $f' \in (\mathcal{O}_U, F \upharpoonright U)$, a morphism $f_{x,U,a} \in (j_!\mathcal{O}_U, F)$ with $(f_{x,U,a})_U : 1_U \mapsto b_{U,a}$. Note that $(f_{x,U,a})_x : (j_!\mathcal{O}_U)_x \longrightarrow F_x$ maps $1_{\mathcal{O}_{X,x}}$ to $a \in F_x$.

Hence $\bigoplus_{x\in X} \bigoplus_{a\in F_x} f_{x,U,a} : \bigoplus_x \bigoplus_a j_!\mathcal{O}_{U=U(x,a)} \longrightarrow F$ is an epimorphism on stalks and hence is an epimorphism in Mod-$\mathcal{O}_X$.

□

Proposition 3.3. *Suppose $\varinjlim_\lambda G_\lambda = G$ in* Mod-$\mathcal{O}_X$ *and suppose that $U \subseteq X$ is compact open. Then the canonical map $g : \varinjlim_\lambda (G_\lambda U) \longrightarrow GU$ is an isomorphism.*

Proof. The sheaf G is the sheafification of the presheaf direct limit, G', of the G_λ and this is given, for $V \subseteq X$ open, by $G'V = \varinjlim (G_\lambda V)$. So the proposition asserts that G' and G agree at compact open sets. Let $g : G' \longrightarrow G$ be the sheafification map in the category PreMod-$\mathcal{O}_X$. We show that g_U is an isomorphism.

Let $s \in G'U$ and suppose that $g_U s = 0$. Then there must be an open cover $\{U_i\}_i$ of U such that, for all i, we have $\mathrm{res}^{G'}_{U,U_i} s = 0$. Since U is compact we may take the cover to be finite: $U_1, ..., U_n$ say. Then, by definition of the restriction maps in the limit, for each i there are λ_i and $a_i \in G_{\lambda_i}U$ such that $(g'_{\lambda_i,\infty})_U(a_i) = s$ (where $g'_{\lambda_i,\infty} : G_{\lambda_i} \longrightarrow G'$ is the canonical map) and $(g'_{\lambda_i,\infty})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i}(a_i) = 0$. Since there are just finitely many U_i we may take λ with $\lambda \geq \lambda_1, ..., \lambda_n$ and also such that $(g_{\lambda_i,\lambda})_U(a_i) = (g_{\lambda_j,\lambda})_U(a_j) = b$, say, for all i, j (since the a_i all map to the same element in the limit) and, furthermore, such that $(g_{\lambda_i,\lambda})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i}(a_i) = 0$ for all i (since each $\mathrm{res}^{G_{\lambda_i}}_{U,U_i} a_i$ maps to 0 in the presheaf limit). But then $\mathrm{res}^{G_\lambda}_{U,U_i}(b) = (g_{\lambda_i,\lambda})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i}(a_i) = 0$ for each i and hence, since G_λ is a sheaf, $b = 0$. Therefore $s = (g'_{\lambda,\infty})_U(b) = 0$ as required.

To see that g_U is onto, take any $t \in GU$. For each $x \in U$ there is an open neighbourhood U_x of x and $t(x) \in G'(U_x)$ such that $g_{U_x} t(x) = \mathrm{res}^G_{UU_x}(t)$. Since U is compact we may take finitely many open sets $U_1, \ldots, U_n$ say, with corresponding $t_i \in G'U_i$, which cover U.

For each i there is λ_i and $s_i \in G_{\lambda_i}U_i$ with $g'_{\lambda_i\infty} s_i = t_i$. Since there are only finitely many λ_i we may take $\lambda \geq \lambda_1, \ldots, \lambda_n$ and we may suppose that $s_i \in G_\lambda U_i$ for each i.

We have, furthermore, that for each pair, i, j, of indices, $\mathrm{res}^{G'}_{U_i,U_i\cap U_j}(t_i) = \mathrm{res}^{G'}_{U_j,U_i\cap U_j}(t_j)$ and hence there is $\mu_{ij} \geq \lambda$ such that $g'_{\lambda\mu_{ij}} \mathrm{res}^{G_\lambda}_{U_i,U_i\cap U_j}(s_i) = g'_{\lambda\mu_{ij}} \mathrm{res}^{G_\lambda}_{U_j,U_i\cap U_j}(s_j)$. So, choosing $\mu \geq \mu_{ij}$ for each i, j and setting $s'_i = g_{\lambda\mu}(s_i)$ we may suppose that for all i, j we have $\mathrm{res}^{G_\mu}_{U_i,U_i\cap U_j}(s'_i) = \mathrm{res}^{G_\mu}_{U_j,U_i\cap U_j}(s'_j)$. Since G_μ is a sheaf, there is $s \in G_\mu U$ such that $\mathrm{res}^{G_\mu}_{U,U_i}(s) = s_i$ for each $i = 1, ..., n$.

Then $(g'_{\mu\infty})_U(s) \in G'U$ with (since G is separated) $g_U\big((g'_{\mu\infty})_U(s)\big) = t$, as required. $\square$

It follows that if X is a noetherian space, that is, if every open subset of X is compact, and if $(G_\lambda)_\lambda$ is a directed system in Mod-$\mathcal{O}_X$ then the direct limit, $\varinjlim G_\lambda$, computed in PreMod-$\mathcal{O}_X$ is a sheaf and hence equals $\varinjlim G_\lambda$, computed in Mod-$\mathcal{O}_X$.

From the adjunction $(j_!\mathcal{O}_U, F) \simeq (\mathcal{O}_U, F \upharpoonright U) \simeq \Gamma(F \upharpoonright U) \simeq FU$ for $U \subseteq X$ open we deduce that $\Gamma_U(-) \simeq (j_!\mathcal{O}_U, -)$ as functors on Mod-$\mathcal{O}_X$. Here Γ denotes the global section functor, $F \mapsto FX$, and Γ_U is the functor $F \mapsto FU$.

