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The Uniformity Principle for Σ-definability

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Abstract

This article is an extended version of the paper published in Korovina and Kudinov (2007, *Lecture Notes in Computer Science*, Vol. 4497, pp. 416–425). The main goal of this research is to develop logical tools and techniques for effective reasoning about continuous data based on Σ-definability. In this article we invent the Uniformity Principleand prove it for Σ-definability over the real numbers extended by open predicates. Using the Uniformity Principle, we investigate different approaches to enrich the language of Σ-formulas in such a way that simplifies reasoning about computable continuous data without enlarging the class of Σ-definable sets. In order to do reasoning about computability of certain continuous data we have to pick up an appropriate language of a structure representing these continuous data. We formulate several major conditions how to do that in a right direction. We also employ the Uniformity Principleto argue that our logical approach is a good way for formalization of computable continuous data in logical terms.

Keywords: Σ-Definability, Uniformity Principle, effective reasoning about continuous data, continuous data types, computable analysis.

1 Introduction

This work is an significant impact to the logical approach to computability over continuous data based on arguments from definability theory and developed in the papers [5–10]. This approach is based on representations of continuous data by suitable structures without the equality test and Σ-definability in extensions of the structures by hereditarily finite sets. In order to logically characterize computable continuous data we have proposed the notion of majorant-computability. One of the main features of the notion of majorant-computability is that on the one side it is independent from concrete representations of the elements of structures on the other side it is flexible, i.e. we can change the language of Σ-formulas to express appropriate computability properties. In this article we investigate different approaches to enrich the language of Σ-formulas in such way that simplifies reasoning about computable continuous data without enlarging the class of Σ-definable sets. The basic idea behind these approaches is the Uniformity Principle for Σ-definability over the real numbers extended by open sets. Informally the Uniformity Principle says that global quantifiers bounded by compact intervals could be effectively reduced to local ones. In the case of Σ_K-formulas over the reals without equality the Uniformity Principle allows to eliminate both existential and universal quantifiers bounded by computable compact sets.

In order to do reasoning about computability of certain continuous data we have to pick up an appropriate language of a structure representing these continuous data. We formulate several major necessary conditions how to do that in a right direction. One of them is topological, which says that computable functions should be continuous. This condition provides correct approximating

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computation of continuous data. Another one is logical, which says that $Th_{\exists}(\mathcal{M})$ should be computably enumerable. This condition provides tools for effective reasoning about computable continuous data based on Σ-definability. We illustrate our arguments by several examples.

The structure of this article is as follows.

In Section 2 we recall basic notions and introduce the language of Σ_K-formulas that is an extension of the language of Σ-formulas by universal and existential quantifiers bounded by computable compact sets.

In Section 3 we prove the Uniformity Principle for Σ-definability over the real numbers extended by open sets. To simplify reasoning about computability of continuous data we propose several ways to enrich the Σ-language. We show that rational numbers, polynomials and computable functions as well can be used in Σ-formulas without enlarging the class of Σ-definable sets. In other words, we can extend the language of Σ-formulas by computable functions, e.g. $\cos, \sin, \exp$ and Uniformity Principle allows eliminate them later. Then we employ the Uniformity Principle to prove that the language of Σ_K-formulas admits elimination of both existential and universal quantifiers bounded by computable compact sets. It is also illustrated how the language of Σ_K-formulas can be used to make reasoning about computable subsets of $\mathbb{R}^n$ in an elegant way.

In Section 4 we show how the Uniformity Principle can be used to make reasoning about topological properties of majorant-computable functionals of the type $F: A \to \mathbb{R}$. In order to do that we consider an arbitrary structure $\mathcal{A} = \langle A, \sigma_P, \neq \rangle$, where A contains more than one element, and σ_P is a finite set of basic predicates. For the structure $\mathcal{A}$, we introduce a topology, called $\tau_{\Sigma}^{\mathcal{A}}$, with the base consisting of the subsets defined by existential formulas. Using the Uniformity Principle we prove that every majorant-computable functional $F: A \to \mathbb{R}$ is continuous. In the case of $A = C[0,1]$ we show how to pick up an appropriate language for the structure of $C[0,1]$ in such a way that majorant-computability of functionals $F: C^n[0,1] \times \mathbb{R}^m \to \mathbb{R}$ coincides with computability in the sense of computable analysis and the theory $Th_{\exists}(C[0,1] \cup \mathbb{R})$ is computably enumerable.

2 Basic definitions and notions

In this article we consider the ordered structure of the real numbers in *finite predicate languages without equality*, $\langle \mathbb{R}, \sigma_P, < \rangle = \langle \mathbb{R}, \sigma_{\mathbb{R}} \rangle$, where σ_P satisfies the following assumption.

Assumption 1
The set σ_P is a finite set of open predicates, i.e. interpreted over the reals as open sets.

We extend the real numbers by the set of hereditarily finite sets HF($\mathbb{R}$), which is rich enough for information to be coded and stored. We construct the set of hereditarily finite sets, HF($\mathbb{R}$) over the reals, as follows [1, 3]:

1. $\mathrm{HF}_0(\mathbb{R}) \rightleftharpoons \mathbb{R}$,
2. $\mathrm{H}F_{n+1}(\mathbb{R}) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_n(\mathbb{R})) \cup \mathrm{HF}_n(\mathbb{R})$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of B.
3. $\mathrm{HF}(\mathbb{R}) = \bigcup_{m \in \omega} \mathrm{HF}_m(\mathbb{R})$.

We define **HF**($\mathbb{R}$) as the following model: $\mathbf{HF}(\mathbb{R}) \rightleftharpoons \big(\mathrm{HF}(\mathbb{R}), U, \sigma_{\mathbb{R}}, \emptyset, \in\big) \rightleftharpoons \big(\mathrm{HF}(\mathbb{R}), \sigma\big)$, where the constant $\emptyset$ stands for the empty set and the binary predicate symbol $\in$ has the set-theoretic interpretation. We also add a primary predicate symbol U naming the set of urelements (the real numbers). The natural numbers 0, 1,... are identified with the (finite) ordinals in HF($\mathbb{R}$) i.e. $\emptyset$, $\{\emptyset, \{\emptyset\}\}, \ldots$, so in particular, $n+1 = n \cup \{n\}$ and the set ω is a subset of HF($\mathbb{R}$).

Remark 2
For better understanding of the Uniformity Principle we slightly modify the definitions of atomic, Δ_0 and Σ-formulas from [5]. It is worth noting that these definitions do not contain negation as a logical operation and give the same class of formulas as in [5].

The atomic formulas include $U(x)$, $\neg U(x)$, $x < y$, $x \in s$, $x \notin s$ where s ranges over sets, and also, for every $Q_i \in \sigma_P$ with the arity n_i, $Q_i(x_1, \ldots, x_{n_i})$, which has the following interpretation:

$$\mathbf{HF}(\mathbb{R}) \models Q_i(x_1, \ldots, x_{n_i}) \text{ if and only if}$$
$$\mathbb{R} \models Q_i(x_1, \ldots, x_{n_i}) \text{ and, for every } 1 \leq j \leq n_i,\ x_j \in \mathbb{R}.$$

The set of Δ_0-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee$, bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$, where $(\exists x \in y)\ \Psi$ means the same as $\exists x (x \in y \wedge \Psi)$ and $(\forall x \in y)\ \Psi$ as $\forall x (x \in y \rightarrow \Psi)$ where y ranges over sets.

