

*Morse Theory for Invariant Functions and its
Application to the n -Body Problem*

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MORSE THEORY FOR INVARIANT
FUNCTIONS AND ITS APPLICATION TO
THE n -BODY PROBLEM.

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

2009

Gemma Lloyd
School of Mathematics

Contents

Abstract	8
Declaration	9
Copyright Statement	10
Acknowledgements	11
1 Introduction	12
2 Preliminaries	15
2.1 Cell Complexes	15
2.2 Simplicial Complexes and Simplicial Maps	16
2.2.1 Barycentric Subdivision	17
2.3 Manifolds	17
2.4 The Chain and Homology Groups of X	18
2.4.1 The Cohomology Groups of X	18
2.5 Group Action	19
2.5.1 G -Complexes and Orbits	19
2.5.2 Regularity	21
2.6 Linear Representations of Finite Groups	21
2.6.1 Character of G on a G -Set	23
2.6.2 Canonical Decomposition of a Representation	23
2.6.3 Negative Representations/Characters	24
2.6.4 Comparing Representations/Characters	26

2.6.5	Frobenius Reciprocity	26
2.6.6	Induced G -Action on the Cohomology	27
2.7	Fibre Bundles and the Gysin Sequence	28
2.7.1	The Gysin Sequence	28
2.8	Arrangements of Hyperplanes	29
3	Critical Point Theorem	32
3.1	Critical Points for Embedded Polyhedral Surfaces	32
3.2	Critical Points for Embedded Polyhedra	34
3.3	Applying a Group Action	37
3.3.1	Critical Points on G -complexes	37
4	G-Lefschetz Number	40
4.1	Lefschetz Fixed Point Theorem	40
4.2	G -Lefschetz Fixed Orbit Theorem	43
5	Morse Theory	50
5.1	Morse Theory	50
5.1.1	The Morse Inequalities	50
5.2	Introducing Flow	55
5.3	G -Morse Theory	56
5.4	Morse-Bott Theory	60
5.4.1	Introduction	60
5.4.2	Non-Degenerate Critical Manifolds	61
5.5	G -Morse-Bott Theory	66
6	The n-Body Problem	71
6.1	Introduction	72
6.2	The Action of a Group on the n -body Problem	74
6.2.1	Action of S_n on the n -Body Problem	75
6.2.2	The Isotropy Subgroup of a Point	76
6.2.3	Action of $S_n \times \mathbb{Z}_2$ on the n -Body Problem	77

6.2.4	Representation on the Unstable Space	77
6.3	The Cohomology of $\mathbb{C}P^{n-2} - \tilde{\Delta}$	78
6.4	Finding the Character on the Cohomology	80
6.5	Gysin Sequence	80
6.5.1	Equivariance of the Gysin Sequence	82
6.6	The 3-Body Problem and S_3	82
6.6.1	Moulton Points	83
6.6.2	Back to the 3-Body Problem Calculation	85
6.7	Further Calculations of the 3-Body Problem	87
6.7.1	Character on the Total Cohomology	88
6.8	What Does All This Look Like?	89
6.9	The 4-Body Problem	90
6.9.1	A Basis for the Moulton Points	92
6.9.2	Looking at $S_4 \times \mathbb{Z}_2$	94
6.10	Critical Points of the 4-Body Problem	98
6.11	The 5-Body Problem	98
6.11.1	Looking at $S_5 \times \mathbb{Z}_2$	103
6.12	Critical Points of the 5-Body Problem	106
6.13	Another Look at the Total Cohomology	106
6.14	Conclusion	109
A	Finding the Character of S_n on the Cohomology of $\mathbb{C}P^{n-2} - \tilde{\Delta}$	110
B	Character Tables	116
B.1	Symmetric Group on 3 Elements, S_3	116
B.2	Symmetric Group on 4 Elements, S_4	116
B.3	Symmetric Group on 5 Elements, S_5	117
B.4	Cyclic Group of Order 2, \mathbb{Z}_2	117
B.5	Cyclic Group of Order 3, \mathbb{Z}_3	117
B.6	Cyclic Group of Order 4, \mathbb{Z}_4	118
B.7	Cyclic Group of Order 5, \mathbb{Z}_5	118

B.8	$\mathbb{Z}_2 \times \mathbb{Z}_2$	118
B.9	$S_4 \times \mathbb{Z}_2$	119
B.10	$S_5 \times \mathbb{Z}_2$	120
B.11	Dihedral Group of Order Eight, D_4	122
C	Finding the Character on the Unstable Manifold	123
C.1	$S_4 \times \mathbb{Z}_2$: Isosceles Points	123
C.2	S_5 : Moulton Points	124
C.3	$S_5 \times \mathbb{Z}_2$: Moulton Points	125
	Bibliography	127

List of Figures

2.1	Barycentric subdivision of the 1 and 2 simplexes.	17
2.2	The barycentric subdivision X' 's inherited \mathbb{Z}_3 -action from X	20
2.3	Commutative diagram for a fibre bundle.	28
3.1	A convex cell structure of D^2	35
3.2	Simplicial structure on a sphere.	37
4.1	Simplicial structure on a sphere.	47
5.1	Orientation changing reflection $(23) \in S_3$	58
5.2	Non-degenerate critical manifolds of the torus.	63
5.3	2-sphere with a Morse–Bott function giving three critical sets.	65
5.4	How $-I$ acts on $H_1(M^b)$	68
5.5	How $-I$ acts on $H_1(M^b, M^a)$	69
5.6	How $-I$ acts on $H_2(M^b, M^a)$	69
5.7	How $-I$ acts on $H_2(M^b, M^a)$ - detailed view.	70
6.1	Example of S_3 action on a critical point in our system.	75
6.2	Action of S_3 on $\mathbb{C}P^1$	76
6.3	How $\sigma = (13)$ fixes a Moulton point of the n -body problem when $n = 3$	77
6.4	Action of $S_4 \times \mathbb{Z}_2$ — complex conjugation.	78
6.5	Relative equilibria for $n = 3$	83
6.6	Unstable space of Moulton points when $n = 3$	85
6.7	How H_M acts on the unstable space of the Moulton points.	85
6.8	Positions of the critical points on $\mathbb{C}P^1$	89

6.9	Positions of the critical points on $\mathbb{C}P^1$ including the gradient flow of $-V$	89
6.10	Known relative equilibria for $n = 4$	91
6.11	A basis for the unstable space of the Moulton points including action of the isotropy subgroup $H_M = \langle(14)(23)\rangle$	92
6.12	Applying the element $\kappa(14)(23)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the basis of the unstable space of the Moulton points — the final positions shown after rotation by π	97
6.13	Applying the element κ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the basis of the unstable space of the Moulton points.	97
6.14	Relative equilibria configurations of four equal masses.	99
6.15	Known relative equilibria for $n = 5$	100
6.16	Basis for the unstable space of the Moulton points when $n = 5$	100
6.17	Relative equilibria configurations of five equal masses.	107
C.1	Basis for the unstable space of the Isosceles points when $n = 4$	124
C.2	How $(23)\kappa$ acts on the unstable space of the Isosceles points.	124
C.3	Basis for the unstable space of the Moulton points when $n = 5$	124
C.4	How $(15)(24)$ acts on the unstable space of the Moulton points.	125
C.5	How $\kappa \in H_M$ acts on the unstable space of the Moulton points.	126
C.6	How $\kappa(15)(24) \in H_M$ acts on the unstable space of the Moulton points — the final positions shown after rotation by π	126

The University of Manchester

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Doctor of Philosophy

Morse Theory for Invariant Functions and its Application to the n -Body Problem.

November 24, 2009

We study various topological properties of G -manifolds and G -complexes where G is a finite group. We achieve this by first observing the work of others, T.F. Banchoff, Milnor and R. Bott, and then extending these ideas and concepts to the situation of G -manifolds and complexes.

We start with embedded polyhedra and a critical point theorem, linking numbers of critical points to the Euler characteristic, which we adapt to encompass G -complexes and G -invariant functions to form a critical orbit theorem. We then move on to the Lefschetz fixed point theorem. We describe the concepts G -trace and G -Euler characteristics which have a direct link to their non- G counterparts. We use these to prove what we call the G -Lefschetz fixed orbit theorem; an extension of the Lefschetz fixed point theorem.

We develop G -Morse and G -Morse-Bott theory from their usual respective theories. We define the notion of orientation representation and G -Morse and G -Poincaré polynomials. The main result of this work is

$$\mathfrak{M}_t^G(f) - P_t^G(M) = (1 + t)Q_t^G(f),$$

where f is a non-degenerate (in the sense of Morse-Bott theory) function on the manifold M , $\mathfrak{M}_t^G(f)$ is the G -Morse polynomial of f , $P_t^G(M)$ is the G -Poincaré polynomial of M and $Q_t^G(f)$ is a polynomial with non-negative coefficients. These polynomials have coefficients in the representation ring.

Finally we apply G -Morse theory to the n -body problem and fully describe the relative equilibria solutions up to and including five particles.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

The development of Morse theory has provided us with a useful tool with which we can discover much about the topology of a manifold. One accomplishes this through the studying of differentiable functions on the manifold. In his work Raoul Bott [7] extended Morse theory to functions with non-degenerate critical sets rather than isolated points, thereby enriching this powerful theory still further. The bulk of this thesis is dedicated to the extension of Morse and Morse-Bott theory and to the specific application of this in describing the relative equilibria solutions of the n -body problem.

Many of the current techniques compare dimensions of vector spaces to numbers of certain objects, usually numbers of critical points, but in some cases traces of functions. The aim of this thesis is, essentially, to replace those numbers with characters of representations after applying a finite group to the spaces in question. This will allow us to clarify new techniques that compare the representations of the group acting on those vector spaces to the representations of group acting on the objects, thereby giving some fairly powerful machinery with which to our spaces.

Although there is a theme that runs throughout, this work draws from many areas of the world of mathematics. We assume little and as such Chapter 2 is solely used for definitions and basic properties of various tools that would otherwise spoil the flow of the main body. The material found in this chapter is mostly well known, however there is some unusual notation which is explained here. Aside from this notation the

majority of this chapter can be found in the literature.

Chapter 3 focuses on the work of T.F. Banchoff in which he defines new indexing procedures for critical points of embedded polyhedra and polyhedral surfaces. We adapt this to G -complexes, with G being a finite group, and define new G -indexes using Banchoff's work as a foundation. It is in this chapter that we first discuss the G -Euler characteristic with further use in Chapters 4 and 5. This uses linear representations of finite groups and the associated character theory.

The Lefschetz fixed point theorem is at the core of Chapter 4. We centre our attention on describing the original theorem and its constituent parts before moving on to deriving a new analogous theorem which tells us whether f has a fixed orbit under G . We discuss the Lefschetz number of a simplicial function f and develop this further with new definitions of G -trace and G -Lefschetz number - again we exploit the theory of linear representations (and characters) of finite groups.

Chapter 5 begins with a study of classical Morse theory from Milnor's viewpoint [19] and we devote some time to the work by Raoul Bott including the main results and proofs thereof. We move on to extend Morse theory to G -manifolds with G -invariant functions, both where we have isolated critical points and where we have non-degenerate critical sets as in Morse-Bott theory. We define the G -Poincaré and G -Morse polynomials to replace their regular counterparts in Morse theory. These polynomials have character or representation coefficients and as such rely heavily on the previously mentioned theory of linear representations of finite groups.

The subject Equivariant Morse Theory is a well-studied area, initially by M. Atiyah and R. Bott [4], however, this approach does not yield any additional information in our setting. When we look at finite groups acting on the cohomology groups of a smooth manifold over \mathbb{R} , we gain no extra information. This is because the equivariant cohomology is equal to the normal cohomology here.

Extensive study of Morse theory has already been accomplished and applications to dynamical systems have become apparent. Through review of work by Smale ([33], [35], [36]) and Palmore ([26]) we see that the relative equilibria solutions of the n -body problem can be found using Morse theory. The configuration space of

the problem (with some restrictions) is isomorphic to complex projective space with the ‘diagonal’ set removed. The differentiable function we use is the potential energy on the n particles. In these restrictions we fix the centre of mass and we restrict to those points $x \in M_n$ where the kinetic energy is equal to one. The potential energy function is restricted to this space accordingly. Smale proves that the relative equilibria of the n -body problem are precisely the critical points of the potential energy function on the configuration space with these restrictions [35]. Using the techniques of Morse theory we are able to deduce the numbers of critical points of each index and therefore the number in each relative equilibria classes associated to these indexes. The theory in its classical form, although useful, has its limitations - we are unable to conclude much about what these configurations look like; despite this Palmore makes suggestions based on these calculations [26].

In Chapter 6 we use the group S_n , the symmetric group on n elements extensively but not exclusively. We apply S_n to the n particles in our system. This group action induces an action on the configuration space of the problem and the cohomology groups thereof. We are able to utilise the work by Arnold since the configuration space of the n -body problem (without the restrictions) is homotopic to the pure or coloured braid space. He gives us a basis for the cohomology groups of this space [3]. Combining this and information gathered from the Gysin sequence we can learn more about the restricted configuration space. We are able to calculate the action of G on the cohomology groups via Arnold’s basis and analysis of the (equivariant) Gysin sequence. Our G -Morse theory allows us to say more about the configurations when the group is S_n ; we recall Palmore’s allusion to the configurations of points of certain index and these suggestions become the starting point for Section 6.9.2 where we look at $S_n \times \mathbb{Z}_2$.

Chapter 2

Preliminaries

In this chapter, we introduce terms and ideas that will be used throughout our discussion. Much of this material is standard and can be found in the literature, though some notation is unusual and is specified here. The main references for this are *Algebraic Topology* by A. Hatcher [14], *Basic Topology* by M. A. Armstrong [2] and *Linear Representations of Finite Groups* by J.P. Serre [31]. Also used, but not extensively are *Arrangements of Hyperplanes* by P. Orlik and H. Terao [25], *A First Course in Abstract Algebra* by J.B. Fraleigh [12], and *Representations and Characters of Groups* by G. James and M. Liebeck [15].

2.1 Cell Complexes

Definition 2.1. A finite, k -dimensional **cell complex** is a space X constructed in the following manner:

- (i) Let X^0 be a discrete set, the 0-cells of X . A 0-cell is called a **vertex** of X .
- (ii) Form the n -skeleton X^n inductively from X^{n-1} by attaching an n -cell e_α^n via the maps $\phi_\alpha : S^{n-1} \rightarrow X^{n-1}$. The n -skeleton X^n is now the quotient space of $X^{n-1} \coprod_\alpha D_\alpha^n$ under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D_\alpha^n$. The cell e_α^n is the homeomorphic image of $D_\alpha^n - \partial D_\alpha^n$ under the quotient map.
- (iii) $X = X^n$ for $n = k$.

Definition 2.2. The **Star** of a vertex $v \in X$ (where X is a cell-complex) is defined to be the subcomplex $\text{Star}(v) \subseteq X$ of cells that contain v . The **open star** consists of the union of the interiors of all cells that contain v .

2.2 Simplicial Complexes and Simplicial Maps

An n -**simplex** is the n -dimensional analogue of the triangle; it is the smallest convex set in \mathbb{R}^m containing $(n + 1)$ points v_0, \dots, v_n that do not lie in a hyperplane of dimension less than n . The standard n -simplex is

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1, t_i \geq 0 \text{ for all } i\},$$

and its vertices are the unit vectors along the coordinate axes. The points v_i are the **vertices** of the simplex and the order in which we take them is important for homology. We shall denote an ordered simplex by $\sigma = (v_0, \dots, v_n)$; this determines the order of the edges (v_i, v_j) according to increasing subscripts. Specifying the order of the vertices determines canonical linear homeomorphism from the standard n -simplex onto any other n -simplex (v_0, \dots, v_n) , namely $(t_0, \dots, t_n) \mapsto \sum_i t_i v_i$. The coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i$ in (v_0, \dots, v_n) . The **barycentre** of the simplex (v_0, \dots, v_n) is the point $b = \sum_i t_i v_i$ whose barycentric coordinates t_i are all equal, $t_i = \frac{1}{n+1}$ for each i . This definition is taken directly from Hatcher [14].

Note: Unconventionally we use round brackets here since square brackets will be later used to indicate the character of a representation, see Section 2.6.

Definition 2.3. [Hatcher [14]] Let K and L be simplicial complexes. A map $f : K \rightarrow L$ is simplicial if it sends each simplex of K to a simplex of L by a linear map taking vertices to vertices. In barycentric coordinates, a linear map of a simplex v_0, \dots, v_n has the form $\sum_i t_i v_i \mapsto \sum_i t_i f(v_i)$. A linear map from a simplex to a simplex is uniquely determined by its values on vertices, this means that a simplicial map is uniquely determined by its values on vertices.

Definition 2.4. If \mathcal{C} is a finite collection of cells, the **Euler characteristic** χ of \mathcal{C} is $\chi(\mathcal{C}) = \sum_{r=0}^k (-1)^r \alpha_r(\mathcal{C})$, where $\alpha_r(\mathcal{C})$ is the number of r -cells in \mathcal{C} .

2.2.1 Barycentric Subdivision

The following definition of barycentric subdivision is taken from Hatcher [14].

The **barycentric subdivision** X' of the simplex $X = (v_0, \dots, v_n)$ is the decomposition of (v_0, \dots, v_n) into the n -simplices (b, w_0, \dots, w_{n-1}) where, inductively, (w_0, \dots, w_{n-1}) is an $(n-1)$ -simplex in the barycentric subdivision of a face $(v_0, \dots, \hat{v}_i, \dots, v_n)$. The induction starts with the case $n=0$ when the barycentric subdivision of (v_0) is defined to be just (v_0) itself. The next two cases $n=1, 2$ are shown in Figure 2.1.

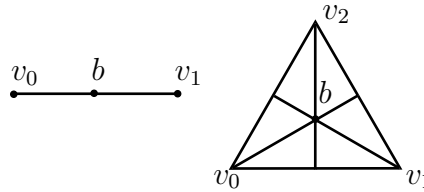


Figure 2.1: Barycentric subdivision of the 1 and 2 simplices.

Essentially this means take for the vertices of X' the simplices of X .

2.3 Manifolds

Although fairly standard, a notion of what we will mean by manifold throughout is needed.

Definition 2.5. An n -dimensional **topological manifold**, M^n is a second countable Hausdorff space in which each point has an open neighbourhood homeomorphic to \mathbb{R}^n .

An n -dimensional **smooth manifold** is a topological manifold M^n together with an atlas whose transition maps are smooth, that is, derivatives of all orders exist.

We can combine the ideas of manifolds and simplicial complexes to give us a

definition for a simplicial manifold. An n -dimensional **simplicial manifold** is a simplicial complex that is homeomorphic to a topological manifold, i.e., a neighbourhood of every vertex looks like \mathbb{R}^n .

In later sections we simply use the word ‘manifold’ since the type of manifold we mean should be clear from the context.

2.4 The Chain and Homology Groups of X

We define $C_r(X)$ to be the free abelian group of r -**chains** of a finite simplicial complex X , where an r -chain is a finite sum of the form $\sum_i m_i \sigma_i$, where $m_i \in \mathbb{Z}$ and the σ_i are the r -simplices of X .

Let X be an k -dimensional simplicial complex with chain groups $C_i(X)$ and boundary map $\partial_i : C_i(X) \rightarrow C_{i-1}(X)$. This forms a chain complex:

$$\cdots \rightarrow C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \rightarrow \cdots .$$

Let $Z_i(X) = \text{Ker } \partial_i$ be the cycles of X and let $B_i(X)$ be the group of i -boundaries of X , i.e., $B_i(X) = \text{Im } \partial_{i+1}$.

We now define the i -**th homology group** of X to be:

$$H_i(X) = Z_i(X)/B_i(X).$$

2.4.1 The Cohomology Groups of X

We consider a chain complex of free abelian groups:

$$\cdots \rightarrow C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \rightarrow \cdots .$$

To dualise this complex, we take the dual of each chain group and replace it in the sequence. This dual, $C_i^* = \text{Hom}(C_i, G)$ is called the cochain group. We also replace each boundary map $\partial : C_i \rightarrow C_{i-1}$ by its dual, the coboundary map, $\delta = \partial^* : C_{i-1}^* \rightarrow C_i^*$. Since $\partial^2 = 0$, it follows that $\delta^2 = 0$ and the **cohomology group** is defined to be:

$$H^i(C) = \text{Ker } \delta / \text{Im } \delta,$$

at C_i^* in the cochain complex:

$$\cdots \leftarrow C_{i+1}^* \xleftarrow{\delta} C_i^* \xleftarrow{\delta} C_{i-1}^* \leftarrow \cdots .$$

Note 2.6. Since we are working over \mathbb{R} , $C_n \cong C_n^* = C^n$.

We use the following in Chapter 6. If M is a smooth manifold then

$$H_{deR}^i(M; \mathbb{R}) \cong H_{\Delta}^i(M; \mathbb{R}) \cong H^i(M; \mathbb{R}),$$

where $H_{deR}^i(M; \mathbb{R})$ is the de Rham cohomology of M , $H_{\Delta}^i(M; \mathbb{R})$ is the singular cohomology of M and $H^i(M; \mathbb{R})$ is the simplicial cohomology, all over \mathbb{R} . The first of these isomorphisms is proved in *Foundations of Differentiable Manifold and Lie Groups* [38] by F. Warner and the second is a standard result, see [14].

2.5 Group Action

In this section we discuss some basic but necessary group-theoretic concepts.

Definition 2.7. Let G be a group and X a set. By a **G-action** on X , we mean a map $\varphi : G \times X \rightarrow X$ with the following properties:

1. $\varphi(e, x) = x$ for all $x \in X$, where e is the identity of G .
2. $\varphi(g_2, \varphi(g_1, x)) = \varphi(g_2 g_1, x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

We sometimes denote the G -action by a dot thus: $g \cdot x$. We call X endowed with an action of G a **G-set**.

2.5.1 G -Complexes and Orbits

Much of what follows can be found in *The Theory of Transformation Groups* by K. Kawakubo.

Let G be a group and let X be a G -set. The **orbit** \mathcal{O}_x of a point $x \in X$ is defined to be $\mathcal{O}_x = \{g \cdot x : g \in G\}$. In Chapter 3 we are concerned only with the orbits of x

when X is a G -complex and x is a vertex of X . In Chapters 5 and 6 the orbits we will be considering will be of critical points (see Section 5.1).

As with a G -set, a G -complex is a complex acted upon by a group G . The group action will preserve the simplicial or cellular structure of the complex and takes i -simplices(cells) to i -simplices(cells). It is easy to see that the barycentric subdivision X' of a G -complex X inherits naturally the G -action on X and becomes a G -complex.

Example 2.8. Let X be a 2-simplex and $G = \mathbb{Z}_3$ the cyclic group of order 3 as shown in Figure 2.2. The action of \mathbb{Z}_3 is rotation about the centre of the simplex, for example $g\sigma_{10} = \sigma_{20}$ and $g\sigma_{11} = \sigma_{21}$. It is clear that $\sigma = (\sigma_{10}, \sigma_{11}, \sigma_{12})$ and $g\sigma = (\sigma_{20}, \sigma_{21}, \sigma_{12}) = (g\sigma_{10}, g\sigma_{11}, g\sigma_{12})$.