Corollary 3.4. *If $U \subseteq X$ is compact open then $j_!\mathcal{O}_U$ is a finitely presented object of* Mod-$\mathcal{O}_X$.

Proof. Let $(G_\lambda)_\lambda$ be a directed system in Mod-$\mathcal{O}_X$ with direct limit G. Then, as just noted, $(j_!\mathcal{O}_U, G) \simeq GU$ and $\varinjlim_\lambda (j_!\mathcal{O}_U, G_\lambda) \simeq \varinjlim(G_\lambda U)$ - and these coincide by 3.3, as required. $\square$

With 3.2 this gives our first result.

Theorem 3.5. *[15], [14] If X has a basis of compact open sets then* Mod-$\mathcal{O}_X$ *is locally finitely presented, with the $j_!\mathcal{O}_U$ for $U \subseteq X$ compact open (or with the U from any basis of such sets) as a generating set of finitely presented objects of* Mod-$\mathcal{O}_X$.

Corollary 3.6. *If X is locally noetherian and $\mathcal{O}_X$ is any sheaf of rings on X then* Mod-$\mathcal{O}_X$ *is locally finitely presented.*

We also have the converse to 3.4.

Proposition 3.7. *If U is open then $j_!\mathcal{O}_U$ is finitely presented in* Mod-$\mathcal{O}_X$ *iff U is compact.*

Proof. If U is compact then we have 3.4, so suppose that U is not compact - say $\{U_i\}_i$ is an open cover with no finite subcover. We may suppose that $\{U_i\}_i$ is closed under finite union. Set $G_i = j_!\mathcal{O}_{U_i}$; these form a directed system under inclusion (see 2.5), so set $G = \varinjlim j_!\mathcal{O}_{U_i}$. Then $G = j_!\mathcal{O}_U$ since from the canonical inclusions $j_!\mathcal{O}_{U_i} \longrightarrow j_!\mathcal{O}_U$ we obtain a map $G = \varinjlim j_!\mathcal{O}_{U_i} \longrightarrow j_!\mathcal{O}_U$ which locally, and hence stalkwise, is an isomorphism and which is, therefore, an isomorphism.

So the identity map of G would factor through some $j_!\mathcal{O}_{U_i}$ if $G = j_!\mathcal{O}_U$ were finitely presented - but since $U_i \neq U$ there can be no such factorisation (recall that if $x \notin U_i$ then $(j_!\mathcal{O}_{U_i})_x = 0$). $\square$

Example 3.8. *If $F \in$ Mod-$\mathcal{O}_X$ is finitely presented and $U \subseteq X$ is open then $F\,|_U$ might not even be finitely generated. Let $X = [0,1]$ and take $U = (0,1)$. Take $\mathcal{O}_X$ to be the sheafification of the constant presheaf k where k is any chosen ring. By 3.4, $\mathcal{O}_X$ is a finitely presented sheaf but, by 3.7, $\mathcal{O}_U$ is not, because U is not compact. Indeed, $\mathcal{O}_U$ is not even finitely generated, as one sees by writing $\mathcal{O}_U$ as the sum over $n \geq 1$ of the sheaves $j_!\mathcal{O}_{(\frac{1}{n}, 1-\frac{1}{n})}$.*

4 The category Mod-$\mathcal{O}_X$: local finite generation

Recall that an object F in an abelian category is **finitely generated** if, whenever $F = \sum_\lambda F_\lambda$ for some subobjects F_λ, we have $F = \sum_{i=1}^n F_{\lambda_i}$ for some $\lambda_1, \dots, \lambda_n$. If the category is locally finitely presented then it is equivalent that F be the image of a finitely presented object.

Suppose that $F \in$ Mod-$\mathcal{O}_X$ and let $s \in FX$. Define a subpresheaf $\langle s\rangle^0$ of F by setting: $\langle s\rangle^0 U = \mathrm{res}^F_{X,U}(s).R_U$ (recall that $R_U = \mathcal{O}_X U$) and with the restriction maps coming from F. This is a separated presheaf; let $\langle s\rangle$ denote the sheafification of $\langle s\rangle^0$ - a subsheaf of F.

More generally, if $U \subseteq X$ is open and $s \in FU$ then we define the subsheaf of F **generated by** s to be $j_!\langle s\rangle$, where $j : U \longrightarrow X$ is the inclusion and $\langle s\rangle \leq F\mid_U$ is defined as above. Recall (2.5) that there is an inclusion $j_!(F\mid_U) \longrightarrow F$ and so, since $j_!$ is (left) exact, $j_!\langle s\rangle$ is indeed a subsheaf of F. Although we call it the sheaf generated by s it need not be a finitely generated sheaf - unless U is compact s might not be a "finitary element" of F. Here, though, is some justification for the terminology.

Lemma 4.1. *Let $F \in$ Mod-$\mathcal{O}_X$, let $U \subseteq X$ be open and let $s \in FU$. Suppose that $G \leq F$ is a subsheaf such that $s \in GU$. Then $j_!\langle s\rangle \leq G$.*

Proof. It is immediate from the definition that $\langle s\rangle^0$ is a subpresheaf of G_U and hence that $\langle s\rangle \leq G_U$. Therefore $j_!\langle s\rangle \leq j_!(G_U) \leq G$. □

Lemma 4.2. *Let $F \in$ Mod-$\mathcal{O}_X$. Then $F = \sum\{j_!\langle s\rangle : U \subseteq X$ is open and $s \in FU\}$ (we write $j_!$ for $(j_U)_!$ where $j_U : U \longrightarrow X$ is the inclusion).*

Proof. By the remarks above, F contains the right hand side. Conversely, given $U \subseteq X$ open and $s' \in FU$ we have $s' \in j_!\langle s'\rangle.U$ (since $s' \in \langle s'\rangle^0 U$) so $s' \in (\sum_U \sum_{s\in FU} j_!\langle s\rangle).U$ as required. □

Note that infinite sums in Mod-$\mathcal{O}_X$ are obtained by first forming the presheaf sum (that is, the algebraic sum of U-sections at each open $U \subseteq X$) and then sheafifying. We say that a sheaf F is **finitely generated** if whenever $F = \sum_\lambda F_\lambda$ with the F_λ subsheaves of F, then there are $\lambda_1, \dots, \lambda_n$ such that $F = F_{\lambda_1} + \cdots + F_{\lambda_n}$. It is immediate that any finitely presented sheaf is finitely generated.