The set of Σ-*formulas* is the closure of the set of Δ_0-formulas under $\wedge, \vee$, $(\exists x \in y)$, $(\forall x \in y)$ and $\exists$, where y ranges over sets.

Remark 3
It is worth noting that all predicates $Q_i \in \sigma_P$ and $<$ occur only positively in Σ-formulas. Hence in Σ-formulas we do not allow equality on the urelements (elements form $\mathbb{R}$).

Remark 4
Through this article we consider also existential formulas in the language $\sigma_{\mathbb{R}}$ with positive occurrences of predicate symbols from $\sigma_{\mathbb{R}}$ without any further references to this restriction.

The set of Σ_K-*formulas* is the closure of the set of Σ-formulas under $\wedge, \vee$, $\exists$, $\exists x \in K$ and $\forall x \in K$, where K is a computable compact subset of $\mathbb{R}^n$.

We define Π-*formulas* as negations of Σ-formulas.

Definition 1
1. A relation $B \subseteq \mathrm{HF}(\mathbb{R})^n$ is Σ-**definable**, if there exists a Σ-formula Φ such that $\mathbf{x} \in B \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \Phi(\mathbf{x})$.

In a similar way, we define the notions of Σ_K-*definable and* Π-*definable sets*. The following theorem reveals algorithmic properties of Σ-formulas over $\mathbf{HF}(\mathbb{R})$.

Theorem 1 (Semantic characterization of Σ-definability)
A set $B \subseteq \mathbb{R}^n$ is Σ-definable if and only if there exists an effective sequence of existential formulas in the language $\sigma_{\mathbb{R}}$, $\{\Phi_s(x_1, \ldots, x_n)\}_{s \in \omega}$, such that

$$(x_1, \ldots, x_n) \in B \leftrightarrow \mathbb{R} \models \bigvee_{s \in \omega} \Phi_s(x_1, \ldots, x_n).$$

The proof of this theorem is based on Gandy's theorem for abstract structures without equality [8] and the technique developed in [7]. It is worth noting that both of the directions of this characterization are important. The right direction gives us an effective procedure that generates existential formulas approximating Σ-relations. The converse direction provides tools for descriptions of the results of effective infinite approximating processes by finite formulas.

3 Uniformity Principle and its applications to effective reasoning about continuous data

In this section we prove the Uniformity Principle for Σ-definability over the reals without equality and show several application of the Uniformity Principle.

It is worth noting that all propositions below do not hold over the reals with equality. Indeed, validity of these propositions over the reals with equality leads to definability of π in $Th(\mathrm{IR})$, which does not hold by quantifier elimination.

3.1 Uniformity Principle for Σ-definability

Now we assume $\sigma_P = \{\mathcal{M}^*_E, \mathcal{M}^*_H, \mathcal{P}^+_E, \mathcal{P}^+_H\}, \sigma = \sigma_P \cup \{<, \emptyset, \in\}$, where $\mathcal{M}^*_E, \mathcal{M}^*_H$ are interpreted as an open epigraph and an open hypograph of multiplication, respectively, and $\mathcal{P}^+_E, \mathcal{P}^+_H$ are interpreted as an open epigraph and an open hypograph of addition, respectively. In sequel we will use the following notations: $x \cdot y < z$ for $\mathcal{M}^*_E(x,y,z)$, $x \cdot y > z$ for $\mathcal{M}^*_H(x,y,z)$, $x+y<z$ for $\mathcal{P}^+_E(x,y,z)$ and $x+y>z$ for $\mathcal{P}^+_H(x,y,z)$. It is worth noting that in Σ-formulas we can also use the expressions $x>0$, $y<1$ (and similar) as notations of the formulas $\exists y \big(x > y \cdot y\big)$, $\exists z > 0 \big(y \cdot z < z\big)$, respectively. In sequel we assume that $||\cdot||$ is the standard norm on IR^n, $[a,b]$ denotes a closed interval, and $\bar{B}(x,\epsilon)$ denotes a closed ball with a centre x and a radius ϵ. The following property of Σ-definable sets over the reals is a straightforward corollary of Theorem 1.

Corollary 1
A set $B \subseteq \mathrm{IR}^n$ is Σ-definable if and only if B is c.e. open.

One of the main goals of this section is to prove that the language of Σ_K-formulas admits elimination of universal quantifiers bounded by computable compact sets. In other words, we are going to show that if we have a formula with quantifier alternations where universal quantifiers are bounded by computable compact sets then we can eliminate all universal quantifiers obtaining a Σ-formula equivalent to the initial one. In order to do that, first we prove the Uniformity Principle for Σ-definability. We extend the given language σ by new predicate symbols P and P'_λ with the following meaning

- P defines an open subset of IR^n;
- $P'_\lambda(a,b,x_2,\ldots,x_n) \leftrightarrow \forall x_1 \in [a,b] P(x_{1\lambda},\ldots,x_{n\lambda})$, where $\lambda : \{1,\ldots,n\} \to \{1,\ldots,n\}$.

The following lemma shows that both languages $\sigma \cup \{P\}$ and $\sigma \cup \{P'_\lambda | \lambda : \{1,\ldots,n\} \to \{1,\ldots,n\}\}$ are subject to Assumption 1.

Lemma 1
If P defines an open subset of IR^n then P'_λ defines an open subset of IR^m, where m depends on λ.

Proof. We give the main idea of the proof for $\lambda = id_{\{1,\ldots,n\}}$. It is sufficient to show that for every closed interval $[c,d]$ the set

$$B_{c,d} = \Big\{\big(a,b,x_2,\ldots,x_n\big) \in [c,d]^{n+1} | [a,b] = \emptyset \vee [a,b] \subset (c,d) \wedge P'_\lambda(a,b,x_2,\ldots,x_n)\Big\}$$

is open. Let us consider $A_{c,d}$, the complement of $B_{c,d}$ in $[c,d]$, which is defined as follows.

$$A_{c,d} = \Big\{ \big(a,b,x_2,\ldots,x_n\big) \in [c,d]^{n+1} \mid [a,b] \subseteq [c,d] \wedge \big(a = c \vee b = d \vee \exists x_1 \in [a,b] \neg P(x_1,\ldots,x_n)\big) \Big\}$$

Since $A_{c,d}$ is a projection of the compact set

$$\Big\{ \big(a,b,x_2,\ldots,x_n\big) \in [c,d]^{n+1} \mid [a,b] \subseteq [c,d] \wedge \big(a = c \vee b = d \vee \neg P(x_1,\ldots,x_n)\big) \Big\},$$

$A_{c,d}$ is compact. So, P'_λ is open as the union of all open sets $B_{c,d}$, where $c \in \mathrm{IR}$ and $d \in \mathrm{IR}$. ■

Let us consider particularly interesting corollaries of Lemma 1. If $f \in C(\mathrm{IR})$, then the sets $P_f^- = \{(x,c) \mid f(x) > c\}$ and $P_f^+ = \{(x,c) \mid f(x) < c\}$ are open. Choosing λ to be identical on $\{1,2\}$ we get the following corollary.