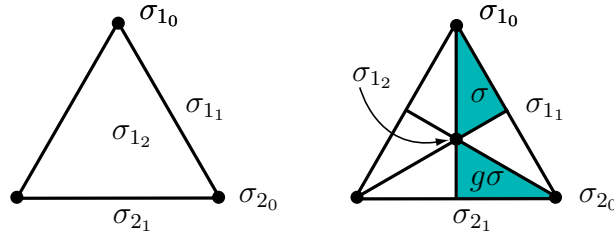


Figure 2.2: The barycentric subdivision X' 's inherited \mathbb{Z}_3 -action from X .

Definition 2.9. Let G be a group. A map f is said to be **G -equivariant** if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and **invariant** if $f(g \cdot x) = f(x)$ for all $g \in G$.

Note 2.10. If $f : X \rightarrow X$ is G -equivariant then so are the induced maps $f_i : C_i(X) \rightarrow C_i(X)$.

Lemma 2.11. Let $f : X \rightarrow Y$ be a G -equivariant map where X and Y are both G -complexes. If \mathcal{O} is an orbit in X then $f(\mathcal{O})$ is an orbit in Y .

Proof. Let $x \in X$ and $y \in Y$ and let $f(x) = y$. The orbits containing x and y are $\mathcal{O}_x = \{g \cdot x : g \in G\}$ and $\mathcal{O}_y = \{g \cdot y : g \in G\}$ respectively.

$$\begin{aligned}
 f(\mathcal{O}_x) &= \{f(g \cdot x) : g \in G\}, x \in X \\
 &= \{g \cdot f(x) : g \in G\}, \text{ since } f \text{ is } G\text{-equivariant} \\
 &= \{g \cdot y : g \in G\} \\
 &= \mathcal{O}_y
 \end{aligned}$$

Since $f : X \rightarrow Y$ is a simplicial map we can conclude that when we have an orbit $\mathcal{O} \subset X$ the image under f of this orbit is an orbit in Y . \square

Note: We suppress the use of a dot to mean group action from here onwards and so $g \cdot x$ becomes simply gx .

2.5.2 Regularity

Let G be a (finite) group and X a G -complex. Let g be an element of G , and σ an r -simplex of X . Then the G -action on X is said to be **regular** if whenever $g\sigma = \sigma$, then σ is pointwise fixed, i.e., $g|_{\sigma} = \text{id}$.

We now prove an important proposition which allows us to use the regularity condition - it is important because we may make a G -action regular by this process.

Proposition 2.12. *If X is a simplicial G -complex then the G -action on the barycentric subdivision of X is regular.*

Proof. Let X be a k -dimensional simplicial complex, so that X is the collection of simplices $\{\sigma_i\}$. The barycentric subdivision X' of X has for its vertex set the simplices $\{\sigma_i\}$ of X . That is, a simplex of X' , $\tau = (\sigma_{i_0}, \dots, \sigma_{i_k})$, where $\sigma_{i_0} \subset \dots \subset \sigma_{i_k}$ and σ_{i_j} is a j -simplex of X .

Since X is a G -complex X' is also a G -complex with induced action:

$$g\tau = (g\sigma_{i_0}, \dots, g\sigma_{i_k}), \text{ where } g\sigma_{i_0} \subset \dots \subset g\sigma_{i_k}.$$

Since X' is a G -complex the G action respects dimension and so if $g\tau = \tau$, then $g\sigma_{i_j} = \sigma_{i_j}$. This means that if $g\tau = \tau$ the vertices σ_{i_j} of τ are fixed and so τ must be point-wise fixed. \square

2.6 Linear Representations of Finite Groups

The following material is standard, much of it can be found in *Linear Representations of Finite Groups* by J. P. Serre [31] or *Representations and Characters of Groups* by G. James and M. Liebeck [15].

Let G be a finite group, V be a vector space (over either \mathbb{C} or \mathbb{R}). A **linear representation** of G in V is a homomorphism ρ from the group G into the group $\mathbf{GL}(V)$. When ρ is given, we say that V is a **representation space** or **representation** of G . Let V have a basis $\{e_i\}$ of n elements and let a be a linear map of V onto itself, with matrix a_{ij} . By the **trace** of a we mean the scalar

$$\mathrm{Tr}(a) = \sum_i a_{ii}.$$

It is the sum of the eigenvalues of a counted with the multiplicities and does not depend on the choice of basis $\{e_i\}$. Let ρ be a linear representation of a finite group G in V . For each $s \in G$, we define

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s).$$

The complex (or real) valued function $\chi_\rho(s)$ is called the **character** of the representation ρ . When ρ is given, we write square brackets to denote the character of the representation, i.e., $[V]$ denotes the character of G on V .

A **class function** on G is a function $\psi : G \rightarrow \mathbb{C}$ such that $\psi(x) = \psi(y)$ whenever x and y are conjugate elements of G (that is, ψ is constant on conjugacy classes).

We now have an important result in character theory which we will use extensively without reference.

Lemma 2.13. *A character is a class function on the group G .*

Proof. Let x and y be conjugate elements of G , then there exists an $a \in G$ such that $x = a^{-1}ya$. Let X, Y and A be the matrix representations, with basis $\{e_i\}$, of x, y and a respectively. Then we can write $X = A^{-1}YA$. To find the character of x we take the trace of the matrix X , giving:

$$\begin{aligned} \mathrm{Tr}(X) &= \mathrm{Tr}(A^{-1}YA) \\ &= \mathrm{Tr}(AA^{-1}Y), \text{ by standard properties of trace} \\ &= \mathrm{Tr}(Y). \end{aligned}$$

Therefore if x and y are conjugate in G , they have the same character. This gives us that character is indeed a class function on G . \square

The **degree** of a representation is given by the dimension of V or the rank of the matrix ρ .

Note: We will be working over the real numbers so whenever we say ‘linear representations of finite groups’ we mean **real** representations of finite groups.

We now state Schur’s Lemma since we will use it extensively in what follows. The precise statement depends on whether the representations are over \mathbb{R} or \mathbb{C} both of these cases are described below.

Lemma 2.14 (Schur’s Lemma, [31]). *Let V_1 and V_2 be irreducible representations over a field k . We consider the set $\text{Hom}_G(V_1, V_2)$ of commuting linear maps. Then*

- i) If V_1 and V_2 are not isomorphic, $\text{Hom}_G(V_1, V_2) = 0$.*
- ii) If $V_1 \simeq V_2$ are vector spaces over \mathbb{C} then $\text{Hom}_G(V_1, V_2) \cong \mathbb{C}$.*
- iii) If instead $V_1 \simeq V_2$ are vector spaces over \mathbb{R} , $\text{Hom}_G(V_1, V_2) \cong \mathbb{R}, \mathbb{C}$ or \mathbb{H} , where \mathbb{H} denotes the quaternions. Correspondingly the representations are of real, complex or quaternionic type.*

2.6.1 Character of G on a G -Set

Associated to a G -set X is a representation called the **permutation representation**. The vector space $\mathbb{R}\{x_1, \dots, x_r\}$ is spanned by the elements of X and the representation is given by the permutation matrices. The trace of each matrix (for each $g \in G$) is the number of 1s on the diagonal, which is the number of elements in X fixed by g . We denote the character of the representation by $[X]$.

2.6.2 Canonical Decomposition of a Representation

Let G be a finite group and V a representation. Let $V = \bigoplus_{j=1}^m U_j$, where the U_j are irreducible representations. We can now use this to define a canonical decomposition. We take V_i to be the direct sum of those U_j such that $U_j \cong W_i$, where the W_i are the distinct irreducible representations of G . Let n_i be the multiplicity of W_i in V_i . Then we have that $V_i = n_i W_i$ and $V = \bigoplus_{i=1}^r V_i$. This decomposition is unique.

Definition 2.15. A representation is said to be **isotypic** if it is a direct sum of isomorphic irreducible representations. For example $V_i = W_i \oplus \cdots \oplus W_i$ is isotypic of type W_i .

The unique decomposition described above is called the **isotypic decomposition** of a representation.

Let $f : V \rightarrow V$ be a linear map. Schur's lemma 2.14 tells us that a linear map that takes a representation to itself and that commutes with the group action takes isomorphic irreducible representations to each other or it is the zero map. So if $V = \bigoplus_{i=1}^r V_i$ where $V_i = n_i W_i$ and the W_i are the distinct irreducible reps of G , then since $f : V \rightarrow V$ commutes with the action of G and takes V_i to V_i we can say that $f = \bigoplus_{i=1}^r f_i$.

Example 2.16. Let $V = V_1 \oplus V_2$ be the isotypic decomposition of V . Then the matrix of $f : V \rightarrow V$ must block diagonal thus

$$F = \left(\begin{array}{c|c} f_1 & 0 \\ \hline 0 & f_2 \end{array} \right),$$

and so $f = f_1 \oplus f_2$ with $f_i : V_i \rightarrow V_i$.

2.6.3 Negative Representations/Characters

Usually we deal with representations as genuine objects such as vector spaces or matrix representations. As such, the notion of a negative representation is a somewhat alien concept. In Section 3.3 and Chapter 5 we take an alternating sum of representations. It is not immediately obvious how to deal with this, but it becomes easier to digest if we instead think of characters. We now recall some character theory.

Proposition 2.17. Let $\rho_1 : G \rightarrow GL(V_1)$ and $\rho_2 : G \rightarrow GL(V_2)$ be representations of G , and let χ_1 and χ_2 be their characters.

1. The character χ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$.
2. The character of the tensor product representation $V_1 \otimes V_2$ is equal to $\chi_1 \cdot \chi_2$.

Proof. Let A_i be the matrix representation of ρ_i , then the representation $V_1 \oplus V_2$ is given by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

and $\text{Tr}(A) = \text{Tr}(A_1) + \text{Tr}(A_2)$, i.e., $\chi(g) = \chi_1(g) + \chi_2(g)$.

Let $\{e_i\}$ be a basis for A_1 and $\{e_j\}$ likewise for A_2 . By definition we have that $\chi_1(g) = \sum_i a_{ii}$ and $\chi_2(g) = \sum_j a_{jj}$. Taking the tensor product gives:

$$\chi(g) = \sum_{i,j} a_{ii}a_{jj} = \chi_1(g) \cdot \chi_2(g). \quad \square$$

We formally define the negative of a representation or character to be its inverse under addition: $\chi + (-\chi) = 0$, the zero representation. This is the character ring which is spanned over \mathbb{Z} by the characters of G , with operations addition and multiplication as defined above.

Another way to think about this is to note that a character of a (real) representation is essentially a vector with real number entries, one for each conjugacy class of G . With this in mind we define a **character vector** to be a vector that has as its entries the character evaluated on each conjugacy class. For example the alternating character of S_3 would have character vector $\chi[1, (12), (123)] = [1, -1, 1]$. When dealing with a negative character we take the appropriate character vector and use vector addition (subtraction).

Example 2.18. Let $G = S_3$ with A_0 being the trivial character, A_1 the alternating character and E the two dimensional character respectively. Let $[V] = A_0 + A_1 + 2E$ and $[U] = A_0 + A_1$ be two characters of G , we can write these as the vectors $\chi_V[1, (12), (123)] = [6, 0, 0]$ and $\chi_U[1, (12), (123)] = [2, 0, 2]$ respectively. We can now calculate $[V] - [U]$ thus:

$$[V] - [U] = (A_0 + A_1 + 2E) - (A_0 + A_1) = 2E$$

or equivalently $\chi[1, (12), (123)] = [6, 0, 0] - [2, 0, 2] = [4, 0, -2]$, which is indeed the character vector associated to $2E$.

Note 2.19. It has become common to define elements of the representation ring as the formal difference of two representations, so an element is of the form $V_1 - V_2$, where V_1 and V_2 are representations in the ring. We, however, will be mainly working with characters and so will use the character ring.

Since irreducible characters are linearly independent, we know that a linear combination $\sum n_i[W_i]$ can be zero if and only if $n_i = 0$ for all i . Thus if any $n_i \neq 0$ then the representation and character are non-zero.

2.6.4 Comparing Representations/Characters

To compare two representations we compare the multiplicities of their summands, the irreducible representations of G . Let $\{[W_1], \dots, [W_r]\}$ be the irreducible characters of G , and let $\sum_i n_i[W_i]$ and $\sum_i m_i[W_i]$ be two characters. We say that $\sum_i n_i[W_i]$ is greater than $\sum_i m_i[W_i]$ if $n_i \geq m_i$ for every i . Otherwise they are incomparable. Equivalently we can say that a representation $\bigoplus_i n_i W_i$ is greater than a representation $\bigoplus_i m_i W_i$ if $n_i \geq m_i$ for every i .

Note 2.20. This process for comparing two representations gives us a partial order on the ring of representations. We can then say that the additive group of the representation ring is a partially ordered abelian group, a fact that we will use later.

2.6.5 Frobenius Reciprocity

In Chapter 6, we will use the Frobenius reciprocity theorem frequently and so it is stated here.

Definition 2.21. Let H be a subgroup of G . Let W be a representation of H and V a representation of G .

- i) The representation of G induced from W will be denoted by $\text{Ind}_H^G(W)$.
- ii) The representation of H given by restriction of V will be denoted by $\text{Res}_H^G(V)$.

We will use $\text{Ind}(W)$ and $\text{Res}(V)$ if G and H are clear.

The representation of H given by restriction of V is clear, but the induced representation is not so apparent.

Note 2.22. Let $\rho : G \rightarrow GL(V)$ be a representation of G and H be a subgroup of G . Let W be a vector subspace of V such that $H\dot{W} = W$ and let $s \in G/H$, a left coset. The vector space $W_s = \rho_s W$ is dependant only on the coset, not the choice of representative. The group G acts on the vector subspaces by permutation, for example for all $g \in G$ we have:

$$\rho_g W_s = \rho_g \rho_s W = \rho_{gs} W = W_{gs},$$

and so the action must be by permutation since G can be partitioned into left cosets.

If we have that $\sum_s W_s = \bigoplus_s W_s$, then

$$\text{Ind}_H^G(W) = \sum_{s \in G/H} W_s.$$

If ψ and ϕ are complex valued functions on G and g is the order of G , we set

$$(\psi|\phi) = \frac{1}{g} \sum_{t \in G} \phi(t) \overline{\psi(t)}.$$

Theorem 2.23 (Frobenius Reciprocity Theorem). *Let $H \leq G$. Let χ be a character of H and ψ a character of G . Then*

$$(\text{Ind}(\chi)|\psi)_G = (\chi|\text{Res}(\psi))_H.$$

Note 2.24. Since isomorphic representations have the same character we use the terms ‘character’ and ‘representation’ interchangeably. Care has been taken to ensure that the correct term has been used in the correct context; however, since the two concepts are inextricably linked, some overlap may be present.

2.6.6 Induced G -Action on the Cohomology

Let G be a finite group and X a k -dimensional simplicial G -complex (or manifold). There is an induced action on the chain groups of X and therefore an induced action on the homology and cohomology groups of X .

We can naturally define the **G-Euler characteristic** of X to be:

$$\chi_G(X) = \sum_{i=1}^k (-1)^i [H_i(X)],$$

where square brackets denote character of G on $H_i(X)$, [37]. This can be rewritten as $\chi_G(X) = \sum_{i=1}^k (-1)^i [C_i(X)]$. A proof of this statement is given in Section 4.2, Corollary 4.10.

Note 2.25. The dimension of the character of the G -Euler characteristic is equal to the Euler characteristic of M .

2.7 Fibre Bundles and the Gysin Sequence

A fibre bundle on a (total) space E with fibre F and base space B is determined by a projection map $p : E \rightarrow B$ such that for each point $x \in B$, there is an open neighbourhood U for which there is a homeomorphism $h : p^{-1}(U) \rightarrow U \times F$ such that the following diagram commutes, where q is projection onto the first factor. For any

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow p & \downarrow q \\ & & U \end{array}$$

Figure 2.3: Commutative diagram for a fibre bundle.

$x \in B$ the fibre F is homeomorphic to the preimage $p^{-1}(x)$. It is usual that a fibre bundle is written as a short exact sequence of spaces thus $F \rightarrow E \xrightarrow{p} B$.

2.7.1 The Gysin Sequence

The Gysin sequence is a long exact sequence in cohomology. If we have a fibre bundle $S^{l-1} \hookrightarrow E \xrightarrow{p} B$, where E is the total space and B is the base space, then there is an exact sequence, the **Gysin sequence**,

$$\dots \rightarrow H^{i-l}(B) \xrightarrow{e^\wedge} H^i(B) \xrightarrow{p^*} H^i(E) \xrightarrow{\alpha} H^{i-l+1}(B) \xrightarrow{e^\wedge} \dots, \quad (2.1)$$

where $\alpha : H^*(E) \rightarrow H^*(B)$ is integration along the fibre and e is a certain ‘Euler class’ in $H^l(B)$. Since $H^i(B) = 0$ for $i < 0$, the initial part of the sequence gives isomorphisms $p^* : H^i(B) \xrightarrow{\cong} H^i(E)$ for $i < (l-1)$ and the more interesting and useful part of the sequence is

$$0 \rightarrow H^{l-1}(B) \xrightarrow{p^*} H^{l-1}(E) \xrightarrow{\alpha} H^0(B) \xrightarrow{e \wedge} H^l(B) \xrightarrow{p^*} H^l(E) \rightarrow \dots .$$

This can be found in Hatcher [14].

Note 2.26. If we apply a finite group G to our fibre bundle $E \xrightarrow{p} B$, it acts on E and B in such a way that it commutes with p . There is an induced action on the cohomology groups of both these spaces and p^* is also equivariant. The Gysin sequence therefore translates to a G -equivariant version if G commutes with both α and with wedge product with e , the Euler class; in which case we have the following exact sequence:

$$0 \rightarrow [H^1(B)] \xrightarrow{p^*} [H^1(E)] \xrightarrow{\alpha} [H^0(B)] \xrightarrow{e \wedge} [H^2(B)] \xrightarrow{p^*} [H^2(E)] \rightarrow \dots .$$

The equivariance of α and $e \wedge$ depends on how each element of G acts on the orientations of E, B and the fibre. The Gysin sequence is natural and so if G preserves the orientation (given by the cohomology) of each of these spaces then the sequence is ‘mapped to itself’.

2.8 Arrangements of Hyperplanes

Let \mathbb{K} be a field and $V_{\mathbb{K}}$ be a vector space of dimension l . A **hyperplane** H in $V_{\mathbb{K}}$ is an affine subspace of dimension $(l-1)$. A **hyperplane arrangement** \mathcal{A} is a finite set of hyperplanes in $V_{\mathbb{K}}$.

Definition 2.27. [Orlik and Terao [24]] Let \mathcal{A} be an arrangement and let $L = L(\mathcal{A})$ be the set of non-empty intersections of elements of \mathcal{A} . Define a partial order on L by $X \leq Y \iff Y \subseteq X$. This is reverse inclusion and so $V_{\mathbb{K}}$ is the unique minimal element.

Define the **Möbius function** $\mu_{\mathcal{A}} = \mu : L \times L \rightarrow \mathbb{Z}$ as follows:

$$\begin{aligned}\mu(X, X) &= 1 && \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X, Z) &= 0 && \text{if } X, Y, Z \in L \text{ and } X < Y, \\ \mu(X, Y) &= 0 && \text{otherwise.}\end{aligned}$$

For fixed X the values of $\mu(X, Y)$ may be computed recursively. For $X \in L$ we define $\mu(X) = \mu(V, X)$.

The **Poincaré polynomial** of a hyperplane arrangement \mathcal{A} is defined as

$$\pi_t(\mathcal{A}) = \sum_{X \in L} \mu(X) (-t)^{r(X)},$$

where $r(X) = \text{codim}(X)$ is a rank function on L , and μ is the Möbius function on the poset described above.

Clearly $\mu(V) = 1$ and $\mu(H) = -1$ for all $H \in L$ and if $r(X) = 2$, then $\mu(X) = |\mathcal{A}_X| - 1$, where $|\mathcal{A}_X|$ is the cardinality of the sub-arrangement $\mathcal{A}_X = \{H \in \mathcal{A} : X \subseteq H\}$. It is not in general possible to give a formula for $\mu(X)$.

It is proved in *Combinatorics and topology of complements of hyperplanes* by P. Orlik and L. Solomon [24] that the Poincaré polynomial of a hyperplane arrangement (over \mathbb{C}) is equal to the Poincaré polynomial of the complement of that hyperplane arrangement.

Example 2.28. Let \mathbb{K} be the complex numbers. For $1 \leq i < j \leq l$, let H_{ij} be hyperplanes defined by $H_{ij} = \text{Ker}(x_i - x_j)$. Let \mathcal{A} be the braid arrangement defined by $Q = 0$, where

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq l} (x_i - x_j).$$

The Poincaré polynomial of the arrangement is

$$\pi_t(\mathcal{A}) = (1+t)(1+2t) \dots (1+(l-1)t).$$

A pure braid is the image of a circle in the complement of the hyperplanes H_{ij} . The variety $N(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H = \{v \in V : Q(\mathcal{A})(v) = 0\}$ is called the **superdiagonal** and its complement $M_n = V \setminus N(\mathcal{A})$ is the **pure (or coloured) braid space** [25].

We now have that the Poincaré polynomial of the pure braid space is

$$P_t(M_n) = \pi_t(\mathcal{A}) = (1+t)(1+2t)\dots(1+(l-1)t).$$

This is precisely the M_n discussed by Arnold [3], see Section 6.3.

When working over \mathbb{R} , the arrangement is simply l choose 2, that is $\frac{l!}{2(l-2)!}$ hyperplanes of dimension l meeting in a line. This means the resulting complement is homotopic to the disjoint union of $\frac{l!}{(n-2)!}$ points.

Chapter 3

Critical Point Theorem

We start by looking at the work of T.F. Banchoff - specifically his work on the critical points of embedded polyhedra and the critical point theorem relating critical points to Euler characteristic. The two main references for this work are Critical Points and Curvature for Embedded Polyhedral Surfaces [6] and Critical Points and Curvature for Embedded Polyhedra [5]. The first gives us a more geometric presentation of the critical point theorem for closed smooth surfaces embedded in \mathbb{R}^3 than is usual and the second is a combinatorial account of the same theorem for a k -dimensional cell complex embedded in \mathbb{R}^n . Our main interest here is summed up by the following question: “What happens to the critical point theorem when we apply a group action to the cell complex?”

3.1 Critical Points for Embedded Polyhedral Surfaces

The following setup is taken almost directly from Banchoff’s paper [6].

Let M^2 be a smooth surface embedded in \mathbb{R}^3 and let f be a linear function taking M^2 to the real line \mathbb{R} . The function f is the projection of \mathbb{R}^3 onto the line determined by a unit vector \mathbf{f} . A point $p \in M^2$ is said to be a **critical point** of f if the tangent plane to M^2 at p is perpendicular to \mathbf{f} . All other points in M^2 are said to be **ordinary points** for f . Banchoff’s **critical point theorem** states that if the height function

f has a finite number of critical points on M^2 then:

(number of local maxima)–

$$(\text{number of saddle points}) + (\text{number of local minima}) = \chi(M^2),$$

where $\chi(M^2)$ is the Euler characteristic of M^2 . Banchoff expresses this theorem as:

$$\sum_{p \text{ critical for } f} i(p, f) = \chi(M^2).$$

He gives each critical point an index by $i(p, f) = 1$ if p is a local maximum or minimum, and $i(p, f) = -1$ if p is a (non-degenerate) saddle point.