Lemma 4.3. *If $F \in$ Mod-$\mathcal{O}_X$, $s \in FX$ and $(V_\lambda)_\lambda$ are open sets with $V = \bigcup_\lambda V_\lambda \supseteq \mathrm{supp}(s)$ then $\langle s\rangle = \sum_\lambda j_!\langle \mathrm{res}^F_{X,V_\lambda} s\rangle$.*

Proof. Arguing as above, the right hand side, G say, is a subpresheaf of $\langle s\rangle$. We have, for each λ, a section in GV_λ which agrees with s on V_λ and hence, since $\langle s\rangle$ is a sheaf, we deduce that $\mathrm{res}^F_{X,V} s \in GV$ and hence, since G is a sheaf, that $s \in GX$. Therefore, by 4.1, $\langle s\rangle \leq G$, as required. □

We define the **support** of a (pre)sheaf F to be the union, $\mathrm{supp}(F) = \{x \in X : F_x \neq 0\}$, of supports of sections of F.

Lemma 4.4. *Suppose that $F \in$ Mod-$\mathcal{O}_X$ is finitely generated. Then there are open subsets $U_1, ..., U_n$ of X and $s_i \in F(U_i)$ such that $\mathrm{supp}(F) = \bigcup_1^n \mathrm{supp}(s_i)$. In particular, $\mathrm{supp}(F)$ is a locally closed subset of X.*

If $F = \sum_i j_!\langle s_i\rangle$ where $s_i \in F(U_i)$ then we may take the U_i and s_i from this representation.

Proof. Take a representation of F as given (we know there is such by 4.2). Since F is finitely generated we have $F = \sum_1^n j_!\langle s_i\rangle$, say. Certainly $\mathrm{supp}(F) \supseteq \bigcup_1^n \mathrm{supp}(s_i)$.

Conversely, if $x \in \mathrm{supp}(F)$ then there is an open set V containing x and $t \in FV$ such that $t_x \neq 0$. We have $FV = \sum_1^n j_!\langle s_i\rangle.V$. If $a \in j_!\langle s_i\rangle.V$ then $\mathrm{supp}(a) \subseteq \mathrm{supp}(s_i)$ (using the definition of $\langle s\rangle^0$). But then $t \in \sum_1^n j_!\langle s_i\rangle.V$ implies $x \in \mathrm{supp}(s_i)$ for some i, as required.

Finally, $\mathrm{supp}(F)$ is locally closed since it is a finite union of closed subsets of open sets. □

Proposition 4.5. *Let $F \in \text{Mod-}\mathcal{O}_X$ be finitely generated. Then* $\mathrm{supp}(F)$ *is compact.*

Proof. We know that $F = \sum_1^n j_!\langle s_i\rangle$ for some $s_i \in F(U_i)$ for some open U_i and then, by the lemma above, $\mathrm{supp}(F) = \bigcup_1^n \mathrm{supp}(s_i)$.

Suppose that $\mathrm{supp}(F)$ is not compact. Then there are open subsets V_λ of X such that $\mathrm{supp}(F) \subseteq V = \bigcup_\lambda V_\lambda$ but no finite number of these cover $\mathrm{supp}(F)$. For each $i = 1, \ldots, n$ the $V_\lambda \cap U_i$ cover $V \cap U_i$ and so, by 4.3, $F = \sum_{i=1}^n \sum_\lambda (j_{V_\lambda \cap U_i, X})_!\langle \mathrm{res}^F_{U_i, V_\lambda \cap U_i}(s_i)\rangle$ (ignoring those where $V_\lambda \cap U_i = \emptyset$).

Since F is finitely generated there is a finite subsum $F = \sum_{k=1}^m (j_{V_{\lambda_k} \cap U_{i_k}, X})_!\langle \mathrm{res}^F_{U_{i_k}, V_{\lambda_k} \cap U_{i_k}}(s_{i_k})\rangle$. By 4.4 above we have $\mathrm{supp}(F) = \bigcup_1^m \mathrm{supp}(s_{i_k}) \cap V_{\lambda_k} \subseteq \bigcup_1^m V_{\lambda_k}$ - contradiction, as required. □

Proposition 4.6. *Let $K \subseteq X$ be locally closed. Then $j_!\mathcal{O}_K$ is finitely generated iff K is compact.*

Proof. If $j_!\mathcal{O}_K$ is finitely generated then, by 4.5, K is locally closed.

For the converse, suppose that K is locally closed and compact. We have $K = U_0 \cap C$ for some open U_0 and closed C. Let $U \subseteq U_0$ be open. Then $j_!\mathcal{O}_K.U = \{s \in \mathcal{O}_K(K \cap U) : \mathrm{supp}(s) \text{ is closed in } U\}$ is generated by $1_{K\cap U}$ since $\mathrm{supp}(1_{K\cap U}) = K \cap U = C \cap U$ is closed in U. Hence $j_!\mathcal{O}_K = \langle 1_K\rangle$.