COROLLARY 2
For every $f \in C(\mathrm{IR})$, the sets

$$E_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]} < z \text{ and } H_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]} > z \text{ are open.}$$

If $f \in C([0,1])$, then applying Corollary 1 to the function

$$g(x) = \begin{cases} f(0), & \text{if } x < 0 \\ f(x), & \text{if } x \in [0,1] \\ f(1), & \text{if } x > 1 \end{cases}$$

we get straightforwardly the following.

COROLLARY 3
For $f \in C([0,1])$, the sets $E_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]\cap[0,1]} < z$ and $H_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]\cap[0,1]} > z$ are open.

THEOREM 2 (Uniformity Principle)
For every Σ-formula φ in the language $\sigma \cup \{P\}$ there exists Σ-formula ψ in the language $\sigma \cup \{P'_\lambda \mid \lambda : \{1,\ldots,n\} \to \{1,\ldots,n\}\}$ such that

$$\mathbf{HF}(\mathrm{IR}) \models \forall x \in [a,b] \varphi(x,x_2,\ldots,x_n) \text{ iff } \mathbf{HF}(\mathrm{IR}) \models \psi(a,b,x_2,\ldots,x_n),$$

where free variables range over IR.

PROOF. First we consider the case of ∃-formulas in the language $\sigma_{\mathrm{IR}} \cup \{P\}$. Using induction on the structure of a ∃-formula φ, we show how to obtain a required formula ψ. Then, based on Theorem 1 we construct a required formula ψ for an arbitrary Σ-formula in the language $\sigma \cup \{P\}$.

Atomic case.

(1) If $\varphi(x_1,\ldots,x_n) \rightleftharpoons P(x_{1\lambda},\ldots,x_{n\lambda})$ then $\psi \rightleftharpoons P'_\lambda$.
(2) If φ does not contain the predicate symbol P, we have a finite number of subcases. We consider non-trivial ones.

(2.1) If $\varphi(x,z) \rightleftharpoons x \cdot x > z$ then

$$\psi(a,b,z) \rightleftharpoons z < 0 \vee a > b \vee \big(a > 0 \wedge b > 0 \wedge a \cdot a > z\big) \vee \big(a < 0 \wedge b < 0 \wedge b \cdot b > z\big).$$

(2.2) If $\varphi(x,z) \rightleftharpoons x \cdot x < z$ then $\psi(a,b,z) \rightleftharpoons a > b \vee \big(a \cdot a < z \wedge b \cdot b < z\big)$.

(2.3) If $\varphi(x,y) \rightleftharpoons x \cdot y > x$ then

$$\psi(a,b,z) \rightleftharpoons a > b \vee \big(a > 0 \wedge b > 0 \wedge y > 1\big) \vee \big(a < 0 \wedge b < 0 \wedge y < 1\big).$$

(2.4) If $\varphi(x) \rightleftharpoons x \cdot x > x$ then $\psi(a,b) \rightleftharpoons a > b \vee \big(a > 1 \wedge b > 1\big) \vee \big(a < 0 \wedge b < 0\big)$.

(2.5) If $y \cdot z < x$ then $\psi(a,b,y,z) \rightleftharpoons y \cdot z < a \vee b < a$. Other atomic subcases can be considered by analogy.

Conjunction.
If $\varphi \rightleftharpoons \varphi_1 \wedge \varphi_2$ and ψ_1, ψ_2 are already constructed for φ_1, φ_2 then $\psi \rightleftharpoons \psi_1 \wedge \psi_1$.

Disjunction.
Suppose $\varphi \rightleftharpoons \varphi_1 \vee \varphi_2$ and ψ_1, ψ_2 are already constructed. Since $[a,b]$ is compact, validity of the formula $\forall x \in [a,b]\big(\varphi_1 \vee \varphi_2\big)$ is equivalent to existence of a finite family of open intervals $\{(\alpha_i, \beta_i)\}_{i=1,\dots,r+s}$ such that $[a,b] \subseteq \bigcup_{i=1}^{r}(\alpha_i, \beta_i)$, for $i = 1,\dots,r$ $\mathrm{IR} \models \varphi_1$ and for $i = r+1,\dots,s$ $\mathrm{IR} \models \varphi_2$. Since φ_1 and φ_2 define open sets, this is equivalent to existence of a finite family of closed intervals $\{[\alpha_i', \beta_i']\}_{i=1,\dots,r+s}$ such that $[a,b] \subseteq \bigcup_{i=1}^{r}[\alpha_i', \beta_i']$, for $i = 1,\dots,r$ $\mathrm{IR} \models \varphi_1$ and for $i = r+1,\dots,s$ $\mathrm{IR} \models \varphi_2$. It is represented by the following formula.

$$\bigvee_{r \in \omega} \bigvee_{r \in \omega} \exists \alpha_1' \dots \exists \alpha_{s+1}' \exists \beta_1' \dots \exists \beta_{s+1}' \left(\bigwedge_{i=1}^{r} \forall x \in [\alpha_i', \beta_i'] \varphi_1 \wedge \bigwedge_{j=r+1}^{s} \forall x \in [\alpha_j', \beta_j'] \varphi_2 \right).$$

By induction hypothesis and Theorem 1, this formula is equivalent to a Σ-formula ψ in the language $\sigma \cup \{P_\lambda' \,|\, \lambda : \{1,\dots,n\} \to \{1,\dots,n\}\}$.

Existential case.
Suppose $\varphi \rightleftharpoons \exists z \varphi_1(z, x_1, \dots, x_n)$. As $[a,b]$ is compact and

$$\{\{x_1 \,|\, \mathrm{IR} \models \varphi_1(z, x_1, \dots, x_n)\}\}_{z \in \mathrm{IR}} = \{V_z\}_{z \in \mathrm{IR}}$$

is its cover, there exists a finite set $J = \{z_1,\dots,z_s\} \subset \mathrm{IR}$ such that $[a,b] \subseteq \bigcup_{z \in J} V_z$. So, validity of the formula $\forall x_1 \in [a,b] \exists z \varphi_1(z, x_1, \dots, x_n)$ is equivalent to existence of the finite set $J = \{z_1,\dots,z_s\}$ such that

$$\mathrm{IR} \models \forall x_1 \in [a,b] \exists z \varphi_1(z, x_1, \dots, x_n) \leftrightarrow \mathrm{IR} \models \forall x_1 \in [a,b] \varphi^s(z_1, \dots, z_s, x_1, \dots, x_n),$$

where $\varphi^s(z_1,\dots,z_s,x_1,\dots,x_n) \rightleftharpoons \varphi_1(z_1,x_1,\dots,x_n) \vee \dots \vee \varphi_1(z_s,x_1,\dots,x_n)$. By induction hypotheses, for every $J = \{z_1,\dots,z_s\}$ there exists a Σ-formula $\psi^s(z_1,\dots,z_s,a,b,x_2,\dots,x_n)$ in the language $\sigma \cup \{P_\lambda' \,|\, \lambda : \{1,\dots,n\} \to \{1,\dots,n\}\}$, which is equivalent to $\forall x_1 \in [a,b] \varphi^s(z_1,\dots,z_s,x_1,\dots,x_n)$. Finally,

$$\begin{aligned} &\mathrm{IR} \models \forall x_1 \in [a,b] \exists z \varphi_1(z, x_1, \dots, x_n) \leftrightarrow \\ &\mathbf{HF}(\mathrm{IR}) \models \textstyle\bigvee_{s \in \omega} \exists z_1 \dots \exists z_s \big(\psi^s(z_1,\dots,z_s,a,b,x_2,\dots,x_n)\big). \end{aligned}$$

A required Σ-formula ψ can be constructed using Theorem 1.

