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A note on the geometry of linear Hamiltonian systems of signature 0 in $\mathbb{R}^4$

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Abstract

It is shown that a linear Hamiltonian system of signature zero on $\mathbb{R}^4$ is elliptic, hyperbolic or mixed according to the number of Lagrangian planes in the null-cone $H^{-1}(0)$, or equivalently the number of invariant Lagrangian planes. A weaker extension to higher dimensions is described.

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Introduction

A linear Hamiltonian system on a vector space V is determined by a quadratic form H, the Hamiltonian, and a bilinear symplectic form ω. In terms of a basis, let S be the symmetric matrix representing H (so $H(\mathbf{v}) = \mathbf{v}^T S\mathbf{v}$) and Ω the nondegenerate skew-symmetric matrix representing ω (so $\omega(\mathbf{u},\mathbf{v}) = \mathbf{u}^T\Omega\mathbf{v}$). The associated linear Hamiltonian dynamical system is then given by $\dot{\mathbf{v}} = L\,\mathbf{v}$ where $L = \Omega^{-1}S$.

The Hamiltonian system (H,ω) or (S,Ω) is *degenerate* if S is degenerate, otherwise it is *non-degenerate*. A nondegenerate system is said to be *elliptic* if all eigenvalues of L are pure imaginary, *hyperbolic* if no eigenvalues of L are pure imaginary, and *mixed* if it is neither elliptic nor hyperbolic. Recall that if H is a non-degenerate quadratic form on $\mathbb{R}^n$, then a basis with coordinates $(x,y) = (x_1\dots,x_\ell,y_1,\dots,y_k) \in \mathbb{R}^\ell \times \mathbb{R}^k$ can be chosen so that

$$H(x,y) = (y_1^2+\cdots+y_k^2) - (x_1^2+\cdots+x_\ell^2).$$

The *signature* of H is then defined to be the difference $k-\ell$, and the *index* is ℓ. In $\mathbb{R}^4$, the signature of a non-degenerate form is necessarily even.

Given a linear Hamiltonian system (H,ω) it is in principle straightforward to determine its type (elliptic/hyperbolic/mixed) simply by calculating the eigenvalues of L. On the other hand, the type depends only on the symplectic geometry of the quadratic form H, and this elementary note is a first attempt to make this dependence explicit. For a system in $\mathbb{R}^4$, it turns out that there is a one-to-one correspondence between the type of equilibrium and the number of Lagrangian planes in the *null-cone* $\mathcal{N} = H^{-1}(0)$ of H. Recall that a subspace U of a symplectic space of dimension $2n$ is *Lagrangian* if it is of dimension n and for any pair of vectors $\mathbf{u},\mathbf{v}\in U$ one has $\omega(\mathbf{u},\mathbf{v}) = 0$. The main result of the paper is:

Theorem 1 *Let $(\mathbb{R}^4,\omega,H)$ be a nondegenerate linear Hamiltonian system, with H of signature 0, and let $\mathcal{N}$ be the null-cone of H. Then*

- *if H is elliptic with simple eigenvalues, $\mathcal{N}$ contains no Lagrangian planes;*
- *if H is elliptic with double eigenvalues and non-zero nilpotent part, $\mathcal{N}$ contains precisely 1 Lagrangian plane;*
- *if H is hyperbolic with a full quadruplet of eigenvalues, $\mathcal{N}$ contains precisely 2 Lagrangian planes;*
- *if H is hyperbolic with a pair of coincident real eigenvalues and non-zero nilpotent part, $\mathcal{N}$ contains precisely 3 Lagrangian planes;*
- *if H is hyperbolic with simple real eigenvalues, $\mathcal{N}$ contains precisely 4 Lagrangian planes.*

In the course of the proof, we also show that if the Hamiltonian system has double eigenvalues and is semisimple, then the null-cone contains infinitely many Lagrangian planes (and "$\infty+2$" if the double eigenvalues are real).

It should perhaps be pointed out that signature 0 is the interesting case. If H has signature ± 4 (so is positive or negative definite) then the system is elliptic, and $\mathcal{N}=\{0\}$ which contains no planes at all, Lagrangian or otherwise. And if H is of signature ± 2, then it is necessarily of mixed type and again $\mathcal{N}$ contains no planes (Lagrangian or otherwise). In fact, as we see in Proposition 2 below, a non-degenerate quadratic form on $\mathbb{R}^{2n}$ whose null-cone contains a subspace of dimension n is necessarily of vanishing signature.