In classical Morse theory (discussed later), a similar index is given by considering the number of negative eigenvalues of the Hessian evaluated at the critical point. However, Banchoff gives a more geometric presentation of this procedure, developing the polyhedral analogue of the critical point theorem. If a point q is ordinary for f , then the horizontal plane to q is not tangential and so divides a ‘small disc neighbourhood’ U of q on M^2 into exactly two pieces, and it meets a ‘small circle’ about q in precisely two points. A critical point is now distinguishable from an ordinary point, since at a local maximum or minimum the tangent plane will not meet a ‘small circle’ at all, and at a non-degenerate saddle point they will meet in four distinct points. Now the index can be defined geometrically as:

$$i(p, f) = 1 - \frac{1}{2}(\text{number of points at which the plane through } p \text{ perpendicular to } f \text{ meets a ‘small circle’ about } p \text{ on } M^2).$$

This agrees with the previous definition, and also gives us that $i(p, f) = 0$ if p is not a critical point. However, this does not give us the required index when we look at the polyhedral case, since the notion of ‘small circle’ is difficult to define precisely.

Definition 3.1. A height function f is said to be **general** for the polyhedral surface M^2 if $f(v) \neq f(w)$ whenever v and w are adjacent vertices in M^2 .

If f is general for M^2 a point q is said to be **ordinary** for f if the plane through q perpendicular to \mathbf{f} cuts the disc neighbourhood $\text{Star}(q)$ (See Definition 2.2) into two pieces.

Note that under this definition any point q in the interior of a face must be ordinary, since no face nor edge can be perpendicular to the vector \mathbf{f} if f is general for M^2 . For vertices, there are critical points corresponding to all the types presented for smooth functions.

This means that the above indexing procedure can be used for smooth surfaces if we use the embedded polygon which is the boundary of the star of the vertex v instead of the ‘small circle’. The number of times the plane perpendicular to \mathbf{f} meets this polygon is then equal to the number of triangles Δ in $\text{Star}(v)$ having one vertex lying above the plane and another below. We can then say that v is the middle vertex of Δ for f , and we may define the index as follows:

$$i(v, f) = 1 - \frac{1}{2}(\text{number of } \Delta \text{ with } v \text{ middle for } f).$$

Again this definition gives index 0 for an ordinary point and corresponds to the definitions given for the smooth case.

The **critical point theorem** then states:

Theorem 3.2. *If f is general for M^2 , then*

$$\sum_{v \in M^2} i(v, f) = \chi(M^2).$$

Proof. The proof uses the fact that for a 2-dimensional surface the Euler characteristic is given by $\chi(M^2) = V - E + T$ where V, E, T are the number of vertices, edges and triangles in the surface respectively, and that for a polyhedral surface, $3T = 2E$. For the full proof, see Banchoff [6]. □

3.2 Critical Points for Embedded Polyhedra

The next paper of Banchoff [5] we will look at extends these ideas to k -dimensional convex cell complexes. Again the set up comes almost directly from the paper being considered.

Definition 3.3. A **convex cell complex** M embedded in \mathbb{R}^n is a finite collection of cells $\{c^r\}$, where c^0 is a point, and each c^r is a bounded convex set with interior

some affine $\mathbb{R}^r \subset \mathbb{R}^n$, such that the boundary ∂c^r of c^r is a union of c^s with $s < r$ and such that if $s < r$ and $c^s \cap c^r \neq \emptyset$, then $c^s \subset \partial c^r$. M is called k -dimensional if there is a c^k in M but no c^{k+1} .

Let M be a k -dimensional convex cell complex. We define a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ to be **general** for M if $f(v) \neq f(w)$ whenever v and w are adjacent in M .

Definition 3.4. Let v be a vertex in a convex cell complex M . If a linear f is general for M , then for any r -cell c^r we can define an **indicator function**:

$$A(c^r, v, f) = \begin{cases} 1 & \text{if } v \in c^r, \text{ and } f(v) \geq f(x) \text{ for all } x \text{ in } c^r, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.5. We take a simple convex cell structure of D^2 with f being the height function as shown in Figure 3.1. We can index the vertices as follows:

$$\begin{aligned} A(c^r, v_1, f) &= \begin{cases} 1 & \text{when } c^r \in \{v_1, e_1, e_4, F_1\} \\ 0 & \text{otherwise} \end{cases} \\ A(c^r, v_2, f) &= \begin{cases} 1 & \text{when } c^r \in \{v_2, e_2\} \\ 0 & \text{otherwise} \end{cases} \\ A(c^r, v_3, f) &= \begin{cases} 1 & \text{when } c^r \in \{v_3\} \\ 0 & \text{otherwise} \end{cases} \\ A(c^r, v_4, f) &= \begin{cases} 1 & \text{when } c^r \in \{v_4, e_3\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

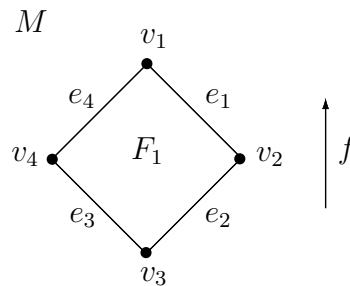


Figure 3.1: A convex cell structure of D^2 .

Lemma 3.6. If a linear map f is general for a convex cell complex M , then for a fixed $c^r \in M$,

$$\sum_{v \in M} A(c^r, v, f) = 1.$$

Proof. A general function f reaches its maximum on c^r at exactly one vertex. \square

Definition 3.7. If a linear f is general for a convex cell complex M , we define the **index** of v with respect to f to be:

$$a(v, f) = \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v, f).$$

We now return to the above example.

Example 3.5 (A). The index of each vertex is as follows:

$$\begin{aligned} a(v_1, f) &= \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v_1, f) = 1 + (-1) \cdot 2 + 1 = 0 \\ a(v_2, f) &= \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v_2, f) = 1 + (-1) \cdot 1 = 0 \\ a(v_3, f) &= \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v_3, f) = 1 \\ a(v_4, f) &= \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v_4, f) = 1 + (-1) \cdot 1 = 0 \end{aligned}$$

Theorem 3.8 (Critical Point Theorem). *If a linear f is general for a convex cell complex M , then*

$$\sum_{v \in M} a(v, f) = \chi(M).$$

Proof.

$$\begin{aligned} \sum_{v \in M} a(v, f) &= \sum_{v \in M} \sum_{r=0}^k (-1)^r \sum_{c^r \in M} A(c^r, v, f) \\ &= \sum_{r=0}^k (-1)^r \sum_{c^r \in M} \left(\sum_{v \in M} A(c^r, v, f) \right) \\ &= \sum_{r=0}^k (-1)^r \alpha_r(M), \text{ where } \alpha_r(M) \text{ is the number of } r\text{-cells in } M. \\ &= \chi(M). \end{aligned}$$

\square

Example 3.5 (B). If we again look at the example above we see that

$$\sum_{v \in M} a(v, f) = 1$$

which is indeed the Euler characteristic of a 2-disc.

3.3 Applying a Group Action

We will be looking at finite group actions on our complexes and will utilise linear representations of finite groups whilst looking at this action.

3.3.1 Critical Points on G -complexes

Definition 3.10. Let G be a finite group and M a k -dimensional convex cell complex. We call M a **G -complex** if, when we apply G to our complex, r -cells are mapped to r -cells. Let f be a linear map on M . We call f **G -invariant** if, under the G -action, everything remains ‘at the same height’, i.e., $f(x) = f(gx)$.

Definition 3.11. Let $\mathcal{V}(M)$ be the **set of all vertices** in a complex M . An orbit of vertices \mathcal{O} is an element of $\mathcal{V}(M)/G$.

Definition 3.12. As an analogue to Definition 3.4 we may now define an **indicator function**. Let \mathcal{C}^r be the set of all r -cells and \mathcal{O} an orbit of vertices then:

$$A_G(\mathcal{O}, r, f) = [\{\sigma : f|_{\sigma} \text{ has its maximum at some } v \in \mathcal{O}, \text{ where } \sigma \in \mathcal{C}^r\}].$$

Here square brackets indicate the character of G on this collection of simplices, see Section 2.6.1.

Example 3.13. Let M be the convex cell complex shown in Figure 3.2, let f again be the height function, and let G be \mathbb{Z}_2 with the action of rotation by π about the axis through v_1 and v_6 .

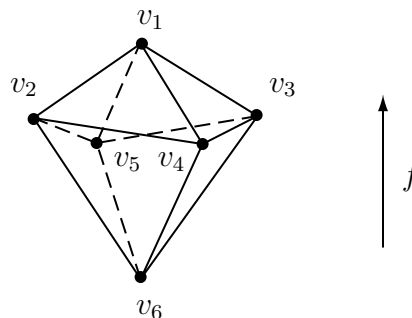


Figure 3.2: Simplicial structure on a sphere.

We fix r and look at each orbit in turn. For this example we will focus on the orbit \mathcal{O}_2 consisting of vertices v_2 and v_3 . To find the character on the r -simplices, we take a basis consisting of the r -simplices in question and write down for each element of G the associated permutation matrix. Taking the trace will now give us the character. The canonical decomposition is easy to find since we are looking for linear combinations of the irreducible characters.

$$\begin{aligned} A_G(\mathcal{O}_2, 0, f) &= [(v_2), (v_3)] &&= A_0 + A_1 \\ A_G(\mathcal{O}_2, 1, f) &= [(v_2, v_4), (v_2, v_5), (v_2, v_6), (v_3, v_4), (v_3, v_5), (v_3, v_6)] &&= 3(A_0 + A_1) \\ A_G(\mathcal{O}_2, 2, f) &= [(v_2, v_4, v_6), (v_2, v_5, v_6), (v_3, v_4, v_6), (v_3, v_5, v_6)] &&= 2(A_0 + A_1). \end{aligned}$$

Here A_0 and A_1 are the trivial and non-trivial representations of \mathbb{Z}_2 respectively. See Appendix B for the full character table.

Lemma 3.14. *If f is general for a simplicial complex M , then for fixed r ,*

$$\sum_{\mathcal{O} \in \mathcal{V}(M)/G} A_G(\mathcal{O}, r, f) = [C_r(M)],$$

where $[C_r(M)]$ is the character of G on $C_r(M)$.

Proof. A general function f reaches its maximum on σ , an r -simplex, at exactly one vertex. □

Definition 3.15. If f is general for a k -dimensional simplicial complex M , define the **index of the orbit** \mathcal{O} , with respect to f to be:

$$a_G(\mathcal{O}, f) = \sum_{r=0}^k (-1)^r A_G(\mathcal{O}, r, f).$$

Example 3.13 (A). We return to the previous example and summarize the results in a table.

$A_G(r, \mathcal{O}, f)$	$r = 0$	$r = 1$	$r = 2$
$\mathcal{O}_1 = \{v_1\}$	A_0	$2(A_0 + A_1)$	$2(A_0 + A_1)$
$\mathcal{O}_2 = \{v_2, v_3\}$	$A_0 + A_1$	$3(A_0 + A_1)$	$2(A_0 + A_1)$
$\mathcal{O}_3 = \{v_4, v_5\}$	$A_0 + A_1$	$A_0 + A_1$	0
$\mathcal{O}_4 = \{v_6\}$	A_0	0	0

So the indexes of the orbits are as follows:

$$\begin{aligned} a_G(\mathcal{O}_1, f) &= \sum_{r=0}^2 (-1)^r A_G(\mathcal{O}_1, r, f) = A_0 \\ a_G(\mathcal{O}_2, f) &= \sum_{r=0}^2 (-1)^r A_G(\mathcal{O}_2, r, f) = 0 \\ a_G(\mathcal{O}_3, f) &= \sum_{r=0}^2 (-1)^r A_G(\mathcal{O}_3, r, f) = 0 \\ a_G(\mathcal{O}_4, f) &= \sum_{r=0}^2 (-1)^r A_G(\mathcal{O}_4, r, f) = A_0. \end{aligned}$$

Theorem 3.16. *Let G be a group and M a G -complex. Then if f is general for M , we have the following:*

$$\sum_{\mathcal{O} \in \mathcal{V}(M)/G} a_G(\mathcal{O}, f) = \chi_G(M).$$

Proof.

$$\begin{aligned} \sum_{\mathcal{O} \in \mathcal{V}(M)/G} a_G(\mathcal{O}, f) &= \sum_{\mathcal{O}} \sum_r (-1)^r A_G(\mathcal{O}, r, f) \\ &= \sum_r (-1)^r \sum_{\mathcal{O}} A_G(\mathcal{O}, r, f) \\ &= \sum_r (-1)^r [C_r(M)] \\ &= \chi_G(M). \end{aligned}$$

□

Example 3.13 (B). We again return to our example. We calculate $\sum_{\mathcal{O} \in \mathcal{V}(M)/G} a_G(\mathcal{O}, f)$ using the calculations from Example 3.13A:

$$\begin{aligned} \sum_{\mathcal{O} \in \mathcal{V}(M)/G} a_G(\mathcal{O}, f) &= (A_0 + A_0) \\ &= 2A_0. \end{aligned}$$

Next we calculate the character on the groups of r -chains:

$$\begin{aligned} \chi_G(M) &= \sum_r^k (-1)^r [C_r(M)] = 4(A_0 + A_1) - 6(A_0 + A_1) + (4A_0 + 2A_1) \\ &= 2A_0. \end{aligned}$$

We indeed have that $\sum_{\mathcal{O} \in \mathcal{V}(M)/G} a_G(\mathcal{O}, f) = \chi_G(M)$.

Chapter 4

G -Lefschetz Number

In 1912 Brouwer [9] proved what is now known as the Brouwer fixed point theorem.

If $f : X \rightarrow X$ is a map and X is homeomorphic to the (closed) unit ball B^n in euclidean space \mathbb{R}^n , then $f(x) = x$ has a solution.

A decade or so later Solomon Lefschetz formulated and proved his fixed point theorem, a version of which we will discuss and prove later. This theorem generalised the Brouwer fixed point theorem to compact manifolds. The theorem states that if a function on a compact manifold $f : X \rightarrow X$ has a non-zero Lefschetz number, that is $\lambda(f) = \sum_i (-1)^i \text{Tr}(f_i, H_i(X))$, then the function f has a fixed point.

In this chapter we will discuss the Lefschetz number and corresponding fixed point theorem. In the latter half we will develop these ideas by extending them to incorporate a group action. This will allow us to establish a Lefschetz fixed orbit theorem.

4.1 Lefschetz Fixed Point Theorem

Definition 4.1. Let X be a k -dimensional simplicial complex and f_i be a simplicial chain map from $C_i(X)$ to $C_i(X)$. This induces a function f_* on the homology groups of X . The number $\lambda(f) = \sum_{i=0}^k (-1)^i \text{Tr}(f_*; H_i(X))$ is called the **Lefschetz number** of f . Recall that we take homology over \mathbb{R} .

Lemma 4.2. *We can write the Lefschetz number as*

$$\lambda(f) = \sum_{i=0}^k (-1)^i \operatorname{Tr}(f_i; C_i(X)).$$

Proof. Let X be a k -dimensional simplicial complex and f_i a simplicial map with $f_i : C_i(X) \rightarrow C_i(X)$. Let $Z_i(X)$ be the group of i -cycles and $B_{i-1}(X)$ the $(i-1)$ -boundaries of X , that is $Z_i(X) = \operatorname{Ker} \partial(C_i(X))$ and $B_{i-1}(X) = \operatorname{Im} \partial(C_{i-1}(X))$. We may write $C_i(X) = Z_i(X) \oplus B_{i-1}(X)$ by the first isomorphism theorem. Since f_i is a simplicial map, we know that it must map boundaries to boundaries and cycles to cycles, so we can write $f_i = f_{Z_i} \oplus f_{B_{i-1}}$, where $f_{Z_i} : Z_i(X) \rightarrow Z_i(X)$ and $f_{B_{i-1}} : B_{i-1}(X) \rightarrow B_{i-1}(X)$. Properties of trace give us that

$$\operatorname{Tr}(f_i; C_i(X)) = \operatorname{Tr}(f_{Z_i}; Z_i(X)) + \operatorname{Tr}(f_{B_{i-1}}; B_{i-1}(X)).$$

If we now let $\operatorname{Tr}(f_i; C_i(X)) = c_i$, $\operatorname{Tr}(f_{Z_i}; Z_i(X)) = z_i$ and $\operatorname{Tr}(f_{B_i}; B_i(X)) = b_i$, we can write:

$$\begin{aligned} \lambda(f) &= \sum_{i=0}^k (-1)^i \operatorname{Tr}(f_i; C_i(X)) \\ &= c_0 - c_1 + \cdots + (-1)^k c_k \\ &= (z_0 + b_{-1}) - (z_1 + b_0) + \cdots + (-1)^k (z_k + b_{k-1}). \end{aligned} \tag{4.1}$$

The i -th homology group of X is $H_i(X) = Z_i(X)/B_i(X)$. As we are working over \mathbb{R} $Z_i(X) = H_i(X) \oplus B_i(X)$ and again using the properties of trace,

$$\operatorname{Tr}(f_i; Z_i(X)) = \operatorname{Tr}(f_*; H_i(X)) + \operatorname{Tr}(f_i; B_i(X)).$$

Let $h_i = \operatorname{Tr}(f_*; H_i(X))$, so that $h_i = z_i - b_i$. Combining this with our previous definition of Lefschetz number we see that

$$\begin{aligned} \lambda(f) &= \sum_{i=0}^k (-1)^i \operatorname{Tr}(f_*; H_i(X)) \\ &= h_0 - h_1 + \cdots + (-1)^k h_k \\ &= (z_0 - b_0) - (z_1 - b_1) + \cdots + (-1)^k (z_k - b_k). \end{aligned} \tag{4.2}$$

It remains for us to show that

$$\sum_{i=0}^k (-1)^i \operatorname{Tr}(f_*; H_i(X)) - \sum_{i=0}^k (-1)^i \operatorname{Tr}(f_i; C_i(X)) = 0.$$

We now take the difference of Equations (4.1) and (4.2) to obtain:

$$\begin{aligned} & (z_0 + b_{-1}) - (z_1 + b_0) + \cdots + (-1)^k(z_k + b_{k-1}) \\ & - [(z_0 - b_0) - (z_1 - b_1) + \cdots + (-1)^k(z_k - b_k)] = b_{-1} + (-1)^k b_k. \end{aligned}$$

Since we assumed that X is k -dimensional, there are no k -boundaries, giving $b_k = 0$. By definition, the boundary of a 0-cell is zero and so $b_{-1} = 0$. Hence $b_{-1} + (-1)^k b_k = 0$ and we can conclude that

$$\lambda(f) = \sum_{i=0}^k (-1)^i \operatorname{Tr}(f; H_i(X)) = \sum_{i=0}^k (-1)^i \operatorname{Tr}(f_i; C_i(X)).$$

□

Theorem 4.3 (Simplicial Lefschetz Fixed Point Theorem). *If X is a finite simplicial complex and $f : X \rightarrow X$ is a simplicial map with non-zero Lefschetz number, then f has a fixed point.*

Proof. Since $\lambda(f)$ is non-zero, one of the $\operatorname{Tr}(f_i)$ must also be non-zero. We now fix i to look at one such non-zero $\operatorname{Tr}(f_i)$. Let F_i be the matrix representation of f_i with basis the i -simplices in X . Since f is simplicial we have that every simplex is mapped to another simplex, possibly itself, therefore all the entries in F_i are in the set $\{1, 0, -1\}$. If we have an entry on the diagonal then a simplex must be mapped to itself in some way. We now consider the barycentre $b = \frac{1}{n+1} \sum_j v_j$ of a fixed simplex (v_0, \dots, v_n) and apply f

$$f \left(\frac{1}{n+1} \sum_j v_j \right) = \frac{1}{n+1} \sum_j f(v_j) = \frac{1}{n+1} \sum_j v_j.$$

This means that the barycentre is fixed and therefore f must have a fixed point. □

Theorem 4.4 (Lefschetz Fixed Point Theorem, [10]). *If X is a compact manifold and $f : X \rightarrow X$ is a map such that $\lambda(f) \neq 0$, then every map homotopic to f has a fixed point.*

Proof. The proof of this result follows from Theorem 4.3 and the Simplicial Approximation Theorem below. □

Theorem 4.5 (Simplicial Approximation Theorem, [14]). *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \rightarrow L$ is homotopic to a map that is simplicial with respect to some iterated subdivision of K .*

4.2 G -Lefschetz Fixed Orbit Theorem

The aim of this section is to produce an analogue of the Lefschetz number in terms of representations, by applying a finite group to our simplicial complex. Note that f is a simplicial map throughout as in the previous section, in addition to being G -equivariant. Specifically we are looking for an index that will pick up when f has a fixed orbit, that is, $f(\mathcal{O}) = \mathcal{O}$. Throughout we assume that G is a finite group and use the definitions and properties outlined in Section 2.6.

Definition 4.6. Let V be a representation decomposed into isotypics V_i of type W_i , see Section 2.6.2. Let $f : V \rightarrow V$ be a G -map. We know by Schur's lemma that $f(V_i) \subseteq V_i$ and so we can decompose f into parts $f = \bigoplus_i f_i$ such that $f_i : V_i \rightarrow V_i$. Now we define the G -trace of f to be

$$\mathrm{Tr}_G(f) = \sum_i \frac{\mathrm{Tr}(f_i)}{\dim W_i} [W_i],$$

where $[W_i]$ is the character of G in W_i .

As with trace, G -trace has some basic properties which we will now formalise.

Proposition 4.7 (Properties of G -trace). *Let $f, g : V \rightarrow V$ and $f_i : V_i \rightarrow V_i$ be G -maps. Then the following properties of G -trace hold:*

- i) $\mathrm{Tr}_G(\mathrm{id}) = [V]$.
- ii) $\mathrm{Tr}_G(fg) = \mathrm{Tr}_G(gf)$.
- iii) $\mathrm{Tr}_G(f_1 \oplus f_2) = \mathrm{Tr}_G(f_1) + \mathrm{Tr}_G(f_2)$.
- iv) $\mathrm{Tr}_G(rf) = r\mathrm{Tr}_G(f)$, for any scalar r .
- v) $\dim(\mathrm{Tr}_G(f)) = \mathrm{Tr}(f)$.