Now we are ready to construct a required formula ψ for a Σ-formula. Suppose φ is a Σ-formula. By Lemma 1 and Theorem 1, there exists an effective sequence of existential formulas $\{\varphi_i\}_{i\in\omega}$ such that $\mathbf{HF}(\mathrm{I\!R})\models\varphi\leftrightarrow\mathbf{HF}(\mathrm{I\!R})\models\bigvee_{i\in\omega}\varphi_i$. As $[a,b]$ is compact and $\{\{x_1|\mathrm{I\!R}\models\varphi_i(x_1,\dots,x_n)\}\}_{i\in\omega}=\{U_i\}_{i\in\omega}$ is its cover, there exist $k\in\omega$ and a finite family $\{U_i\}_{i\leq k}$ such that $[a,b]\subseteq\bigcup_{i\leq k}U_i$. So,

$$\mathrm{I\!R}\models\forall x_1\in[a,b]\varphi(x_1,\dots,x_n)\leftrightarrow$$
$$\mathbf{HF}(\mathrm{I\!R})\models\bigvee_{k\in\omega}\forall x_1\in[a,b]\bigvee_{i\leq k}\varphi_i(x_1,\dots,x_n).$$

By induction hypotheses, for every $k\in\omega$ there exists $\psi_k(a,b,x_2,\dots,x_n)$ in the language $\sigma\cup\{P'_\lambda|\lambda:\{1,\dots,n\}\to\{1,\dots,n\}\}$ which is equivalent to $\forall x_1\in[a,b]\bigvee_{i\leq k}\varphi_i(x_1,\dots,x_n)$. A required Σ-formula ψ can be constructed using Theorem 1. ■

It is worth noting that the Uniformity Principle holds for any finite extension of σ by open predicates.

COROLLARY 4
For every Σ-formula φ in the language σ there exists a Σ-formula ψ in the language σ such that

$$\mathbf{HF}(\mathrm{I\!R})\models\forall x\in[a,b]\varphi(x,y_1,\dots,y_n)\text{ iff }\mathbf{HF}(\mathrm{I\!R})\models\psi(a,b,y_1,\dots,y_n),$$

where free variables range over $\mathrm{I\!R}$.

3.2 *Extension of Σ-language by computable functions*

In this subsection we show that rational numbers, polynomials, computable real numbers and computable real-valued functions as well can be used in Σ-formulas without enlarging the class of Σ-definable sets. In other words we can extend the language of Σ-formulas by computable functions, e.g. cos, sin, exp and Uniformity Principle allows eliminate them later.

PROPOSITION 1
For every Σ-formula $\varphi(\bar{y},\bar{z})$ in the language σ and computable total real-valued functions $f_1(\bar{x}),\dots,f_n(\bar{x})$ there exists a Σ-formula ψ in the language σ such that $\mathbf{HF}(\mathrm{I\!R})\models\varphi(f_1(\bar{x}),\dots,f_n(\bar{x}),\bar{z})$ iff $\mathbf{HF}(\mathrm{I\!R})\models\psi(\bar{x},\bar{z})$.

PROOF. Let $\varphi(\bar{y})$ be a Σ-formula and $f_1(\bar{x}),\dots,f_n(\bar{x})$ be computable functions. It is easy to note that

$$\mathbf{HF}(\mathrm{I\!R})\models\varphi(f_1(\bar{x}),\dots,f_n(\bar{x}),\bar{z})\text{ iff }\mathbf{HF}(\mathrm{I\!R})\models\exists a_1\dots\exists a_n\exists b_1\dots\exists b_n$$
$$\forall y_1\in[a_1,b_1]\dots\forall y_n\in[a_n,b_n]\big(\bigwedge_{1\leq i\leq n}\big(f_i(\bar{x})<b_i\wedge f_i(\bar{x})>a_i\big)\wedge\varphi(\bar{y},\bar{z})\big).$$

In [10] we have shown that f_i is computable if and only if $f_i(\bar{x})<z$ and $f_j(\bar{x})>z$ are Σ-definable. So, we can construct a required formula ψ using Corollary 4. ■

COROLLARY 5
For every Σ-formula $\varphi(\bar{y})$ in the language σ and rational numbers $q_1,\dots,q_n$ there exists a Σ-formula ψ in the language σ such that $\mathbf{HF}(\mathrm{I\!R})\models\varphi(q_1,\dots,q_n,\bar{z})$ iff $\mathbf{HF}(\mathrm{I\!R})\models\psi(\bar{x},\bar{z})$.

COROLLARY 6
For every Σ-formula $\varphi(\bar{y})$ in the language σ and polynomials $p_1(\bar{x}),\dots,p_n(\bar{x})$ with rational coefficients there exists a Σ-formula ψ in the language σ such that $\mathbf{HF}(\mathrm{I\!R})\models\varphi(p_1(\bar{x}),\dots,p_n(\bar{x}),\bar{z})$ iff $\mathbf{HF}(\mathrm{I\!R})\models\psi(\bar{x},\bar{z})$.

3.3 Elimination of quantifiers bounded by computable compact sets for Σ_K-language

In this subsection we prove than the Σ_K-language over the reals without equality admits elimination of both universal and existential quantifiers bounded by computable compact sets.

Proposition 2
Suppose B is Π-definable and $B \subseteq [-q,q]^n$ for some rational q. For every Σ-formula φ in the language σ there exists a Σ-formula ψ in the language σ such that $\mathbf{HF}(\mathbb{R}) \models \forall x \in B \varphi(x,\bar{y})$ iff $\mathbf{HF}(\mathbb{R}) \models \psi(\bar{y})$, where free variables range over $\mathbb{R}$.

Proof. Suppose $B \subseteq [-q,q]^n$ is definable by a Π-formula η. It is easy to see that $\forall x \in B \varphi(x,\bar{y})$ is equivalent to the formula

$$\forall x \in [-q,q]^n \left(\neg\eta(x) \vee \varphi(x,\bar{y})\right). \tag{1}$$

By Corollary 4 and Corollary 6, the formula (1) is equivalent to a Σ-formula. ∎

Proposition 3
Suppose K is a co-semicomputable compact set. For every Σ-formula φ in the language σ there exists a Σ-formula ψ in the language σ such that $\mathbf{HF}(\mathbb{R}) \models \forall x \in K \varphi(x,\bar{y})$ iff $\mathbf{HF}(\mathbb{R}) \models \psi(\bar{y})$, where free variables range over $\mathbb{R}$.