In higher dimensions, the type of equilibrium continues to determine the number of Lagrangian planes in the null-cone of a signature zero quadratic Hamiltonian; however the one-to-one correspondence no longer holds. This is discussed in the final section §4.

1 Linear geometry of the null-cone of a quadratic form

Although we do not need the full generality, we discuss arbitrary non-degenerate quadratic forms in this section as the arguments are sufficiently simple and short. Let $H:\mathbb{R}^n\to\mathbb{R}$ be a quadratic form of index ℓ, and denote its null-cone $H^{-1}(0)$ by $\mathcal{N}$. We assume that $2\ell\leq n$, as otherwise we can replace H by $-H$, which has the same null-cone. After a suitable change of basis we can write $H:\mathbb{R}^\ell\oplus\mathbb{R}^k\to\mathbb{R}$ (where $k=n-\ell$) so that $H(x,y)=\|y\|^2-\|x\|^2$. Let $\mathcal{N}=H^{-1}(0)$, and let $\Gamma\subset\mathcal{N}$ be a linear subspace of $\mathbb{R}^n$. Let $\Pi:\mathbb{R}^\ell\oplus\mathbb{R}^k\to\mathbb{R}^\ell$ be the Cartesian projection on to the first factor. Note that since $\Gamma\subset\mathcal{N}$ we know that $\Pi|_\Gamma$ is injective, for otherwise $\Gamma\cap\mathbb{R}^k\neq\{0\}$ which would contradict the obvious fact that $\mathcal{N}\cap\mathbb{R}^k=\{0\}$. It follows that any linear subspace of the null-cone has dimension at most ℓ. Furthermore, since $\Gamma\cap\mathbb{R}^k=\{0\}$ it follows that Γ can be represented as the graph of a linear map $\mathbb{R}^\ell\to\mathbb{R}^k$, with matrix M. That is, $\Gamma=\Gamma_M=\{(x,y)\in\mathbb{R}^\ell\oplus\mathbb{R}^k\mid y=Mx\}$.

Suppose now $\Gamma=\Gamma_M\subset\mathcal{N}$. Then $H(x,Mx)=\|Mx\|^2-\|x\|^2\equiv 0$ (for all $x\in\mathbb{R}^\ell$). It follows that the ℓ columns of M are orthonormal vectors in $\mathbb{R}^k$. In particular, if $k=\ell$, M is an orthogonal matrix. Consequently,

Proposition 2 *Let H be a quadratic form in n variables of index $\ell\leq n/2$. Then the null-cone of H contains no linear spaces of dimension greater than ℓ, and the set of linear spaces of dimension ℓ contained in the null-cones is in 1-1 correspondence with the set of orthonormal ℓ-frames in $\mathbb{R}^{n-\ell}$. In the case that $n=2\ell$, so H is of signature zero, this set can be identified with the group of orthogonal $\ell\times\ell$ matrices.*

The set of orthonormal ℓ-frames in $\mathbb{R}^N$ is called a Stiefel manifold, denoted $V_{\ell,N}$. If $\ell < N$ then $V_{\ell,N}$ is connected, while if $\ell = N$ it has two connected components, distinguished by the orientation of the frame (or by the determinant of M). Both the Stiefel manifold and the set of linear spaces of dimension ℓ in the null-cone have natural manifold structures (the latter as a submanifold of the appropriate Grassmannian), and it can be shown that the 1-1 correspondence of the proposition above is in fact a diffeomorphism.

In particular, in the case $n = 4$ and $\ell = 2$, the set of planes in $\mathcal{N}$ can be identified with the union of two circles. Indeed, if $(x,y) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ is such that $\|y\|^2 = \|x\|^2 \neq 0$ then there are precisely two orthogonal matrices M, M' with $y = Mx = M'x$, and $\det(M) = 1 = -\det(M')$. Conversely, given any $M, M' \in \mathbf{O}(2)$ with $\det(M) = 1 = -\det(M')$ then $\Gamma_M \cap \Gamma_{M'}$ consists of a single line through the origin. The two circles therefore lie in different connected components of the Stiefel manifold $V_{2,2}$.

Finally, recall that the Grassmannian $G_{2,4}$ of planes in $\mathbb{R}^4$ is of dimension 4, while the Lagrange-Grassmannian Λ_2 is a submanifold of dimension 3. We are interested in the intersection between Λ_2 and the 1-dimensional family in $G_{2,4}$ of planes in $\mathcal{N}$, which could generically be expected to be finite.