Proof. i) We decompose V into isotypics of type W_i so that

$$[V] = \sum_i [V_i] = \sum_i n_i [W_i],$$

where square brackets, as always, denote the character of G . We now consider our definition of G -trace with f the identity,

$$\begin{aligned} \mathrm{Tr}_G(\mathrm{id}) &= \sum_i \frac{\mathrm{Tr}(\mathrm{id}_i)}{\dim W_i} [W_i] \\ &= \sum_i n_i \frac{\dim W_i}{\dim W_i} [W_i], \quad \text{since } \mathrm{Tr}(\mathrm{id}_i) = n_i \dim W_i \\ &= \sum_i n_i [W_i] \\ &= [V]. \end{aligned}$$

ii) Let $V = \bigoplus V_i$ be the isotypic decomposition of V . By Schur's Lemma we can write $(fg)_i : V_i \rightarrow V_i$.

$$\begin{aligned} \mathrm{Tr}_G(fg) &= \sum_i \frac{\mathrm{Tr}((fg)_i)}{\dim W_i} [W_i] \\ &= \sum_i \frac{\mathrm{Tr}((gf)_i)}{\dim W_i} [W_i], \quad \text{by standard properties of trace} \\ &= \mathrm{Tr}_G(gf). \end{aligned}$$

iii) We may decompose V_1 and V_2 into isotypics (Section 2.6.2), so that $V_1 = \bigoplus_i V_{1_i} = \bigoplus_i m_i W_i$ and $V_2 = \bigoplus_i V_{2_i} = \bigoplus_i n_i W_i$. If we now take the direct sum of V_1 and V_2 , we obtain $V_1 \oplus V_2 = \bigoplus_i (m_i + n_i) W_i$. Since f_1 acts on V_1 and f_2 acts on V_2 the direct sum of f_1 and f_2 maps $V_1 \oplus V_2$ to itself. The G -trace is then given by the following:

$$\mathrm{Tr}_G(f_1 \oplus f_2) = \sum_i \frac{\mathrm{Tr}(f_{1_i} \oplus f_{2_i})}{\dim W_i} [W_i],$$

where f_{j_i} is the direct summand of f_j that permutes the copies of W_i within V_{j_i} .

Then by properties of trace

$$\mathrm{Tr}(f_{1_i} \oplus f_{2_i}) = \mathrm{Tr}(f_{1_i}) + \mathrm{Tr}(f_{2_i}),$$

and we have that

$$\begin{aligned}
\mathrm{Tr}_G(f_1 \oplus f_2) &= \sum_i \frac{\mathrm{Tr}(f_{1_i} \oplus f_{2_i})}{\dim W_i} [W_i] \\
&= \sum_i \frac{\mathrm{Tr}(f_{1_i}) + \mathrm{Tr}(f_{2_i})}{\dim W_i} [W_i] \\
&= \sum_i \frac{\mathrm{Tr}(f_{1_i})}{\dim W_i} [W_i] + \sum_i \frac{\mathrm{Tr}(f_{2_i})}{\dim W_i} [W_i] \\
&= \mathrm{Tr}_G(f_1) + \mathrm{Tr}_G(f_2).
\end{aligned}$$

iv) Let r be a scalar. The G -trace of rf is as follows:

$$\begin{aligned}
\mathrm{Tr}_G(rf) &= \sum_i \frac{\mathrm{Tr}(rf_i)}{\dim W_i} [W_i] \\
&= \sum_i \frac{r\mathrm{Tr}(f_i)}{\dim W_i} [W_i] \quad \text{by standard properties of trace} \\
&= r \sum_i \frac{\mathrm{Tr}(f_i)}{\dim W_i} [W_i] \\
&= r\mathrm{Tr}_G(f).
\end{aligned}$$

v) We evaluate $\mathrm{Tr}_G(f)$ at e to obtain the dimension of $\mathrm{Tr}_G(f)$, where e denotes the identity element of G .

$$\begin{aligned}
\dim(\mathrm{Tr}_G(f)) &= \mathrm{Tr}_G(f)(e) \\
&= \sum_i \frac{\mathrm{Tr}(f_i)}{\dim W_i} [W_i](e), \\
&= \mathrm{Tr}(f) \frac{\dim W_i}{\dim W_i} \\
&= \mathrm{Tr}(f).
\end{aligned}$$

□

We are now in a position to define G -Lefschetz number. We use the term ‘number’ here loosely as we are, in fact, dealing with representations and characters.

Definition 4.8. Let X be a k -dimensional simplicial complex. The **G -Lefschetz number** of a map $f : X \rightarrow X$ is defined to be:

$$\lambda_G(f) = \sum_{i=0}^k (-1)^i \mathrm{Tr}_G(f_*; H_i(X)).$$

Recall (from Note 2.10) that if $f : X \rightarrow X$ is G -equivariant then so are the induced maps $f_i : C_i(X) \rightarrow C_i(X)$.

Lemma 4.9. *We can write the G -Lefschetz number of f as:*

$$\lambda_G(f) = \sum_{i=0}^k (-1)^i \operatorname{Tr}_G(f; C_i(X)).$$

Proof. Following the proof of Lemma 4.2 we write $C_i(X) = Z_i(X) \oplus B_{i-1}(X)$ and $Z_i(X) = H_i(X) \oplus B_i(X)$. Again we have that $f : Z_i(X) \rightarrow Z_i(X)$ and $f : B_i(X) \rightarrow B_i(X)$, and so by Proposition 4.7(iii):

$$\begin{aligned} \operatorname{Tr}_G(f; C_i(X)) &= \operatorname{Tr}_G(f; Z_i(X)) + \operatorname{Tr}_G(f; B_{i-1}(X)), \text{ and} \\ \operatorname{Tr}_G(f; H_i(X)) &= \operatorname{Tr}_G(f; Z_i(X)) - \operatorname{Tr}_G(f; B_i(X)). \end{aligned}$$

Now, if we let $c_i = \operatorname{Tr}_G(f; C_i(X))$, $z_i = \operatorname{Tr}_G(f; Z_i(X))$, $b_i = \operatorname{Tr}_G(f; B_i(X))$ and $h_i = \operatorname{Tr}_G(f; H_i(X))$, then we are left with the same situation as before (albeit with representations rather than numbers) and we can conclude that:

$$\lambda_G(f) = \sum_{i=0}^k (-1)^i \operatorname{Tr}_G(f; H_i(X)) = \sum_{i=0}^k (-1)^i \operatorname{Tr}_G(f; C_i(X)).$$

□

Corollary 4.10. *Let $f : X \rightarrow X$, and X a k -dimensional simplicial complex, then*

i) $\lambda_G(f) = \chi_G(X)$, when $f = \operatorname{id}$, and

$$ii) \chi_G(X) = \sum_{i=0}^k (-1)^i [C_i(X)].$$

Proof.

i) We set $f = \operatorname{id}$ and by Definition 4.8 we have

$$\begin{aligned} \lambda_G(\operatorname{id}) &= \sum_{i=0}^k (-1)^i \operatorname{Tr}_G(\operatorname{id}; H_i(X)) \\ \lambda_G(\operatorname{id}) &= \sum_{i=0}^k (-1)^i [H_i(X)] \quad \text{by Proposition 4.7(i)}. \end{aligned}$$

Which is by definition the G -Euler characteristic of M .

ii) We have from part (i) that $\lambda_G(\operatorname{id}) = \chi_G(X)$ and we have

$$\begin{aligned} \lambda_G(f) &= \sum_{i=0}^k (-1)^i \operatorname{Tr}_G(f; C_i(X)) \quad \text{by Lemma 4.9,} \\ &= \sum_{i=0}^k (-1)^i [C_i(X)] \quad \text{by Proposition 4.7(i),} \end{aligned}$$

and so we are able to say that $\chi_G(X) = \sum_{i=0}^k (-1)^i [C_i(X)]$. \square

Example 4.11. As in Example 3.13, let X be the 2-sphere with the simplicial structure shown in Figure 4.1. Let f be rotation by π and set $G = \mathbb{Z}_2$ with action given by rotation by π .

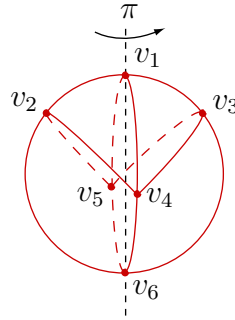


Figure 4.1: Simplicial structure on a sphere.

We look at the zero-chains of X : $C_0(X) = \text{span}\{v_1, v_2, v_3, v_4, v_5, v_6\}$. These split into orbits. Letting κ be the generator of \mathbb{Z}_2 , we can summarize these actions in a table.

Orbit	Mapping under κ	Character	Mapping under f
$\mathcal{O}_1 = \{v_1\}$	$\mathcal{O}_1 = \{v_1\}$	A_0	$\mathcal{O}_1 = \{v_1\}$
$\mathcal{O}_2 = \{v_2, v_3\}$	$\mathcal{O}_2 = \{v_3, v_2\}$	$A_0 + A_1$	$\mathcal{O}_2 = \{v_2, v_3\}$
$\mathcal{O}_3 = \{v_4, v_5\}$	$\mathcal{O}_3 = \{v_5, v_4\}$	$A_0 + A_1$	$\mathcal{O}_3 = \{v_4, v_5\}$
$\mathcal{O}_4 = \{v_6\}$	$\mathcal{O}_4 = \{v_6\}$	A_0	$\mathcal{O}_4 = \{v_6\}$

The total character for $C_0(X)$ is therefore $4A_0 + 2A_1$. We are looking for the G -trace, so we need to look at each isotypic representation separately. We take a basis for each isotypic representation: for the A_0 part $\{v_1, v_2 + v_3, v_4 + v_5, v_6\}$ and for the A_1 part $\{v_2 - v_3, v_4 - v_5\}$. Next we look at the G -trace on each part and sum over all types to get the G -Lefschetz number of f .

$$F_G(A_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad F_G(A_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

From this we can easily see that $\text{Tr}_G(f; C_0(X)) = 4A_0 - 2A_1$.

Theorem 4.12 (The Simplicial G -Lefschetz Fixed Orbit Theorem). *If X is a finite simplicial G -complex and $f : X \rightarrow X$ is a G -equivariant simplicial map with $\lambda_G(f) \neq 0$, then f has a fixed orbit, that is $f(\mathcal{O}_x) = \mathcal{O}_x$ for some x a point in X .*

Proof. Let $C_i(X)$ be the \mathbb{R} -vector space with basis $\{\sigma_1, \dots, \sigma_k\}$ consisting of all i -simplices of X . We may decompose $C_i(X)$ thus:

$$C_i(X) = \mathbb{R}\mathcal{O}_1 \oplus \dots \oplus \mathbb{R}\mathcal{O}_r,$$

where each $\mathbb{R}\mathcal{O}_j$ is the \mathbb{R} -vector subspace of $C_i(X)$ spanned by the orbit \mathcal{O}_j of i -simplices. Let $\{W_j\}_j$ denote the distinct irreducible summands of $C_i(X)$, then we may decompose each the orbit vector spaces thus $\mathbb{R}\mathcal{O}_l = \bigoplus_j n_{lj}W_j$. Let the ‘ W_j -part’ of each orbit be denoted $\mathcal{O}_l|_{W_j} = n_{lj}W_j$ and let $V_j = \bigoplus_l \mathcal{O}_l|_{W_j}$. Then in the notation of Definition 4.6, $f_j : V_j \rightarrow V_j$. Since f is a simplicial map the matrix representation F_j of the map f_j with basis $\{\mathcal{O}_l|_{W_j}\}_l$ has only entries from the set $\{1, 0, -1\}$. If we have that $\lambda_G(f) \neq 0$, it follows from the definition that F_j must have a non-zero entry on its diagonal; in turn this corresponds to an orbit of i -simplices that is mapped to itself in some way. We now consider such a fixed orbit of i -simplices and let $\sigma = (v_0, \dots, v_n)$ be a simplex representative for the orbit, with $g\sigma = \sigma'$. The G -action on the barycentre of σ , $b_\sigma = \frac{1}{1+n} \sum_i v_i$, gives that

$$\begin{aligned} gb_\sigma &= \frac{1}{1+n} \sum_i gv_i \\ &= \frac{1}{1+n} \sum v'_i \\ &= gb_{\sigma'}. \end{aligned}$$

This means that the barycentres of the simplices in this orbit also form an orbit under the action of G . If we apply f to this orbit we obtain:

$$f(gb_\sigma) = \frac{1}{1+n} \sum_i f(gv_i) = \frac{1}{1+n} \sum_i f(v'_i) = f(b_{\sigma'}),$$

and so the orbit \mathcal{O}_{b_σ} is fixed under f .

Note 4.13. Since the orbit of barycentres is fixed under the map f we can say that were the G -action trivial, these points would be fixed points of f .

□

Example 4.12 (A). We refer back to Example 4.11.

i) Let us look at the 1-chains of X , $C_1(X) \cong \mathbb{R}^{12}$. The orbits are: $\mathcal{O}_1 = \{v_1v_2, v_1v_3\}$, $\mathcal{O}_2 = \{v_1v_4, v_1v_5\}$, $\mathcal{O}_3 = \{v_2v_4, v_3v_5\}$, $\mathcal{O}_4 = \{v_2v_5, v_3v_4\}$, $\mathcal{O}_5 = \{v_2v_6, v_3v_6\}$ and $\mathcal{O}_6 = \{v_4v_6, v_5v_6\}$. Under f every orbit is mapped to itself. This means that each orbit has $A_0 + A_1$ as its character. Let e_1 and e_2 be the two edges in an orbit. Then a basis that will split each of these orbits into the two isotypics is $\{e_1 + e_2, e_1 - e_2\}$. The two representative permutation matrices are both I_6 , therefore the trace of these matrices is 6 and so the G -trace on $C_1(X)$ is $6A_0 + 6A_1$.

ii) Now we look at the 2-chains of X , $C_2(X) \cong \mathbb{R}^8$. There are five orbits and they are: $\mathcal{O}_1 = \{v_1v_2v_4, v_1v_3v_5\}$, $\mathcal{O}_2 = \{v_1v_2v_5, v_1v_3v_4\}$, $\mathcal{O}_3 = \{v_2v_4v_6, v_3v_5v_6\}$ and $\mathcal{O}_4 = \{v_2v_5v_6, v_3v_4v_6\}$. It is easy to see that the G -trace on $C_2(X)$ is $4A_0 + 4A_1$.

iii) Putting this and Example 4.11 together, we get that the G -Lefschetz number is:

$$\lambda_G(f) = (4A_0 + 2A_1) - (6A_0 + 6A_1) + (4A_0 + 4A_1) = 2A_0.$$

We can conclude from this that f has a fixed orbit.

We end this chapter with a conjecture, the proof of which will probably include a G -equivariant version of the Simplicial Approximation Theorem which is also stated below.

Conjecture 4.14 (G -Lefschetz Fixed orbit Theorem). *If X is a compact G -manifold and $f : X \rightarrow X$ is a G -equivariant map such that $\lambda_G(f) \neq 0$, then every G -equivariant map homotopic to f has a fixed orbit.*

Conjecture 4.15 (G -Simplicial Approximation Theorem). *If K is a finite simplicial G -complex and L is an arbitrary simplicial G -complex, then any G -equivariant map $f : K \rightarrow L$ is homotopic to an equivariant map that is simplicial with respect to some iterated barycentric subdivision of K .*

Chapter 5

Morse Theory

5.1 Morse Theory

The material in this section is taken from Morse Theory by Milnor [19].

Let f be a smooth real valued function on a k -dimensional manifold M . A point $x \in M$ is a **critical point** if, when we choose a local coordinate system (x_1, \dots, x_k) in a neighbourhood of x , all the first order partial derivatives evaluated at x are zero. That is

$$\left. \frac{\partial f}{\partial x_1} \right|_x = \dots = \left. \frac{\partial f}{\partial x_k} \right|_x = 0.$$

A critical point x of f is **non-degenerate** if and only if the Hessian matrix, $\left(\left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right|_x$ is non-singular. A function is a **Morse function** if it has only isolated non-degenerate critical points.

The **index** $\text{ind}(x)$ of the point x is defined to be the number of negative eigenvalues of the Hessian of f evaluated at x .

5.1.1 The Morse Inequalities

The Morse Inequalities describe the relationship between the critical points of a Morse function on M and the topology of M (where M is a manifold). We will take homology groups over the real numbers.

Theorem 5.1 (Morse inequalities). *Let f be a Morse function on a k -dimensional manifold M . Let c_i be the number of critical points of index i and b_i the i -th Betti number of M - that is, the rank of the i -th homology group $H_i(M)$. Then for each j*

$$b_j - b_{j-1} + \cdots + (-1)^j b_0 \leq c_j - c_{j-1} + \cdots + (-1)^j c_0,$$

with equality when $j = k$.

For a non-degenerate f on a manifold M , Morse introduced the following polynomial:

$$\mathfrak{M}_t(f) = \sum_p t^{\lambda_p}, \quad p \in C(f).$$

Here $C(f)$ is the set of critical points of f , and λ_p is the index of p . We call $\mathfrak{M}_t(f)$ the **Morse polynomial** of f . The **Poincaré polynomial** of M is given by

$$P_t(M) = \sum_{j=0}^k t^j b_j,$$

where b_j is the j -th Betti number of M .

We can now restate Theorem 5.1 as follows. For every non-degenerate f there exists a polynomial $Q_t(f) = q_0 + q_1 t + \cdots + q_n t^n$ with non-negative coefficients q_i such that

$$\mathfrak{M}_t(f) - P_t(M) = (1+t)Q_t(f). \quad (5.1)$$

To see that these are equivalent, write $d_j = c_j - b_j$, where c_j and b_j are as above. We consider the Morse inequalities at each stage to obtain the coefficients q_i :

$$\begin{aligned} q_0 &:= & d_0 &\geq 0 \\ q_1 &:= & d_1 - d_0 &\geq 0 \\ q_2 &:= & d_2 - d_1 + d_0 &\geq 0 \\ &\vdots & & \vdots \\ q_k &:= & d_k - d_{k-1} + \cdots \pm d_0 &\geq 0, \end{aligned}$$

then $\mathfrak{M}_t(f) - P_t(M) = (1+t)Q(t)$ because

$$\begin{aligned}
 (1+t)Q(t) &= (1+t) \sum_{i=0}^k (q_i t^i) \\
 &= \sum_{i=0}^k (q_{i-1} + q_i) t^i \\
 &= \sum_{i=0}^k (d_i t^i) \\
 &= \sum_{i=0}^k (c_i t^i - b_i t^i) \\
 &= \mathfrak{M}_t(f) - P_t(M).
 \end{aligned}$$

We now follow Milnor's proof in Morse Theory [19]. The following definition is an extension of a similar definition in Milnor's book; the original takes pairs of spaces to the integers, and we extend this in order to deal later with representations.

Definition 5.2. A function S from pairs of spaces to a partially ordered abelian group is said to be **subadditive** if whenever we have that $X \supset Y \supset Z$, we have:

$$S(X, Z) \leq S(X, Y) + S(Y, Z).$$

If equality holds, S is **additive**.

Lemma 5.3. *Let S be subadditive and let $X_0 \subset \dots \subset X_n$. Then*

$$S(X_n, X_0) \leq \sum_{i=1}^n S(X_i, X_{i-1}).$$

If S is additive then equality holds.

Proof. We work by induction on n . For $n = 1$, equality holds. For $n = 2$, we have the definition of (sub)additivity:

$$S(X_2, X_0) \leq S(X_2, X_1) + S(X_1, X_0).$$

Suppose the statement is true for $n = m$. Then $S(X_m, X_0) \leq \sum_{i=1}^m S(X_i, X_{i-1})$, again with equality if S is additive. Therefore

$$S(X_{m+1}, X_0) \leq S(X_m, X_0) + S(X_{m+1}, X_m) \leq \sum_{i=1}^{m+1} S(X_i, X_{i-1})$$

with equality holding for S additive. The result is true for all m . □

Let M be a k -dimensional manifold, and f a differentiable function on M with isolated non-degenerate critical points. Let M^a be the set of all points $x \in M$ such that $f(x) \leq a$. Let $a_1 < \cdots < a_i$ be such that M^{a_i} contains exactly i critical points, a_i is a regular value of f (that is not a critical point), and $M^{a_i} = M$. Let c^r be an r -cell and \dot{c}^r the boundary of c^r defined as $\dot{c}^r = \{x \in \mathbb{R} : |x| = 1\}$.

$$\begin{aligned} H_j(M^{a_i}, M^{a_{i-1}}) &= H_j(M^{a_{i-1}} \cup c^{\lambda_i}, M^{a_{i-1}}), \\ &\quad \text{where } \lambda_i \text{ is the index of the critical point} \\ &= H_j(c^{\lambda_i}, \dot{c}^{\lambda_i}), \text{ by homotopy and excision} \\ &= \begin{cases} \mathbb{R} & \lambda_i = j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note : $M^{a_0} = \emptyset$ and we have that $\emptyset = M^{a_0} \subset \cdots \subset M^{a_i} = M$ so we can apply Lemma 5.3 to this.

Lemma 5.4. *The j -th Betti number $b_j(X, Y)$ of the pair (X, Y) is a subadditive function.*

Proof. Consider the exact sequence of vector spaces:

$$\xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} C \xrightarrow{d} \cdots \rightarrow D \rightarrow 0.$$

Since the sequence is exact, we note that $\text{rank } a + \text{rank } b = \text{rank } A$, giving:

$$\begin{aligned} \text{rank } a &= \text{rank } A - \text{rank } b \\ &= \text{rank } A - \text{rank } B + \text{rank } c \\ &= \text{rank } A - \text{rank } B + \text{rank } C - \text{rank } d \\ &\quad \vdots \\ &= \text{rank } A - \text{rank } B + \text{rank } C + \cdots \pm \text{rank } D. \end{aligned} \tag{5.2}$$

Since $\text{rank } a$ is greater than or equal to zero we have that the right hand side above is also greater than or equal to zero.

We consider the following exact sequence

$$\text{Ker } d_{i+1} \xrightarrow{d_{i+1}} H_i(Y, Z) \rightarrow H_i(X, Z) \rightarrow H_i(X, Y) \xrightarrow{d_i} \text{Im } d_i$$

and let $K = \text{rank}(\text{Ker } d_{i+1})$ and $I = \text{rank}(\text{Im } d_i)$ we find that

$$K + b_i(X, Z) + I = b_i(X, Y) + b_i(Y, Z).$$

Since the rank of a space is always greater than or equal to zero we have that betti number is subadditive, that is $b_i(X, Z) \leq b_i(X, Y) + b_i(Y, Z)$. \square

Take $S = b_j(M^{a_i}, M^{a_{i-1}})$ to be the j -th Betti number of the pair $(M^{a_i}, M^{a_{i-1}})$. We then obtain

$$b_j(M, \emptyset) = b_j(M) \leq \sum_{i=1}^l b_j(M^{a_i}, M^{a_{i-1}}) = c_j,$$

where c_j is the number of critical points of index j .