Proof. It is easy to see that $\forall x \in K \varphi(x,\bar{y})$ is equivalent to the formula

$$\forall x \in [-q,q]^n \left(x \notin K \vee \varphi(x,\bar{y})\right) \tag{2}$$

for some rational q, which can be find effectively by K. By properties of co-semicomputable closed sets, the distance function d_K is lower semicomputable [2], and, as a corollary, $\{x \mid x \notin K\} = \{x \mid d_K(x) > 0\}$ is Σ-definable. By Corollary 2, the formula (2) is equivalent to a Σ-formula. ∎

Proposition 4
Suppose K is a semicomputable compact set. For every Σ-formula φ in the language σ there exists a Σ-formula ψ in the language σ such that

$$\mathbf{HF}(\mathbb{R}) \models \exists x \in K \varphi(x,\bar{y}) \text{ iff } \mathbf{HF}(\mathbb{R}) \models \psi(\bar{y}),$$

where free variables range over $\mathbb{R}$.

Proof. Let us note that $\exists x \in K \varphi(x,\bar{y})$ is equivalent to the formula

$$\exists x' \exists \epsilon > 0 \left(\varphi(x',\bar{y}) \wedge d_K(x') < \epsilon \wedge \forall z \in \bar{B}(x',\epsilon)\varphi(z,\bar{y})\right). \tag{3}$$

By properties of semicomputable closed sets, the distance function d_K is upper semicomputable [2], and, as a corollary, the set $\{(x',\epsilon) \mid d_K(x') < \epsilon\}$ is Σ-definable. By the Uniformity Principle, the formula (3) is equivalent to a Σ-formula. ∎

Theorem 3
For every Σ_K-formula $\varphi(x)$ in the language σ there exists Σ-formula $\psi(x)$ such that $\mathbf{HF}(\mathbb{R}) \models \varphi(x) \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \psi(x)$.

3.4 Effective reasoning about computable subsets of $\mathbb{R}^n$

Now we show how the Uniformity Principle can be used for reasoning about computable subsets of $\mathbb{R}^n$.

We start with investigation of the boundaries of computable compact sets. In [4] it has been proven that the boundary operator defined on the closed sets over the reals is Σ^0_2-complete in Borel hierarchy. In contrast, the following theorem shows that in special cases it is possible to prove computability of boundaries.

THEOREM 4
Suppose $K \subset \mathbb{R}^n$ is a computable regular compact set, D is its interior, Γ is its boundary and $d_K : \mathbb{R}^n \to \mathbb{R}$ is its distance function. Then the following assertions are equivalent:

1. Γ is Π—definable.
2. D is Σ—definable.
3. Γ is computable.

PROOF. We give only main ideas of the proof.
$1 \to 2$. Suppose that Γ is Π-definable. It is clear that the formula $d_\Gamma(x) > d_K(x)$ defines D. By properties of co-semicomputable closed sets, D is Σ-definable.

$2 \to 3$. In order to show that Γ is computable it is sufficient to prove that the epigraph and the hypograph of its distance function d_Γ are Σ-definable. Indeed,

$$d_\Gamma(x) < \epsilon \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \exists y \in D \exists z \notin K \big(||x-y|| < \epsilon \wedge ||x-z|| < \epsilon\big) \text{ and}$$
$$d_\Gamma(x) > \epsilon \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \bar{B}(x,\epsilon) \subset D \vee \bar{B}(x,\epsilon) \subset \mathbb{R}^n \setminus K.$$

So, Γ is computable.

$3 \to 1$. Since Γ is computable, the distance function d_Γ is upper and lower semicomputable. So, $\Gamma = \{x | d_\Gamma(x) = 0\}$ is Π-definable. ■

THEOREM 5
Suppose $K \subset \mathbb{R}^n$ is a computable regular compact set, Γ is its boundary and every component of Γ is a smooth variety of codimension 1. Then Γ is computable.

PROOF. It is sufficient to show that Γ is Π-definable. Since Γ is smooth variety of codimension 1, for any $z \in \Gamma$ the following statement holds:

$$\exists y \in D \exists x \notin K (x - z = z - y \wedge B(y, ||y-z||) \subset D \wedge B(x, ||x-z||) \subset \mathbb{R}^n \setminus K).$$

So, $\psi(z) \rightleftharpoons z \in K \wedge \exists y \in K \forall t > 0 \big(t \leq 1 \vee d_K(z - t(y-z)) \geq t||y-z||\big)$ defines Γ. Since K is co-semicomputable, K is Π-definable. So, Γ is Π-definable by Theorem 3. ■

Now using the Uniformity Principle we prove that computability of the closer of a regular open set and its boundary follows form co-semicomputability of the boundary of this regular open set.

THEOREM 6
Let $D \subset \mathbb{R}^n$ be regular open set such that D and $\mathbb{R}^n \setminus cl(D)$ have a finite number of connected components. Then from co-semicomputability of $\Gamma = \partial D$ it follows that Γ and $K = cl(D)$ are computable.

First we need the following lemma.

LEMMA 2
If $A \subset \mathbb{R}^n$ is Σ-definable then every connected component of A is also Σ-definable.

PROOF. Suppose A_i is a connected component of A. Let us choose some $x_0 \in A_i \cap Q^n$. By properties of connected sets, it follows that the meta-formula

$$\exists m \in \omega \exists x_1 \dots \exists x_m \in \mathbb{R}^n \bigwedge_{i=0}^{m-1} \forall y \in [x_i, x_{i+1}] y \in A \wedge \forall y \in [x_m, x] \in A$$

which is equivalent to the Σ-formula

$$\bigvee_{m \in \omega} \exists x_1 \dots \exists x_{m-1} \exists r_0 > 0 \dots \exists r_m > 0 \Bigg(\bigwedge_{i < m-1} r_i + r_{i+1} > ||x_{i+1} - x_i|| \wedge$$
$$r_{m-1} + r_m > ||x - x_{m-1}|| \wedge \forall y \in \Gamma \bigwedge_{i \le m-1} ||x_i - y|| > r_i \wedge ||x - y|| > r_m \Bigg)$$

defines A_i. By Theorem 1 and Theorem 3, A_i is Σ-definable. ■

PROOF (Theorem 6). Since Γ is co-semicomputable, $A = \mathbb{R}^n \setminus \Gamma$ is c.e. open, and hence it is Σ-definable. By assumption, A has a finite number of connected components. Let $A_0, \dots, A_m$ be all connected components of A. By Lemma 2, $D = \bigcup_{i \in J} A_i$, where $J \subseteq \{0, \dots, m\}$, as well as $\mathbb{R}^n \setminus K = \bigcup_{i \notin J} A_i$ are Σ-definable. In order to show that Γ is computable it is sufficient to prove that the epigraph of its distance function d_γ is Σ-definable. Indeed,

$$d_\Gamma(x) < \epsilon \leftrightarrow (\exists y \in D)(\exists z \in \mathbb{R}^n \setminus K) ||x - y|| < \epsilon \wedge ||x - z|| < \epsilon.$$

So, Γ is computable.

