On Σ-representability of countable structures over real numbers, complex numbers and quaternions

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Σ-DEFINABILITY OF COUNTABLE STRUCTURES OVER REAL NUMBERS, COMPLEX NUMBERS, AND QUATERNIONS

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We study Σ-definability of countable models over hereditarily finite ($\mathbb{HF}$-) superstructures over the field $\mathbb{R}$ of reals, the field $\mathbb{C}$ of complex numbers, and over the skew field $\mathbb{H}$ of quaternions. In particular, it is shown that each at most countable structure of a finite signature, which is Σ-definable over $\mathbb{HF}(\mathbb{R})$ with at most countable equivalence classes and without parameters, has a computable isomorphic copy. Moreover, if we lift the requirement on the cardinalities of the classes in a definition then such a model can have an arbitrary hyperarithmetical complexity, but it will be hyperarithmetical in any case. Also it is proved that any countable structure Σ-definable over $\mathbb{HF}(\mathbb{C})$, possibly with parameters, has a computable isomorphic copy and that being Σ-definable over $\mathbb{HF}(\mathbb{H})$ is equivalent to being Σ-definable over $\mathbb{HF}(\mathbb{R})$.

1. PRELIMINARY INFORMATION

We will be working with finite predicate signatures only. This restriction, yet, will not inhibit us to speak about operations and constants. Thus, for instance, an operation F can be viewed as its graph, i.e., the predicate $P_F = \{\langle \bar{x}, f(\bar{x})\rangle \mid \bar{x} \in \mathtt{dom}(F)\}$, and we may think of a signature constant as a unary predicate true on just that constant. This conception agrees with the idea of constants treated as functions with zero number of elements.

Here we use the basic definitions and notions from the theory of admissible sets (see [1, 2]) and computable model theory (theory of constructive models); see, e.g., [3]. We consider the field $\mathbb{R}$ of reals in the predicate signature $\sigma = \langle +, \cdot, <, 0, 1\rangle$, the field $\mathbb{C}$ of complex numbers, and also the skew field $\mathbb{H}$ of quaternions in the predicate signature $\sigma_1 = \langle +, \cdot, 0, 1\rangle$, where constants likewise are viewed as predicates, in accordance with the convention made above. Sometimes, it might be convenient to work with signatures in which $+$, $\cdot$, 0, and 1 are operations. These cases will be specially announced. A corresponding operational version of the signature σ is denoted by σ_f. In passing to operational signatures and back, note, the class of Σ-definable relations is kept fixed (see [2]).

For intervals, common designations are used: $[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$, $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$, etc. The relation $\{\bar{a} \mid \mathfrak{M} \models \varphi(\bar{a}, \bar{p})\}$ is also denoted by $\varphi(\bar{x}, \bar{p})^{\mathfrak{M}}[\bar{x}]$.

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The following definition is due to Ershov (see [1, 4]).

Definition 1.1. A model $\mathfrak{M} = \langle M, \mathbf{P}_0^{n_0}, \dots, \mathbf{P}_k^{n_k} \rangle$ of a finite signature is said to be Σ-*definable* over an admissible set $\mathbb{A}$ if there exist a finite tuple of parameters $\bar{p} \in \mathbb{A}$ and Σ-formulas

$$\begin{aligned} & W(x, \bar{z}), \\ & E^+(x, y, \bar{z}),\ E^-(x, y, \bar{z}), \\ & P_i^+(x_1, \dots, x_{n_i}, \bar{z}),\ P_i^-(x_1, \dots, x_{n_i}, \bar{z}),\ i = 1, \dots, k, \end{aligned}$$

such that:

(1) for all $i = 1, \dots, k$,

$$\begin{aligned} & P_i^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] \cap P_i^-(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] = \varnothing, \\ & P_i^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] \cup P_i^-(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] = \big(W(x, \bar{p})^{\mathbb{A}}[x]\big)^{n_i}; \end{aligned}$$

(2) $E^+(x, y, \bar{p})^{\mathbb{A}}[x, y]$ is a congruence relation on

$$\langle W(x, \bar{p})^{\mathbb{A}}[x]; P_1^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}], \dots, P_k^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] \rangle,$$

and

$$\begin{aligned} & E^+(x, y, \bar{p})^{\mathbb{A}}[x, y] \cap E^-(x, y, \bar{p})^{\mathbb{A}}[x, y] = \varnothing, \\ & E^+(x, y, \bar{p})^{\mathbb{A}}[x, y] \cup E^-(x, y, \bar{p})^{\mathbb{A}}[x, y] = \big(W(x, \bar{p})^{\mathbb{A}}[x]\big)^2; \end{aligned}$$

(3) $\langle W(x, \bar{p})^{\mathbb{A}}[x]; P_1^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}], \dots, P_k^+(\bar{x}, \bar{p})^{\mathbb{A}}[\bar{x}] \rangle /_{E^+(x,y,\bar{p})^{\mathbb{A}}[x,y]} \cong \mathfrak{M}$.
In this case we say that the above formulas $W(x, \bar{p})$, $E^+(x, y, \bar{p})$, $E^-(x, y, \bar{p})$, $P_i^+(\bar{x}, \bar{p})$, and $P_i^-(\bar{x}, \bar{p})$, $i = 1, \dots, k$, *define the model* $\mathfrak{M}$ *with parameters* $\bar{p}$.

If a tuple of parameters in our definition is empty then we speak of Σ-*definability without parameters*. We also say that a model $\mathfrak{M}$ is *definable over* $\mathbb{A}$ *with an equivalence having* some particular *property* (e.g, an equivalence all of whose classes are at most countable) if there exists a Σ-definition of $\mathfrak{M}$ over $\mathbb{A}$ in which the equivalence $E^+(x, y, \bar{p})^{\mathbb{A}}[x, y]$ has this property.

If we allow the formulas W, E^+, E^-, P_i^+, and P_i^- in the definition to be arbitrary first-order formulas then we obtain a definition of *elementary definability* (with or without parameters, respectively, with an equivalence having some property, etc.).

Definition 1.2. Suppose that $\mathfrak{A}$ is a submodel of $\mathfrak{B}$. We call it a 1-*submodel* (and denote this fact by $\mathfrak{A} \leqslant_1 \mathfrak{B}$) if, for any $\exists$-formulas $\varphi(x_1, \dots, x_n)$ and any $a_1, \dots, a_n \in \mathfrak{A}$,

$$\mathfrak{B} \models \varphi(a_1, \dots, a_n) \Rightarrow \mathfrak{A} \models \varphi(a_1, \dots, a_n).$$

Obviously, $\mathfrak{B} \preccurlyeq \mathfrak{A}$ implies $\mathfrak{A} \leqslant_1 \mathfrak{B}$.

The next definition is also due to Ershov.

Definition 1.3. Let $\mathbb{A}$ and $\mathbb{B}$ be admissible sets and $\mathbb{B}$ be an end extension of $\mathbb{A}$. We say that $\mathbb{A}$ is a Σ-*substructure of* $\mathbb{B}$ (and denote this fact by $\mathbb{A} \leqslant_\Sigma \mathbb{B}$) if, for all Σ-formulas $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_n \in \mathbb{A}$,

$$\mathbb{B} \models \varphi(a_1, \dots, a_n) \Rightarrow \mathbb{A} \models \varphi(a_1, \dots, a_n).$$

THEOREM 1.4. Let $\mathfrak{A} \leqslant \mathfrak{B}$. Then the following equivalence holds:

$$\mathfrak{A} \leqslant_1 \mathfrak{B} \Leftrightarrow \mathbb{HF}(\mathfrak{A}) \leqslant_\Sigma \mathbb{HF}(\mathfrak{B}).$$

Proof. We prove a less trivial part, namely, the statement (⇒). A proof for the other part is obvious and so left to the reader. Suppose $\mathbb{HF}(\mathfrak{B}) \models \exists \bar{z}\psi(\bar{z}, \bar{a})$, where ψ is a Δ_0-formula and $\bar{a} \in \mathbb{HF}(\mathfrak{A})$. Fix a tuple $\bar{b}$ so that $\mathbb{HF}(\mathfrak{B}) \models \psi(\bar{b}, \bar{a})$. Let c be the transitive closure of a set $\{\bar{a}, \bar{b}\}$. Treating c as a structure for our language of KPU, we see that $c \models \psi(\bar{b}, \bar{a})$ because c is an initial substructure of $\mathbb{HF}(\mathfrak{B})$. This structure c is finite and contains a finite family of pairwise distinct urelements $p_1, \dots, p_k$, $q_1, \dots, q_l$, and we assume that $\bar{q} = q_1, \dots, q_l$ form a support of the parameters $\bar{a}$ and $\bar{p} = p_1, \dots, p_k$ are the remaining urelements. It is clear that $q_1, \dots, q_l \in \mathfrak{A}$.