Lemma 5.5. *The Euler characteristic $\chi(X, Y) = \sum_{j=0}^k (-1)^j b_j(X, Y)$ of the pair (X, Y) is an example of an additive function.*

Proof. We consider the argument in the proof of Lemma 5.4 and apply this to the long exact sequence of homology groups of the triple (X, Y, Z) where $X \supset Y \supset Z$:

$$0 \rightarrow H_k(Y, Z) \rightarrow H_k(X, Z) \rightarrow H_k(X, Y) \rightarrow H_{k-1}(Y, Z) \rightarrow \cdots \rightarrow H_0(X, Y) \rightarrow 0.$$

in this case we have $\text{rank } a = 0$ and so we have:

$$0 = b_k(Y, Z) - b_k(X, Z) + b_k(X, Y) - b_{k-1}(Y, Z) + \cdots - (-1)^k b_0(X, Y),$$

which leads us to conclude that the Euler characteristic is an additive function, that is $\chi(X, Z) = \chi(X, Y) + \chi(Y, Z)$. \square

We may now apply Lemma 5.3 with $S = \chi(X, Y)$ to $M^{a_0} \subset \cdots \subset M^{a_l}$ to obtain:

$$\begin{aligned} \chi(M) &= \sum_{i=1}^l \chi(M^{a_i}, M^{a_{i-1}}) \\ &= \chi(M^{a_l}, M^{a_{l-1}}) + \cdots + \chi(M^{a_1}, M^{a_0}) \\ &= \sum_{j=0}^k (-1)^j b_j(M^{a_l}, M^{a_{l-1}}) + \cdots + \sum_{j=0}^k (-1)^j b_j(M^{a_1}, \emptyset) \\ &= c_0 - \cdots + (-1)^k c_k \end{aligned} \tag{5.3}$$

Hence we have proved the following:

Theorem 5.6 (Weak Morse Inequalities). *Let c_i be the number of critical points of index i of a function f on M , and b_i the i -th Betti number of M . Then*

$$i) \quad b_i(M) \leq c_i,$$

$$ii) \quad \sum_{i=0}^k (-1)^i b_i(M) = \sum_{i=0}^k (-1)^i c_i.$$

Lemma 5.7. *The function S_j is subadditive where*

$$S_j(X, Y) = b_j(X, Y) - b_{j-1}(X, Y) + \cdots + (-1)^j b_0(X, Y).$$

Proof. We again consider the argument in the proof of Lemma 5.4 and apply this to the long exact sequence

$$\xrightarrow{\partial} H_j(Y, Z) \rightarrow H_j(X, Z) \rightarrow H_j(X, Y) \xrightarrow{\partial} H_{j-1}(Y, Z) \rightarrow \cdots,$$

we obtain

$$\text{rank } \partial = b_j(Y, Z) - b_j(X, Z) + b_j(X, Y) - b_{j-1}(Y, Z) + \cdots \geq 0.$$

Collecting terms gives $S_j(Y, Z) - S_j(X, Z) + S_j(X, Y) \geq 0$. □

We now apply this function to the spaces $\emptyset = M^{a_0} \subset \cdots \subset M^{a_l} = M$ to obtain the Morse Inequalities:

$$S_j(M) \leq \sum_{i=0}^l S_j(M^{a_i}, M^{a_{i-1}})$$

$$b_j - b_{j-1} + \cdots + (-1)^j b_0 \leq c_j - c_{j-1} + \cdots + (-1)^j c_0$$

This proves Theorem 5.1.

5.2 Introducing Flow

Here we follow the paper *Morse Theory, the Conley Index and Floer Homology* by Dietmar Salamon [30], however the original approach was devised by Smale in *Morse inequalities for a dynamical system* [33].

We consider the gradient flow $\dot{x} = -\nabla f$ of a smooth function $f : M \rightarrow \mathbb{R}$ on a smooth manifold M . The flow of \dot{x} will be denoted $\phi^s \in \text{Diff}(M)$, and is defined by

$$\frac{d}{ds}\phi^s = -\nabla f \circ \phi^s, \quad \phi^0 = \text{id}.$$

The rest points of ϕ^s (where $\frac{d}{ds}\phi^s = 0$) are the critical points of f , and f decreases along the curves ϕ^s . If f is a Morse function then the Hessian of f is non-singular at every critical point. We may define the **unstable set** at a critical point x to be

$$W_x^- = \{y \in M : \lim_{s \rightarrow -\infty} \phi^s(y) = x\}.$$

This is a submanifold of M , and its dimension is the **index** of the critical point, i.e., $\text{ind}(x) = \dim(W_x^-)$. We will call W_x^- the **unstable manifold** of f at x . This is equivalent to defining $\text{ind}(x)$ to be the number of negative eigenvalues of the Hessian of f evaluated at x . We now take the tangent space of W_x^- and let this, the **unstable space**, be denoted TW_x^- .

Note 5.8. There are many (equivalent) ways to define the index λ of a critical point x of f . For example the index of x in f is

- i) equal to the dimension of the unstable manifold $\lambda = \dim W_x^-$,
- ii) the positive number k such that $H_k(W_x^-, \partial W_x^-) \neq 0$,
- iii) the positive number k such that $H_k(N, N^-) \neq 0$ where N is a neighbourhood of x and $N^- = \{y \in N : f(y) \leq f(x)\}$.

The last two are seen to be equivalent since the pair (N, N^-) is homotopic to $(W_x^-, \partial W_x^-)$.

5.3 G -Morse Theory

We now apply a group action to our manifold. Since f is G -invariant, critical points are mapped to critical points of the same index. As always, our group G is finite.

Recall from Section 2.5.1 that the orbit of a point x is defined to be $\mathcal{O}_x = \{gx : g \in G\}$ and that in this chapter we are interested only in orbits of critical points.

The **index of an orbit** \mathcal{O} is defined to be the index of x where $x \in \mathcal{O}$, and will be denoted $\lambda_{\mathcal{O}}$.

Since we are only concerned with the orbits, we can naturally define the tangent space of the unstable manifold of the orbit to be

$$TW_{\mathcal{O}}^{-} = \bigoplus_{x \in \mathcal{O}} TW_x^{-},$$

which we will call the **unstable space** of the orbit. Let $[TW_{\mathcal{O}}^{-}]$ denote the character of the **orientation representation** of G on the sum of the orientations of the TW_x^{-} , thus:

$$[TW_{\mathcal{O}}^{-}] = \bigoplus_{x \in \mathcal{O}} [or(TW_x^{-})],$$

where $or(V)$ is the orientation vector space of V . This is a representation of dimension $|\mathcal{O}| = |G/G_x|$.

We now discuss an equivalent definition of the orientation representation. Recall from Note 5.8 that we may define the index of a critical point in terms of a small neighbourhood of x ; we now apply method (iii) of Note 5.8 to an orbit of critical points. Take a small neighbourhood N of the orbit \mathcal{O} and define $N^{-} = \{y \in N : f(y) \leq f(\mathcal{O})\}$. We earlier defined the index of \mathcal{O} to be the index of x where $x \in \mathcal{O}$, equivalently we can define the index of the orbit to be the positive k such that $H_k(N, N^{-})$ is non-zero. There is an induced action of G on the homology group and we call the representation $H_k(N, N^{-})$ the orientation representation of G , denoted $TW_{\mathcal{O}}^{-}$.

Example 5.9. Let $TW_{\mathcal{O}}^{-}$ be the direct sum of three copies of \mathbb{R} with the action of $G = S_3$, the symmetric group of degree three. Elements of cycle type c_3 act as rotation and those of type c_2 as reflection. Notice that in this example this reflection changes the orientation as in Figure 5.1. For a basis we choose the three copies of \mathbb{R} themselves with orientation as shown (Fig. 5.1). Since under the rotation none of the components are fixed we obtain a character value of zero and the identity gives a character value of three, we need only look at the reflection, see Figure 5.1. The

matrix representation of (23) acting on $TW_{\mathcal{O}}^-$ is

$$\rho_{(23)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The trace and therefore character value is -1. Now we know the character vector is $\chi[e, c_2, c_3] = [3, -1, 0]$ which gives the representation $[TW_{\mathcal{O}}^-] = A_1 + E$ where A_1 is the alternating character and E is the two dimensional character of S_3 .

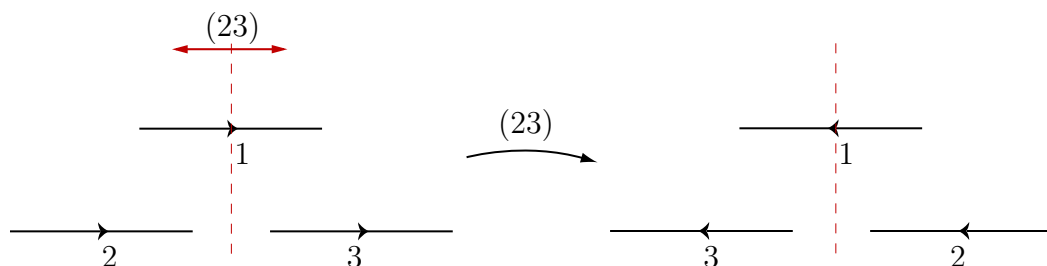


Figure 5.1: Orientation changing reflection $(23) \in S_3$.

Notation. We denote the set of orbits of index i by $\mathcal{O}r_i$.

Theorem 5.10 (G -Morse Inequalities). *Let G be a finite group and f be a G -invariant Morse function on M , a k -dimensional G -manifold. Let $[H_i(M)]$ be the character of G acting on the i -th homology group of M , and $[TW_{\mathcal{O}}^-]$ the orientation character of G acting on $TW_{\mathcal{O}}^-$. The G -Morse inequalities for each j are*

$$\begin{aligned} [H_j(M)] - [H_{j-1}(M)] + \cdots + (-1)^j [H_0(M)] \\ \leq \sum_{\mathcal{O} \in \mathcal{O}r_j} [TW_{\mathcal{O}}^-] - \sum_{\mathcal{O} \in \mathcal{O}r_{(j-1)}} [TW_{\mathcal{O}}^-] + \cdots + (-1)^j \sum_{\mathcal{O} \in \mathcal{O}r_0} [TW_{\mathcal{O}}^-], \end{aligned}$$

with equality when $j = k$.

Recall that to compare two representations or characters we compare the multiplicities of the isotypic decomposition, see Section 2.6.4.

Earlier we defined M^a to be the set of all points $x \in M$ such that $f(x) \leq a$, where f is a differentiable function on M , a k -dimensional manifold. Now we let $a_1 < \cdots < a_l$ be such that M^{a_i} contains i **orbits** of critical points, that is:

$$a_0 < f(\mathcal{O}_1) < a_1 < f(\mathcal{O}_2) < a_2 < \cdots < f(\mathcal{O}_l) < a_l.$$

Let $\lambda_{\mathcal{O}_i}$ be the index of \mathcal{O}_i , and $|\mathcal{O}_i|$ the number of points in orbit i . Then we define $C^{\lambda_{\mathcal{O}_i}} = c^{\lambda_i} \Pi \cdots \Pi c^{\lambda_i}$, $|\mathcal{O}_i|$ times with boundary $\dot{C}^{\lambda_{\mathcal{O}_i}}$. Again $M^{a_0} = \emptyset$ and $M^{a_l} = M$.

We can now say:

$$\begin{aligned} [H_j(M^{a_i}, M^{a_{i-1}})] &= [H_j(C^{\lambda_{\mathcal{O}_i}}, \dot{C}^{\lambda_{\mathcal{O}_i}})] \text{ by excision} \\ &= \begin{cases} [TW_{\mathcal{O}_i}^-] & \lambda_{\mathcal{O}_i} = j \text{ by excision} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\emptyset = M^{a_0} \subset \cdots \subset M^{a_l} = M$ as defined above, let $S_j = [H_j(M^{a_1}, M^{a_{l-1}})]$, and apply Lemma 5.3 to obtain

$$[H_j(M)] \leq \sum_{i=1}^l [H_j(M^{a_i}, M^{a_{i-1}})] = \sum_{\mathcal{O} \in \mathcal{O}_{r_j}} [TW_{\mathcal{O}}^-].$$

Recall from Corollary 4.10 that the G -Euler characteristic may be defined as $\chi_G(X) = \sum_{r=0}^k (-1)^r [H_r(X)]$. As with the Euler characteristic, the G -Euler characteristic is an additive function. By Lemma 5.3 and Note 2.20 we may write

$$\begin{aligned} \chi_G(M) &= \sum_{i=1}^l \chi_G(M^{a_i}, M^{a_{i-1}}) \\ &= \sum_{i=1}^l \sum_{j=0}^k (-1)^j [H_j(M^{a_i}, M^{a_{i-1}})] \\ &= \sum_{\mathcal{O} \in \mathcal{O}_{r_0}} [TW_{\mathcal{O}}^-] - \sum_{\mathcal{O} \in \mathcal{O}_{r_1}} [TW_{\mathcal{O}}^-] + \cdots + (-1)^k \sum_{\mathcal{O} \in \mathcal{O}_{r_k}} [TW_{\mathcal{O}}^-]. \end{aligned}$$

We have shown the following:

Theorem 5.11 (Weak G -Morse Inequalities).

$$i) [H_j(M)] \leq \sum_{\mathcal{O} \in \mathcal{O}_{r_j}} [TW_{\mathcal{O}}^-],$$

$$ii) \sum_{j=0}^k (-1)^j [H_j(M)] = \sum_{j=0}^k (-1)^j \sum_{\mathcal{O} \in \mathcal{O}_{r_j}} [TW_{\mathcal{O}}^-].$$

Lemma 5.12. *The function S_j is subadditive when*

$$S_j(X, Y) = [H_j(X, Y)] - [H_{j-1}(X, Y)] + \cdots + (-1)^j [H_0(X, Y)].$$

Proof. Recall the proof of Lemma 5.4. We consider the exact sequence

$$\xrightarrow{\partial} H_j(Y, Z) \rightarrow H_j(X, Z) \rightarrow H_j(X, Y) \xrightarrow{\partial} H_{j-1}(Y, Z) \rightarrow \cdots,$$

of representations of G on the homology of the triple $X \supset Y \supset Z$, and apply Equation 5.2 to this sequence. In this case we mean character of G acting on the image of the boundary map ∂ when we say $\text{rank } \partial$.

$$[H_j(Y, Z)] - [H_j(X, Z)] + [H_j(X, Y)] - [H_j(Y, Z)] + \cdots \geq 0.$$

Collecting terms gives the result, namely, $S_j(Y, Z) - S_j(X, Z) + S_j(X, Y) \geq 0$. \square

We now use Lemma 5.12 to prove the G -Morse inequalities by applying the sub-additive function S_j to the spaces $M^{a_0} \subset \cdots \subset M^{a_i}$ and hence Theorem 5.10 is proven.

5.4 Morse-Bott Theory

Morse-Bott theory is the generalization of Morse theory to the situation where f has non-degenerate critical manifolds instead of having isolated non-degenerate critical points. In this section we follow *Lectures on Morse Theory, Old and New* by Raoul Bott [7].

5.4.1 Introduction

Recall from Section 5.1.1 that for a non-degenerate f on a k -dimensional manifold M , we have the Morse polynomial of f given by

$$\mathfrak{M}_t(f) = \sum_p t^{\lambda_p}, \quad p \in C(f).$$

Here $C(f)$ is the set of critical points of f , and λ_p is the index of p . The Poincaré polynomial of M is given by

$$P_t(M) = \sum_{j=0}^k t^j b_j,$$

where b_j is the j -th Betti number of M . The Morse inequalities give that for every non-degenerate f there exists a polynomial $Q_t(f) = q_0 + q_1 t + \cdots + q_k t^k$ with non-negative coefficients q_i such that

$$\mathfrak{M}_t(f) - P_t(M) = (1 + t)Q_t(f). \tag{5.4}$$

If $Q_t(f) = 0$ then the function f is called a **perfect** Morse function.

Note 5.13. (a) The factor of $(1+t)$ comes from the fact that both $\mathfrak{M}_t(f)$ and $P_t(M)$ give the Euler characteristic when we evaluate the polynomials at $t = -1$. Since $\mathfrak{M}_t(f) = P_t(M)$ at $t = -1$, the right hand side must be equal to zero and so the polynomial on the right must have the required factor.

(b) The polynomial $Q_t(f)$ can be seen to have non-negative coefficients by considering the Morse inequalities at each stage. See discussion Section 5.1.1.

5.4.2 Non-Degenerate Critical Manifolds

Definition 5.14. A submanifold $N \subset M$ is a **non-degenerate critical manifold** if:

- i) each point $x \in N$ is a critical point of f ,
- ii) the Hessian of f is non-degenerate in the normal direction to N . That is, when we choose a local coordinate system $(x_1, \dots, x_l, x_{l+1}, \dots, x_k)$ in a neighbourhood of $x \in N$, N is given by the $(k - l)$ equations $x_{l+1} = 0, \dots, x_k = 0$. Then the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \Big|_x, \text{ for } i, j = l + 1, \dots, k,$$

is non-singular.

We can decompose the normal bundle into negative and positive parts $\nu N = \nu^+ N \oplus \nu^- N$, where $\nu_x^+ N$ and $\nu_x^- N$ are spanned by the positive and negative Eigen-directions of the Hessian of f (evaluated at $x \in N$) and are disk bundles over N . The fibre dimension of $\nu^- N$ will be denoted λ_N , and will be called the **index** of N relative to f .

Theorem 5.15 (Bott [7]). *If $f : M \rightarrow \mathbb{R}$ is non-degenerate in the sense that all its critical sets are non-degenerate critical manifolds, then the Morse polynomial is given by*

$$\mathfrak{M}_t(f) = \sum_{N \in M} P_t(\nu^- N, \partial \nu^- N),$$

and the Morse inequalities, $\mathfrak{M}_t(f) - P_t(M) = (1+t)Q_t(f)$, hold.

Again we can write this as: For every non-degenerate f there exists a polynomial $Q_t(f) = q_0 + q_1t + \dots + q_k t^k$ with non-negative coefficients such that

$$\mathfrak{M}_t(f) - P_t(M) = (1+t)Q_t(f). \quad (5.5)$$

If $Q_t(f) = 0$ then the function f is called a **perfect** Morse–Bott function.

Note 5.16. This is an inequality of polynomials and is similar to an inequality of representations in that to compare two polynomials we compare the coefficients. So let $p(t) = \sum_i \alpha_i t^i$ and $q(t) = \sum_i \beta_i t^i$ be two polynomials. We say that $p(t) \geq q(t)$ if $\alpha_i \geq \beta_i$ for all i , otherwise they are incomparable.

Remark 5.17. We have defined the Morse polynomial in terms of the homology of the pair $(\nu^-N, \partial\nu^-N)$; via the Thom isomorphism we can define the Morse polynomial in terms of the homology of the manifold N , since:

$$P_t(\nu^-N, \partial\nu^-N) = \begin{cases} t^{\lambda_N} P_t(N) & \text{if the fibre bundle is orientable,} \\ t^{\lambda_N} P_t(N; o(\nu^-N)) & \text{if the fibre bundle is non-orientable,} \end{cases}$$

where $o(\nu^-N)$ is the orientation bundle of ν^-N (see *Differential Forms in Algebraic Topology* [8], Theorems 6.17 and 7.10).

Example 5.18. Let M be a torus obtained by rotating a circle in the (x, y) -plane about the z -axis. Let f be the height function. There are two non-degenerate critical manifolds N_0 and N_1 , both circles, as shown in Figure 5.2.

The minima are the non-degenerate circle N_0 . The Morse polynomial for this is $\mathfrak{M}_t^{N_0}(f) = t^0(1+t)$. Similarly, the maxima are the non-degenerate circle N_1 , and the Morse polynomial for this is $\mathfrak{M}_t^{N_1}(f) = t^1(1+t)$. The Morse polynomial for the whole torus is $\mathfrak{M}_t(f) = (1+t) + t(1+t)$. The Poincaré polynomial of M is $P_t(M) = 1 + 2t + t^2$. The Morse inequalities are then given by Equation 5.4 as

$$\begin{aligned} \mathfrak{M}_t(f) - P_t(M) &= (1+t)Q_t(f) \\ (1+t)^2 - (1+2t+t^2) &= 0. \end{aligned}$$

This shows that f is a perfect Morse–Bott function on the torus.

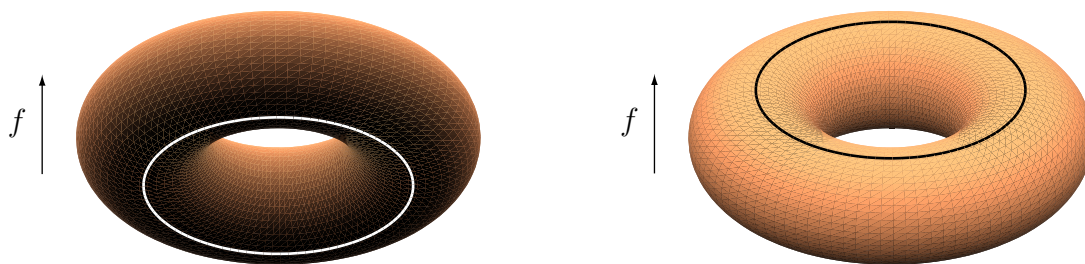
(a) Circle of minima, N_0 .(b) Circle of maxima, N_1 .

Figure 5.2: Non-degenerate critical manifolds of the torus.

It is worth noting that as we pass a critical level a_i we attach a thickened version of the negative disk bundle over N to $M^{a_{i-1}}$ to obtain M^{a_i} , where $a_{i-1} < f(N) < a_i$ and $M^{a_i} = \{x \in M : f(x) \leq a_i\}$.

We now give a proof (by Bott [7]) of Theorem 5.15, .

Proof. Let M^a be the set of points in M such that $f(x) \leq a$. Let $a_1 < \dots < a_l$ be such that M^{a_i} contains exactly i critical manifolds and $M = M^{a_l}$.

We define the Morse polynomial of the half space M^{a_i} to be

$$\mathfrak{M}_t^{a_i}(f) = \sum_{N \subset M^{a_i}} t^{\lambda_N} P_t(N),$$

where $P_t(N)$ is the Poincaré polynomial of a critical manifold N in M^{a_i} . **Note:** For ease we use the Thom isomorphism and assume the negative disk bundles are all orientable. The proof in the general case runs similarly.

We define the Poincaré polynomial of the half space M^{a_i} to be

$$P_t(M^{a_i}) = \sum_{j=0}^k t^j \dim H_j(M^{a_i}).$$

The Morse inequalities become $\mathfrak{M}_t^{a_i}(f) \geq P_t(M^{a_i})$.

We consider what happens as we move from $M^{a_{i-1}}$ to M^{a_i} . The change in the Morse polynomial is clear and is $\Delta \mathfrak{M}_t(f) = t^{\lambda_N} P_t(N)$, where N is the critical manifold such that $a_{i-1} < f(N) < a_i$. The change in the Poincaré polynomial ΔP_t is either $t^{\lambda_N} P_t(N)$ or $-t^{(\lambda_N-1)} P_t(N)$.

We can see this by considering the boundary of the thickened version of the negative disk bundle (which we will denote $N \times S^{\lambda_N-1}$) over N . The cycle carried by this space either bounds a chain in M^{a_i-1} or not.

Case 1: we cap the boundary by $N \times e^{\lambda_N}$, the thickened version of the negative disk bundle, to create a new non-trivial homology class in M^{a_i-1} . Thus P_t changes by $t^{\lambda_N} P_t(N)$.

Case 2: $N \times e^{\lambda_N}$ has as its boundary the non-trivial cycle $N \times S^{(\lambda_N-1)}$ in M^{a_i-1} . Hence P_t is decreased by $t^{(\lambda_N-1)} P_t(N)$.