Now we show that K is computable. Since $\mathbb{R}^n \setminus K$ is Σ-definable, K is semicomputable. Co-semicomputability follows from Σ-definability of $d_K(x) < \epsilon$ by the formula $x \in D \vee d_\Gamma(x) < \epsilon$. So, K is computable. ■

4 The Uniformity Principle and majorant-computability

In order to do reasoning about computability of certain continuous data we have to pick up an appropriate language of a structure representing these continuous data. There are two major conditions how to do that in a right direction. The first one is topological, which states that computable functions should be continuous. This condition provides correctness of approximating computation. The second one is logical, which says that $Th_\exists(\mathcal{M})$ should be computably enumerable. This condition provides effectiveness of reasoning about continuous data based on Σ-definability. In this section we illustrate how these conditions work.

4.1 Continuity of majorant-computable functionals

First we will employ the Uniformity Principle to make reasoning about topological properties of majorant-computable functionals of the type $f : A \to \mathbb{R}$.

Suppose we have an arbitrary model $\mathcal{A}=\langle A,\sigma_{\mathcal{A}}\rangle=\langle A,\sigma_P,\neq\rangle$, where A contains more than one element, σ_P is a finite set of basic predicates.

The topology $\tau_\Sigma^{\mathcal{A}}$ is formed by the base consisting the subsets defined by existential formulas.

Let $\sigma_{\mathrm{IR}}\cap\sigma_{\mathcal{A}}=\emptyset$. In order to recall the notion of majorant-computability of functionals $f:A\to\mathrm{IR}$ we extend the structure $\mathcal{R}=\mathrm{IR}\cup A$ by the set of hereditarily finite sets $\mathrm{HF}(\mathcal{R})$ and consider Σ-definability in $\mathbf{HF}(\mathcal{R})\rightleftharpoons\big(\mathrm{HF}(\mathcal{R}),U_1,U_2,\sigma_{\mathrm{IR}},\sigma_{\mathcal{A}},\in,\emptyset\big)$, where the predicate symbol U_1 naming the set of the real numbers and the predicate symbol U_2 naming A.

It is worth noting some properties of Σ-definable sets in $\mathbf{HF}(\mathcal{R})$.

PROPOSITION 5
If a set $B\subseteq\mathcal{R}^n$ is Σ-definable then there exists an effective sequence of existential formulas in the language $\sigma_{\mathrm{IR}}\cup\sigma_{\mathcal{A}}$, $\{\Phi_s(x)\}_{s\in\omega}$, such that

$$x\in B\leftrightarrow\mathcal{R}\models\bigvee_{s\in\omega}\Phi_s(x).$$

COROLLARY 7
Every Σ-subset of $\mathcal{R}$ is open.

PROOF. The claim follows from Proposition 5. ■

COROLLARY 8
A set $B\subseteq A$ is open in the topology $\tau_\Sigma^{\mathcal{R}}$ if and only if A is open in the topology $\tau_\Sigma^{\mathcal{A}}$.

PROOF. $\to$). Suppose $B\subseteq A$ and $B\in\tau_\Sigma^{\mathcal{R}}$. By the definition of $\tau_\Sigma^{\mathcal{R}}$, there exists an existential formula ψ such that

$$b\in B\leftrightarrow\mathcal{R}\models\psi(b).$$

Without loss of generality suppose $\psi(b)\rightleftharpoons\exists a\exists r\big(\nu(a,b)\wedge\phi(r)\wedge U_1(r)\wedge U_2(a)\big)$, where a ranges over A and r ranges over IR. By quantifier elimination, we can effectively check validity of the formula $\exists r\phi(r)$. If $\mathrm{IR}\models\exists r\phi(r)$ then $\psi(b)\leftrightarrow\exists a\nu(a,b)$. If $\mathrm{IR}\not\models\exists r\phi(r)$ then $\psi(b)\leftrightarrow\bot$. So, there exists an existential formula φ such that

$$b\in B\leftrightarrow\mathcal{A}\models\varphi(b).$$

Therefore $B\in\tau_\Sigma^{\mathcal{A}}$.
$\leftarrow$). The claim follows from the inclusion $\tau_\Sigma^{\mathcal{A}}\subseteq\tau_\Sigma^{\mathcal{R}}$. ■

DEFINITION 2
A functional $F:A\to\mathrm{IR}$ is called **majorant-computable** if there exists a Σ-formula $\Phi(s,a,y)$ and a Π-formula $\Psi(s,a,y)$ such that the following conditions hold.

1. For all $s\in\omega$, $a\in A$, the formulas $\Phi(s,a,\cdot)$ and $\Psi(s,a,\cdot)$ define non-empty intervals $<\alpha_s,\beta_s>$ and $[\delta_s,\gamma_s]$.
2. For all $a\in A$, the sequences $\{<\alpha_s,\beta_s>\}_{s\in\omega}$ and $\{[\delta_s,\gamma_s]\}_{s\in\omega}$ decrease monotonically and $<\alpha_s,\beta_s>\subseteq[\delta_s,\gamma_s]$ for all $s\in\omega$.
3. For all $a\in\mathrm{dom}(F)$, $F(a)=y\leftrightarrow\bigcap_{s\in\omega}<\alpha_s,\beta_s>=\{y\}\leftrightarrow\bigcap_{s\in\omega}[\delta_s,\gamma_s]=\{y\}$ holds; for all $a\notin\mathrm{dom}(F)$, $||\cap_{s\in\omega}[\delta_s,\gamma_s]||>1$.

The formulas $\Phi(s,\cdot,\cdot)$ and $\Psi(s,\cdot,\cdot)$ define effective sequences $\{\Phi_s\}_{s\in\omega}$ and sequences $\{\Psi_s\}_{s\in\omega}$. The sequence $\{\Phi_s\}_{s\in\omega}$ is called a *sequence of* Σ*-approximations* for F. The sequence $\{\Psi_s\}_{s\in\omega}$ is called a *sequence of* Π*-approximations* for F. As we can see, the process which carries out the computation is represented by two effective procedures. These procedures produce Σ-formulas and Π-formulas that define approximations closer and closer to the result.

Below we will write $\varphi_1(a,\cdot)<\varphi_2(a,\cdot)$ if $\mathbf{HF}(\mathcal{R})\models\varphi_1(a,y)\wedge\varphi_2(a,z)\to y<z$ for all real numbers y,z. The following theorem connects a majorant-computable functional with validity of finite formulas in the set of hereditarily finite sets, $\mathbf{HF}(\mathcal{R})$.

THEOREM 7 [6]
For every functional $F:A\to\mathrm{IR}$ the following assertions are equivalent:.