Now suppose that $j_!\mathcal{O}_K = \sum_\lambda F_\lambda$ for some subsheaves F_λ. We may suppose that the sum is directed. Let $x \in K$. Since $\sum_\lambda F_\lambda$ is the sheafification of the presheaf sum of the F_λ there is an open set U_x containing x (without loss of generality $U_x \subseteq U_0$) such that $1_{U_x \cap K}(\in j_!\mathcal{O}_K.U_x)$ belongs to the presheaf sum of the $F_\lambda U_x$ and hence (since the sum is directed) belongs to $F_\lambda U_x$ for some λ.

As x varies over K we get a cover $(U_x)_{x\in K}$ and so, by compactness, some finite subset, $U_{x_1}, ..., U_{x_n}$, covers K. Set $U_1 = U_{x_1} \cup ... \cup U_{x_n} \subseteq U_0$. The sum is directed so we may choose λ such that $1_{K\cap U_{x_i}} \in F_\lambda(U_{x_i})$ for $i = 1, ..., n$ and hence such that $1_K = 1_{K\cap U_1} \in F_\lambda(U_1)$ (since F_λ is a sheaf). Therefore $j_!\mathcal{O}_K = \langle 1_K\rangle \leq F_\lambda$, as required. □

Proposition 4.7. *Let $U \subseteq X$. Suppose that there is a finitely generated sheaf $F \in \text{Mod-}\mathcal{O}_X$ such that there is a non-zero homomorphism $f : F \longrightarrow j_!\mathcal{O}_U$. Then U contains a compact locally closed set. If $x \in X$ is such that the morphism of stalks $f_x : F_x \longrightarrow \mathcal{O}_{X,x}$ is non-zero then this compact locally closed set may be taken to contain x.*

Proof. Let $F' = \mathrm{im}(f)$. Being an image of a finitely generated object, F' is finitely generated. Since F' is a non-zero subfunctor of $j_!\mathcal{O}_U$, we have $\emptyset \neq$

$\mathrm{supp}(F') \subseteq \mathrm{supp}(j_!\mathcal{O}_U) \subseteq U$ and, by 4.5 above, $\mathrm{supp}(F')$ is compact. The set $\mathrm{supp}(F')$ is also locally closed by 4.4. Finally, if $f_x \neq 0$ then $x \in \mathrm{supp}(F')$. □

A Grothendieck abelian category $\mathcal{C}$ is **locally finitely generated** if every object is an epimorphic image of a direct sum of finitely generated objects. From the above result it follows that if Mod-$\mathcal{O}_X$ is locally finitely generated then for every $x \in X$ and open set U containing x there is a compact locally closed set K with $x \in K \subseteq U$.

We strengthen this as follows. Say that X is **locally compact** if for every $x \in X$ and for every open set U containing x there is an open set V containing x and a compact locally closed set K with $x \in V \subseteq K \subseteq U$.

Theorem 4.8. *Suppose that* Mod-$\mathcal{O}_X$ *is locally finitely generated. Then X is locally compact.*

Proof. Given x and U, consider $j_!\mathcal{O}_U$. By assumption there is an epimorphism from a direct sum of finitely generated sheaves to $j_!\mathcal{O}_U$ and this must be surjective on stalks. Hence there is $F \in$ Mod-$\mathcal{O}_X$ finitely generated and $f : F \longrightarrow j_!\mathcal{O}_U$ such that $f_x : F_x \longrightarrow (j_!\mathcal{O}_U)_x = \mathcal{O}_x$ has $1_x \in \mathcal{O}_x$ in its image (and hence which is surjective at x). Therefore there is an open set V' with $x \in V' \subseteq U$ and sections $s' \in FV'$ and $t' \in j_!\mathcal{O}_U.V'$ with $f_{V'}s' = t'$ and $t' \mapsto 1_x$ under the canonical map $\mathcal{O}_UV' \longrightarrow \mathcal{O}_x$. Since $(t' - 1_{V'})_x = 0$ there is an open set V with $x \in V \subseteq V'$ and with $\mathrm{res}_{V'V}t' = \mathrm{res}_{V'V}(1_{V'}) = 1_V$. Thus there is an open set V with $x \in V \subseteq U$ and a section $(\mathrm{res}^F_{V'V}(s') =)\ s$ with $f_Vs = 1_V \in \mathcal{O}_VV$. So we have $V \supseteq \mathrm{supp}(s) \supseteq \mathrm{supp}(f_Vs) = V$, that is $\mathrm{supp}(s) = V$.

Now, F finitely generated implies that fF is a finitely generated subsheaf of $j_!\mathcal{O}_U$ and so, by 4.6, $\mathrm{supp}(fF) \subseteq \mathrm{supp} j_!\mathcal{O}_U = U$ is a compact locally closed subset of X which contains V. □

Whether or not Mod-$\mathcal{O}_X$ is locally finitely generated depends on $\mathcal{O}_X$, not just on X.

Example 4.9. *Let X be the closed interval $[0,1]$ and let $\mathcal{O}_X$ be the sheaf of continuous functions from X to $\mathbb{R}$. We show that the $j_!\mathcal{O}_K$ with $K = [e,f], 0 \leq e < f \leq 1$ are generating (these are finitely generated by 4.6). Since, by 3.2, the $i_!\mathcal{O}_U$ with $U = (c,d)$ are generating it is enough to show that, given such an open $U, x \in U$ and $g \in \mathcal{O}_{X,x} = (i_!\mathcal{O}_U)_x$, there is such a set K and $\phi : j_!\mathcal{O}_K \longrightarrow i_!\mathcal{O}_U$ with ϕ_x having g in its image.*

Choose a continuous function $h_1 : [0,1] \longrightarrow \mathbb{R}$ such that $\mathrm{supp}(h)$, $= K$ *say, is a closed subinterval of U and such that $h_1 \restriction V = 1$ (the constant function) on some open set $V \subseteq K$ with $x \in V$. Then choose a function, f, on $[0,1]$ which has germ g at x and replace h_1 by $h = fh_1$.*

Note that $h \in i_!\mathcal{O}_U.X$, in fact, $h \in (i_!\mathcal{O}_U)^K.X$ and hence h gives a section in $j^!(i_!\mathcal{O}_U).K$. Under the adjunction $(j^!i_!\mathcal{O}_U.K \simeq)(\mathcal{O}_K, j^!i_!\mathcal{O}_U) \simeq (j_!\mathcal{O}_K, i_!\mathcal{O}_U))$ this gives a morphism $\phi : j_!\mathcal{O}_K \longrightarrow i_!\mathcal{O}_U$ with $\phi_xh_x = g$, as required.