The inequalities at the next level follow from these assertions, since either

$$\Delta(\mathfrak{M}_t(f) - P_t) = 0, \quad \text{or} \quad \Delta(\mathfrak{M}_t(f) - P_t) = t^{\lambda_N-1} P_t(N)(1+t).$$

By induction the Morse inequalities hold. □

Remark 5.19. Since there is a factor of $(1+t)$ in $\Delta(\mathfrak{M}^{a_i}(f) - P_t)$ for either case, we can say that for every non-degenerate function f on M there exists a polynomial $Q_t(f)$ with non-negative coefficients such that

$$\mathfrak{M}_t(f) - P_t(M) = (1+t)Q_t(f).$$

Corollary 5.20 (Bott [7]). *Let f be non-degenerate and let $a < b$ be two regular (non-critical) values of f . Let $C(f)$ be the critical set of f . Then if*

$$\mathfrak{M}_t(f)_a^b = \sum_N t^{\lambda_N} P_t(N), \quad \text{where } N \in C(f) \text{ and } N \in M^b - M^a,$$

and

$$P_t(M^b, M^a) = \sum_j t^j \dim H_j(M^b, M^a)$$

then the Morse inequalities still hold, that is $\mathfrak{M}_t(f)_a^b \geq P_t(M^b, M^a)$. These inequalities are called the **relative Morse inequalities**.

Proof. We obtain the relative inequalities from the proof above. □

We now have another example which we will come back to later in this chapter.

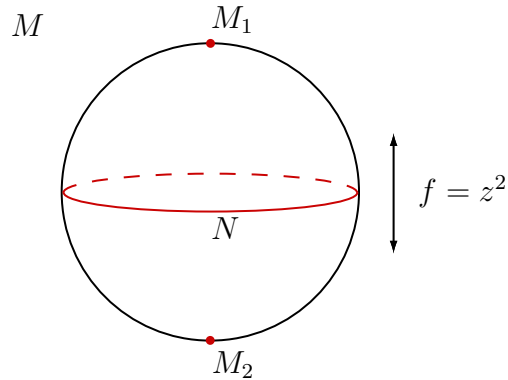


Figure 5.3: 2-sphere with a Morse–Bott function giving three critical sets.

Example 5.21. Let M be a 2-sphere with the function $f = z^2$ as shown in Figure 5.3.

- a) We have three critical sets, M_1 , M_2 and N . M_1 and M_2 are isolated critical points and N is a circle of critical points.

Critical Set, L	$\text{ind}(L)$	$P_t(L)$	$\mathfrak{M}_t^L(f)$
M_1	2	1	t^2
M_2	2	1	t^2
N	0	$1 + t$	$1 + t$

The Morse polynomial of the function is $\mathfrak{M}_t(f) = 2t^2 + t + 1$ and the Poincaré polynomial of the space is $P_t(M) = 1 + t^2$. These polynomials satisfy the Morse inequalities and we can easily see that $Q_t(f) = t$.

- b) if we now look at $-f$, we have the same critical sets, but their indexes have changed.

Critical Set, L	$\text{ind}(L)$	$P_t(L)$	$\mathfrak{M}_t^L(f)$
M_1	0	1	1
M_2	0	1	1
N	1	$1 + t$	$t(1 + t)$

The Morse polynomial of the function is $\mathfrak{M}_t(f) = 2 + t + t^2$ and the Poincaré polynomial of the space is $P_t(M) = 1 + t^2$. These polynomials satisfy the Morse inequalities and we can easily see that $Q_t(f) = 1$.

5.5 G -Morse-Bott Theory

In this section we look to extend Bott's work by seeking a G -invariant version of Morse–Bott theory. We now apply a finite group G to our manifold M . Since f is G -invariant, critical manifolds of the same index are mapped to each other and critical manifolds in an orbit are isomorphic.

Definition 5.22. Let \mathcal{O}_N be the orbit of the critical manifold $N \subset M$. We define the **G-Poincaré polynomial** of the orbit \mathcal{O}_N , to be $P_t^G(\mathcal{O}_N) = \sum_j t^j [H_j(\mathcal{O}_N)]$, where square brackets denote the character of G on $H_j(\mathcal{O}_N)$.

Theorem 5.23. *Let f be a non-degenerate (in the sense that all its critical sets are non-degenerate critical manifolds) G -invariant function on M . Let \mathcal{O}_N be an orbit of critical manifolds of f in M . If the **G-Morse polynomial** is given by*

$$\mathfrak{M}_t^G(f) = \sum_{\mathcal{O}_N} P_t^G(\mathcal{O}_{\nu^-N}, \mathcal{O}_{\partial\nu^-N}),$$

where \mathcal{O}_{ν^-N} and $\mathcal{O}_{\partial\nu^-N}$ are the orbits of the negative disk bundle and its boundary respectively, and the **G-Poincaré polynomial** is given by

$$P_t^G(M) = \sum_{j=0}^k t^j [H_j(M)],$$

then the Morse inequalities hold, that is $\mathfrak{M}_t^G(f) \geq P_t^G(M)$.

Furthermore for every non-degenerate G -invariant function f , there is a polynomial $Q_t^G(f) = q_0 + q_1t + \dots + q_nt^n$ with non-negative coefficients such that

$$\mathfrak{M}_t^G(f) - P_t^G(M) = (1+t)Q_t^G(f). \quad (5.6)$$

Remark 5.24. Similarly to Remark 5.17 we may use the Thom isomorphism to define the G -Morse polynomial in terms of the homology of the orbit \mathcal{O}_N of the manifold N :

$$P_t^G(\mathcal{O}_{\nu^-N}, \mathcal{O}_{\partial\nu^-N}) = \begin{cases} t^{\lambda_N} P_t^G(\mathcal{O}_N) & \text{if the fibre bundle is orientable,} \\ t^{\lambda_N} P_t^G(\mathcal{O}_N; o(\nu^-N)) & \text{if the fibre bundle is non-orientable,} \end{cases}$$

where $o(\nu^-N)$ is the orientation bundle of ν^-N .

Proof. Let M^a be all the points x in M such that $f(x) \leq a$. Let $a_1 < \cdots < a_l$ be such that M^{a_i} contains exactly i **orbits** of critical manifolds and $M = M^{a_l}$.

We define the **G-Morse polynomial** of the half space M^{a_i} to be

$$\mathfrak{M}_G^{a_i}(f) = \sum_{\mathcal{O}_N \in M^{a_i}} t^{\lambda_N} P_t^G(\mathcal{O}_N).$$

Note: For ease we again use the Thom isomorphism and assume the negative disk bundles are all orientable. The proof in the general case runs similarly.

The G -Poincaré polynomial of the half space M^{a_i} is defined as

$$P_t^G(M^{a_i}) = \sum_j t^j [H_j(M^{a_i})].$$

The G -Morse inequalities of the half space are then given by $\mathfrak{M}_G^{a_i}(f) \geq P_t^G(M^{a_i})$.

We consider what happens as we move from $M^{a_{i-1}}$ to M^{a_i} and proceed inductively. Let $a_{i-1} < f(\mathcal{O}_N) < a_i$, then the change in G -Morse polynomial is $\Delta \mathfrak{M}_t^G(f) = t^{\lambda_N} P_t^G(\mathcal{O}_N)$ and the change in the G -Poincaré polynomial is $\Delta P_t^G = t^{\lambda_N} P_t^G(\mathcal{O}_N)$ or $-t^{\lambda_N-1} P_t^G(\mathcal{O}_N)$ by the same argument as in the proof of Theorem 5.15.

The inequalities at the next level follow from these assertions since either

$$\Delta(\mathfrak{M}_t^G(f) - P_t^G) = 0, \quad \text{or} \quad \Delta(\mathfrak{M}_t^G(f) - P_t^G) = t^{\lambda_N-1} P_t^G(\mathcal{O}_N)(1+t).$$

By induction we can say that the Morse inequalities hold for the whole of M .

If we set $t = -1$ and evaluate both $\mathfrak{M}_t^G(f)$ and $P_t^G(M)$ we obtain the G -Euler characteristic in both cases, and similarly to Note 5.13(a) we can say that there must be a factor of $(1+t)$ in the difference.

The G -Morse inequalities at each level give that the coefficients of $Q_t^G(f)$ are non-negative and so the coefficients of $Q_t^G(f)$ are genuine characters. \square

Remark 5.25. If we replace each character coefficient in $Q_t^G(f)$ with its dimension, the resulting polynomial is in fact $Q_t(f)$ (see Equation 5.4).

Example 5.26. Let M be a 2-sphere with the function $f = z^2$ as in Example 5.21 and as shown in Figure 5.3. Let G be Z_2 with action $-I$, minus the identity. The G -Poincaré polynomial of M is

$$P_t^G(M) = \sum_{j=0}^k t^j [H_j(M)] = [H_0(M)] + t^2 [H_2(M)],$$

since $H_1(S^2) = 0$.

- a) We have two orbits of critical manifolds, $\mathcal{O}_1 = \{M_1, M_2\}$ and $\mathcal{O} = N$. We need to find $P_t^G(\mathcal{O})$ for each orbit.

Let A_0 be the trivial character and A_1 the alternating character of \mathbb{Z}_2 . For \mathcal{O}_1 the character of the orientation representation is easy to find since the points in the orbit are fixed under I and are interchanged by $-I$. The character vector is $\chi[I, -I] = [2, 0]$ which corresponds to character of $A_0 + A_1$ giving us that $P_t^G(\mathcal{O}_1) = (A_0 + A_1)t^2$.

For \mathcal{O}_2 we need to look at $H_i(M^b, M^a)$, where $a < f(\mathcal{O}_2) < b$. Since \mathcal{O}_2 is a circle of minima, we can say that $M^a = \emptyset$. We look at $H_i(M^b)$ and what happens to the orientation as we apply $-I$. We only need to look at $[H_1(M^b)]$ since it is clear that $[H_0(M^b)] = A_0$, where A_0 is the trivial character. We turn to Figure 5.4 to see what happens to the orientation of $[H_1(M^b)]$. It is clear from the figure that

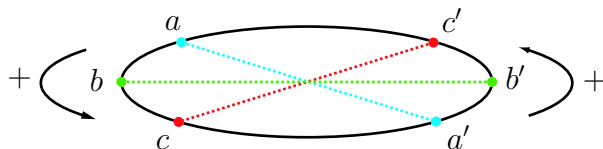


Figure 5.4: How $-I$ acts on $H_1(M^b)$.

the orientation of $[H_1(M^b)]$ is fixed by $-I$ (arrows indicated with $+$) and that the character we are looking for is $\chi[I, -I] = [1, 1]$ which is the trivial character, so this means that $P_t^G(\mathcal{O}_2) = A_0 + A_0t$. Putting all this together gives us

$$\mathfrak{M}_t^G(f) = A_0 + A_0t + (A_0 + A_1)t^2.$$

Now we compare $\mathfrak{M}_t^G(f)$ and $P_t^G(M)$ using Theorem 5.23.

$$\begin{aligned} (1+t)Q_t^G(f) &= \mathfrak{M}_t^G(f) - P_t^G(M) \\ &= (A_0 + A_0t + (A_0 + A_1)t^2) - ([H_0(M)] + t^2[H_2(M)]) \\ &= A_0 + A_1t(1+t) + A_1t^2 - ([H_0(M)] + t^2[H_2(M)]) \\ &= (1+t)tA_0 + A_0 + A_1t^2, \end{aligned}$$

giving us that $[H_0(M)] = A_0$ and $[H_2(M)] = A_1$. Therefore we have that $P_t^G(M) = A_0 + A_1 t^2$ and $Q_t^G(f) = A_0 t$.

b) If instead we look at the function $f = -z^2$ on the manifold we have the same critical manifolds as in part a, except that now the isolated points are minima and the maxima are the circle N . We label the orbits in the same way as above.

Again for the first orbit \mathcal{O}_1 , we have the character $A_0 + A_1$ which gives us a Poincaré polynomial of $P_t^G(\mathcal{O}_1) = A_0 + A_1$.

The second orbit is slightly more work, we look at the pair (M^b, M^a) , where $a < f(\mathcal{O}_2) < b$ again. It is easy to see that $H_0(M^b, M^a) = 0$ and $H_i(M^b, M^a) \cong \mathbb{R}$ for $i = 1, 2$ and so we are now left to calculate the character of the orientation representation on these. We choose an orientation and see what happens under the group action. We start by looking at $[H_1(M^b, M^a)]$, see Figure 5.5. From this it is

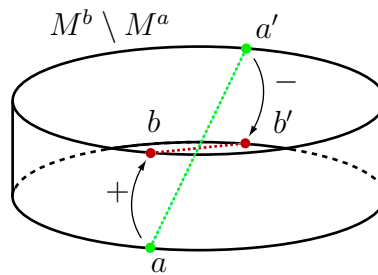


Figure 5.5: How $-I$ acts on $H_1(M^b, M^a)$.

easy to see that the orientation changes and so the character is $[H_1(M^b, M^a)] = A_1$.

Now, looking at $[H_2(M^a, M^b)]$ we use the same method as before - see Figure 5.6.

We need to remember that we are viewing the tube from the “outside” and we

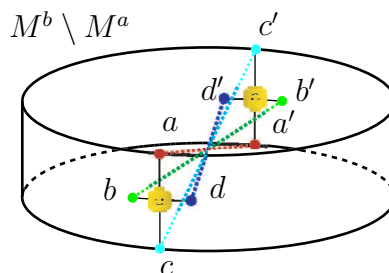


Figure 5.6: How $-I$ acts on $H_2(M^b, M^a)$.

can now compare the orientation before and after - see Figure 5.7. From these

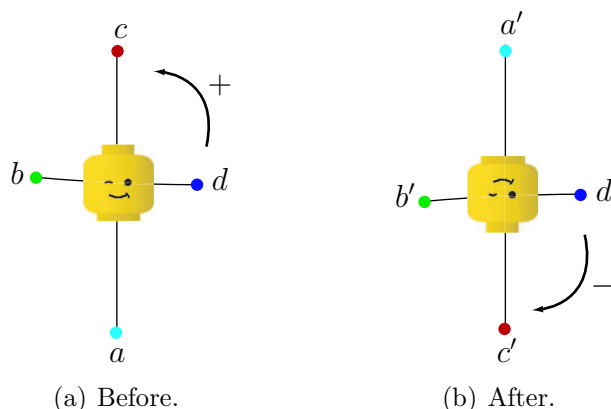


Figure 5.7: How $-I$ acts on $H_2(M^b, M^a)$ - detailed view.

diagrams we can see that the orientation has been reversed and so the appropriate character is $[H_2(M^b, M^a)] = A_1$.

Putting this together we find that $P_t^G(\mathcal{O}_2) = A_1t + A_1t^2$ and that the G -Morse polynomial is

$$\mathfrak{M}_t^G(f) = (A_0 + A_1) + A_1t + A_1t^2.$$

We now compare the G -Morse and G -Poincaré polynomials:

$$\begin{aligned} (1+t)Q_t^G(f) &= \mathfrak{M}_t^G(f) - P_t^G(M) \\ &= ((A_0 + A_1) + A_1t + A_1t^2) - ([H_0(M)] + [H_2(M)]t^2) \\ &= A_1(1+t) + A_0 + A_1t^2 - ([H_0(M)] + [H_2(M)]t^2) \end{aligned}$$

This gives that $[H_0(M)] = A_0$ and $[H_2(M)] = A_1$. Therefore we have that $P_t^G(f) = A_0 + A_1t^2$ and $Q_t^G(f) = A_1$.

Note 5.27. In Chapter 6 we will use cohomology rather than homology. This means that we will use a slightly different, yet equivalent, definition for the Poincaré and G -Poincaré polynomials. We use the fact that over a field $H^i(M) \cong H_i(M)$ to redefine the polynomials thus:

$$P_t(M) = \sum_{j=0}^k H^j(M)t^j, \quad \text{and} \quad P_t^G(M) = \sum_{j=0}^k [H^j(M)]t^j.$$

Chapter 6

The n -Body Problem

The n -body problem originally arose from the study of celestial mechanics, the modern study of which began over 300 years ago. Both Kepler and Newton did much to further the study of planetary motion. After Newton, Lagrange attempted to solve the 3-body problem discovering, in the process, the Lagrange relative equilibria classes and the Lagrangian points — the positions in an orbital configuration at which a small mass will be stationary relative to two larger masses if affected only by gravitational forces.

Since then the problem has been solved completely for three bodies (see Wintner [39]), the solutions being the two Lagrange configurations plus three collinear configurations first discovered by Euler.

Stephen Smale's *Topology and Mechanics I* [34] and *II* [35], and *Problems on the Nature of Relative Equilibria* [36] are our starting point for the study of the relative equilibria of the n -body problem. Here we find the first insight into the nature of the classes of the relative equilibria for $n \geq 3$.

In the mid-seventies Palmore wrote a series of papers *Classifying Relative Equilibria of the n -Body Problem I, II, III* [26, 27, 28] and a fourth paper entitled *New Relative Equilibria of the n -Body Problem* [29] in which he discussed many results including a minimum for the number of relative equilibria, descriptions of the relative equilibria of the four-body problem and degeneracies of the general case — for the 4-body in particular. These degeneracies were further considered by Carlos Simó

in the paper *Relative Equilibrium Solutions in the Four Body Problem* [32], giving manifolds of degeneracy and analysis of linear stability.

Rick Moeckel tells us that there are finitely many relative equilibria of the four-body problem in *Relative Equilibria of the Four-Body Problem* [21] for any four masses.

In what follows we aim to describe relative equilibria solutions of the planar n -body problem using the work from Chapter 5. Since one can regard the potential energy function V of the n -body problem as a Morse function on a manifold we may use the techniques developed in the previous chapter. Firstly we must define what we mean by the n -body problem and relative equilibria and then we will apply a group to the system to enable us to make use of G -Morse theory.

6.1 Introduction

This introductory section is taken almost directly from Stephen Smale's paper *Problems on the Nature of Relative Equilibria in Celestial Mechanics* [36]; he describes the planar n -body problem in the context of relative equilibria and modern topology as follows.

We start by looking at the planar n -body problem since questions of relative equilibria reduce to the planar case. Given the masses, $m_1, \dots, m_n > 0$, one has as the configuration space, the subset of $(\mathbb{R}^2)^n$ defined by

$$M = \left\{ x = (x_1, \dots, x_n) \in (\mathbb{R}^2)^n : \sum m_i x_i = 0 \right\}.$$

Here \mathbb{R}^2 is the Euclidean plane and we suppose that the centre of mass is fixed at the origin. **Note:** This space has dimension $2n - 2$.

The tangent bundle is given by

$$T(M) = M \times M = \left\{ (x, v) : \sum m_i x_i = 0, \sum m_i v_i = 0 \right\}$$

and the kinetic energy, $K : T(M) \rightarrow \mathbb{R}$, is given by $K(x, v) = K(v) = \frac{1}{2} \sum m_i \|v_i\|^2$ where $\| \cdot \|$ is the Euclidean norm. The potential energy is the function $V : (M - \Delta) \rightarrow$

\mathbb{R} given by

$$V(x) = - \sum_{i < j} \frac{m_i m_j}{\|x_i - x_j\|},$$

and the diagonal Δ is the union $\bigcup_{i < j} \Delta_{ij}$ where $\Delta_{ij} = \{x \in M : x_i = x_j\}$. These are the ‘collision’ points and are where the potential function tends to minus infinity. Then, via Newton’s equations, K and V define the dynamics of this mechanical system.

The group of rotations $SO(2) \cong S^1$ acting on \mathbb{R}^2 induces an action on $(\mathbb{R}^2)^n$, M and $T(M)$, leaving invariant K , V and Δ .

A point $x \in (M - \Delta)$ is said to be a **relative equilibrium** ($x \in R_e$), if there is a 1-parameter group of rotations ϕ_t of \mathbb{R}^2 such that $(\phi_t(x_1), \dots, \phi_t(x_n))$ satisfies Newton’s equations. Note that R_e is invariant under the action S^1 acting on $(M - \Delta)$ and also under non-zero scalar multiplication. The quotient $\bar{R}_e = R_e / \{S^1, R^+\}$ is called the set of classes of relative equilibria. We are interested in the nature of \bar{R}_e for a given choice of masses.

Our starting point for considering the relative equilibria is the following theorem.

Theorem 6.1 (Theorem C. [35]). *Let $S_K = \{x \in M : K(x) = 1\}$. Then $x \in S_K - \Delta$ is a relative equilibrium if and only if x is a critical point of the restriction of the potential energy V to $S_K - \Delta$.*

Proof. It is well known that the augmented potential V_ξ has a critical point at z if and only if z is a relative equilibrium with angular velocity ξ (see [18], Proposition 4.2).

So for $K = \frac{1}{2} \sum_i m_i |v_i|^2$ and with $M = \{z = (x_i, \dots, x_n) : \sum_i m_i x_i = 0\}$ we have

$$V_\xi(z) = V(x) - \frac{1}{2} \sum_i |\xi x_i|^2, \quad \text{where} \quad \xi = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

is an infinitesimal rotation. Note that $|\xi x_i|^2$ is dot product and so $|\xi x_i|^2 = x_i^T \xi^T \xi x_i$.

We are concerned with critical points, the points at which the first order partial derivatives are zero:

$$\frac{\partial V_\xi}{\partial x_i} = \frac{\partial V}{\partial x_i} - m_i \xi^T \xi x_i.$$

Those are the points at which $\frac{\partial V}{\partial x_i} = m_i \omega^2 x_i$, for all i , where ω is a constant.

In our setting this means that the potential restricted to the sphere $V|_{S_K}$ has a critical point at $z = (x_1, \dots, x_n)$ if and only if z is a relative equilibrium. That is, using the method of Lagrange multipliers,

$$\begin{aligned} dV(z) &= \lambda dK(z) \\ \Leftrightarrow \quad \frac{\partial V}{\partial x_i} &= \lambda m_i x_i, \quad i = 1, \dots, n. \end{aligned}$$

This is precisely what we had earlier, in fact

$$\sum_i x_i \frac{\partial V}{\partial x_i} = \lambda \sum_i m_i |x_i|^2 = \lambda,$$

and so $\lambda = \omega^2$ and $\lambda > 0$. □

Note 6.2. S_K is topologically a sphere of dimension $2n - 3$.

The symmetry group S^1 acts on S_K leaving Δ and V invariant. The quotient S_K/S^1 can be seen to be complex projective space $\mathbb{C}P^{n-2}$ and $\tilde{\Delta} = \Delta/S^1$ the union of projective subspaces. Let $\tilde{V} : (\mathbb{C}P^{n-2} - \tilde{\Delta}) \rightarrow \mathbb{R}$ be the map induced from V .

Remark 6.3. The collision set Δ is not necessarily a collection of isolated points, neither is it necessarily a manifold. We are, therefore, unable to use Morse-Bott Theory. Instead we look at $-V$ and ‘build up from the bottom’. This approach makes the calculations easier for two reasons.

1. We need not worry about the collision set.
2. An index i point now has index $(k - i)$, where k is the dimension of the space. This reduces the indexes and therefore dimensions of the unstable spaces that we have to work with.

6.2 The Action of a Group on the n -body Problem

In what follows we will be applying a group to our system — either the symmetric group on n elements S_n , or $S_n \times \mathbb{Z}_2$. This section explains the group action on the bodies and the related spaces. We must first require that all particles have equal mass and we let this mass be one.

6.2.1 Action of S_n on the n -Body Problem

We first note that $-V$ is invariant under the action of the group, this follows from the definition of $-V$.

The action of S_n on the particles is by permutation — a relative equilibrium of a certain class will be mapped to another (possibly itself) equilibrium of that class as $-V$ is invariant under the action of S_n if all the masses are equal. We will only be considering the case when all masses are equal. This also gives that the collision set Δ_{ij} is fixed under the action. For example see Figure 6.1 which shows $(13) \in S_3$ acting as a rotation (in the dotted line) on a Lagrange configuration of three particles.