Let $\mathfrak{B}^*$ be a restriction of the model $\mathfrak{B}$ to the set $\{p_1, \dots, p_k, q_1, \dots, q_l\}$. Denote the diagram of $\mathfrak{B}^*$ by $D(\mathfrak{B}^*)$; it is obviously finite. We have $\mathfrak{B} \models \exists \bar{u}\,(\bigwedge D(\mathfrak{B}^*))^{\bar{p}}_{\bar{u}}$. By the hypothesis, this implies $\mathfrak{A} \models \exists \bar{u}\,(\bigwedge D(\mathfrak{B}^*))^{\bar{p}}_{\bar{u}}$. Fix some witnesses $\bar{u}_0$ in $\mathfrak{A}$ for the existential quantifier. Then a mapping, which is the identity on q_i and takes $\bar{p}$'s termwise to $\bar{u}_0$, is an isomorphism from $\mathfrak{B}^*$ to a finite submodel $\mathfrak{A}^* \leqslant \mathfrak{A}$; the mapping also preserves $q_1, \dots, q_k$. We can extend this isomorphism to an isomorphism onto some initial substructure c' in $\mathbb{HF}(\mathfrak{A})$ which preserves the parameters $\bar{a}$. This implies $c' \models \exists \bar{y}\psi(\bar{y}, \bar{a})$, whence $\mathfrak{A} \models \exists \bar{y}\psi(\bar{y}, \bar{a})$, as required. The theorem is completed.

In what follows, we need the concept of a *set-theoretic term*, or *s-term*, introduced by Ershov (see, e.g., [1, 4]). We define it by induction as follows:

(1) $\varnothing$ is an s-term;

(2) each variable x_i, $i < \omega$, is an s-term;

(3) if $t_0, \dots, t_l$ are s-terms then $\{t_0, \dots, t_l\}$ is an s-term;

(4) there are no other s-terms.

It is easy to see that s-terms express exactly those sets that can be represented as finite expressions built up from $\varnothing$ and variables $x_0, x_1, \dots$ using finitely many curly brackets {, } and a comma; for example: $\{x_0, \{x_1, \varnothing\}\}$. These terms will be used to represent hereditarily finite sets via their urelements.

Denote by $\mathtt{sp}\,(a)$ the *support* of a (see [2]), i.e., the set of all urelements involved in the construction of a. Thus, for instance, if m is an urelement then $\mathtt{sp}\,(\{\{m\}, m, \varnothing\}) = \{m\}$.

2. SOME GENERAL PROPERTIES OF Σ-DEFINABILITY

Here we give some general results used in the paper.

The following statement is rather obvious and we so omit its proof.

PROPOSITION 2.1. If a structure $\mathfrak{B}$ is computable with an oracle $H \subseteq \omega$, and a model $\mathfrak{A}$ is Σ-definable over $\mathbb{HF}(\mathfrak{B})$, then $\mathfrak{A}$ has an H-computable isomorphic copy.

The theorem below is well known.

THEOREM 2.2. There exists a uniform effective procedure which, given any Σ-formula $\varphi(x_1, \dots, x_n)$ and any s-terms $\tau_1(\bar{y}), \dots, \tau_n(\bar{y})$, outputs a computable family of $\exists$-formulas $(\psi_i(\bar{y}))_{i<\omega}$ of the language of $\mathfrak{M}$ such that, for any model $\mathfrak{M}$ and any tuple $\bar{p}$ of its elements of appropriate length,

$$\mathbb{HF}(\mathfrak{M}) \models \varphi(\tau_1(\bar{p}), \dots, \tau_n(\bar{p})) \Leftrightarrow \mathfrak{M} \models \bigvee_{i<\omega} \psi_i(\bar{p}).$$

Proof. We give a sketchproof. It is well known that $\varphi(x_1, \dots, x_n)$ is equivalent to some Σ_1-formula $\exists z\phi(z, x_1, \dots, x_n)$, where ϕ is a Δ_0-formula [2]. It is clear that the formula $\exists z\phi(z, x_1, \dots, x_n)$ is itself equivalent to an infinite disjunction of formulas $\exists \bar{u}\,\phi(\theta(\bar{u}), x_1, \dots, x_n)$ taken over all s-terms θ. It remains to realize that any Δ_0-formula of the form $\phi(\theta(\bar{u}), \tau_1(\bar{p}), \dots, \tau_n(\bar{p}))$ can be effectively reduced to an equivalent quantifier-free first-order formula. For this, we need only use induction on the complexity of formulas and s-terms occurring in the formulas. The theorem is completed.

THEOREM 2.3. If an $\exists$-theory for a model $\mathfrak{M}$ is computably enumerable then an $\exists$-theory of any model which is Σ-definable without parameters over $\mathbb{HF}(\mathfrak{M})$ is computably enumerable as well.

Proof. Let φ be an arbitrary $\exists$-formula. First, we reduce it to an equivalent $\exists$-formula all of whose atomic subformulas have one of the following forms:

$$P(x_1,\dots,x_n),\quad \neg P(x_1,\dots,x_n),$$
$$x=c,\ \neg(x=c),\quad x=y,\quad \neg(x=y),$$
$$F(x_1,\dots,x_n)=y,\quad \neg(F(x_1,\dots,x_n)=y),$$

where $x_1,\dots,x_n,y,z$ are arbitrary variables and c are constants.

Now, if in the resulting formula we replace all occurrences of the given atomic subformulas with Σ-subformulas (for positive occurrences) or their negations (for negative occurrences) that represent corresponding operations and predicates, we will obtain a Σ-formula φ' over $\mathbb{HF}(\mathfrak{M})$ whose truth in $\mathbb{HF}(\mathfrak{M})$ is equivalent to being true for φ. It remains to apply Theorem 2.2.

The next theorem, together with Proposition 2.1, will be our basic tool in estimating the complexity of a Σ-definable model. Actually, the theorem says that under some extra conditions, a Σ-definition of a model over one admissible set can be replaced with a Σ-definition over another admissible set, which is generally simpler than the former.

THEOREM 2.4. Let $\mathbb{A}$ and $\mathbb{B}$ be admissible sets and $\mathbb{A}\leqslant_\Sigma\mathbb{B}$. Assume that a model $\mathfrak{M}$ is Σ-definable over $\mathbb{B}$ with parameters from $\mathbb{A}$, and that each equivalence class in this representation contains at least one element of $\mathbb{A}$. Then the same formulas with the same parameters define the same model $\mathfrak{M}$ over $\mathbb{A}$.

Proof. Suppose that Σ-formulas such as

$$W(x,\bar z),\quad E^+(x,y,\bar z),\quad E^-(x,y,\bar z),$$
$$P_i^+(x_1,\dots,x_{n_i},\bar z),\quad P_i^-(x_1,\dots,x_{n_i},\bar z),\quad i=1,\dots,k,$$

define over $\mathbb{B}$ a model $\mathfrak{M}=\langle M,\mathbf{P}_0^{n_0},\dots,\mathbf{P}_k^{n_k}\rangle$ with parameters $\bar p$ from $\mathbb{A}$. The hypotheses of the theorem readily imply that $W(x,\bar p)^{\mathbb{A}}[x]=W(x,\bar p)^{\mathbb{B}}[x]\cap\mathbb{A}$, and the mapping

$$a/_{E^+(x,y,\bar p)^{\mathbb{B}}[x,y]}\mapsto\left(a/_{E^+(x,y,\bar p)^{\mathbb{B}}[x,y]}\right)\cap W(x,\bar p)^{\mathbb{A}}[x],\ \ a\in W(x,\bar p)^{\mathbb{A}}[x],$$

is an isomorphism of the models

$$\left\langle W(x,\bar p)^{\mathbb{B}}[x];P_1^+(\bar x,\bar p)^{\mathbb{B}}[\bar x],\dots,P_k^+(\bar x,\bar p)^{\mathbb{B}}[\bar x]\right\rangle/_{E^+(x,y,\bar p)^{\mathbb{B}}[x,y]},$$
$$\left\langle W(x,\bar p)^{\mathbb{A}}[x];P_1^+(\bar x,\bar p)^{\mathbb{A}}[\bar x],\dots,P_k^+(\bar x,\bar p)^{\mathbb{A}}[\bar x]\right\rangle/_{E^+(x,y,\bar p)^{\mathbb{A}}[x,y]}.$$

The first of these models is isomorphic to $\mathfrak{M}$. The theorem is completed.

COROLLARY 2.5. Assume $\mathfrak{M}\leqslant_1\mathfrak{N}$ and the basic set of $\mathfrak{M}$ is a subset of ω. Suppose also that the model $\mathfrak{A}$ is Σ-definable over $\mathbb{HF}(\mathfrak{N})$ with parameters from $\mathbb{HF}(\mathfrak{M})$ and that each equivalence class in this representation contains at least one element of $\mathbb{HF}(\mathfrak{M})$. Then $\mathfrak{A}$ is isomorphic to a model which is computable in the diagram of $\mathfrak{M}$. In particular, if $\mathfrak{M}$ has an isomorphic computable copy then $\mathfrak{A}$ has a computable presentation. The same conclusion holds for the case of definability with parameters from $\mathbb{HF}(\mathfrak{M})$.

Proof. Theorem 1.4 implies that $\mathbb{HF}(\mathfrak{M}) \leqslant_\Sigma \mathbb{HF}(\mathfrak{N})$. By Theorem 2.4, $\mathfrak{A}$ is Σ-definable over $\mathbb{HF}(\mathfrak{M})$. By Proposition 2.1, $\mathfrak{A}$ has a computable presentation in the diagram of $\mathfrak{M}$. The theorem is completed.