The same argument applies if we replace $[0,1]$ by, for instance, $\mathbb{R}$ and/or take $\mathcal{O}_X$ to be the sheaf of smooth functions from X to $\mathbb{R}$ (see e.g. [9, p. 158] for the construction of a smooth function like h_1 above).

We will see, 5.5, that this category of sheaves is not, however, locally finitely presented.

Example 4.10. *Let $X = [0,1]$ (or $\mathbb{R}$) and let $\mathcal{O}_X$ be the "constant" (i.e., sections are constant over connected sets) sheaf with values in a chosen ring Then* Mod-$\mathcal{O}_X$ *is not locally finitely generated.*

For suppose that we have a finitely generated $\mathcal{O}_X$-module, F, and a non-zero morphism $F \longrightarrow i_!\mathcal{O}_U$ where U is any non-empty proper open subset of X. By 4.5, $\mathrm{supp}(F) = K$ *is compact, hence closed. Without loss of generality, F is cyclic (factor out all but one, suitably chosen, element in a generating set in the sense of this section). Then there is an epimorphism $j_!\mathcal{O}_K \longrightarrow F$ and hence a non-zero morphism $j_!\mathcal{O}_K \longrightarrow i_!\mathcal{O}_U$. Therefore $i_!\mathcal{O}_U$ has a non-zero section s in an open neighbourhood V of K but with support contained in K (see the description, 5.1, of the functor $(j_!\mathcal{O}_K, -)$ at the beginning of the next section). This open neighbourhood V is a union of disjoint open intervals and, on each interval, each section is constant so, if non-zero, has support the whole of that interval. Therefore* $\mathrm{supp}(s)$ *is a non-empty union of disjoint open intervals and so, since* $\mathrm{supp}(s)$ *is a proper subset of V, cannot be closed in V - contradiction, as required.*

5 More on finitely presented and finitely generated sheaves

Recall that if $K \subseteq X$ is locally closed then $j_!\mathcal{O}_K$ is finitely presented if the functor $(j_!\mathcal{O}_K, -)$ commutes with direct limits. We give an alternative description of this functor: by 2.3 we have, for $F \in$ Mod-$\mathcal{O}_X$, $(j_!\mathcal{O}_K, F) \simeq (\mathcal{O}_K, j^!F)$. Hence $(j_!\mathcal{O}_K, F) \simeq j^!F(K) = j^*F^K(K)$.

Proposition 5.1. *Let K be locally closed. Then $(j_!\mathcal{O}_K, F) \simeq j^*F^K K = \varinjlim_{U \supseteq K}(F^K U)$ where the direct limit is taken over all open $U \subseteq X$ containing K.*

Proof. (Cf. [9, p. 149])

First note that, by definition of j^*, we have $s \in j^*F^K K$ iff for each $x \in K$ there is an open neighbourhood V_x of x in X and a section $t \in F^K V_x$ such that s and t agree on $K \cap V_x$.

For each open set $U \subseteq X$ with $K \subseteq U$ there is a natural map $F^K U \longrightarrow j^*F^K K$ given by restriction and hence we have a canonical map $h : \varinjlim_{U \supseteq K}(F^K U) \longrightarrow j^*F^K K$.

Suppose that $a \in \varinjlim_{U \supseteq K}(F^K U)$ with $ha = 0$. Take $U \supseteq K$ and $t \in F^K U$ such that $r_{U,\infty}t = a$, where $r_{U,\infty} : F^K U \longrightarrow \varinjlim_{V \supseteq K}(F^K V)$ is the canonical map. Note that $hr_{U,\infty}$ is restriction to K so, since $hr_{U,\infty}t = 0$, and since $\mathrm{supp}(t) \subseteq K$, we have $t = 0$ and hence $a = r_{U\infty}t = 0$, as required.

To see that h is onto, let $s \in j^*F^K K$. For each $x \in K$ choose an open neighbourhood V_x of x in X and $t(x) \in F^K V_x$ such that s and $t(x)$ agree on $K \cap V_x$. Note that if $y \in K$ then $t(x)$ and $t(y)$ agree not just on $(K \cap V_x) \cap (K \cap V_y)$ but, since $\mathrm{supp}(t(x)), \mathrm{supp}(t(y)) \subseteq K$, on $V_x \cap V_y$ and hence there is $t \in F^K(V_x \cup V_y)$ which agrees with $t(x)$ on V_x and with $t(y)$ on V_y - hence which agrees with s on $(V_x \cup V_y) \cap K$.

Therefore, if $V = \bigcup\{V_x : x \in K\}$ then the compatible sections $t(x)$ yield a section $t \in F^K V$ which agrees with s on K. That is, $hr_{V\infty}t = s$, as required. □

So $(j_!\mathcal{O}_K, -)$ is the functor "germs of sections with support in K".