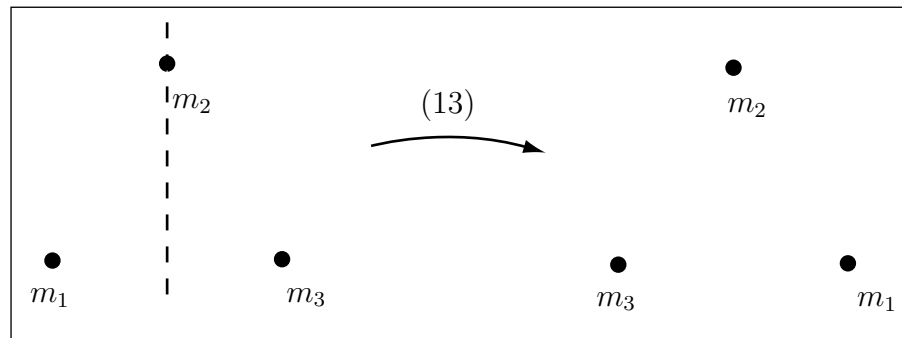


Figure 6.1: Example of S_3 action on a critical point in our system.

The induced action on the configuration space M , is again by permutation; this time permutation of the \mathbb{R}^2 in accordance with the permutation of the masses. For example, let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in M$ be a point in the configuration space and σ be an element of S_n , then σ acting on the point \mathbf{z} is:

$$\sigma(\mathbf{z}) = \sigma(z_1, z_2, \dots, z_n) = (z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}).$$

We now consider the action of S_n on $\mathbb{C}P^{n-2}$. Let $[z_1 : z_2 : \dots : z_n]$ be a point of $\mathbb{C}P^{n-2}$ — the action of S_n is the same as for M , i.e., $\sigma[z_1 : z_2 : \dots : z_n] = [z_{\sigma(1)} : z_{\sigma(2)} : \dots : z_{\sigma(n)}]$.

Example 6.4. The elements of S_n act by rotation on $\mathbb{C}P^{n-2}$; elements give a rotation of $\frac{2\pi}{l}$ where l is the order of the element. Figure 6.2 shows first an even element and then an odd element when $n = 3$, where the axes of rotation are shown as a dotted

line. The labels for the points on $\mathbb{C}P^1$ are as follows: the Moulton points M_{ijk} , where the masses lie on the line $y = 0$ and are ordered as $x_i < x_j < x_k$; the Lagrange points L_{ijk} , the triangular configuration with masses labelled in an anticlockwise direction; and the collision points Δ_{ij} , where $z_i = z_j$ — here $i, j, k \in \{1, 2, 3\}$. See Section 6.6 for an explanation of these configurations.

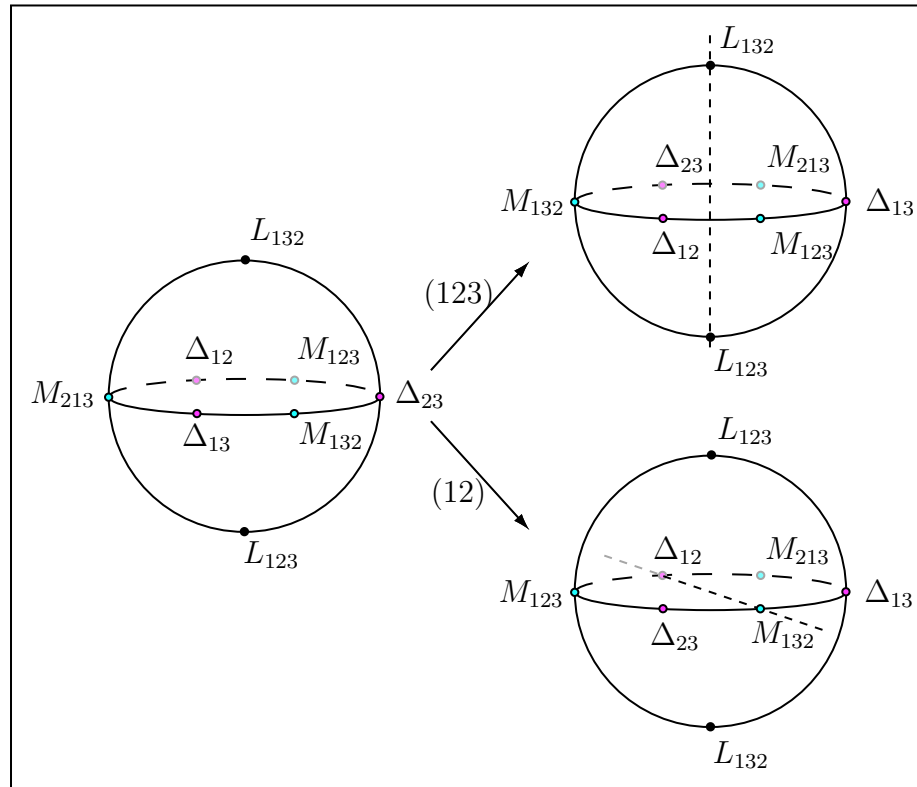


Figure 6.2: Action of S_3 on $\mathbb{C}P^1$.

6.2.2 The Isotropy Subgroup of a Point

We now look at the fixed points of $\mathbb{C}P^{n-2}$ under the action of S_n . Let \mathbf{z} be a point in $\mathbb{C}P^{n-2}$ and $\sigma \in S_n$. We know that $(z_1, z_2, \dots, z_n) = e^{i\theta}(z_1, z_2, \dots, z_n)$, and that under the action $\sigma(\mathbf{z}) = \sigma(z_1, z_2, \dots, z_n) = (z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)})$. We are looking for the fixed points of $\mathbb{C}P^{n-2}$ and so we want

$$(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}) = e^{i\theta}(z_1, z_2, \dots, z_n).$$

All such σ form the isotropy subgroup of the point. We now consider a specific example where the point \mathbf{z} is a critical point of $-V$.

Example 6.5. Let $n = 3$, $\sigma = (13)$ and \mathbf{z} be a Moulton point where all the masses lie on a line (see Section 6.6). We have

$$\begin{aligned}\sigma(z_1, z_2, z_3) &= (z_3, z_2, z_1) \\ &= e^{i\theta}(z_1, z_2, z_3),\end{aligned}$$

which gives that $z_1 = e^{i\theta}z_3$ and $z_3 = e^{i\theta}z_1$. We now have $z_1 = e^{2i\theta}z_1$ giving $\theta = \pi$ therefore $z_1 = -z_3$ and $z_2 = -z_2 = 0$. The isotropy subgroup of the Moulton points when $n = 3$ is \mathbb{Z}_2 ; with $\mathbb{Z}_2 = \langle (13) \rangle$ for this particular point — see Figure 6.3.

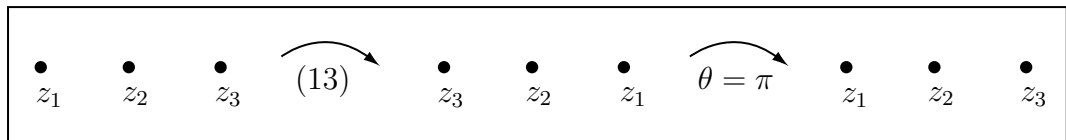


Figure 6.3: How $\sigma = (13)$ fixes a Moulton point of the n -body problem when $n = 3$.

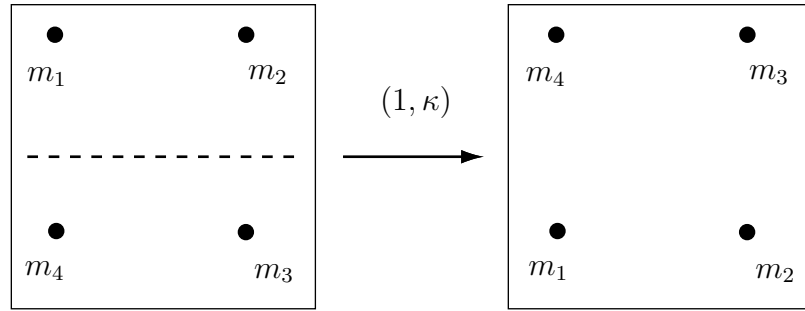
6.2.3 Action of $S_n \times \mathbb{Z}_2$ on the n -Body Problem

The action of $S_n \times \mathbb{Z}_2$ is an extension of the action by S_n — the S_n part again acts by permutation as described earlier and the \mathbb{Z}_2 acts by complex conjugation, see Example 6.6.

Example 6.6. Let $n = 4$ and the positions of the particles be as follows: m_1 at z , m_2 at w , m_3 at \bar{w} , and m_4 at \bar{z} , where z, \bar{z}, w, \bar{w} are all distinct points in \mathbb{C} . We now apply $(1, \kappa) \in S_4 \times \mathbb{Z}_2$ and since κ is complex conjugation we have that: $\kappa(m_1, m_2, m_3, m_4) = (m_4, m_3, m_2, m_1)$. This is shown in Figure 6.4 below.

6.2.4 Representation on the Unstable Space

In the rest of this chapter, we will be interested in finding the orientation representation of G on the tangent space of orbits. We are able to find this representation using the Frobenius reciprocity theorem (Theorem 2.23). The representation of the isotropy subgroup on the tangent space induced to G is equal to the representation

Figure 6.4: Action of $S_4 \times \mathbb{Z}_2$ — complex conjugation.

of G on the tangent space restricted to the isotropy subgroup. So we may find the representation of G by inducing from the appropriate representation of the isotropy subgroup.

Example 6.7. Let the isotropy subgroup be $H = \mathbb{Z}_4$, and the representation of \mathbb{Z}_4 on the tangent space be a_0 , the trivial representation. Then the representation of $G = S_4$ on the tangent space is

$$\text{Ind}_{\mathbb{Z}_4}^{S_4}(a_0) = A_0 + E + T_1,$$

where A_0, E, T_1 are representations of S_4 — the full character table can be found in Appendix B.6.

6.3 The Cohomology of $\mathbb{C}P^{n-2} - \tilde{\Delta}$

The homology of $\mathbb{C}P^{n-2} - \Delta$ was described by Palmore [26] via the inductive formula

$$H_i(\mathbb{C}P^{n-2} - \tilde{\Delta}_{n-2}) \cong H_i(\mathbb{C}P^{n-3} - \tilde{\Delta}_{n-3}) \oplus H_{i-1}(\mathbb{C}P^{n-3} - \tilde{\Delta}_{n-3})^{n-1},$$

where $0 \leq i \leq (n-2)$ and for $i > (n-2)$, $H_i(\mathbb{C}P^{n-2} - \tilde{\Delta}_{n-3}) = 0$. Here n is the number of particles and $n \geq 3$.

This is not a very useable formula since we don't have an obvious extension to find the representation of the group acting on the homology. Instead, we can relate our space to the pure (or coloured) braid group.

The following is taken from *The cohomology ring of the coloured braid group* by Arnol'd [3]. We are looking at the complement of the pure braid space M_n of ordered

sets of pairwise different points of a plane. The space M_n is described as the complex affine space \mathbb{C}^n with “eliminated diagonals”,

$$M_n = \{z \in \mathbb{C}^n : z_k \neq z_l, \forall k \neq l\}.$$

We denote the exterior graded ring of the pure braid group by $A(n)$. It is generated by the one-dimensional elements $\omega_{kl} = \omega_{lk}$, $1 \leq k \neq l \leq n$ satisfying the relationship

$$\omega_{kl}\omega_{lm} + \omega_{lm}\omega_{mk} + \omega_{mk}\omega_{kl} = 0.$$

Theorem 6.8 (Arnol'd [3]).

(i) *The cohomology ring of M_n is isomorphic to $A(n)$ and the isomorphism*

$$H(M_n, \mathbb{Z}) \cong A(n)$$

is set up by the formulas
$$\omega_{kl} = \frac{1}{2\pi i} \frac{dz_k - dz_l}{z_k - z_l}.$$

(ii) *The cohomology groups of the pure braid group are torsion free.*

(iii) *The Poincaré polynomial of the manifold M_n is*

$$P_t(M_n) = (1+t)(1+2t) \dots (1+(n-1)t).$$

Proof. For proof see Arnol'd [3]. □

Arnol'd also gives, for each p , an additive basis of the cohomology ring $H^p(M_n)$ which consists of all products of the form

$$\omega_{k_1 l_1} \omega_{k_2 l_2} \dots \omega_{k_p l_p}, \quad \text{where} \quad k_s < l_s, \quad l_1 < l_2 < \dots < l_p,$$

multiplication in this ring is wedge product. The elements satisfy the following relationships:

1. $\omega_{kl} = \omega_{lk}$, $1 \leq k \neq l \leq n$, and
2. $\omega_{kl}\omega_{lm} + \omega_{lm}\omega_{mk} + \omega_{mk}\omega_{kl} = 0$.

Remarks 6.9.

1. It is easy to see that M_n is homotopic to $E = S^{2n-3} - \Delta$ and so $H^p(M_n) \cong H^p(E)$.
2. Let q be a change of coefficients map $q : H^n(E, \mathbb{Z}) \rightarrow H^n(E, \mathbb{R})$, then the kernel of q is the torsion part of $H^n(E, \mathbb{Z})$. By Theorem 6.8 (ii), $H^p(M_n, \mathbb{Z})$ is torsion free, q is an injection, and $\dim H^p(E, \mathbb{R}) = \text{rank } H^p(M_n, \mathbb{Z})$.
3. This basis gives us a method to find the representation of the induced group action on the cohomology group — we apply the group to this basis and find the resulting character and therefore representation.

6.4 Finding the Character on the Cohomology

To find the character on the p th-cohomology group of E , we take the basis described above and apply the group to it. Since we are using the symmetric group on n elements we permute the indices in the natural way, for example $\omega_{12} \xrightarrow{(23)} \omega_{13}$. Some of the elements in S_n will take us outside the basis and so we must correct for this. Since an element of the form $\omega_{k_1 l} \omega_{k_2 l}$ is not in the basis, Relation 2 from above becomes:

$$\omega_{km} \omega_{lm} = \omega_{kl} \omega_{km} - \omega_{kl} \omega_{lm}, \text{ where } k < l < m,$$

since we must have $k_s < l_s$ and $l_1 < l_2 < \dots < l_p$. To work out the character we take the trace of the matrix representation. We take the character on each conjugacy class of G and calculate the multiplicities of each irreducible character.

See Appendix A for a MATLAB program, written in collaboration with David Szotten, which accomplishes this.

6.5 Gysin Sequence

Recall the Gysin sequence from Section 2.7. Here we have that $E = S^{2n-3} - \Delta$, $B = \mathbb{C}P^{n-2} - \tilde{\Delta}$ and $l = 1$. This gives:

$$0 \rightarrow H^0(E) \xrightarrow{\alpha} H^0(B) \rightarrow 0$$

$$0 \rightarrow H^1(B) \xrightarrow{p^*} H^1(E) \xrightarrow{\alpha} H^0(B) \xrightarrow{e\wedge} H^2(B) \xrightarrow{p^*} H^2(E) \xrightarrow{\alpha} \dots \quad (6.1)$$

This sequence gives us a zero Euler class which can be seen by the following argument. Since B is connected we know that $H^0(B) \cong \mathbb{R}$. Now, using exactness at $H^0(B)$ in Equation 6.1, there are only two options: **either** $\alpha = 0$ and $e\wedge$ is injective ($\text{Im}(\alpha)=0$ and $\text{Ker}(e\wedge)=0$) **or** $e\wedge = 0$ and α is surjective ($\text{Ker}(e\wedge) = \mathbb{R}$ and $\text{Im}(\alpha) = \mathbb{R}$). Arnol'd tells us that the set of ω_{jk} with $j < k$ generate an additive basis of $H^1(E)$ and if we let $z_j = e^{i\theta} z_j^0$, we can then apply α to an element of $H^1(E)$ thus:

$$\begin{aligned} \alpha(\omega_{jk}) &= \oint \frac{1}{2\pi i} \frac{d(z_j - z_k)}{z_j - z_k} \\ &= \oint \frac{1}{2\pi i} \frac{d(e^{i\theta} z_j^0 - e^{i\theta} z_k^0)}{e^{i\theta} z_j^0 - e^{i\theta} z_k^0} \\ &= \oint \frac{1}{2\pi i} \frac{ie^{i\theta}(z_j^0 - z_k^0)d\theta}{e^{i\theta}(z_j^0 - z_k^0)} \\ &= \oint \frac{id\theta}{2\pi i} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} \\ &= 1. \end{aligned}$$

Since the image of α is not zero we conclude that $e\wedge = 0$. This in turn means we have a zero Euler class. These calculations are valid in the G -equivariant case and so we have an exact sequence of characters:

$$0 \rightarrow [H^0(E)] \xrightarrow{\alpha} [H^0(B)] \rightarrow 0$$

$$0 \rightarrow [H^1(B)] \xrightarrow{p^*} [H^1(E)] \xrightarrow{\alpha} [H^0(B)] \xrightarrow{e\wedge} [H^2(B)] \xrightarrow{p^*} [H^2(E)] \xrightarrow{\alpha} \dots$$

Remark 6.10. Since this sequence has a zero Euler class, the sequence splits into a series of short exact sequences and we have $e_i = b_i + b_{i+1}$ where e_i is the rank of $H_i(E)$ and b_i is similarly defined.

The Poincaré polynomial of E is given by

$$P_t(E) = \sum_i e_i t^i = \sum_i (b_i + b_{i-1}) t^i = (1+t) \sum_i b_i t^i = (1+t) P_t(B).$$

By Theorem 6.8 and Remarks 6.9 we know that $P_t(E) = (1+t)(1+2t) \dots (1+(n-1)t)$, and so we have

$$P_t(B) = (1+2t) \dots (1+(n-1)t).$$

6.5.1 Equivariance of the Gysin Sequence

Recall from Section 2.7 that the Gysin sequence is equivariant if both $e\wedge$ and α are G -equivariant. It is clear that $e\wedge = 0$ is equivariant. We now consider α . Let $X(z_1, \dots, z_n) = (iz_1, \dots, iz_n)$ be the vector field generating the S^1 -action and let $\omega_{jk} = \frac{1}{2\pi i} \frac{d(z_j - z_k)}{z_j - z_k}$ be a differential form and write $\omega_{jk}(X)$ for the contraction of the form, then

$$\alpha(\omega_{jk}) = \oint \omega_{jk} = 1.$$

If we now take a 2-form $\omega_{jk} \wedge \omega_{lm}$ and apply α we see that

$$\oint \omega_{jk} \wedge \omega_{lm} = \oint \omega_{jk}(X)\omega_{lm} - \omega_{jk}\omega_{lm}(X) = \omega_{lm} - \omega_{jk}.$$

It is easy to see that since for $\sigma \in S_n$, $\sigma\omega_{jk} = \omega_{\sigma(jk)}$ we have

$$\sigma\alpha(\omega_{jk} \wedge \omega_{lm}) = \sigma(\omega_{lm} - \omega_{jk}) = \omega_{\sigma(lm)} - \omega_{\sigma(jk)},$$

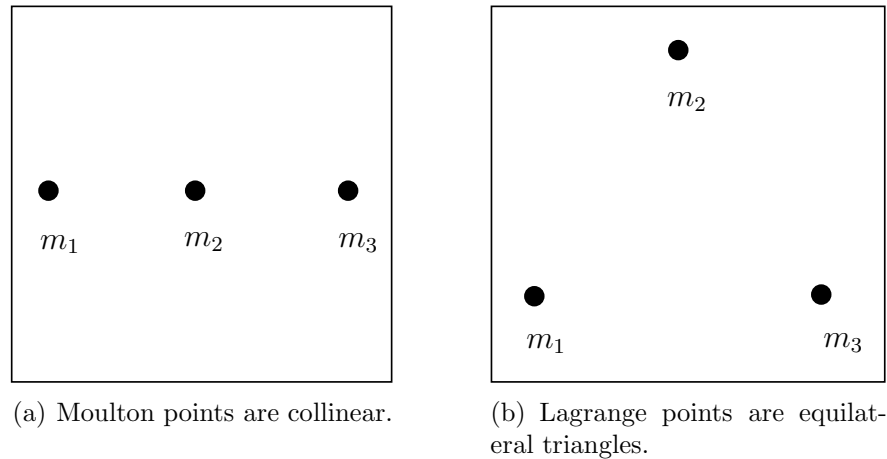
and $\alpha\sigma(\omega_{jk} \wedge \omega_{lm}) = \alpha(\omega_{\sigma(jk)} \wedge \omega_{\sigma(lm)}) = \omega_{\sigma(lm)} - \omega_{\sigma(jk)}$, giving us that $\alpha\sigma = \sigma\alpha$ and so α is equivariant for a 2-form. We now proceed by induction on the degree of the form to see that α is equivariant. This all means that we are able to use the equivariant Gysin sequence

$$0 \rightarrow [H^{l-1}(B)] \xrightarrow{p^*} [H^{l-1}(E)] \xrightarrow{\alpha} [H^{l-2}(B)] \xrightarrow{e\wedge} [H^l(B)] \xrightarrow{p^*} [H^l(E)] \rightarrow \dots$$

6.6 The 3-Body Problem and S_3

The relative equilibria classes of the 3-body problem are well known. These classes have been shown to be the only classes and so the 3-body problem is considered to be solved (see Wintner [39]). It is, however, worth pursuing this example to enable us to clarify the techniques we will employ for looking at higher numbers of particles. We are going to look at the case where all particles have the same mass so that we can apply the symmetric group to the system.

There are two types of relative equilibria when $n = 3$, Moulton points (M) and Lagrange (L) configurations, see Figure 6.5. From Smale [36] we know that the

Figure 6.5: Relative equilibria for $n = 3$.

Lagrange points have index 0, the Moulton points have index 1 and that there are two and three of each type respectively. These indexes are with respect to $-V$.

We apply S_3 (the symmetric group on three elements) to our system and use G -Morse theory to help describe the critical points. The action of S_3 on the particles is by permutation of the masses, see Section 6.2.1.

We first work out the isotropy subgroups for each class of relative equilibria and by looking at small perturbations of the configurations we get a basis for the tangent space of the unstable manifold. We can then work out the character and therefore which representation of the isotropy subgroup to induce from.

6.6.1 Moulton Points

The following discussion holds for Moulton points of any number of particles.

We start by remembering that the centre of mass of our system is fixed and so perturbations in the basis of the unstable space must also fix the centre of mass. There are two classes of perturbation: *i*) where all the particles move along the x -axis; and *ii*) where they all move in the y -direction. Any perturbation can be decomposed into a sum of perturbations of each type. This gives a direct sum decomposition of \mathbb{R}^{2n} . We have fixed the centre of mass and so there is one type of each that should be removed. We are now looking at \mathbb{R}^{2n-2} . The two sets of decompositions (x and y

directions) each have dimension $(n - 1)$. Recall that the dimension of the unstable space is equal to the index of the critical point and that when we work with $-V$ we know that a critical point of index i , with respect to V , has an index of $(k - i)$ with respect to $-V$, where k is the dimension of the manifold. Our manifold has dimension $k = 2n - 3$ and the Moulton points have index of $(n - 1)$, which means that the unstable space of the Moulton points has dimension $(n - 2)$ with respect to $-V$.