1. The functional F is majorant-computable.
2. There exists Σ-formulas $\varphi_1(a,y)$, $\varphi_2(a,y)$ such that $\varphi_1(a,\cdot)<\varphi_2(a,\cdot)$ and

$$F(a)=y\leftrightarrow\forall z_1\forall z_2\big(\varphi_1(a,z_1)<y<\varphi_2(a,z_2)\big)\wedge$$
$$\{z\,|\,\varphi_1(a,z)\}\cup\{z\,|\,\varphi_2(a,z)\}=\mathrm{IR}\setminus\{y\}.$$

PROOF. $\to$) Let $F:A\to\mathrm{IR}$ be majorant-computable. By Definition 2, there exists a sequence $\{F_s\}_{s\in\omega}$ of Σ-approximations for F and a sequence $\{\Psi_s\}_{s\in\omega}$ of Π-approximations for F. Let

$$\varphi_1(a,y)\rightleftharpoons(\exists s\in\omega)\big(y\notin[\delta_s,\gamma_s]\wedge\big(\exists z\in<\alpha_s,\beta_s>\big)\big(y<z\big)\big)$$

and

$$\varphi_2(a,y)\rightleftharpoons(\exists s\in\omega)\big(y\notin[\delta_s,\gamma_s]\wedge\big(\exists z\in<\alpha_s,\beta_s>\big)\big(y>z\big)\big).$$

By construction, φ_1 and φ_2 are the sought formulas.

$\leftarrow$) Let φ_1 and φ_2 satisfy the requirements of the theorem. Let us construct approximations in the following way:

$$\Phi_s(a,y)\rightleftharpoons\exists z\exists v\big(\varphi_1(a,z)\wedge\varphi_2(a,v)\wedge y\in(z,v)\wedge v-z<1/s\big),$$
$$\Psi_s(a,y)\rightleftharpoons\forall z\big(\varphi_1(a,z)\to z-y\leq 1/s\big)\wedge\forall z\big(\varphi_2(a,z)\to y-z\leq 1/s\big).$$

■

THEOREM 8
If $F:(A,\tau_\Sigma^{\mathcal{A}})\to(\mathrm{IR},\tau_{||\cdot||})$ is majorant-computable then it is continuous.

PROOF. It is sufficient to show that the preimage of an open interval with rational endpoints is open in $\tau_\Sigma^{\mathcal{A}}$. It is easy to see that $F^{-1}((q_1,q_2))$ is Σ-definable by the formula $\psi(a,q_1,q_2)\rightleftharpoons\varphi_1(a,q_1)\wedge\varphi_2(a,q_2)$, where φ_1,φ_2 are defined in the theorem above. Using Corollary 5, we can eliminate the rational numbers q_1,q_2 from the formula ψ. By Corollary 7 and Corollary 8, $F^{-1}((a,b))$ is open in $\tau_\Sigma^{\mathcal{A}}$. ■

4.2 *Majorant-computability and computability*

Now we illustrate on an example how we can prove computability of continuous data using the language of Σ_K-formulas, the Uniformity Principle for Σ-definability and majorant-computability.

Let $f\in C[0,1]$. We extend the language σ by two predicates $Q(x_1,x_2,z)\rightleftharpoons f|_{[x_1,x_2]}<z$ and $P(x_1,x_2,z)\rightleftharpoons f|_{[x_1,x_2]}>z$.

PROPOSITION 6
For every $\lambda:\{1,2,3\}\to\{1,2,3\}$ there exist Σ-formulas ψ^- and ψ^+ in the language $\sigma\cup\{P,Q\}$, which do not depend on the choice of f and

$$\mathbf{HF}(\mathrm{IR})\models P'_\lambda(a,b,x_2,x_3)\leftrightarrow\psi^-(P,a,b,x_2,x_3) \text{ and}$$
$$\mathbf{HF}(\mathrm{IR})\models Q'_\lambda(a,b,x_2,x_3)\leftrightarrow\psi^+(Q,a,b,x_2,x_3).$$

PROOF. We show how to construct the required formulas ψ^- for some λ. If $\lambda=id_{\{1,2,3\}}$ then $\psi^-(a,b,x_2,x_3)\rightleftharpoons b<a\vee P(a,y,z)$. If $\lambda=\{<1,1>,<2,1>,<3,3>\}$ then $\psi^-(a,b,x_2,x_3)\rightleftharpoons P(a,b,z)$. In the non-trivial case, where $\lambda=\{<1,1>,<2,2>,<3,1>\}$, we have

$$\mathbf{HF}(\mathrm{IR})\models\forall x_1\in[a,b]P(x_1,x_2,x_1)\leftrightarrow$$
$$\mathbf{HF}(\mathrm{IR})\models x_2<a\vee b<a\vee\big(P(a,x_2,a)\wedge\theta(a,b,x_2)\big),$$

where

$$\begin{aligned}\theta(a,b,x_2)\rightleftharpoons\ &\big(x_2<b\wedge\big(P(a,x_2,x_2)\vee\textstyle\bigvee_{m\in\omega}\exists t_1\dots\exists t_m\big(a=t_0<\dots\\ &\dots<t_m<t_{m+1}=x_2\wedge\textstyle\bigwedge_{i\le m}P(t_i,t_{i+1},t_{i+1})\big)\big)\big)\vee\\ &\big(P(b,x_2,b)\wedge\big(P(a,b,b)\vee\textstyle\bigvee_{m\in\omega}\exists t_1\dots\exists t_m\big(a=t_0<\dots\\ &\dots<t_m<t_{m+1}=b\wedge\textstyle\bigwedge_{i\le m}P(t_i,t_{i+1},t_{i+1})\big)\big)\big).\end{aligned}$$

Using this equivalence and Theorem 1 we can effectively construct ψ^-.

■

In [6] we have shown that a functional $F:C[0,1]^n\to\mathrm{IR}$ is majorant-computable iff it is computable in the sense of computable analysis [11]. Now we are going to generalize this result to functionals $F:C[0,1]^n\times\mathrm{IR}^m\to\mathrm{IR}$. It is worth noting that in the proof we essentially use the Uniformity Principle.

THEOREM 9
For every functional $F:C[0,1]^n\times\mathrm{IR}^m\to\mathrm{IR}$ the following assertions are equivalent:

1. The functional F is majorant-computable.
2. The functional F is computable.

PROOF. Without loss of generality let us consider the case $n=m=1$. For simplicity of notation, we will give the construction only for that case, since the main ideas are already contained here. Let $F:C[0,1]\times\mathrm{IR}\to\mathrm{IR}$ be a majorant-computable functional. For $f\in C[0,1]$ we denote $E_f(x_1,x_2,z)\rightleftharpoons f|_{[x_1,x_2]\cap[0,1]}<z$ and $H_f(x_1,x_2,z)\rightleftharpoons f|_{[x_1,x_2]\cap[0,1]}>z$. Let us define $G:C[0,1]^2\to\mathrm{IR}$ by the rule $G(f,g)=F(f,g(½))$. We show that G is also majorant-computable. It is easy to see that

$$G(f,g)<y\leftrightarrow$$
$$\mathbf{HF}(\mathrm{IR})\models\exists x_1\exists x_2\Big(x_1<x_2\wedge\forall x\in[x_1,x_2]F(f,x)<y\wedge x_1<g(\tfrac{1}{2})<x_2\Big),$$
$$G(f,g)>y\leftrightarrow$$
$$\mathbf{HF}(\mathrm{IR})\models\exists x_1\exists x_2\Big(x_1<x_2\wedge\forall x\in[x_1,x_2]F(f,x)>y\wedge x_1<g(\tfrac{1}{2})<x_2\Big).$$

By the the Uniformity Principleand Theorem 1 [6], the formulas $\forall x\in[x_1,x_2]F(f,x)<y$, $\forall x\in[x_1,x_2]F(f,x)>y$ are equivalent to $\forall x\in[x_1,x_2]\varphi^-(E_f,H_f,x,y)$ and $\forall x\in[x_1,x_2]\varphi^+(E_f,H_f,x,y)$ for some Σ-formulas φ^-, φ^+, and, by Proposition 6, to Σ-formulas ψ^-, ψ^+.

