Momentum maps and ...

Montaldi, James

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Momentum maps and . . .

James Montaldi

University of Manchester

UK-Japan Winter School,
Manchester, January MMX

Momentum maps — what are they?

G (Lie group) acting on (M, ω) (symplectic manifold)

$\xi \in \mathfrak{g} \rightsquigarrow \xi_M$ — symplectic vector field on M (so $\mathcal{L}_{\xi_M}\omega = 0$)

Question: Is ξ_M Hamiltonian?

A vector field X on M is *Hamiltonian* if $\omega(X, -) = \mathrm{d}h$ for some $h \in C^\infty(M)$

Answer: It depends .... (see later)

Assume action is Hamiltonian: ξ_M is Hamiltonian for every $\xi \in \mathfrak{g}$.

what are they . . . ctd

For each $\xi \in \mathfrak{g}$ we have a function $h_\xi \in C^\infty(M)$ (unique up to constant).

Dependence on ξ is linear (for suitable choices of constants)

This gives a map — the *momentum map*

$$\mathbf{J} : M \longrightarrow \mathfrak{g}^*$$

The defining equation is, for $m \in M$ and $v \in T_m M$,

$$\langle \mathrm{d}\mathbf{J}_m(v), \xi \rangle = \omega(\xi_M(m), v).$$

Immediate consequence (the *bifurcation lemma*):

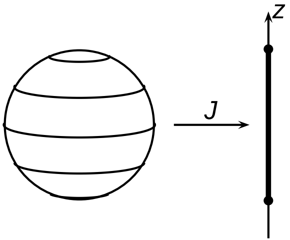
$$\mathrm{Image}(\mathrm{d}\mathbf{J}_m) = \mathfrak{g}_m^\circ \subset \mathfrak{g}^*$$

. . . responsible for the famous polytope structure of the image

Examples

1. S^1 on S^2 — rotation about z-axis:

$$\mathbf{J}(x,y,z) = z$$



2. $SO(3)$ on S^2 — all rotations:

$$\mathbf{J}(x,y,z) = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \in \mathfrak{so}(3)^* \simeq \mathfrak{so}(3) \simeq \mathbb{R}^3$$

This is just the “inclusion” map $S^2 \hookrightarrow \mathbb{R}^3$.

Examples ctd

3. Let V be a symplectic repn of G. Momentum map is given by

$$\langle \mathbf{J}(v), \xi \rangle = \tfrac{1}{2}[\xi.v, v]$$

4. Cotangent bundle: G acts on Q and hence on $M = T^*Q$. Then,

$$\langle \mathbf{J}(q,p), \xi \rangle = \langle p, \xi_Q(q) \rangle .$$

5. Generalizing $\mathrm{SO}(3)$ on S^2: let G act on $\mathfrak{g}^*$ by ”coadjoint repn” (dual of adjoint rep). Let $\mathcal{O}$ be an orbit—it's a symplectic manifold; the momentum map is the *inclusion* $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

6. $\mathrm{SE}(2)$ on the plane: $J(z) = (z, \frac{1}{2}|z|^2) \in \mathbb{R}^2 \times \mathfrak{so}(2)^*$ (using $\mathbb{R}^2 \simeq \mathbb{C}$)

Existence

TM	$\overset{\omega}{\longleftrightarrow}$	T^*M
tangent vector		linear form
X vector field		$\alpha = \iota_X \omega$ 1-form
X symplectic vector field		α closed 1-form
X *Hamiltonian vector field*		α *exact 1-form*

So, is $X = \xi_M$ Hamiltonian?

- Obstruction is $[\alpha] \in H^1(M, \mathbb{R})$.
- If $\omega = d\beta$ then $\mathbf{J}_\xi = \iota_{\xi_M}\beta$, so obstruction is $[\omega] \in H^2(M, \mathbb{R})$.
- If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ then momentum map exists, so obstruction is in $H^1(\mathfrak{g}, \mathbb{R})$.
  Indeed, one defines $J_{[\xi,\eta]} := \{J_\xi, J_\eta\}$ (Poisson brackets)

So if *any* of these cohomology groups vanishes, the momentum map exists.