Proposition 5.2. *Let $K \subseteq X$ be compact and locally closed, say $K = U \cap C$ with U open and C closed. Suppose that for every downwards-directed system $(K_\lambda)_\lambda$ of closed subsets of U with $\bigcap_\lambda K_\lambda = K$ there exists U' open and λ with $U' \cap K_\lambda = K$. Let $((G_\lambda)_\lambda, (g_{\lambda\mu})_{\lambda\leq\mu})$ be a directed system in* Mod-$\mathcal{O}_X$ *and set $G = \varinjlim_\lambda G_\lambda$ in* Mod-$\mathcal{O}_X$ *with limit maps $g_{\lambda\infty} : G_\lambda \longrightarrow G$. Then $G^K K = (\varinjlim_\lambda (G_\lambda^K))K$. Furthermore if g' is the canonical map from the presheaf limit, $\mathrm{p}\varinjlim_\lambda (G_\lambda^K)$ to G^K then g'_K is monic.*

Proof. Let V be an open set containing K. Note that the $G_\lambda^K V$ form a directed system since, if $f : G_1 \longrightarrow G_2$ is a morphism of sheaves and $s \in G_1 V$, then $\mathrm{supp}(fs) \subseteq \mathrm{supp}(s)$. For the same reason we have $(g_{\lambda\infty})_V . G_\lambda^K V \leq G^K V$ for all V and all λ and hence there is induced, for each V, a canonical map $g'_V : \varinjlim_\lambda (G_\lambda^K V) \longrightarrow G^K V$ and these fit together to give a map $g' : G' = \mathrm{p}\varinjlim_\lambda (G_\lambda^K) \longrightarrow G^K$ where $\mathrm{p}\varinjlim$ denotes the presheaf direct limit.

We have a commutative diagram as shown where $G' \longrightarrow G'' = \mathrm{p}\varinjlim_\lambda G_\lambda$ is the natural map induced by the inclusions $G_\lambda^K \longrightarrow G_\lambda$.

$$\begin{array}{ccc} G'' = p\varinjlim_\lambda G_\lambda & \xrightarrow{g''} & G \\ \uparrow & & \uparrow \\ G' = p\varinjlim_\lambda G_\lambda^K & \xrightarrow[g']{} & G^K \end{array}$$

Since direct limit is left exact the left-hand map is an inclusion and so g' is just the restriction of $g'' : G'' = \mathrm{p}\varinjlim_\lambda (G_\lambda) \longrightarrow G$, that is, is sheafification.

We show that g'_K is monic. By 5.1 every element of $G'K$ is represented by a section $s \in G'V$ for some open $V \supseteq K$; note that $\mathrm{supp}(s) \subseteq K$. Suppose that $g'_V s = 0$. Then there is an open cover $(V_i)_i$ of V such that $\mathrm{res}^{G'}_{VV_i} s = 0$ for each i. Since K is compact there are $V_1, \dots, V_n$, say, which cover K. Replace the original choice of V by $V_1 \cup \dots \cup V_n$ and s by its restriction to this set.

For $i = 1, \dots, n$ choose λ_i and $a_i \in G_{\lambda_i}^K(V)$ such that $(g'_{\lambda_i\infty})_V a_i = s$ and such that $(g'_{\lambda_i\infty})_{V_i} \mathrm{res}^{G_{\lambda_i}^K}_{VV_i} a_i = 0$ (because $\mathrm{res}^{G'}_{VV_i} s = 0$ there are such λ_i and a_i). Here $g'_{\lambda,\infty}$ denotes the limit presheaf map $G_\lambda^K \longrightarrow G'$.

Since $a_1, \dots, a_n$ all map to s in the limit and since a_i restricts to 0 on V_i in the limit and since there are just finitely many of these, there is $\lambda \geq \lambda_1, \dots, \lambda_n$ such that for all i, j we have $(g_{\lambda_i\lambda})_V a_i = (g_{\lambda_j\lambda})_V a_j = b$ say and such that $\mathrm{res}^{G_\lambda^K}_{VV_i}(b) = (g_{\lambda_i\lambda})_{V_i} \mathrm{res}^{G_{\lambda_i}^K}_{VV_i}(a_i) = 0$. So, since G_λ^K is a sheaf, $b = 0$ and hence $s = (g'_{\lambda\infty})_V b = 0$. It follows that g'_K is indeed monic.

Now we show that if we have a section of $G^K K$, represented, using 5.1, by, say, $t \in G^K V$ where V is an open neighbourhood of K (without loss of generality, $V \subseteq U$) then there is some section of $\varinjlim_\lambda (G_\lambda^K)$ over some open neighbourhood V'' of K contained in V which maps to $\mathrm{res}^G_{V,V''}(t)$.

Set $L = \mathrm{supp}(t)$ - a closed, hence compact, subset of K.

Since $t \in GV$ and $G = \varinjlim G_\lambda$ there is, for each $x \in L$, an open neighbourhood $V_x (\subseteq V)$ of x and $t_x \in \mathrm{p}\varinjlim G_\lambda . V_x$ such that $(g'')_{V_x} t_x = \mathrm{res}^G_{V,V_x}(t)$.

Finitely many of these V_x suffice to cover L, say $V_1, \dots, V_n$ (writing V_1 for V_{x_1} etc.). For each $i = 1, \dots, n$ there is λ_i and $s_i \in G_{\lambda_i} V_i$ such that $(g''_{\lambda_i\infty})_{V_i} s_i = t_i$, where $g''_{\lambda\infty} : G_{\lambda_i} \longrightarrow \mathrm{p}\varinjlim G_\lambda$ is the map to the presheaf limit. We will show that we may take the s_i to have support contained in K.

Write $s_i^\lambda = (g_{\lambda_i,\lambda})_{V_i} s_i \in G_\lambda V_i$ for each $\lambda \geq \lambda_i$ and set $L_\lambda = \bigcup_i^n \mathrm{supp}(s_i^\lambda)$ for each $\lambda \geq \lambda_1, \ldots, \lambda_n$.

We claim that L_λ is a closed subset of $V' = V_1 \cup \cdots \cup V_n$. Let $y \in V' \setminus L_\lambda$. If $y \in V_i$ then, since $(s_i)_y = 0$, hence $(s_i^\lambda)_y = 0$, there is an open neighbourhood of y contained in $V_i \setminus \mathrm{supp}(s_i^\lambda)$. Taking the intersection of these neighbourhoods over those i such that $y \in V_i$, we obtain an open neighbourhood of y which is disjoint from L_λ, as required.