2 Invariant Lagrangian planes

Any Lagrangian plane in $\mathcal{N}$ is invariant under the Hamiltonian dynamics, so that this "symplectic geometry" of the null-cone is intimately related to the dynamics:

Lemma 3 *Let (M, ω, H) be any (smooth) Hamiltonian system, and let L be a Lagrangian submanifold of M. Then L is invariant if and only if it is contained in a level set of the Hamiltonian.*

Proof L is invariant if and only if it is tangent to the characteristic direction of the Hamiltonian system at each of its points. Since L is Lagrangian, this is equivalent to the tangent space to L at each point annihilating $\mathrm{d}H$, which in turn is equivalent to L being contained in a level set of H. (This argument continues to hold even at critical points of H.) □

In 4 dimensions a converse to Lemma 3 holds:

Lemma 4 *If $\dim(M) = 4$, and if L is an invariant submanifold of dimension 2 contained in a level set of the Hamiltonian then it is Lagrangian.*

Proof Let $x \in L$. Then as L is invariant, $T_xL = \langle X_H(x), \mathbf{u}\rangle$ (where X_H is the Hamiltonian vector field and $\mathbf{u}$ is some vector). Then $\omega(\mathbf{u}, X_H(x)) = \mathrm{d}H(\mathbf{u}) = 0$ since L is in a level set of H. Thus L is Lagrangian. □

Lemma 5 *Suppose (V, ω, H) is a linear Hamiltonian system, which decomposes into the direct sum of two subsystems $(V_1, \omega_1, H_1) \oplus (V_2, \omega_2, H_2)$, and let $\mathcal{N}_j$ be the null-cone of H_j. Suppose that there are no eigenvalues common to the two subsystems H_1 and H_2. Then every Lagrangian subspace $L \subset \mathcal{N}$ can be decomposed as a direct sum $L = L_1 \oplus L_2$ such that each L_j is a Lagrangian subspace of V_j contained in $\mathcal{N}_j$.*

Proof This involves the lemma above that Lagrangian subspaces in the null-cone are invariant. In general, any subspace invariant under a linear transformation is a direct sum of (generalized) eigenspaces. Since any (generalized) eigenspace of (V, ω, H) is contained in either V_1 or V_2, it follows that $L = L_1 \oplus L_2$ with $L_j \subset V_j$ and invariant. That each L_j is Lagrangian in V_j follows easily by contradiction, and since L_j is Lagrangian in V_j and invariant it is contained in $\mathcal{N}_j$, by Lemma 3. □

3 Lagrangian planes in the null-cone of a Hamiltonian

In this section we prove Theorem 1, proceeding case by case. The argument in each case is either based on Lemma 5 (if the system is a product of 2-dimensional systems) or on a calculation involving a normal form for the Hamiltonian (see [A]). Given a quadratic form H on $\mathbb{R}^4$ of index 2, we know from Section 1 that the null-cone $\mathcal{N}$ contains 2 circles of planes. Rather than use a basis adapted to the quadratic form, we use a symplectic basis in order to make the Lagrangian nature of a given plane transparent. Recall that if a plane is given by $y = Mx$ and the symplectic form is $\omega = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2$, then the plane is Lagrangian if and only if M is symmetric (in which case M is the hessian matrix of the generating function determining the plane).

The proof in the hyperbolic and non-semisimple elliptic cases consist in taking the Hamiltonian $H(q, p)$ in some normal form, substituting $y = Mx$ for an arbitrary symmetric matrix M, where y and x are suitable (symplectic) choices of p's and q's, and finally determining for which such M the restriction of H to the graph of $y = Mx$ vanishes. In order to consider all possible Lagrangian planes it is necessary to consider 4 cases: (i) $y = (p_1, p_2)$ and $x = (q_1, q_2)$; (ii) $y = (q_1, q_2)$ and $x = (-p_1, -p_2)$; (iii) $y = (q_1, p_2)$ and $x = (-p_1, q_2)$; and (iv) $y = (p_1, q_2)$ and $x = (q_1, -p_2)$ (the signs are chosen so that the symplectic form $\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2$). In the calculations we will always take $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, often without further reference.

3.1 The elliptic cases: imaginary eigenvalues

The semisimple system If H has distinct imaginary eigenvalues $\pm i\lambda_1$ and $\pm i\lambda_2$, (λ_1, λ_2 positive and distinct) the result follows from Lemma 5, for each of the two "modes" (eigenspaces for $\pm i\lambda_1$ and for $\pm i\lambda_2$) is symplectic and has no invariant Lagrangian subspaces.