3. Σ-DEFINABILITY OVER $\mathbb{R}$

3.1. Definability with parameters. The Σ-definability with parameters for countable structures over $\mathbb{HF}(\mathbb{R})$ looks trivial, since each diagram of a countable model may be coded by some real number and the expressive power of Σ-formulas is quite enough to restore this model given that number. Namely, we have

THEOREM 3.1. Each countable model $\mathfrak{M}$ of a finite language is Σ-definable over $\mathbb{HF}(\mathbb{R})$ with a trivial equivalence and a single parameter.

Proof. Without loss of generality, we may assume that the basic set of $\mathfrak{M}$ coincides with ω. It will be convenient to work with binary decompositions of reals. To make such decompositions unique, we exclude from our consideration the decompositions ending in $111\ldots$ (since $1 = 0{,}11111\ldots$ for instance).

LEMMA 3.2. A function $f(r, n)$, which outputs the nth element $a_n \in \{0, 1\}$ in the binary decomposition $r = a{,}a_0a_1 \ldots a_n \ldots$ given any $r \in \mathbb{R}$ and any $n \in \omega$, is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters.

Proof. We give an idea behind the proof. It is easy to see that the following two functions are Σ-definable without parameters: the function $\mathtt{sg} : \mathbb{R} \to \mathbb{R}$, which determines whether our real r is zero, positive, or negative, i.e.,

$$\mathtt{sg}\,(r) = \begin{cases} 0 & \text{if } r \leqslant 0, \\ 1 & \text{otherwise,} \end{cases}$$

and the function $N : \omega \to \mathbb{R}$, which establishes a correspondence between the natural numbers in ω and the same natural numbers, but now treated as elements of $\mathbb{R}$.

We define a function $\mathtt{I} : \mathbb{R} \to \omega$ which, given an $r \in \mathbb{R}$, computes a natural number $m \in \omega$ such that $r = \pm m, a_0a_1 \ldots$, and

$$\mathtt{I}\,(r) = m \overset{df}{\Leftrightarrow} \Big(N(m) \leqslant r(-1)^{1+\mathtt{sg}\,(r)} < N(m+1)\Big).$$

Now the function $f(n, r)$ can be defined by recursion using the following property:

$$f(n,r) = k \overset{df}{\Leftrightarrow} (k \in \{0,1\}) \wedge \Bigg[N(\mathtt{I}\,(r)) + \sum_{i<n} \frac{N(f(i,r))}{N(2^{i+1})} + \frac{N(k)}{N(2^{n+1})}$$
$$\leqslant r(-1)^{1+\mathtt{sg}\,(r)} < N(\mathtt{I}\,(r)) + \sum_{i<n} \frac{N(f(i,r))}{N(2^{i+1})} + \frac{N(k+1)}{N(2^{n+1})}\Bigg].$$

The lemma is proved.

There exists a real r for which the characteristic function of the diagram of a model $\mathfrak{M}$ coincides with $f(r, 2n)$ (multiplication by 2 is added to avoid problems with sets whose characteristic function tends to 1). Thus, all basic predicates of $\mathfrak{M}$ are definable by Σ-formulas with parameter r, which yields Σ-definability of $\mathfrak{M}$ over $\mathbb{HF}(\mathfrak{M})$. The theorem is completed.

3.2. Definability without parameters and without restrictions on the equivalence. Inasmuch as a theory for $\mathbb{R}$ is decidable, Theorem 2.3 implies that an $\exists$-theory for any model Σ-definable over $\mathbb{HF}(\mathbb{R})$ is computably enumerable. On the other hand, the next theorem shows that such structures can be rather complicated.

THEOREM 3.3. For each $\delta < \omega_1^{\mathtt{CK}}$, there exists a countable model $\mathfrak{M}$ such that:

(1) $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters;

(2) for each $H \subseteq \omega$ such that $\mathfrak{M}$ is isomorphic to an H-computable model, $\mathbf{0}^{(\delta)} \leqslant_T H$.

Proof. Recall that any nonempty set $T \subseteq \omega^{<\omega}$ is called *a tree* if it is closed under initial segments, i.e., for arbitrary α and β such that α is an initial segment of β and $\beta \in T$, it is true that $\alpha \in T$. It follows from the definition that the empty sequence $\varnothing$ always belongs to T.

Fix a one-to-one Gödel numbering of the set ω^ω, i.e., some one-to-one onto mapping

$$m \in \omega \mapsto \gamma_m \in \omega^{<\omega}$$

such that given a number m we can effectively restore the sequence γ_m. If we identify the elements of ω^ω with their Gödel numbers we can speak about computably enumerable trees, recursive subsets of ω^ω, etc.

A known *linear Kleene–Brouwer ordering* $<_{\mathrm{KB}}$ on a set $\omega^\omega \cup \omega^{<\omega}$ is defined as follows (see [5]):

$\alpha <_{\mathrm{KB}} \beta$ iff either β is an initial segment of α, or β is not an initial segment of α while α is lexicographically less than β.

It is well known (and can be easily checked) that a tree T has an infinite branch iff T is not well ordered by $<_{\mathrm{KB}}$ (see [5]). If T is computably enumerable then $<_{\mathrm{KB}}$ is a computable ordering on T.

We will need one more fact from computability theory.

PROPOSITION 3.4 [5]. For each $\delta < \omega_1^{\mathtt{CK}}$, there exists a computable tree with a single infinite branch; the Turing degree of this branch equals $\mathbf{0}^{(\delta)}$.

Fix a computably enumerable tree T such as in Proposition 3.4 and denote its single infinite branch by $\boldsymbol{\xi}$. Without loss of generality, we may assume that the only infinite branch of T is neither leftmost nor rightmost in this tree. Fix a computable enumeration of T and denote the set of elements enumerated by the end of step t by T_t. The enumeration process can be thought of as organized so that each T_i is a tree, $T_0 = \{\varnothing\}$, and $|T_{t+1} \setminus T_t| = 1$ for any $t \in \omega$. Thus, we obtain a sequence of finite trees T_t, $t \in \omega$, such that

$$\{\varnothing\} = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_k \subseteq \ldots \subseteq \bigcup_{t\in\omega} T_t = T.$$

Now, fix some one-to-one computable function h so that $T = \{\gamma_{h(i)} \mid i < \omega\}$. A computable sequence $(a_{\gamma,i})_{\gamma\in T, i\in\omega}$ of rational numbers is defined by steps as follows.

Step 0. Let $a_{\varnothing,0} = 0$.

Step t+1. This step consists of two substeps:

Substep A (further growing sprouts). For each $\alpha \in T^t$, we search for a maximal m so that $a_{\alpha,m}$ is already defined, and then act in accordance with which one of the cases below is realized.

Case 1. There exists $\beta \in T^t$ such that no member that is strictly between $a_{\alpha,m}$ and $a_{\beta,0}$ in our sequence is defined at that moment; in this case we let $a_{\alpha,m+1} = \frac{1}{2}(a_{\alpha,m} + a_{\beta,0})$.

Case 2. An element $a_{\alpha,m}$ is maximal among all members in our sequence defined at this moment; in that event we set $a_{\alpha,m+1} = a_{\alpha,m} + 1$.

Substep B (sowing new sprouts). Let β be the unique element in $T^{t+1} \setminus T^t$. Define $a_{\beta,0}$ as follows: if there exist $\alpha_0, \alpha_1 \in T^t$ for which $\alpha_0 <_{\mathrm{KB}} \beta <_{\mathrm{KB}} \alpha_1$, and under $<_{\mathrm{KB}}$, there are no elements lying between α_0 and α_1 in T^t, then we find a maximal m such that $a_{\alpha_0,m}$ is defined, and put $a_{\beta,0} = \frac{1}{2}(a_{\alpha_0,m} + a_{\alpha_1,0})$; if there are no such β_0 and β_1, then either there is a $<_{\mathrm{KB}}$-smallest $\alpha \in T^t$ with $\beta <_{\mathrm{KB}} \alpha$, or there is a $<_{\mathrm{KB}}$-greatest $\alpha \in T^t$ with $\alpha <_{\mathrm{KB}} \beta$. In the former case we let $a_{\beta,0} = a_{\alpha,0} - 1$. In the latter case we find a maximal $m \in \omega$ for which $a_{\alpha,m}$ is defined, and put $a_{\beta,0} = a_{\alpha,m} + 1$. Pass to the next step.

The construction is completed.

It follows immediately from the construction that

$$a_{\alpha,i} < a_{\beta,j} \Leftrightarrow (\alpha <_{\mathrm{KB}} \beta) \vee (\alpha = \beta \wedge i < j),$$

i.e., elements $a_{\alpha,i}$, $\alpha \in T$, $i < \omega$, are ordered as $\omega \times L_{\mathrm{KB}}$, where L_{KB} is a linear Kleene–Brouwer ordering on T. This means that the ordering on such elements is isomorphic to a sum $L_0 + L_1$, where

$$L_0 = \{a_{\alpha,i} \mid \alpha <_{\mathrm{KB}} \boldsymbol{\xi},\ i < \omega\},$$
$$L_1 = \{a_{\alpha,i} \mid \boldsymbol{\xi} < \alpha,\ i < \omega\}.$$

The order L_0 is well founded, but L_1 is not, although each $x \in L_1$ defines a well-ordered set $\{y \in L_1 \mid x < y\}$. By construction, for any $\alpha \in T$ and any $i < \omega$, there are no $\beta \in T$ and $j < \omega$ such that $a_{\alpha,i} < a_{\beta,j} < a_{\alpha,i+1}$.