Equivariance

Souriau (1968): *There is an action of G on $\mathfrak{g}^*$ which renders* **J** *equivariant*.

Basically, this is the coadjoint action (can always be arranged for compact groups, semisimple groups and cotangent actions)

– but sometimes need to add a *cocycle* $\theta : G \to \mathfrak{g}^*$

One consequence is that one can define the *orbit momentum map* **j**:

$$\begin{array}{ccc} M & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{\mathbf{j}} & \mathfrak{g}^*/G \; (\simeq \mathfrak{t}^*_+ \text{ if } G \text{ compact}) \end{array}$$

This can be very useful in studying the dynamics of symmetric Hamiltonian systems.

Coadjoint orbits

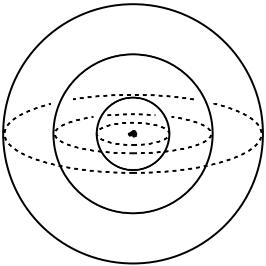
For $A \in SO(3)$ and $\mu \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$,

$$\mathrm{Coad}_A \mu = A\mu$$

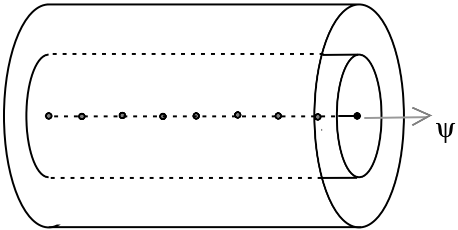
$SE(2) \simeq \mathbb{R}^2 \rtimes S^1$. Then

$$\mathrm{Coad}_{(\mathbf{u},R)}(\nu, \psi) = (R\nu, \psi + R\nu.u).$$

Orbits:



SO(3)



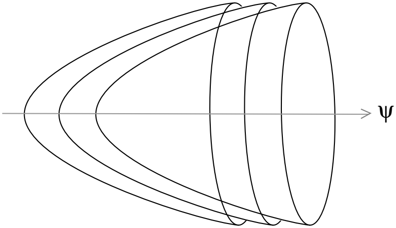
SE(2)

Coadjoint orbits

SE(2) with a non-zero cocycle

$$\theta(R,\mathbf{u}) = (\mathrm{i}\mathbf{u},\ \tfrac{1}{2}|\mathbf{u}|^2)$$

(identifying $\mathbb{R}^2 \simeq \mathbb{C}$) — the orbits become paraboloids



For SL(2) (semisimple, non-compact) we have $\mathfrak{sl}(2) \simeq \mathbb{R}^3$; orbits are level sets of $x^2 + y^2 - z^2$:

- the origin
- the two regular sheets of the cone
- one sheeted hyperbolae
- each of the sheets of the 2-sheeted hyperbolae

Point Vortices on the sphere

Points $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathrm{S}^2$, of vorticities $\kappa_1, \ldots, \kappa_N$. $M = \mathrm{S}^2 \times \cdots \times \mathrm{S}^2$, with symplectic form

$$\omega = \kappa_1 \omega_1 + \cdots \kappa_N \omega_N$$

Momentum map is then

$$\mathbf{J}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \kappa_1 \mathbf{x}_1 + \cdots \kappa_N \mathbf{x}_N.$$

Clearly equivariant with the standard $\mathrm{SO}(3)$ actions.

Point Vortices in the plane

Points $z_1, \ldots, z_N \in \mathbb{R}^2 = \mathbb{C}$, of vorticities $\kappa_1, \ldots, \kappa_N$.

$M = \mathbb{C}^N$, with symplectic form

$$\omega = \kappa_1 \omega_1 + \cdots + \kappa_N \omega_N$$

Momentum map is then

$$\mathbf{J}(z_1, \ldots, z_N) = \left(\mathrm{i} \sum_j \kappa_j z_j,\ \tfrac{1}{2} \textstyle\sum_j \kappa_j |z_j|^2 \right).$$

Equivariance is, for $\mathbf{z} = (z_1, \ldots, z_N)$

$$J(Rz_1 + \mathbf{u}, \ldots, Rz_1 + \mathbf{u}) = \mathrm{Coad}_{(R,\mathbf{u})} \mathbf{J}(\mathbf{z}) + \Lambda(\mathrm{i}\mathbf{u},\ \tfrac{1}{2}|\mathbf{u}|^2),$$

where $\Lambda = \sum \kappa_j \in \mathbb{R}$ (total vorticity) — it's a cocycle.