Since

$$x_1<g(\tfrac{1}{2})<x_2\leftrightarrow\mathbf{HF}(\mathrm{IR})\models\exists u\exists v\Big(u<\tfrac{1}{2}<v\wedge E_g(u,v,x_1)\wedge H_g(u,v,x_1)\Big),$$

by Corollary 6, the formula $x_1<g(½)<x_2$ is equivalent to a Σ-formula in the language $\sigma\cup\{E_g,H_g\}$. As we can see the relations $G(f,g)<y$, $G(f,g)>y$ can be represented by Σ-formulas in the language

$\sigma \cup \{E_f, H_f, E_g, H_g\}$. So, G is majorant-computable. In [6] we have shown that a functional $H\colon C[0,1]^n \to \mathbb{IR}$ is computable iff it is majorant-computable. Hence, G is computable. Since $F(f,x) = G(f,\lambda z.x)$, F is computable as composition of computable functions.

If F is computable, then G is also computable. By Theorem 3 [6], there exist two Σ-formulas φ^- and φ^+ such that

$$G(f,g) < y \leftrightarrow \mathbf{HF}(\mathbb{IR}) \models \varphi^+(E_f, H_f, E_g, H_g, y),$$
$$G(f,g) > y \leftrightarrow \mathbf{HF}(\mathbb{IR}) \models \varphi^-(E_f, H_f, E_g, H_g, y).$$

If we substitute E_g by $U \rightleftharpoons [0,1] \times (x, +\infty)$ and H_g by $U \rightleftharpoons [0,1] \times (-\infty, x)$ then we get

$$F(f,x) < y \leftrightarrow \mathbf{HF}(\mathbb{IR}) \models \varphi^+(E_f, H_f, U, V, y),$$
$$F(f,x) > y \leftrightarrow \mathbf{HF}(\mathbb{IR}) \models \varphi^-(E_f, H_f, U, V, y).$$

By Theorem 1 [6], F is majorant-computable. ■

4.3 Computably enumerability of $Th_\exists(C[0,1] \cup \mathbb{IR})$

For effective reasoning about computable continuous data based on Σ-definability we have to choose an appropriate structure $\mathcal{M}$ representing these data that has the computable enumerable theory $Th_\exists(\mathcal{M})$.

In this subsection we show that computably enumerability of $Th_\exists(C[0,1] \cup \mathbb{IR})$ follows from computably enumerability of $Th(\mathbb{IR})$.

Let us denote $\sigma_{C[0,1]} = \{E_f(x_1,x_2,z), H_f(x_1,x_2,z)\}$, where $E_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]\cap[0,1]} < z$ and $H_f(x_1,x_2,z) \rightleftharpoons f|_{[x_1,x_2]\cap[0,1]} > z$. First we need the following proposition.

PROPOSITION 7
If $A \subseteq \mathbb{IR}^n$ is Σ-definable in $\mathbf{HF}(C[0,1] \cup \mathbb{IR})$ then A is Σ-definable in $\mathbf{HF}(\mathbb{IR})$.

PROOF. Let A be Σ-definable in $\mathbf{HF}(C[0,1] \cup \mathbb{IR})$. Using Proposition 5, without loss of generality, we assume that there exists an effective sequence of existential formulas in the language $\sigma_{\mathbb{IR}} \cup \sigma_{C[0,1]}$, $\{\Phi_s(f,x)\}_{s\in\omega}$, such that

$$x \in A \leftrightarrow \big(\mathbb{IR} \cup C[0,1]\big) \models \bigvee_{s\in\omega} \exists f\, \Phi_s(f,x).$$

Since A is open and the set of piecewise linear functions with rational coefficients is dense in $C[0,1]$, the existence of the continuous function f is equivalent to the existence of a piecewise linear function. So,

$$(C[0,1] \cup \mathbb{IR}) \models \exists f\, \Phi_s(f,x) \leftrightarrow$$
$$\mathbf{HF}(\mathbb{IR}) \models \textstyle\bigvee_{r\in\omega} \exists b_0 \exists a_1 \dots b_{r+1} \big(0 < a_1 < \dots < 1 \wedge \Phi_s^r\big),$$

where Φ_s^r is obtained from Φ_s by substitution of formulas $E_f(c,d,z)$ by the following

$$\exists k \exists j \Big(0 < k < j < r+1 \wedge a_k < c \wedge d < a_j \wedge \textstyle\bigwedge_{s=k}^{j} b_s < z\Big) \vee$$
$$\exists i < k+1 \Big(a_i > d \wedge \textstyle\bigwedge_{s=0}^{i} b_s < z\Big) \vee$$
$$\exists k > 0 \Big(a_k < c \wedge \textstyle\bigwedge_{s=1}^{r+1} b_s < z\Big) \vee$$
$$\textstyle\bigwedge_{s=0}^{r+1} b_s < z,$$

and formulas $H_f(c,d,z)$ by the following

$$\exists k \exists j \Big(0<k<j<r+1 \wedge a_k<c \wedge d<a_j \wedge \bigwedge_{s=k}^{j} b_s>z\Big) \vee$$
$$\exists i<k+1 \Big(a_i>d \wedge \bigwedge_{s=0}^{i} b_s>z\Big) \vee$$
$$\exists k>0 \Big(a_k<c \wedge \bigwedge_{s=1}^{r+1} b_s>z\Big) \vee$$
$$\bigwedge_{s=0}^{r+1} b_s>z.$$

By Theorem 1 and Corollary 5, A is Σ-definable in **HF**(IR). ■

THEOREM 10
The theory $Th_\exists(C[0,1] \cup \mathrm{IR})$ is computably enumerable.

PROOF. Since the procedure described in the theorem above is effective and uniform, computably enumerability of $Th_\exists(C[0,1] \cup \mathrm{IR})$ can be reduced to computably enumerability of $Th(\mathrm{IR})$. ■

REMARK 5
It is worth noting that the Uniformity Principleallows us to extend the language $\sigma_{C[0,1]}$ by new constant symbols for computable functions e.g. sin, cos, exp. We still have computably enumerability of $Th_\exists(C[0,1] \cup \mathrm{IR})$. Indeed, for example, $E_{\sin}(x,y,z) \rightleftharpoons \forall a \in [x,y] \sin(a) < z$ is Σ-definable in **HF**(IR) by the Uniformity Principle and properties of computable real-valued functions.

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