Put

$$S_i = \bigcup_{a,b \in L_i} [a,b), \quad i = 0,1.$$

Clearly, $S_0 \cap S_1 = \varnothing$ and each element of S_0 is smaller than each one of S_1.

LEMMA 3.5. For any $x \in S_0 \cup S_1$, there exist $\alpha \in T$ and $i < \omega$ such that $x \in [a_{\alpha,i}, a_{\alpha,i+1})$.

Proof. The argument for S_0 being similar, we only give a proof for S_1. Assume that some $x_0 \in S_1$ does not have the required property.

First, we prove that the Kleene–Brouwer ordering has a smallest $\alpha_0 \in T$ with the property $x_0 < a_{\alpha_0,0}$. Inasmuch as $x_0 \in S_1$, the conditions $a_{\beta_0,i_0}, a_{\beta_1,i_1} \in L_1$ and $a_{\beta_0,i_0} \leqslant x_0 < a_{\beta_1,i_1}$ hold for appropriate $\beta_0, \beta_1 \in T$ and $i_0, i_1 < \omega$. The elements of L_1 that are greater than a_{β_0,i_0} are well ordered. Therefore, L_1 contains a smallest element $a_{\alpha_0,i}$ with the property $x_0 < a_{\alpha_0,i}$. By assumption, x_0 is not contained in any interval of the form $[a_{\alpha_0,i}, a_{\alpha_0,i+1})$. Hence $x_0 < a_{\alpha_0,0}$.

Fix a step t_0 after which $a_{\alpha_0,0}$ and some element $a_{\beta,j} \in L_1$ with $a_{\beta,j} < x_0$ have already entered our construction. At succeeding steps, by assumption, no new element can be added between x_0 and $a_{\alpha_0,0}$. For each step $t > t_0$, let b_t be the leftmost neighbor of x_0 among the elements of L_1 defined by the end of this step. By construction, it follows easily by induction that $|a_{\alpha_0,0} - b_{t+1}| \leqslant \frac{1}{2}\,|a_{\alpha_0,0} - b_t|$ for all $t > t_0$. Thus, the sequence $(|a_{\alpha_0,0} - b_t|)_{t<\omega, t>t_0}$ tends to zero and is bounded by a positive number $a_{\alpha_0,0} - x_0$, which is a contradiction. The lemma is proved.

LEMMA 3.6. Between sets S_0 and S_1 is exactly one real $\mathbf{r}_0 \notin S_0 \cup S_1$.

Proof. Indeed, at each finite step, we have finite families $\{\ell_0^0 < \dots \ell_m^0\} \subseteq L_0$ and $\{\ell_0^1 < \dots \ell_n^1\} \subseteq L_0$ consisting of elements of the form $a_{\alpha,i}$ constructed by the end of that step; moreover, $\ell_0^0 < \dots \ell_m^0 < \ell_0^1 < \dots \ell_n^1$. Our construction guarantees that the distance between the last element ℓ_m^0 in L_0 and the first element ℓ_0^1 in L_1 becomes at least two times shorter at each step. Therefore, there is at most one such $\mathbf{r}_0$. And this $\mathbf{r}_0$ exists since L_0 has no maximal element and L_1 has no minimal one. The lemma is proved.

The argument above implies that the set $S = S_0 \cup S_1 \cup \{\mathbf{r}_0\}$ is an interval of the form $[a,b)$ or $(-\infty, b)$, where $a \in \mathbb{Q}$ and $a, b \in \mathbb{Q} \cup \{\infty\}$; hence this set is Δ-definable over $\mathbb{HF}(\mathbb{R})$.

Now we define $\mathfrak{M}$. The basic set M of this model is the quotient of the disjoint sum of S and ω modulo an equivalence η, which informally can be defined as an equivalence whose classes are exactly the sets $[a_{\alpha,i}, a_{\alpha,i+1})$, $\{\mathbf{r}_0\}$, and $\{m\}$, $m < \omega$. We show that this equivalence and its complement are Σ-definable without parameters in $\mathbb{HF}(\mathbb{R})$. In fact,

$$\begin{aligned} x\,\eta\,y \Leftrightarrow {} & [x, y \in S\ \wedge ((x = y) \vee \exists \alpha \in T\, \exists t \in \omega\, (x, y \in [a_{\alpha,t}, a_{\alpha,t+1})))] \\ & \vee [x, y \in \omega\ \wedge (x = y)]\,. \end{aligned}$$

On the other hand,

$$\begin{aligned}\neg(x\,\eta\,y) \Leftrightarrow\ & \neg(x \in S \leftrightarrow y \in S)\\ & \vee [x, y \in S \wedge \exists\alpha \in T\, \exists t \in \omega\, (x \in [a_{\alpha,t}, a_{\alpha,t+1}) \leftrightarrow y \notin [a_{n,t}, a_{n,t+1}))]\\ & \vee [(x, y \in \omega)\ \wedge (x \neq y)].\end{aligned}$$

Clearly, the expressions in the right parts of the definitions above are equivalent to Σ-formulas.

Now we give a list of basic predicates for our model together with their definitions.

N^1, a unary predicate distinguishing ω;

$\lhd^2$, $x \lhd y \overset{df}{\Leftrightarrow} \neg N(x)\ \wedge \neg N(y)\ \wedge (x \text{ is to the left of } y)$;

R^2, a binary predicate defined thus:

$$R(m, y) \Leftrightarrow \neg N(y) \wedge y = [a_{\gamma_{h(m)},0}, a_{\gamma_{h(m)},1});$$

s^2, a binary predicate distinguishing the usual successor function on N;

c, a constant whose value is $0 \in \omega$.

It is not hard to verify that the so defined model $\mathfrak{M} = \langle N, \lhd, R, s, c\rangle$ is countable and is Σ-definable over $\mathbb{HF}(\mathbb{R})$.

Assume that $H \subseteq \omega$ and our model is isomorphic to some H-computable model $\mathfrak{M}^*$. We show that, given an arbitrary $\alpha \in T$, we can effectively (from H) determine whether $\alpha < \boldsymbol{\xi}$. First, we search for $m \in \omega$ such that $\alpha = \gamma_{h(m)}$. Next, we search for y^* for which $\mathfrak{M}^* \models R(s^m(c), y^*)$; this y^* is an isomorphic image for $[a_{\gamma_{h(m)},0}, a_{\gamma_{h(m)},1}) = [a_{\alpha,0}, a_{\alpha,1})$. Denote by p the isomorphic image of $\{\mathbf{r}_0\} \in \mathfrak{M}$ in $\mathfrak{M}^*$. Obviously, the condition $y^* \lhd p$ is then equivalent to $\alpha < \boldsymbol{\xi}$. Note also that the negation of $\alpha < \boldsymbol{\xi}$ is equivalent to $\boldsymbol{\xi} < \alpha$.

Without loss of generality, we may assume that for each initial segment of β in the branch $\boldsymbol{\xi}$, there exists an extension α which is lexicographically less than $\boldsymbol{\xi}$. Note that an arbitrary β is an initial segment of $\boldsymbol{\xi}$ iff there exist α_0 and α_1 such that β is in their common initial segment and $\alpha_0 <_{\mathrm{KB}} \boldsymbol{\xi} <_{\mathrm{KB}} \alpha_1$. This gives us an H-computable procedure for enumerating all initial segments of $\boldsymbol{\xi}$, and hence the H-computability of $\boldsymbol{\xi}$. Hence $\mathbf{0}^{(\delta)} \leqslant H$. The theorem is completed.

Remarks. (1) The model constructed in the proof of Theorem 3.3 has a $\mathbf{0}^{(\delta)}$-computable isomorphic copy. Indeed, without the element $\mathbf{r}_0/\eta$, this model is computable. To add $\mathbf{r}_0/\eta$, all that we need is to know how to answer the questions "$\alpha <_{\mathrm{KB}} \boldsymbol{\xi}$?" and "$\boldsymbol{\xi} <_{\mathrm{KB}} \alpha$?" To do this, it suffices to have an oracle $\boldsymbol{\xi}$ whose Turing degree equals $\mathbf{0}^{(\delta)}$. Thus, for an arbitrary oracle H, $\mathfrak{M}$ is H-computable iff $\mathbf{0}^{(\delta)} \leqslant H$.

(2) Theorem 3.3 supplies us with an example of a structure $\mathfrak{M}$ and its element $a = \mathbf{r}_0/\eta$ such that $\mathfrak{M}$ is Σ-definable without parameters over $\mathbb{HF}(\mathbb{R})$ while the same model with a distinguished element a is no longer so definable. Otherwise, by Theorem 2.3, an $\exists$-theory for $\langle\mathfrak{M}, a\rangle$ would be computable, which is impossible for $\delta > 0$. In fact, it is easy to see that

$$\gamma_{h(n)} <_{\mathrm{KB}} \boldsymbol{\xi} \Leftrightarrow \exists x_0 \ldots x_n, y \left[x_0 = c \wedge \bigwedge_{i<n} s(x_i, x_{i+1}) \wedge R(x_n, y) \wedge y \lhd a\right];$$

$$\boldsymbol{\xi} <_{\mathrm{KB}} \gamma_{h(n)} \Leftrightarrow \exists x_0 \ldots x_n, y \left[x_0 = c \wedge \bigwedge_{i<n} s(x_i, x_{i+1}) \wedge R(x_n, y) \wedge a \lhd y\right].$$

Our proof shows how the computable enumerability of the $\exists$-theory for $\langle\mathfrak{M}, a\rangle$ can be used to derive the computable enumerability of $\boldsymbol{\xi}$, which is untrue even for $\delta > 0$.