A 1-parameter family

In current work (on point vortices) with T. Tokieda, we embed the three groups $\mathrm{SO}(3)$, $\mathrm{SE}(2)$, $\mathrm{SL}(2)$ in a 1-parameter family parametrized by λ as follows. At the level of Lie algebras, let $\lambda \in \mathbb{R}$ and

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These satisfy the commutation relations

$$[X_1, X_2] = \lambda X_3, \quad [X_3, X_1] = X_2, \quad [X_3, X_2] = -X_1.$$

For each $\lambda \in \mathbb{R}$, write $\mathfrak{g}_\lambda$ for the corresponding Lie algebra. Then $\mathfrak{g}_\lambda$ is the Lie algebra of the group of automorphisms of the quadratic form $x^2 + y^2 + \lambda z^2$, and is isomorphic to

$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{se}(2)$	$\mathfrak{so}(3)$

λ-family ctd ...

Consider the cocycle $\vartheta : \mathfrak{g}_\lambda \to \mathfrak{g}_\lambda^*$ defined by

$$\vartheta(aX_1 + bX_2 + cX_3) = \tfrac{1}{2}(-b,\ a,\ 0)$$

And use this to define the (infinitesimal) action of $\mathfrak{g}_\lambda$ on $\mathfrak{g}_\lambda^* \simeq \mathbb{R}^3$ by

$$X \cdot \mu = X\mu + \vartheta(X).$$

[Exercises: (1) check this is an action (this is where the cocycle condition comes from in general). (2) for $\lambda \neq 0$ every cocycle is exact—find the corresponding $\nu \in \mathfrak{g}_\lambda^*$ for which $\vartheta = \delta\nu$.]

λ-family ctd ...

Instead of preserving $x^2 + y^2 + \lambda z^2$ (as before), this modified action preserves the inhomogeneous form $x^2 + y^2 + \lambda z^2 - 2z$. So the orbits are spheres (or ellipsoids)/paraboloids/hyperboloids, corresponding to the sign of λ . [The ellipsoids are spheres if we use the invariant metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \lambda \mathrm{d}z^2$.]

Using the cocycle ϑ (and following Souriau) one can modify the natural Poisson structure on $\mathfrak{g}_\lambda^*$ to make **J** into a Poisson map:

$$\{f, g\}(\mu) = \langle \mu, [\xi, \eta] \rangle - \langle \vartheta(\xi), \eta \rangle$$

where $\xi = \mathrm{d}f(\mu)$ and $\eta = \mathrm{d}g(\mu)$ (both in $\mathfrak{g}_\lambda$). The symplectic leaves then coincide with the orbits of the modified coadjoint action.

Back to point vortices

For each value of λ there is one orbit $M_\lambda \subset \mathfrak{g}_\lambda^* \simeq \mathbb{R}^3$ passing through the origin. This is a sphere (ellipsoid), paraboloid or hyperboloid accordingly as λ is positive, zero, or negative. We use this as the space for the point vortices. The phase space is $(M_\lambda)^n$, and the momentum map $\mathbf{J}_\lambda : (M_\lambda)^n \to \mathfrak{g}_\lambda^*$ is given by

$$\mathbf{J}_\lambda(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum \kappa_j \mathbf{x}_j.$$

This is Poisson (with the modified structure from the previous slide) and equivariant.

This is the starting point for studying the dynamics of point vortices using the curvature of the surface as a parameter (the curvature of M_λ is 4λ). But I have no time to continue further.

Remark This 1-parameter family of Lie algebras was developed to better understand a paper of Y. Kimura (see References), and details will appear in a joint paper with T. Tokieda.

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