So the L_λ form a downwards-directed system of closed subsets of V' with, note, intersection L. Hence the $K \cup L_\lambda$ form a downwards-directed system of closed subsets of V' with intersection K and so, by hypothesis, there is an open set $V'' \subseteq V'$ and λ with $V'' \cap (K \cup L_\lambda) = K$. Replacing each V_i by $V'' \cap V_i$ and each s_i by $\mathrm{res}^{G_\lambda}_{V_i, V'' \cap V_i}((g_{\lambda_i,\lambda})_{V_i} s_i)$, we may assume now that $\mathrm{supp}(s_i) \subseteq K$ and hence that $s_i \in G_\lambda^K V_i$.

It remains to show that the s_i are locally eventually compatible and hence that, together, they correspond to an element of $(\varinjlim_\lambda (G_\lambda^K))K$. Given $x \in K$, if $x \in V_i \cap V_j$ then, since $\mathrm{res}^G_{V_i, V_i \cap V_j}((g' g''_{\lambda_i \infty})_{V_i} s_i) = \mathrm{res}^G_{V, V_i \cap V_j}(t) = \mathrm{res}^G_{V_j, V_i \cap V_j}((g' g''_{\lambda_j \infty})_{V_j} s_j)$, the restrictions in $\mathrm{p}\varinjlim G_\lambda^K$ of $(g''_{\lambda_i \infty})_{V_i} s_i$ and $(g''_{\lambda_j \infty})_{V_j} s_j$ must agree in some neighbourhood of x. Hence the s_i glue together to form a section $s \in (\varinjlim G_\lambda^K) V''$ representing t, as required. $\square$

Of course, we would like the stronger result that, in the above situation, $G^K K$ is actually equal to the limit $\varinjlim(G_\lambda^K K)$ of the $G_\lambda^K K$, for then every morphism from $j_! \mathcal{O}_K$ to G would lift through the direct system, and we would deduce that $j_! \mathcal{O}_K$ is finitely presented.

In more detail, we have $(j_! \mathcal{O}_K, G) \simeq j^* G^K K = \varinjlim_{U \supseteq K} (G^K U)$ by 5.1, the correspondence being, to a morphism $f : j_! \mathcal{O}_K \longrightarrow G$ we assign the adjoint morphism $f' : \mathcal{O}_K \longrightarrow j^* G^K$ and to this we assign the image, $f' 1_K$ of $1_K \in \mathcal{O}_K K$ - this image will be represented by a section of G, over some open neighbourhood V of K, with support contained in K. So, to show that every morphism from $j_! \mathcal{O}_K$ to G lifts through some G_λ it would be enough to show that every K-germ of a section of G^K lifts to a K-germ of a section of G_λ^K for some λ.

Proposition 5.3. *Suppose that X is Hausdorff and locally compact in the sense defined before 4.8. Let K satisfy the hypotheses of 5.2; then the morphism g'_K as there is an isomorphism. Hence $j_! \mathcal{O}_K$ is finitely presented.*

Proof. We continue with the notation of the proof of 5.2. Since j^* is a left adjoint it preserves colimits and hence $G|_K = j^* G = \varinjlim (G_\lambda |_K)$. By 3.4, $\mathcal{O}_K$ is a finitely presented object of Mod-$\mathcal{O}_K$ and so the morphism from $\mathcal{O}_K$ to $G|_K$ corresponding to the section $t|_K \in G^K K$ (in the notation of the proof of 5.2) lifts through some global section u_1 of $G_\lambda |_K$ for some λ.

Now we use [9, III.2.2] which has the Hausdorff hypothesis and gives that u_1 is represented by some section, u say, of $G_\lambda V_0$ for some open neighbourhood V_0 of K in X. We may assume that $V'' \subseteq V_0$ (V'' as in the proof of 5.2). This means that, in the notation at the end of the proof of 5.2, each $(g''_{\lambda_i \infty})_{V_i} s_i$ agrees with $\mathrm{res}^{G''}_{V_0, V_i}(g''_{\lambda,\infty})_{V_i}(u)$ already in the presheaf limit and hence there is $\mu \geq \lambda_1, \ldots, \lambda_n$ such that $\mathrm{res}^{G_\mu^K}_{V_i, V_i \cap V_j}((g_{\lambda_i \mu})_{V_i} s_i) = \mathrm{res}^{G_\mu^K}_{V_j, V_i \cap V_j}((g_{\lambda_j \mu})_{V_j} s_j)$ for each i, j, and so the $(g_{\lambda_i\mu})_{V_i}s_i$ glue together to form a section $s \in G_\mu^K$ with $(g''g''_{\mu\infty})_{V''}s = t\,|_{V''}$, as required. $\square$

We show that the condition of 5.2 is necessary for $j_!\mathcal{O}_K$ to be finitely presented.

Lemma 5.4. *Let $K \subseteq X$ be locally closed. If $j_!\mathcal{O}_K$ is finitely presented then K is compact and also for every open set $U \supseteq K$ and for every downwards-directed set $(K_\lambda)_\lambda$ of closed subsets of U with $\bigcap_\lambda K_\lambda = K$, there is λ and an open set V with $U \supseteq V \supseteq K$ and $K_\lambda \cap U = K$.*

Proof. We know, by 4.6, that K must be compact since $K = \mathrm{supp}(j_!\mathcal{O}_K)$ and $j_!\mathcal{O}_K$ is finitely generated.