For our further reasoning, we will need the following:

PROPOSITION 3.7. Every quantifier-free formula $\varphi(x, \bar{a})$ of a signature σ_f with parameters $\bar{a} \in \mathbb{R}$ defines a union of a finite number of open intervals and a finite set of elements algebraic over $\bar{a}$. In particular, if the set defined by this formula is nonempty then it contains at least one element algebraic over $\bar{a}$.

Proof. We reduce the formula $\varphi(x, \bar{a})$ to a disjunctive normal form. Each subformula like $\lambda(x)$ or $\neg\lambda(x)$, where $\lambda(x)$ is atomic, defines either a finite set of elements algebraic over $\bar{a}$ or a union of a finite family of intervals. By this token, each disjunctive member of this form defines a union of a finite number of intervals and a finite set of elements algebraic over $\bar{a}$. This implies that the entire formula, too, defines a union of a finite number of intervals and a finite set of elements algebraic over $\bar{a}$. The proposition is proved.

We have shown that the countable models that are Σ-definable over $\mathbb{HF}(\mathbb{R})$ can have an arbitrary hyperarithmetical complexity. It appears that hyperarithmetics is an upper bound for this complexity, which is shown by the following:

THEOREM 3.8. Assume that a countable model $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters. Then $\mathfrak{M}$ has a hyperarithmetical isomorphic copy.

Proof. First, we need some injective coding of reals by means of functions in ω^ω. We consider usual decimal decompositions of reals, excluding the decompositions ending in $999\ldots$. A real number r having a decimal decomposition $r = \pm a, a_0 a_1 a_2 \ldots$ is coded by a function $f \in \omega^\omega$ such that $f(0)$ codes a sign of r, $f(0) = \mathtt{sg}\,(r)$, $f(1) = a$, $f(2) = a_0$, $f(3) = a_1$, $f(4) = a_2$, $\ldots$. It is easy to see that this coding is injective and that the set of all possible codes for reals forms an arithmetical subset in ω^ω. If f is a code for r then f_n denotes the rational number

$$f_n = (-1)^{f(0)} \left[f(1) + \sum_{i<n} \frac{f(i)}{10^{i+1}} \right],$$

which is the nth decimal approximation for r. Clearly, f_n is a computable function of f and n, and $\lim\limits_{n\to\infty} f_n = r$.

A real is said to be *hyperarithmetical* if its coding function is hyperarithmetical. An element $a = \tau(\bar{p}) \in \mathbb{HF}(\mathbb{R})$, where $\bar{p}$ are urelements and τ is an s-term, is said to be *hyperarithmetical* if all elements of $\bar{p}$ are hyperarithmetical. We fix some Σ-definition of $\mathfrak{M}$ and use the perfect set theorem to prove that each equivalence class of this definition contains a hyperarithmetical element, and moreover, the choice of hyperarithmetical representatives of the classes can be bounded by some $A \in \mathbb{HYP}(\omega)$. Thereafter, it will remain to show that each subset $A \in \mathbb{HYP}(\omega)$ consisting of reals is extendable to some hyperarithmetical subfield $\mathbb{R}' \leqslant_1 \mathbb{R}$.

For formulas that Σ-define the model $\mathfrak{M}$, we adopt the same notation as was used in Definition 1.1. An arbitrary triple $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x}, \bar{y})\rangle$, consisting of a tuple of rational numbers $\bar{q} = q_1, \ldots, q_n$, a tuple of reals $\bar{\alpha} = \alpha_1, \ldots, \alpha_m$, and some s-term $\theta(x_1, \ldots, x_n, y_1, \ldots, y_m)$, with fixed linear order on the variables, is said to be *correct* provided that the following two conditions hold:

(1) $\theta(\bar{q}, \bar{\alpha}) \in W(x)^{\mathbb{HF}(\mathbb{R})}[x]$;

(2) all projections of the set $\left\{\bar{\beta} \mid E(\theta(\bar{q}, \bar{\alpha}), \theta(\bar{q}, \bar{\beta}))\right\}$ onto the coordinate axis contain no nontrivial open intervals.

LEMMA 3.9. For each equivalence class S of E and each s-term θ, if S contains at least one element of the form $\theta(\bar{\gamma})$, then there exist a linear ordering of its variables $x_1, \ldots, x_n$, a number $k \leqslant n$, and tuples $\bar{q} = q_1, \ldots, q_k$ and $\bar{\alpha} = \alpha_{k+1}, \ldots, \alpha_n$ such that:

(1) $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ is a correct triple;

(2) $\theta(\bar{q}, \bar{\alpha}) \in S$.

Proof. We fix some $\theta(\bar{\gamma}) \in S$ and an initial ordering of the s-term θ on $x_1, \ldots, x_n$. Assume that there is a number $i \in \{1, \ldots, n\}$ such that the ith projection of the set

$$\{\bar{y} \in \mathbb{R}^n \mid \mathbb{HF}(\mathbb{R}) \models E(\theta(\bar{y}), \theta(\bar{\gamma}))\}$$

contains a nontrivial open interval. Then we choose a minimal i with this property and pick up some rational number q_1 in this interval.

Consider a set such as

$$\{\langle y_1, \ldots, y_{i-1}, q_1, y_{i+1}, , \ldots, y_n\rangle \in \mathbb{R}^n \mid$$
$$\mathbb{HF}(\mathbb{R}) \models E(\theta(y_1, \ldots, y_{i-1}, q_1, y_{i+1}, \ldots, y_n), \theta(\bar{\gamma}))\},$$

choose a minimal j so that the jth projection of this set contains a nontrivial open interval, and pick up some q_2 in that interval. Then we consider a set such as

$$\{\langle y_1, \ldots, y_{i-1}, q_1, y_{i+1}, \ldots, y_{j-1}, q_2, y_{j+1}, \ldots, y_n\rangle \in \mathbb{R}^n \mid$$
$$\mathbb{HF}(\mathbb{R}) \models E(\theta(y_1, \ldots, y_{i-1}, q_1, y_{i+1}, \ldots, y_{j-1}, q_2, y_{j+1} y_n), \theta(\bar{\gamma}))\},$$

etc., until possible. Once this process has been terminated, we change the order of variables so that the variables with the numbers $i, j, \ldots$ that we have selected precede all the remaining ones. As a result, we obtain a new ordering of variables $x'_1, \ldots, x'_n$ and a tuple $\bar{q} = q_1, q_2, \ldots$ of rational numbers. Let $\bar{\alpha}$ be an arbitrary tuple of reals such that if elements of a tuple $\bar{q}\hat{\ }\bar{\alpha}$ are substituted in θ for $x'_1, \ldots, x'_n$, respectively, then we arrive at an element in the class S.

Obviously, the so obtained triple $\langle \bar{q}, \bar{\alpha}, \theta(x'_1, \ldots, x'_n)\rangle$ is correct. The lemma is proved.

LEMMA 3.10. For each correct tuple $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$, the set

$$\left\{\bar{\beta} \mid \mathbb{HF}(\mathbb{R}) \models E(\theta(\bar{q}, \bar{\beta}), \theta(\bar{q}, \bar{\alpha}))\right\} \tag{1}$$

is at most countable.

Proof. Each projection of this set onto any one of its coordinates does not contain nontrivial open intervals. On the other hand, the ith projection is defined on $\mathbb{HF}(\mathbb{R})$ by the following condition for x:

$$\exists x_1 \ldots x_{i-1} x_{i+1} \ldots x_n \in \mathbb{R}\, E(\theta(\bar{q}, x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n), \theta(\bar{q}, \bar{\alpha})).$$

By Theorem 2.2, this condition can be rewritten thus:

$$\exists x_1 \ldots x_{i-1} x_{i+1} \ldots x_n \bigvee_{i<\omega} \psi_i(x_1, , \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n, \bar{q}, \bar{\alpha})$$

for an appropriate computable sequence of $\exists$-formulas $(\psi_i)_{i<\omega}$, which is equivalent to the condition

$$\bigvee_{i<\omega} \exists x_1, \ldots, x_{i-1} x_{i+1}, \ldots, x_n \psi_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n, \bar{q}, \bar{\alpha}).$$

Using quantifier elimination in $\mathbb{R}$, we may replace all members in this infinite disjunction with equivalent quantifier-free formulas of a signature σ_f; thereafter, we see that our condition is equivalent to

$$\bigvee_{i<\omega} \zeta_i(x, \bar{q}, \bar{\alpha}),$$

where all ζ_i are quantifier-free formulas of the signature σ_f.

By Proposition 3.7, the infinite disjunction defines a union of an at most countable family of one-element sets and an at most countable family of open intervals. By the condition of being correct, there are no nonempty open intervals in this union. Hence we are left with just one version—where the infinite disjunction defines an at most countable set. Further, each point $\bar{\beta}$ in set (1) is completely defined by its projections, and for each projection, there are only countably many possibilities. This means that the number of elements in (1) is at most countable.