If on the other hand, $\lambda_1 = \lambda_2$ and the system is semisimple, then every non-zero point is contained in a periodic orbit, and that periodic orbit spans a plane in $\mathbb{R}^4$ which consists entirely of periodic orbits, so is invariant. Since the initial point lies in $\mathcal{N}$, the orbit is in $\mathcal{N}$ and so therefore is the plane it spans. That this 2-dimensional plane is Lagrangian follows from Lemma 4.

Consequently, if $\lambda_1 = \lambda_2$ every point in $\mathcal{N}$ is contained in a Lagrangian plane in $\mathcal{N}$, and so there are infinitely many such planes (forming one of the two families of planes in $\mathcal{N}$; the other family consists of symplectic planes).

The non-semisimple elliptic system Generically, an elliptic Hamiltonian with double eigenvalues has a nontrivial nilpotent part. For a normal form we take

$$H_\pm = \pm\tfrac{1}{2}(q_1^2 + q_2^2) + \lambda(p_2 q_1 - p_1 q_2)$$

for which the associated linear system has eigenvalues $\pm i\lambda$, with multiplicity 2 (we assume $\lambda \neq 0$). The two normal forms $H_\pm$ are symplectically inequivalent.

We show that the only Lagrangian plane in the null-cone is $\{q_1 = q_2 = 0\}$. There are 4 cases to consider as was pointed out above.

(i) $y = (p_1, p_2)$ and $x = (q_1, q_2)$. With $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ we substitute $y = Mx$ into the normal form for the Hamiltonian, to obtain

$$2H_\pm = (\pm 1 + 2\lambda\beta)q_1^2 + 2\lambda(\gamma - \alpha)q_1 q_2 + (\pm 1 - 2\lambda\beta)q_2^2$$

It is clear that this cannot vanish identically, so there are no Lagrangian planes in this portion of $\mathcal{N}$.

(ii) $y = (q_1, q_2)$ and $x = (-p_1, -p_2)$. With $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ we again substitute $y = Mx$ into the normal form for the Hamiltonian, to obtain the more complicated expression

$$2H_\pm = (\pm(\alpha^2+\beta^2)+2\lambda\beta)p_1^2 + 2(\pm\beta(\alpha+\gamma)+\lambda(\gamma-\alpha))p_1p_2 + (\pm(\beta^2+\delta^2)-2\lambda\beta)p_2^2$$

Suppose this vanishes identically; then if $\beta = 0$ it follows that $\alpha = \gamma = 0$, while if $\beta \neq 0$ the coefficients of p_1^2 and p_2^2 cannot both vanish. The only Lagrangian plane in $\mathcal{N}$ therefore corresponds to $M = 0$ which is therefore $\{q_1 = q_2 = 0\}$.
(iii) $y = (q_1, p_2)$ and $x = (-p_1, q_2)$, and (iv) $y = (p_1, q_2)$ and $x = (q_1, -p_2)$. Similar computations to those above show that there are no Lagrangian planes of this form.

The single Lagrangian plane found in this part is the limit of the (symplectic) normal modes in the semisimple case above, as the eigenvalues approach equality, and indeed on this plane the motion is periodic with period $2\pi/\lambda$.

3.2 The hyperbolic case: a quadruplet of eigenvalues

Here we take

$$H = \kappa(p_1q_1 + p_2q_2) + \lambda(p_1q_2 - p_2q_1).$$

The eigenvalues of the linear system are $\pm\kappa \pm i\lambda$, and we assume $\kappa\lambda \neq 0$. We consider the 4 cases as before:
(i) $y = (p_1, p_2)$ and $x = (q_1, q_2)$. Substituting $y = Mx$, with as usual $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, into the Hamiltonian gives

$$H = (\kappa\alpha - \lambda\beta)q_1^2 + (2\kappa\beta + \lambda(\alpha-\gamma))q_1q_2 + (\kappa\gamma + \lambda\beta)q_2^2.$$

This only vanishes identically if $\alpha = \beta = \gamma = 0$, corresponding to the plane $\{p_1 = p_2 = 0\}$.
(ii) $y = (q_1, q_2)$ and $x = (-p_1, -p_2)$. Substituting this $y = Mx$ (same M as before) into the Hamiltonian gives

$$H = (\kappa\alpha + \lambda\beta)p_1^2 + (2\kappa\beta + \lambda(\gamma-\alpha))p_1p_2 + (\kappa\gamma - \lambda\beta)p_2^2.$$

Again, this only vanishes identically if $\alpha = \beta = \gamma = 0$, corresponding to the plane $\{q_1 = q_2 = 0\}$.
(iii) $y = (q_1, p_2)$ and $x = (-p_1, q_2)$ and (iv) $y = (p_1, q_2)$ and $x = (q_1, -p_2)$. Repeating similar calculations shows in these cases that there are no other solutions.