Suppose that we have an open set $U \supseteq K$ and downwards-directed system $(K_\lambda)_\lambda$ of closed subsets of U with intersection K. We have, by 2.9, $\varinjlim j_!\mathcal{O}_{K_\lambda} = j_!\mathcal{O}_K$. If $j_!\mathcal{O}_K$ is finitely presented then $\mathrm{id}_{j_!\mathcal{O}_K}$ lifts through some $j_!\mathcal{O}_{K_\lambda}$, that is, the canonical epimorphism $j_!\mathcal{O}_{K_\lambda} \longrightarrow j_!\mathcal{O}_K$ splits and hence, by 2.7, K is open in K_λ. That is, there is V (without loss of generality $V \subseteq U$) with $K_\lambda \cap V = K$, as required. $\square$

We finish by showing that Mod-$\mathcal{O}_X$, where $\mathcal{O}_X$ is the sheaf of continuous real-valued functions on $[0,1]$, is not locally finitely presented (recall, 4.9, that it is locally finitely generated). In fact, only certain properties of this ringed space are needed so we begin by assuming just that X is a space such that:

$(*)$ every compact subset of X is closed

(for instance, if X is Hausdorff then we have this).

Suppose also that:

$(**)$ every proper closed subset C of X is a directed intersection $C = \bigcap_\lambda C_\lambda$ of compact closed subsets C_λ such that C is not an open subset of any C_λ.

Certainly this is so for $X = [0,1]$ or, more generally, for X any locally closed subset of $\mathbb{R}^n$. [Proof for $[0,1]$: For each n set $U_n = \bigcup\{B_{\frac{1}{n+1}}(x) : x \in X \setminus C$ is such that $B_{\frac{1}{n}}(x) \cap C = \emptyset\}$ ($B_\epsilon(x)$ denotes the open ball of radius ϵ centred at x). Then $U_n \cap C = \emptyset$ and C is strictly contained in $C_n = U_n^{\mathrm{c}}$ and $\bigcap_n C_n = C$. Now, if there were n and an open set V such that $C = C_n \cap V$ then $\bigcap_{m \geq n} C_m \setminus C$ would be non-empty (since each $C_m \setminus C$ would be closed and since X is compact) - contradiction.]

Then, for such a space X and any ringed space $\mathcal{O}_X$ on X, we have, by 5.4, that if K is locally closed and $j_!\mathcal{O}_K$ is finitely presented in Mod-$\mathcal{O}_X$ then $K = X$ (and $\mathcal{O}_X$ will be finitely presented iff X is compact, by 3.4).

Now, suppose further that:

$(***)$ every stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ is a (not necessarily commutative) local ring and every section which is, at each point, in the radical, is in fact zero.

This is true in our example since if $f \in \mathcal{O}_{[0,1]}(V)$ is non-zero, say $f(x) \neq 0$ for some $x \in V$, then $f_x \in \mathcal{O}_{[0,1],x}$ is invertible.

Suppose that F is a cyclic sheaf in the sense that it is generated over $\mathcal{O}_X$ by a single section (see the previous section). Then, we claim, $F \simeq j_!\mathcal{O}_K$ for some locally closed $K \subseteq X$. For say $F = j_!\langle s\rangle$ where $s \in FU$ some open $U \subseteq X$. Let $K = \mathrm{supp}(s)$. Then there is a morphism $f : j_!\mathcal{O}_K \longrightarrow F$ with $f_U : (j_!\mathcal{O}_K)U \longrightarrow FU$ taking "1_K" to s and so f is an epimorphism. Let $G = \ker(f)$, so we have the exact sequence $0 \longrightarrow G \longrightarrow j_!\mathcal{O}_K \longrightarrow F \longrightarrow 0$. At $x \in K$ this is $0 \longrightarrow G_x \longrightarrow \mathcal{O}_{X,x} \longrightarrow F_x \longrightarrow 0$ so, since $F_x \neq 0$ and $\mathcal{O}_{X,x}$ is local, we have $G_x \leq \mathrm{rad}\, \mathcal{O}_{X,x}$. Thus, for every open V, $t \in GV$ and $x \in V$, we have $t_x \in \mathrm{rad}\, \mathcal{O}_{X,x}$, noting that $t_x = 0$ for $x \in V \setminus K$. So, by assumption, $t = 0$. Hence $G = 0$ and so $F \simeq j_! \mathcal{O}_K$, as required.

In particular, if F is a cyclic finitely presented sheaf, then $F \simeq \mathcal{O}_X$.

It follows that every finitely presented sheaf, F, is generated by global sections, for if $F = \sum_1^n j_! \langle s_i \rangle$ where $s_i \in FU_i$ then, without loss of generality, $s_1 \notin \sum_2^n j_! \langle s_i \rangle U_1$. So $F' = F / \sum_2^n j_! \langle s \rangle$ is finitely presented and cyclic, generated by the image of s_1. By the above, $F' \simeq \mathcal{O}_X$ so, since the image of s_1 generates F', it must have support equal to X. Hence $\mathrm{supp}(s_1) = X$ and so s_1 must be a global section. The same applies to any member of a minimal generating set and so we have the claim (which may be compared with the fact, see [9, III.3.9, III.2.9], that every sheaf of modules over this ringed space is soft).

It follows that every non-zero sheaf generated by finitely presented sheaves must have a non-zero global section. The final assumption we need is:

$(\ast\ast\ast\ast)$ there is a non-zero sheaf which has no non-zero global sections.

Certainly not every sheaf in Mod-$\mathcal{O}_X$, where $\mathcal{O}_X$ is the sheaf of continuous fuctions on $[0, 1]$, has a non-zero global section (for instance, consider $j_! \mathcal{O}_U$ where U is a proper open subset of X) and so we conclude that Mod-$\mathcal{O}_X$ is not locally finitely presented. Therefore we have the following for $X = [0, 1]$ among other spaces (indeed, if $X = \mathbb{R}$ then we see that there is no non-zero finitely presented $\mathcal{O}_X$-module).

Proposition 5.5. *Let $\mathcal{O}_X$ be the sheaf of real-valued continuous functions on the closed real unit interval. Then* Mod-$\mathcal{O}_X$ *is not locally finitely presented.*

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