LEMMA 3.11. For any $\bar{q}$ and any $\theta(\bar{x})$, each equivalence class of E contains an at most countable number of elements $\bar{\beta}$ for which $\langle \bar{q}, \bar{\beta}, \theta(\bar{x})\rangle$ is correct.

Fix a tuple $\bar{\alpha}$ such that $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ is a correct triple (if it exists). For each such tuple $\bar{\beta}$, the hypothesis implies, in particular, that $\mathbb{HF}(\mathbb{R}) \models E(\theta(\bar{q}, \bar{\alpha}), \theta(\bar{q}, \bar{\beta}))$. The lemma now follows from Lemma 3.10.

Lemma 3.11 and the fact that the number of classes in $E(x, y)^{\mathbb{HF}(\mathbb{R})}[x, y]$ is countable readily yield the following:

LEMMA 3.12. There exists an at most countable number of correct triples.

It remains to define a coding for some set of triples containing a set of correct triples, write down a definition for the code of a correct triple, and apply the perfect set theorem. Clearly, any triple $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ prone to be correct should satisfy some conditions on the lengths of tuples: namely, $\mathtt{lh}\,(\bar{q}) + \mathtt{lh}\,(\bar{\alpha}) = \mathtt{lh}\,(\bar{x})$. The proof of Lemma 3.10 implies that effective quantifier elimination can be applied to real-closed fields to prove that, given any $j = 1, \ldots, \mathtt{lh}\,(\bar{x}) - \mathtt{lh}\,(\bar{q})$ and any (not necessarily correct) triple satisfying this condition, we can effectively construct a sequence of quantifier-free formulas $\zeta_i^{j,\bar{q},\theta(\bar{x})}(x, \bar{y}, \bar{z})$ of the signature σ_f, so that the jth projection of the set

$$\left\{\bar{\beta} \mid E(\theta(\bar{q}, \bar{\alpha}), \theta(\bar{q}, \bar{\beta}))\right\}$$

is defined by the infinite formula

$$\bigvee_{i<\omega} \zeta_i^{j,\bar{q},\theta(\bar{x})}(x, \bar{q}, \bar{\alpha});$$

moreover, the corresponding algorithm does not depend on $\bar{\alpha}$. Thus, $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ is correct iff the following condition is satisfied:

$$W(\theta(\bar{q}, \bar{\alpha})) \wedge \bigwedge_{j=1}^{\mathtt{lh}\,(\bar{x})-\mathtt{lh}\,(\bar{q})} \left[\forall q_1, q_2 \in \mathbb{Q} \exists r \in \mathbb{R} \left(q_1 < q_2 \to q_1 < r < q_2 \wedge \neg \bigvee_{i<\omega} \zeta_i^{j,\bar{q},\theta(\bar{x})}(r, \bar{q}, \bar{\alpha})\right)\right]. \quad (2)$$

Having an appropriate coding for triples $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ by means of functions, we can (uniformly in $\bar{q}$ and $\theta(\bar{x})$) effectively reduce the last-mentioned condition to a Σ_1^1-form, which does not depend on $\bar{\alpha}$. Here is an example of such a coding. A triple $\langle \bar{q}, \bar{\alpha}, \theta(\bar{x})\rangle$ can be coded, for instance, by a function $h : \omega \to \omega$ whose values are arranged as follows: $h(0) = \mathtt{lh}\,(\bar{x})$ and $h(1) = \mathtt{lh}\,(\bar{q})$; $h(2), \ldots, h(2 + \mathtt{lh}\,(\bar{q}) - 1)$ are equal to respective codes of the rationals $q_1, \ldots q_{\mathtt{lh}\,(\bar{q})}$; the next value is the code of a term θ, followed by the code of a tuple of variables $\bar{x}$, followed by a sequence of codes for signs of the reals $\alpha_1, \ldots, \alpha_{\mathtt{lh}\,(\alpha)}$, followed by the parts of $\alpha_1, \ldots, \alpha_{\mathtt{lh}\,(\alpha)}$ before decimal commas, followed by $a_1^0, \ldots, a_{\mathtt{lh}\,(\alpha)}^0$, $a_1^1, \ldots, a_{\mathtt{lh}\,(\alpha)}^1, \ldots$, where $a_i^0 \alpha_i^1 a_i^2 \ldots$ is the mantissa of α_i. Replace, then, $W(\theta(\bar{q}, \bar{\alpha}))$ by an appropriate infinite computable disjunction, which may be found uniformly in θ. We need to know how given a code of a triple we can reduce (2) to a Σ_1^1-form.

First, we show how to represent the second infinite disjunction in (2) as an arithmetical predicate for the code of a triple h, the code of a number r, and other natural parameters. The first disjunction is transformed in a similar way, and then standard methods (see [5]) will apply to reduce the entire expression to a Σ^1_1-form. We may assume that each quantifier-free formula $\zeta_i^{j,\bar{q},\theta(\bar{x})}(r,\bar{q},\bar{\alpha})$ is a conjunction of conditions like

$$F(r,\bar{\alpha},\bar{q})=0,\quad G(r,\bar{\alpha},\bar{q})\neq 0,\quad H(r,\bar{\alpha},\bar{q})>0, \tag{3}$$

where F, G, H, $\bar{q}$, $\bar{\alpha}$ are found (uniformly) effectively from h and other parameters, which are natural numbers. The equality $F(r,\bar{\alpha},\bar{q})=0$ can be written in the form

$$\forall e\exists n\forall m>n\left(|F(r_m,\bar{\alpha}_m,\bar{q})|<\frac{1}{e+1}\right),$$

where r_m and $\bar{\alpha}_m$ are mth decimal approximations for r and $\bar{\alpha}$, respectively. The inequality $H(r,\bar{\alpha},\bar{q})>0$ can be written in the form

$$\exists k\exists n\forall m>n\left(H(r_m,\bar{\alpha}_m,\bar{q})>\frac{1}{k+1}\right).$$

From this, we see that the conjunction of conditions (3) may be rewritten with a quantifier prefix of bounded length, with all quantifiers running over ω. Further steps in the derivation of a Σ^1_1-form are rather obvious.

Now we recall the perfect set theorem (here, we combine [2, Thm. IV.4.4 and Cor. IV.4.7]).

THEOREM 3.13. Let $\mathfrak{M}$ be a countable structure of a finite signature and P be a Σ^1_1-predicate on it. If P contains less than 2^ω elements then there exists $P'\in\mathbb{HYP}(\mathfrak{M})$ such that $P\subseteq P'$.

From this theorem we conclude that all codes of correct triples are hyperarithmetical, and moreover, are contained in some set of $\mathbb{HYP}(\omega)$. This implies that each equivalence class of our representation contains some element $\theta(\bar{\beta})$, with all $\bar{\beta}$ bounded by some fixed element of $\mathbb{HYP}(\omega)$.

It remains to realize that for each $S\in\mathbb{HYP}(\omega)$ consisting of codes of reals, there exists a hyperarithmetical subfield $\mathbb{R}'\subseteq\mathbb{R}$ such that $\mathbb{R}'\leqslant_1\mathbb{R}$.

LEMMA 3.14. The relation $<$, addition, multiplication, the operation of taking the opposite real, as well the partial inversion operation on hyperarithmetical real numbers identified with their codes defined above, are Σ-definable without parameters over $\mathbb{HYP}(\omega)$.

Proof. In fact, Σ-definitions of the above operations can be easily derived from the following relations:

$$\begin{aligned}
r<p &\Leftrightarrow (r\neq p)\wedge\forall n\in\omega(r_n\leqslant p_n);\\
r+p=q &\Leftrightarrow \forall k\in\omega\exists n_0\in\omega\forall n>n_0((k+1)\cdot|r_n+p_n-q_n|<1);\\
r\cdot p=q &\Leftrightarrow \forall k\in\omega\exists n_0\in\omega\forall n>n_0((k+1)\cdot|r_n\cdot p_n-q_n|<1);\\
r=\frac{1}{p} &\Leftrightarrow p\neq 0\wedge(p\times r=1).
\end{aligned}$$

Further, if we consider these relations as formulas with functional parameters we see that the reals in $\mathbb{HYP}(\omega)$ are closed under all of the operations mentioned. Indeed, let $p+q=r$ and $p,q\in\mathbb{HYP}(\omega)$. Using the above formulas, we obtain

$$r(m)=n\Leftrightarrow\exists f\,(p+q=f\wedge f(m)=n)\Leftrightarrow\forall f\,(p+q=f\rightarrow f(m)=n)\,.$$

Hence r is hyperarithmetical w.r.t. p and q. By [6, Cor. XXXI], r is hyperarithmetical. The other operations can be treated similarly. The lemma is proved.