Thus the null-cone contains precisely 2 Lagrangian planes. These two planes are in fact the stable and unstable manifolds of the vector field: every initial point in $\{q_1 = q_2 = 0\}$ tends to the origin as $t \to \infty$ (the stable manifold), while every initial point in $\{p_1 = p_2 = 0\}$ tends to the origin as $t \to -\infty$ (the unstable manifold).

Further calculations show that the two Lagrangian planes belong to the same family of planes in $\mathcal{N}$. Indeed, write $H = p_1(\kappa q_1 + \lambda q_2) + p_2(-\lambda q_1 + \kappa q_2) \equiv p_1Q_1 + p_2Q_2$. Thus,

$$4H = (p_1+Q_1)^2 - (p_1-Q_1)^2 + (p_2+Q_2)^2 - (p_2-Q_2)^2.$$

In these coordinates, $\begin{pmatrix} p_1+Q_1 \\ p_2+Q_2 \end{pmatrix} = M\begin{pmatrix} p_1-Q_1 \\ p_2-Q_2 \end{pmatrix}$ lies in $\mathcal{N}$ iff $M \in \mathbf{O}(2)$. The two Lagrangian planes found above correspond to $M = -I$ and $M = I$ respectively, and both are in $\mathbf{SO}(2)$.

3.3 The hyperbolic cases: real eigenvalues

Semisimple cases Here the normal form is

$$H = \lambda_1 p_1 q_1 + \lambda_2 p_2 q_2,$$

and the associated linear system has eigenvalues $\pm\lambda_1, \pm\lambda_2$, and we can assume $\lambda_1 > 0, \lambda_2 > 0$. In the 2-dimensional hyperbolic system with $H = \lambda pq$ there are two invariant (Lagrangian) lines. It then follows from Lemma 5 that if $\lambda_1 \neq \lambda_2$ there are precisely 4 invariant Lagrangian planes in the 4-dimensional system, as required. We now show this again by direct calculation, as the calculation is required for the case $\lambda_1 = \lambda_2$.

Taking the four cases for Lagrangian planes in turn:

(i) $y = (p_1, p_2)$ and $x = (q_1, q_2)$. With $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ as usual, we substitute $y = Mx$ into the Hamiltonian to obtain

$$H = \lambda_1 \alpha q_1^2 + (\lambda_1 + \lambda_2)\beta q_1 q_2 + \lambda_2 \gamma q_2^2.$$

Since $\lambda_1, \lambda_2 > 0$, this vanishes if and only if $M = 0$, corresponding to the plane $\{p_1 = p_2 = 0\}$.

(ii) $y = (q_1, q_2)$ and $x = (-p_1, -p_2)$. Again, since $\lambda_1, \lambda_2 > 0$, the Hamiltonian restricted to $q = Mp$ vanishes if and only if $M = 0$, corresponding to the plane $\{q_1 = q_2 = 0\}$.

(iii) $y = (q_1, p_2)$ and $x = (-p_1, q_2)$. Substituting $y = Mx$ gives

$$H = \lambda_1 \alpha p_1^2 + (\lambda_2 - \lambda_1)\beta p_1 q_2 - \lambda_2 \gamma q_2^2.$$

Provided $\lambda_1 \neq \lambda_2$, this vanishes again only when $M = 0$, corresponding to the plane $\{q_1 = p_2 = 0\}$.

(iv) $y = (p_1, q_2)$ and $x = (q_1, -p_2)$. Again, provided $\lambda_1 \neq \lambda_2$, the restriction of the Hamiltonian vanishes again only when $M = 0$, corresponding to the plane $\{p_1 = q_2 = 0\}$.

If $\lambda_1 = \lambda_2$, so the system has a double real eigenvalue and is semisimple, then in cases (i) and (ii) the same conclusion applies, while in cases (iii) and (iv) the planes Γ_M are in the null-cone whenever $\alpha = \gamma = 0$ (for all β). That is, of the two circles of planes in $\mathcal{N}$ (Section 1), one consists entirely of Lagrangian planes while the other contains exactly two Lagrangian planes.