Using Σ-collection, we see that the subfield $\mathbb{R}_0$ generated by S also belongs to $\mathbb{HYP}(\omega)$; so it has a hyperarithmetical isomorphic copy. Due to elimination of quantifiers in $\mathbb{R}$ in the signature σ_f, we may assume that each formula $\varphi(x, \bar{y})$ is quantifier free. This means that if $\mathbb{R} \models \exists x\varphi(x, \bar{a})$ then there exists x_0 which is algebraic over $\bar{a}$ and is such that $\mathbb{R} \models \varphi(x_0, \bar{a})$. Thus, if $\mathbb{R}' \leqslant \mathbb{R}$ is a real-closed field then $\mathbb{R}' \preccurlyeq \mathbb{R}$. It is well known that each effective computable ordered field has a computable real-closed extension (see [3, Thm. 2.3.6]). If we apply a relativized version of this result to $\mathbb{R}_0$ we obtain a subfield $\mathbb{R}' \leqslant \mathbb{R}$ which is isomorphic to some hyperarithmetical structure and satisfies the extra condition that each equivalence class in this representation contains at least one element of $\mathbb{HF}(\mathbb{R}')$ (since we have started from the set S). The theorem then follows immediately from Corollary 2.5.

In fact we have thus proved the following result.

THEOREM 3.15 (a version of the basis theorem). Suppose that E is an equivalence relation that is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters and whose number of equivalence classes is less than 2^ω. Then there exists a set $S \in \mathbb{HYP}(\omega)$ such that each equivalence class of E contains at least one element a with the property that $\mathtt{sp}\,(a) \subseteq S$. In particular, the number of equivalence classes of E is at most countable.

For other basis theorems, we refer the reader to [5, 7]. Note that Theorem 3.3 implies that the hyperarithmetical bound in Theorem 3.15 cannot be improved up to some prefixed hyperarithmetical level. Indeed, the construction plugged in the proof readily implies that the infinite branch $\boldsymbol{\xi}$ Turing reduces to $\mathbf{r}_0$.

3.3. Definability without parameters, with at most countable equivalence classes. In what follows, we denote the field of algebraic reals by $\mathbb{R}_{alg}$.

PROPOSITION 3.16. Let an at most countable set S be Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters. Then $S \subseteq \mathbb{HF}(\mathbb{R}_{alg})$.

Proof. Assume that S is definable by some Σ-formula $\varphi(x)$. Take an arbitrary element $\theta(\bar{p}) \in S$, where $\bar{p}$ are urelements and $\theta(\bar{y})$ is an s-term, with $\bar{y} = y_1, \ldots, y_n$. By Theorem 2.2, $\varphi(\theta(\bar{y}))$ is equivalent to an infinite enumerable disjunction $\bigvee\limits_{i<\omega} \psi_i(\bar{y})$ in which all ψ_i are quantifier-free formulas of the signature σ_f. Projections of this set onto the coordinate planes are defined by infinite formulas of the form

$$\exists y_1 \ldots y_{j-1} y_j \ldots y_n \bigvee_{i<\omega} \psi_i(\bar{y}), \quad j = 1, \ldots, n,$$

which in turn are equivalent to appropriate formulas $\bigvee\limits_{i<\omega} \exists y_1 \ldots y_{j-1} y_j \ldots y_n \psi_i(\bar{y})$ of the signature σ_f.

We reduce all formulas in this disjunction to equivalent quantifier-free ones. The outcome is a disjunction of quantifier-free formulas of the signature σ_f, each of which defines an at most countable set. Reducing each of these to a disjunctive normal form, we arrive at a disjunction of formulas of the form

$$\begin{gathered} f_1(x) = g_1(x) \wedge \ldots \wedge f_k(x) = g_k(x) \\ \wedge f_1'(x) \neq g_1'(x) \wedge \ldots \wedge f_l'(x) \neq g_l'(x) \\ \wedge f_1''(x) > g_1''(x) \wedge \ldots \wedge f_m''(x) > g_m''(x), \end{gathered}$$

where all f_i, g_i, f_i', g_i', f_i'', g_i'' are polynomials over $\mathbb{Q}$. We may therefore assume that all formulas ψ_i are of just this kind. Each such formula defines an at most countable set. The form of this formula shows that it defines a finite set of algebraic reals in that event. In particular, this means that all elements of $\bar{p}$ are algebraic, and so $\theta(\bar{p}) \in \mathbb{HF}(\mathbb{R}_{alg})$. The theorem is completed.

This statement, together with Theorem 1.4, Corollary 2.5, the existence of a computable isomorphic copy for $\mathbb{R}_{alg}$, and the property $\mathbb{R}_{alg} \preccurlyeq \mathbb{R}$, implies the following:

THEOREM 3.17. Let $\mathfrak{M}$ be an at most countable structure of a finite signature. Then the statements below are equivalent:

(1) $\mathfrak{M}$ is Σ-definable without parameters over $\mathbb{HF}(\mathbb{R})$ so that all equivalence classes in this presentation are at most countable;

(2) $\mathfrak{M}$ has an isomorphic computable copy.

THEOREM 3.18. Suppose that a (not necessarily countable) model $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters. Then there exists a computable model $\mathfrak{M}^*$ such that $\mathfrak{M}^* \leqslant_1 \mathfrak{M}$.

Proof. We claim that the desired model is one that is definable by the same formulas over $\mathbb{HF}(\mathbb{R}_{alg})$. We identify $\mathfrak{M}$ with its isomorphic presentation in the form

$$\langle W(x)^{\mathbb{HF}(\mathbb{R})}[x]; P_1^+(\bar{x})^{\mathbb{HF}(\mathbb{R})}[\bar{x}], \ldots, P_k^+(\bar{x})^{\mathbb{HF}(\mathbb{R})}[\bar{x}]\rangle/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R})}[x,y]},$$

such as in the basic definition. By the property $\mathbb{R}_{alg} \preccurlyeq \mathbb{R}$ and Theorem 1.4, we see that the same formulas define some computable model $\mathfrak{M}^*$ over $\mathbb{HF}(\mathbb{R}_{alg})$.

Our present goal is to show that $\mathfrak{M}^* \leqslant_1 \mathfrak{M}$. Let

$$\mathfrak{M} \models \exists x_1, \ldots, x_n \varphi\big(x_1, \ldots, x_n,\, \theta_1(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R})}[x,y]}, \ldots, \theta_k(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R})}[x,y]}\big),$$

where $\theta_1(\bar{p}), \ldots, \theta_n(\bar{p}) \in W(x)^{\mathbb{HF}(\mathbb{R}_{alg})}[x]$ and φ is a quantifier-free formula. Replace in φ all positive occurrences of subformulas $\mathbf{P}_i(\bar{x})$ with corresponding formulas $P_i^+(\bar{x})$, and all negative occurrences with $P_i^-(\bar{x})$; replace all positive occurrences of equalities $A = B$ with $E^+(A, B)$, and all negative occurrences with $E^-(A, B)$. The result is a Σ-formula

$$\varphi^*\,(x_1, \ldots, x_n,\, y_1, \ldots, y_k)$$

satisfying the conditions

$$\mathbb{HF}(\mathbb{R}) \models \exists x_1, \ldots, x_n \left(\bigwedge_{i=1}^{n} W(x_1) \wedge \varphi^*\,(x_1, \ldots, x_n,\, \theta_1(\bar{p}), \ldots, \theta_k(\bar{p}))\right).$$

Inasmuch as the last expression is a Σ-expression, the fact that $\mathbb{R}_{alg} \preccurlyeq \mathbb{R}$, along with Theorem 1.4, implies

$$\mathbb{HF}(\mathbb{R}_{alg}) \models \exists x_1, \ldots, x_n \left(\bigwedge_{i=1}^{n} W(x_1) \wedge \varphi^*\,(x_1, \ldots, x_n,\, \theta_1(\bar{p}), \ldots, \theta_k(\bar{p}))\right),$$

which in turn entails

$$\mathfrak{M}^* \models \exists x_1, \ldots, x_n \varphi\left(x_1, \ldots, x_n,\, \theta_1(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R}_{alg})}[x,y]}, \ldots, \theta_k(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R}_{alg})}[x,y]}\right).$$

It remains to note that the mapping

$$\theta(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R}_{alg})}[x,y]} \mapsto \theta(\bar{p})/_{E^+(x,y)^{\mathbb{HF}(\mathbb{R})}[x,y]}$$

is an isomorphic embedding. That the basic set of $\mathfrak{M}^*$ is nonempty follows from $\mathbb{HF}(\mathbb{R}) \models \exists x W(x)$, which yields $\mathbb{HF}(\mathbb{R}_{alg}) \models \exists x W(x)$ in view of $\mathbb{HF}(\mathbb{R}_{alg}) \leqslant_\Sigma \mathbb{HF}(\mathbb{R})$. The theorem is completed.

COROLLARY 3.19. Assume that a model $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters and that one of the following conditions is satisfied:

(1) $\mathfrak{M}$ is generated by its signature constants;

(2) $\mathfrak{M}$ has no proper submodels;

(3) each 1-submodel of $\mathfrak{M}$ is isomorphic to $\mathfrak{M}$.

Then $\mathfrak{M}$ is isomorphic to a computable model.

4. Σ-DEFINABILITY OVER $\mathbb{HF}(\mathbb{C})$

THEOREM 4.1. For an arbitrary countable model $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{C})$;

(2) $\mathfrak{M}$ is isomorphic to a computable model.

Proof. We prove the less obvious part of the theorem. Assume that a countable model $\mathfrak{M}$ is Σ-definable over $\mathbb{HF}(\mathbb{C})$. Fix an arbitrary transcendency basis X of $\mathbb{C}$ over $\mathbb{Q}$, i.e., $\mathbb{C} = \overline{\mathbb{Q}(X)}$, where $\overline{}$ denotes an algebraic closure. Let X_0 be any countable subset of X.

LEMMA 4.2. $\mathbb{HF}(\overline{\mathbb{Q}(X_0)}) \preccurlyeq \mathbb{HF}(\mathbb{C})$.

Proof. It suffices to show that if $\varphi(x,\bar{y})$ is an arbitrary formula and $\mathbb{HF}(\mathbb{C}) \models \exists x\, \varphi(x,\bar{p})$, where $\bar{p} \in \mathbb{HF}(\overline{\mathbb{Q}(X_0)})$, then $\mathbb{HF}(\mathbb{C}) \models \varphi(x,\bar{p})$ for an appropriate $x \in \mathbb{HF}(\overline{\mathbb{Q}(X_0)})$. Fix some $x_0 \in \mathbb{HF}(\mathbb{C})$ so that $\mathbb{HF}(\mathbb{C}) \models \varphi(x_0,\bar{p})$. Let A be a finite subset of X_0 for which $\mathtt{sp}\,(\bar{p}) \subseteq \overline{\mathbb{Q}(A)}$. Let B be a finite subset of X such that $\mathtt{sp}\,(x_0) \subseteq \overline{\mathbb{Q}(B)}$. Take any permutation θ of X which is the identity on the elements of A satisfying the condition $\theta(B \setminus A) \subseteq X_0$. The permutation θ can be extended to an automorphism of $\mathbb{C}$ that fixes all elements of $\mathtt{sp}\,(\bar{p})$; this automorphism, in turn, is extendable to some automorphism θ^* of the structure $\mathbb{HF}(\mathbb{C})$ with the property that $\theta^*(\bar{p}) = \bar{p}$. We have $\mathbb{HF}(\mathbb{C}) \models \varphi(\theta^*(x_0),\bar{p})$ and $\theta^*(x_0) \in \mathbb{HF}(\overline{\mathbb{Q}(X_0)})$, which proves the lemma.

We come back to the proof of the theorem. Since $\mathfrak{M}$ is countable, we can take a countable subset $X_0 \subseteq X$ so that each equivalence class of the Σ-definition of $\mathfrak{M}$ contains a representative a such that $\mathtt{sp}\,(a) \subseteq X_0$ and the supports of all the parameters used are contained in X_0. The theorem then follows from Lemma 4.2, Theorem 1.4, Corollary 2.5, and the existence of a computable copy for $\overline{\mathbb{Q}(X_0)}$.

Similarly to the above, we can show that an arbitrary countable model $\mathfrak{M}$ is elementary definable over $\mathbb{HF}(\mathbb{C})$ iff it has an arithmetical copy. This implies that the field $\mathbb{R}$ of reals is not definable over $\mathbb{HF}(\mathbb{C})$, even with parameters. Indeed, if $\mathbb{R}$ were definable over $\mathbb{HF}(\mathbb{C})$, possibly with parameters, then this presentation could be transformed into a presentation of some nonarithmetical model, such as in the proof of Theorem 3.3, with the same parameters. We have arrived at a contradiction with the remark made above.

5. Σ-DEFINABILITY OVER $\mathbb{HF}(\mathbb{H})$

Here we prove that the property of being Σ-definable over $\mathbb{HF}(\mathbb{H})$ is equivalent to being Σ-definable over $\mathbb{HF}(\mathbb{R})$. Roughly speaking, an arbitrary model has a Σ-presentation over $\mathbb{HF}(\mathbb{R})$ with some family of the two basic properties considered in the paper (the property that classes of a presentation are at most countable and the existence or nonexistence of parameters) iff it has a presentation with the same properties over $\mathbb{HF}(\mathbb{R})$. More exactly, we have

THEOREM 5.1. (1) Any countable model is Σ-definable over $\mathbb{HF}(\mathbb{H})$ with a trivial equivalence and a single parameter.

(2) Any countable model is Σ-definable over $\mathbb{HF}(\mathbb{H})$ with an equivalence whose classes are all at most countable and without parameters iff it has a computable presentation.

(3) Any countable model is Σ-definable over $\mathbb{HF}(\mathbb{H})$ without parameters iff it is Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters.

Proof. The theorem derives from the following two remarks.

First, the structure $\mathbb{H}$ is obviously Σ-definable over $\mathbb{HF}(\mathbb{R})$ without parameters and with trivial equivalence. Therefore, we can transform each Σ-presentation of an arbitrary model $\mathfrak{M}$ over $\mathbb{HF}(\mathbb{H})$ into its

Σ-presentation over $\mathbb{HF}(\mathbb{R})$ while preserving the cardinalities of the corresponding classes in the presentation; moreover, if the initial presentation had no parameters then the resulting presentation would have none either. This immediately implies ($\Rightarrow$) in item (3). Using Theorem 3.17, we derive (2).

Second, $\mathbb{R}$ is distinguished in $\mathbb{H}$ by some $\exists$- and $\forall$-formulas simultaneously, and the ordering $<$ is clearly Δ-definable over $\mathbb{R}$. Indeed, consider arbitrary elements $a, b, c \in \mathbb{H}$ satisfying the same relations as the basis imaginary units i, j, k, and namely, the quantifier-free formula

$$\begin{aligned}\eta(x, y, x) \stackrel{df}{=} & (x \neq y \neq z \neq x) \wedge (x^2 = y^2 = z^2 = -1)\\ & \wedge(xy = z) \wedge (yz = x) \wedge (zx = y)\\ & \wedge(yx = -z) \wedge (zy = -x) \wedge (xz = -y).\end{aligned}$$

We verify that 1, a, b, and c are linearly independent over $\mathbb{R}$ in that event. Suppose $\alpha a + \beta b + \gamma c + \delta = 0$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Multiplying this equality by a from the left and from the right and adding the results, we obtain $-2\alpha + 2\delta a = 0$. If $\delta \neq 0$ then we derive $-1 = a^2 = \left(\frac{\alpha}{\delta}\right)^2 > 0$, which is a contradictory statement. Thus, $\delta = \alpha = 0$. Doing the same with b and c, we see that all the remaining coefficients are equal to zero.

This means that each x in $\mathbb{H}$ is uniquely representable as $x = \alpha a + \beta b + \gamma c + \delta$. Furthermore, $xa + ax = -2\alpha + 2a\delta$. Similarly, $xb + bx = -2\beta + 2b\delta$. Now, if $xa + ax = yb + by$ for some $y = \alpha' a + \beta' b + \gamma' c + \delta'$ then $-2\alpha + 2a\delta = -2\beta' + 2b\delta'$. Since 1, a, b, and c are linearly independent, we obtain $\delta = \delta' = 0$, i.e., $xa + ax \in \mathbb{R}$. On the other hand, it is obvious that each real number is representable as $xa + ax = yb + by$ for appropriate $x, y \in \mathbb{H}$. Thus,

$$t \in \mathbb{R} \Leftrightarrow \exists u \exists v \exists w \exists x \exists y\, (\eta(u, v, w) \wedge ((t = xu + ux) \wedge (t = yv + vy)))\,.$$

We verify that

$$t \notin \mathbb{R} \Leftrightarrow \exists u \exists v \exists w\, (\eta(u, v, w) \wedge (ut \neq tu \vee vt \neq tv \vee wt \neq tw))\,.$$

First, we consider ($\Rightarrow$). Take standard imaginary units $i, j, k \in \mathbb{H}$ to be u, v, w. Assume that $t = \alpha + i \cdot \beta + j \cdot \gamma + k \cdot \delta$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The condition $t \notin \mathbb{R}$ implies that at least one of the elements β, γ, or δ differs from 0. To be specific, let $\beta \neq 0$. Then $tj - jt = k\beta - i\delta \neq 0$.

Next, we handle the reverse implication ($\Leftarrow$). Fix the required elements u, v, and w. Similarly to the above, we can prove that these elements together with 1 form a basis $\mathbb{H}$ over $\mathbb{R}$. Assume for definiteness that $ut \neq tu$. Then the condition that $t \in \mathbb{R}$ fails, since all elements of $\mathbb{R}$ commute with all elements of $\mathbb{H}$.

The last remark easily implies that each model that is Σ-presentable with parameters over $\mathbb{HF}(\mathbb{R})$ is definable over $\mathbb{HF}(\mathbb{H})$ with the same parameters and the same equivalence, proving (1).

We argue for ($\Leftarrow$) in (3). Suppose that some structure $\mathfrak{M}$ is definable over $\mathbb{HF}(\mathbb{R})$. Replacing in the formulas of this definition all occurrences of $U(x)$ with a Δ-formula defining $\mathbb{R}$ in $\mathbb{H}$ and relativizing all unbounded existential quantifiers and formulas to the Δ-set $\{x \mid \mathtt{sp}(x) \subseteq \mathbb{R}\} = \mathbb{HF}(\mathbb{R})$, we arrive at a Σ-presentation of $\mathfrak{M}$ without parameters, now over $\mathbb{HF}(\mathbb{H})$. The theorem is completed.

The results given in Sec. 4 were obtained jointly by both authors. All other results are due to A. S. Morozov.

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