The planes $\{p_1 = p_2 = 0\}$ and $\{q_1 = q_2 = 0\}$ (cases (i) and (ii)) are respectively the stable and unstable manifolds of the vector field. On the other hand, the remaining invariant planes are all saddles. These last ones are not the only invariant planes with saddles (for example $\{q_1 = p_1 = 0\}$ is an invariant symplectic plane with a saddle point), but they are distinguished by the fact that the flow on a Lagrangian plane does not need to be area-preserving.

Non-semisimple case A normal form is

$$H = \lambda(p_1 q_1 + p_2 q_2) + p_1 q_2.$$

The eigenvalues of the corresponding system are $\pm\lambda$.

(i) $y = (p_1, p_2)$ and $x = (q_1, q_2)$. Substituting $y = Mx$ as usual, one obtains

$$H = \lambda \alpha q_1^2 + (\alpha + 2\lambda\beta) q_1 q_2 + (\lambda\gamma + \beta) q_2^2.$$

This vanishes identically if and only if $M = 0$, corresponding to the plane $\{p_1 = p_2 = 0\}$ (the unstable manifold).

(ii) $y = (q_1, q_2)$ and $x = (-p_1, -p_2)$. In this case we obtain $H = \lambda\alpha p_2^2 + (\gamma + 2\lambda\beta) p_1 p_2 + (\lambda\alpha + \beta) p_1^2$, which vanishes only when $M = 0$, corresponding to the plane $\{q_1 = q_2 = 0\}$ (the stable manifold).

(iii) $y = (q_1, p_2)$ and $x = (-p_1, q_2)$. In this case there are no solutions.

(iv) $y = (p_1, q_2)$ and $x = (q_1, -p_2)$. Here the only solution is $\{p_1 = q_2 = 0\}$. (On this plane the dynamics is a simple area-preserving saddle, with eigenvalues $\pm\lambda$.)

4 Higher dimensional systems

In higher dimensions the number of Lagrangian planes in a null-cone does not distinguish between different types of equilibrium. For example, in $\mathbb{R}^8$ with its standard symplectic structure, the two Hamiltonians

$$\begin{aligned} H_1 &= (p_1^2+q_1^2)-2(p_2^2+q_2^2)+3(p_3^2+q_3^2)-4(p_4^2+q_4^2) \\ H_2 &= (p_1^2+q_1^2)-2(p_2^2+q_2^2)+p_3q_3+2p_4q_4 \end{aligned}$$

are both of signature 0. The first is elliptic while the second has two pairs of imaginary eigenvalues and two pairs of real ones, yet neither has any invariant Lagrangian subspaces.

The fact that neither has any Lagrangian subspaces in its null-cone follows from the following argument.

Let (V,ω,H) be a linear Hamiltonian system with signature$(H)=0$, and suppose that the system is of codimension at most 1. Then (V,ω,H) splits into a direct sum of symplectic "eigenspaces"

$$(V,\omega,H)=\bigoplus_{j=1}^{r}(V_j,\omega_j,H_j),$$

such that the eigenvalue-quadruplets $\{\lambda_j,-\lambda_j,\overline{\lambda}_j,-\overline{\lambda}_j\}$ of the different components are distinct. The codimension hypothesis implies that each component is of dimension 2 or 4.

Define a function $\delta(j)$ by

- If j is such that $\dim V_j=2$ and λ_j is imaginary then $\delta(j)=0$;
- If j is such that $\dim V_j=4$ and λ_j is imaginary (so of algebraic multiplicity 2 and geometric multiplicity 1), then $\delta(j)=1$;
- If j is such that $\dim V_j=4$ and λ_j has non-zero real and imaginary parts, then $\delta(j)=2$;
- If j is such that $\dim V_j=4$ and λ_j is real (so of algebraic multiplicity 2 and geometric multiplicity 1), then $\delta(j)=3$.
- If j is such that $\dim V_j=2$ and λ_j is real then $\delta(j)=2$;

Then it follows immediately from Theorem 1 and Lemma 5 that the number of Lagrangian planes in the null-cone $\mathcal{N}$ is $\prod_{j=1}^{r}\delta(j)$.

In the two examples H_1 and H_2 above, both have the p_1,q_1 subspace as an eigenspace, for which the corresponding value of δ is 0. Consequently, neither has a Lagrange plane in $\mathcal{N}$.

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