We look at the perturbations in the x -direction. It is easy to verify that the function $f = -V$ is convex. We must show that

$$\frac{\partial^2 f}{\partial x_k \partial x_l} < 0 \quad k \neq l, \quad \frac{\partial^2 f}{\partial x_k^2} > 0, \quad \text{and} \quad \sum_l \frac{\partial^2 f}{\partial x_k \partial x_l} = 0 \quad \text{for fixed } k.$$

We can easily show that

- i) $\frac{\partial^2 f}{\partial x_k \partial x_l} = \frac{-2}{|x_k - x_l|^3} < 0$, for all $l \neq k$, and
- ii) $\frac{\partial^2 f}{\partial x_k^2} = \sum_{j \neq k} \frac{2}{|x_k - x_j|^3} > 0$ for every k .

Which allows us to say

$$\sum_l \frac{\partial^2 f}{\partial x_k \partial x_l} = \frac{\partial^2 f}{\partial x_k^2} + \sum_{l \neq k} \frac{\partial^2 f}{\partial x_k \partial x_l} = \sum_{j \neq k} \frac{2}{|x_k - x_j|^3} + \sum_{l \neq k} \frac{-2}{|x_k - x_l|^3} = 0.$$

We are now in a position to use the following lemma which is a variant of Gershgorin's theorem.

Lemma 6.11 (Lemma 1. [22]). *Let $A = (a_{kl})$ be a symmetric $N \times N$ matrix satisfying $a_{kl} < 0$ for $k \neq l$, and $a_{kk} > 0$, $\sum_{l=1}^N a_{kl} = 0$ for each k . Then 0 is a simple eigenvalue of A and all other eigenvalues are strictly positive.*

Proof. See *Vortex Dynamics on a Cylinder* by J. Montaldi, A Soulière and T. Tokieda [22]. □

This tells us the function $-V$ is convex and is positive definite on the x -perturbation subspace.

We now look at the y -perturbations. Any y -perturbation, with one exception, strictly increases the pairwise distances and so decreases the function $-V$. This means that $-V$ is negative definite on this subspace. The exception is the y -perturbation that is an infinitesimal rotation about the origin of the whole configuration — here all distances are unchanged and so the eigenvalue of the hessian in this direction is zero. The unstable space is given by the y -perturbations and we can find a basis for the tangent space by finding $(n - 2)$ linearly independent perturbations that lie in the space. A basis for the unstable space when $n = 3$ is shown in Figure 6.6. This is a basis since the perturbation lies in the space and we are only looking for one non-zero perturbation. One can now see how the isotropy subgroup acts on this and its orientation. The isotropy subgroup for the Moulton point M_{123} is $H_M = \langle (13) \rangle \cong \mathbb{Z}_2$

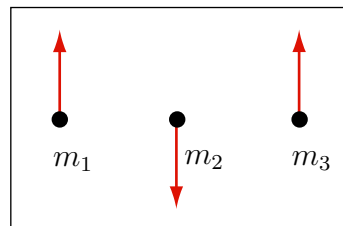


Figure 6.6: Unstable space of Moulton points when $n = 3$.

and it is orientation reversing - see Figure 6.7 for details, (where $\rho \in SO(2)$ indicates rotation by π).

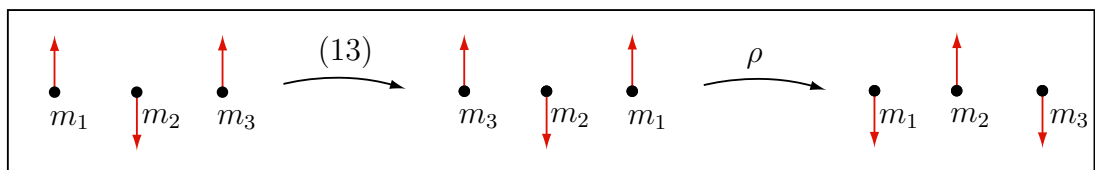


Figure 6.7: How H_M acts on the unstable space of the Moulton points.

6.6.2 Back to the 3-Body Problem Calculation

Notation. Throughout the rest of this chapter the characters of isotropy subgroups H will be denoted by a lowercase letter to avoid confusion with a similarly named character of G , see Appendix B for character tables.

For the Moulton points we calculate $\text{Ind}_{H_M}^{S_3}(a_1)$ where a_1 is the alternating character of \mathbb{Z}_2 . Since the Lagrange points have an index of zero, it is easy to see that the orientation is preserved and so we calculate $\text{Ind}_{H_L}^{S_3}(a_0)$ using Frobenius reciprocity (see Section 2.6.5). Here $H_L = \langle(123)\rangle \cong \mathbb{Z}_3$ is the isotropy subgroup of the Lagrange points and a_0 is the trivial representation of \mathbb{Z}_3 . The following table summarises these results.

Type	Index	Isotropy Subgroup	Induction	Character
Lagrange	0	$H_L = \langle(123)\rangle \cong \mathbb{Z}_3$	$\text{Ind}_{H_L}^G(a_0)$	$A_0 + A_1$
Moulton	1	$H_M = \langle(13)\rangle \cong \mathbb{Z}_2$	$\text{Ind}_{H_M}^G(a_1)$	$A_1 + E$

Where A_0 is the trivial character, A_1 the alternating character and E the two dimensional character of S_3 , (for the character table see Appendix B.1).

We know that the Poincaré polynomial of $B = \mathbb{C}P^1 - \tilde{\Delta}$ is $P_t(B) = 1 + 2t$ (since $\mathbb{C}P^1 - \tilde{\Delta}$ is homotopic to a 2-sphere with 3 points removed) and so this gives Betti numbers of $b_0 = 1, b_1 = 2, b_2 = 0$. Combining this information with the equivariant Morse inequalities and letting square brackets denote the character of S_3 , we see that

$$(i) [H_0(B)] \leq \sum_{\text{ind}(\mathcal{O})=0} [TW_{\mathcal{O}}^-] = A_0 + A_1,$$

(ii) and

$$\begin{aligned} [H_1(B)] - [H_0(B)] &= \sum_{\text{ind}(\mathcal{O})=1} [TW_{\mathcal{O}}^-] - \sum_{\text{ind}(\mathcal{O})=0} [TW_{\mathcal{O}}^-] \\ &= A_1 + E - (A_0 + A_1) \\ &= E - A_0. \end{aligned}$$

The character on $H_0(B)$ is trivial so these equations give us that $[H_1(B)] = E$.

When we move on to four or five masses it will be necessary for us to use another method to find the representation on the cohomology since there will be greater ambiguity with just the Betti numbers as a guide. For four or five masses we will need to utilise the Gysin sequence and as such we include the relevant calculations for the 3-body problem.

6.7 Further Calculations of the 3-Body Problem

For $n = 3$ we have $E = S^3 - \Delta$ and $B = \mathbb{C}P^1 - \tilde{\Delta}$, the Poincaré polynomials of which are $P_t(E) = (1+t)(1+2t) = 1+3t+2t^2$ and $P_t(B) = 1+2t$ respectively. The Gysin sequence is

$$0 \rightarrow [H^1(B)] \xrightarrow{p^*} [H^1(E)] \xrightarrow{\alpha} [H^0(B)] \xrightarrow{e^\wedge} [H^2(B)] \xrightarrow{p^*} [H^2(E)] \xrightarrow{\alpha} [H^1(B)] \rightarrow 0.$$

Recall from Section 6.5 that we have zero Euler class, that is, $e = 0$.

Notation. We use E to denote both the total space and the two dimensional representation of S_3 and later S_4 , however it should be clear from the context which E we are talking about.

Using the program given in Appendix A, we find the characters on the cohomology groups of E to be:

Cohomology Group	Dimension	Character
$H^0(E)$	1	A_0
$H^1(E)$	3	$A_0 + E$
$H^2(E)$	2	E

(6.2)

We now put these into the exact sequence (in blue) and use exactness to calculate the characters on the cohomology of B , $[H^i(B)]$, shown in red,

$$0 \rightarrow [H^1(B)] \xrightarrow{p^*} [H^1(E)] \xrightarrow{\alpha} [H^0(B)] \xrightarrow{e^\wedge} [H^2(B)] \xrightarrow{p^*} [H^2(E)] \xrightarrow{\alpha} [H^1(B)] \rightarrow 0.$$

We summarise this in the following table:

Cohomology Group	Dimension	Character
$H^0(B)$	1	A_0
$H^1(B)$	2	E

This agrees with our earlier calculations.

6.7.1 Character on the Total Cohomology

In *Coxeter group actions on the complement of hyperplanes and special involutions* G. Felder and A.P. Veselov give us a convenient method for checking that our program (Appendix A) does indeed give the correct character of G on the cohomology groups of $E = S^{2n-3} - \Delta$. This result was originally due to G. Lehrer in his paper *Coxeter Group Actions on Complements of Hyperplanes* [16].

Proposition 6.12 (Proposition 2 [11], Proposition (5.6) [16]). *Let M_n be the configuration space of n distinct points of the complex plane with standard action of S_n , the symmetric group, upon it. The character on the total cohomology is:*

$$[H^*(M_n)] = 2\text{Ind}_{\langle s \rangle}^{S_n}(1),$$

where $\text{Ind}_{\langle s \rangle}^{S_n}(1)$ denotes the induced representation of S_n from the trivial representation of the subgroup \mathbb{Z}_2 generated by the simple reflection $s = (12)$.

Proof. See Felder and Veselov [11]. □

Since $E \cong M_n$ (see Section 2.8) we are able to use this proposition to check our calculations. We first calculate the induced character of S_3 :

$$\text{Ind}_{\langle s \rangle}^{S_3}(1) = A_0 + E.$$

Recall from Table 6.2 that $[H^0(E)] = A_0$, $[H^1(E)] = A_0 + E$, and $[H^2(E)] = E$ which gives a total cohomology of:

$$\begin{aligned} [H^*(E)] &= [H^0(E)] + [H^1(E)] + [H^2(E)] \\ &= 2A_0 + 2E. \end{aligned}$$

Which is equal to $2\text{Ind}_{\langle s \rangle}^{S_3}(1)$ and so we conclude that it is likely that the calculations are correct.

Felder and Veselov give us a second result (Equation 6.3) - a representation of the twisted action of S_n on the total cohomology of the configuration space M_n . This twisted action is given by complex conjugation and the representation is given by

$$H_\epsilon^*(M_n) = \bigoplus_k \epsilon^k \otimes H^k(M_n) = \rho, \quad (6.3)$$

where ϵ is the alternating representation and ρ the regular representation of S_n . We will see later, in Section 6.7.1, that this result is a consequence of the first (Proposition 6.12) and so unfortunately gives us no additional information.

6.8 What Does All This Look Like?

By analysing how S_3 acts on the critical points, we see that the points can be thought of as being in positions on $\mathbb{C}P^1 \cong S^2$ as shown in Figure 6.8. As pointed out by Smale [36] the Moulton points lie on $\mathbb{R}P^1$ in $\mathbb{C}P^1$, as do the collision points.

The Moulton points are shown in **cyan**, the collision points are shown in **fuchsia** and the Lagrange points are **black**.

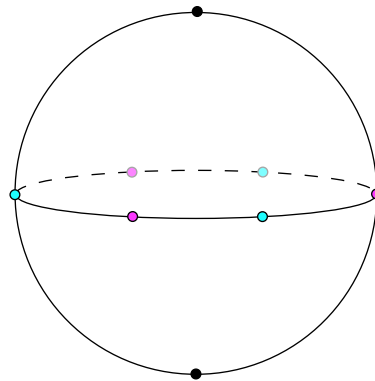


Figure 6.8: Positions of the critical points on $\mathbb{C}P^1$.

We can easily add in the flow to see the unstable manifolds, Figure 6.9.

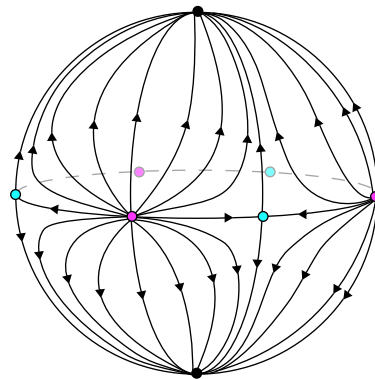


Figure 6.9: Positions of the critical points on $\mathbb{C}P^1$ including the gradient flow of $-V$.

6.9 The 4-Body Problem

We again require that our bodies are of equal mass and we set this mass to be one. We start by looking at the Morse inequalities for hints at the number of each index of points. For four particles there are three relative equilibria classes that are known, they are:

- twelve Moulton classes (M) with index 2 [23],
- six Square classes (S), where the masses are at the vertices of a square with index zero [26], and
- eight Equilateral classes (Eq) of three masses at the vertices of equilateral triangle with the fourth mass located at the origin, also with index two [27].

The indexes are given with respect to $-V$. Figure 6.10 shows these configurations. The Poincaré polynomial of $B = \mathbb{C}P^2 - \tilde{\Delta}$ is $P_t(B) = (1 + 2t)(1 + 3t) = 1 + 5t + 6t^2$, giving Betti numbers $b_0 = 1, b_1 = 5$ and $b_2 = 6$. The Morse inequalities suggest that (assuming that there are no more than the listed index zero and index two points) there are at least ten, but no more than twenty four critical points of index one. So we have two possibilities, namely twelve or twenty four, since the number of classes must divide the order of S_4 . If there are twelve classes the configuration must have a rotational symmetry of order two (but not four), and a collinear configuration is the only possible way this can happen. We conclude that there must be 24 classes — Palmore agrees [26] and suggests that an equilibrium configuration with this index has at most one axis of symmetry (and therefore no rotational symmetry) and will have the shape of an isosceles triangle with one mass at each vertex and a fourth in the interior on the axis of symmetry. Palmore reiterates this statement [29] and also tells us that 96 relative equilibria arise from two pairs of scalene triangle configurations with a mass at each vertex and one in the interior. This has been disputed by many people with Long and Sun [17] proving that (in the four-body problem) a relative equilibrium configuration with three equal masses forming a triangle and the fourth (with any mass) in the interior must either be an equilateral triangle, with

fourth mass at the geometric centre, or an isosceles triangle, with fourth mass on the axis of symmetry. We end this introductory discussion noting that Albouy recently classified the four body problem of equal masses in his paper *The symmetric central configurations of four equal masses* [1], in which he proves that the four classes of relative equilibria are precisely those mentioned above.

We now turn to the G -cohomology to utilise the methods from Chapter 5. Similarly

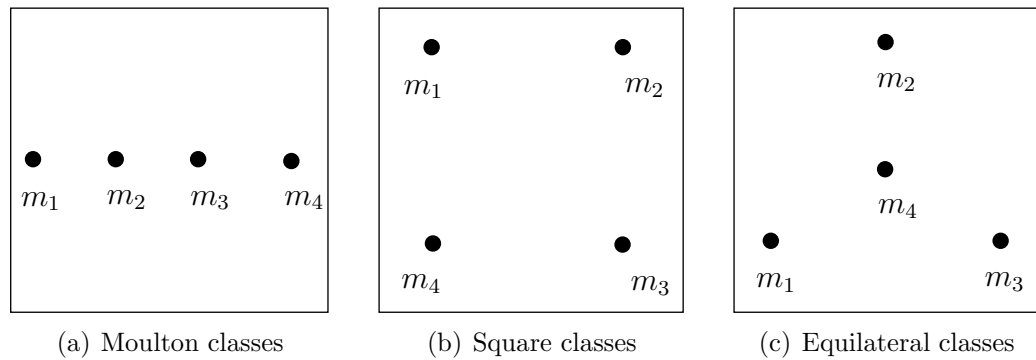


Figure 6.10: Known relative equilibria for $n = 4$.

to when $n = 3$, we apply S_4 , the symmetric group on four elements to our system as a permutation of the masses. The isotropy subgroups of these points are shown in the following table.

Type	Index	Isotropy Subgroup
Square	0	$H_S = \langle (1234) \rangle \cong \mathbb{Z}_4$
Moulton	2	$H_M = \langle (14)(23) \rangle \cong \mathbb{Z}_2$
Equilateral	2	$H_{Eq} = \langle (123) \rangle \cong \mathbb{Z}_3$

Since the index of the Square classes are zero we see that H_S and H_{Eq} must be orientation preserving, hence we can induce (using the Frobenius reciprocity theorem — Theorem 2.23) from the trivial representation of the isotropy subgroup. For the index two points we need to look at how the relevant isotropy subgroup acts on the unstable manifold.

6.9.1 A Basis for the Moulton Points

Figure 6.11 gives a basis for the unstable space of the Moulton configuration and the resulting points when we apply elements of the isotropy subgroup (where $\rho \in SO(2)$ indicates the appropriate rotation). This is a genuine basis for the unstable space,

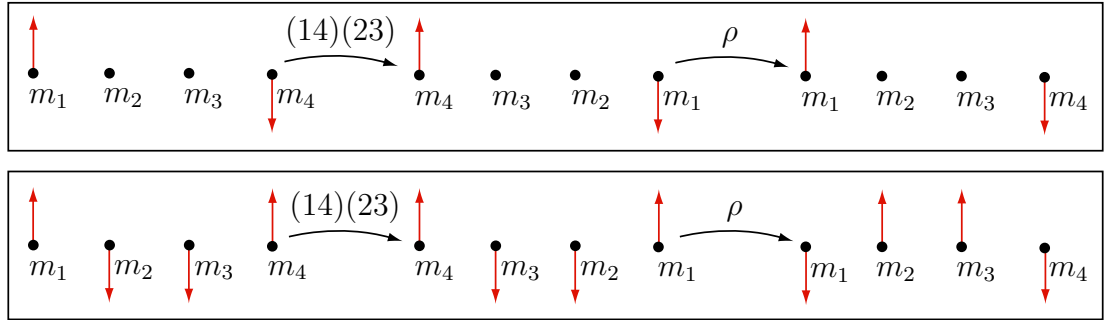


Figure 6.11: A basis for the unstable space of the Moulton points including action of the isotropy subgroup $H_M = \langle (14)(23) \rangle$.

see Section 6.6.1.

We now look at the matrix representations of this basis under the action of their isotropy subgroup. Let

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the matrix representation (for the generator of \mathbb{Z}_2) for the Moulton points which has a determinant of minus one. We take the determinant rather than trace as we are looking for the orientation representation. This means that the character to induce from is a_1 of \mathbb{Z}_2 . The characters of S_4 acting on the unstable spaces of the critical points are shown in table (6.4).

Type	Index	Isotropy Subgroup	Character
Square	0	$H_S = \langle (1234) \rangle \cong \mathbb{Z}_4$	$\text{Ind}_{H_S}^{S_4}(a_0) = A_0 + E + T_1$
Isoceles	1	$H_I = \langle 1 \rangle$	$\text{Ind}_{H_I}^{S_4}(1) = A_0 + A_1 + 2E + 3T_1 + 3T_2$
Equilateral	2	$H_{E_q} = \langle (123) \rangle \cong \mathbb{Z}_3$	$\text{Ind}_{H_{E_q}}^{S_4}(a_0) = A_0 + A_1 + T_1 + T_2$
Moulton	2	$H_M = \langle (14)(23) \rangle \cong \mathbb{Z}_2$	$\text{Ind}_{H_M}^{S_4}(a_1) = 2T_1 + 2T_2$

(6.4)

These inductions were computed using GAP [13].

We need to calculate the character on the cohomology group of the space $B = \mathbb{C}P^2 - \tilde{\Delta}$; to do this we use the Gysin sequence. We know from the Poincaré polynomial that the Betti numbers of B are $b_0 = 1, b_1 = 5$ and $b_2 = 6$ and from the Poincaré polynomial of E , $P_t(E) = 1 + 6t + 11t^2 + 6t^3$, we know the dimensions of the cohomology groups of E . Using the program in Appendix A we find the characters on the cohomology groups of E to be:

Cohomology group	Dimension	Character
$H^0(E)$	1	A_0
$H^1(E)$	6	$A_0 + E + T_2$
$H^2(E)$	11	$E + T_1 + 2T_2$
$H^3(E)$	6	$T_1 + T_2$

(6.5)

Again we use that the Gysin sequence is G -equivariant. Using the discussion from Section 6.7.1 we can check these calculations — we show that $[H^*(E)] = 2\text{Ind}_{(s)}^{S_4}(1)$, where $s = (12)$:

$$\begin{aligned} \text{Ind}_{(s)}^{S_4}(1) &= A_0 + E + T_1 + 2T_2 \\ [H^*(E)] &= A_0 + (A_0 + E + T_2) + (E + T_1 + 2T_2) + (T_1 + T_2) \\ &= 2(A_0 + E + T_1 + 2T_2). \end{aligned}$$

It is therefore reasonable for us to conclude that these are the correct characters.

Putting the characters from Table 6.5 (shown in blue) into the Gysin sequence and using exactness, we can calculate the characters on the cohomology of B (shown in red). These characters are summarised in Table 6.6.

$$\begin{aligned} 0 &\rightarrow [H^0(B)] \rightarrow [H^0(E)] \rightarrow 0 \\ 0 &\rightarrow [H^1(B)] \xrightarrow{p^*} [H^1(E)] \xrightarrow{\alpha} [H^0(B)] \xrightarrow{e^\wedge} [H^2(B)] \xrightarrow{p^*} [H^2(E)] \\ &\xrightarrow{\alpha} [H^1(B)] \xrightarrow{e^\wedge} [H^3(B)] \xrightarrow{p^*} [H^3(E)] \xrightarrow{\alpha} [H^2(B)] \xrightarrow{e^\wedge} 0 \end{aligned}$$

Cohomology group	Dimension	Character
$H^0(B)$	1	A_0
$H^1(B)$	5	$E + T_2$
$H^2(B)$	6	$T_1 + T_2$

(6.6)

Using the information from Tables (6.4) and (6.6) we can construct the G -Poincaré and G -Morse polynomials. They are

$$P_t^G(B) = A_0 + t(E + T_2) + t^2(T_1 + T_2), \quad \text{and}$$

$$\mathfrak{M}_t^G(-V) = A_0 + E + T_1 + t(A_0 + A_1 + 2E + 3T_1 + 3T_2) + t^2(A_0 + A_1 + 3T_1 + 3T_2),$$

respectively. We now set $t = -1$ and refer to Theorem 5.10:

$$\begin{aligned} \mathfrak{M}_{t=-1}^G(-V) &= A_0 + E + T_1 - (A_0 + A_1 + 2E + 3T_1 + 3T_2) + (A_0 + A_1 + 3T_1 + 3T_2) \\ &= A_0 - E + T_1 \\ P_{t=-1}^G(B) &= A_0 - (E + T_2) + (T_1 + T_2) \\ &= A_0 - E + T_1 \end{aligned}$$

This shows that $P_{t=-1}^G(B) = \mathfrak{M}_{t=-1}^G(-V)$.

6.9.2 Looking at $S_4 \times \mathbb{Z}_2$

To see what more we can learn about the symmetry of the critical points, we set $G = S_4 \times \mathbb{Z}_2$. The representations of $S_4 \times \mathbb{Z}_2$ are related to those of S_4 — each representation of S_4 splits into two new representations. We will give each irreducible representation of S_4 a superscript depending on how the \mathbb{Z}_2 -part acts — a superscript of a plus indicates action of the identity I and a superscript of a minus indicates action of minus the identity $-I$. For example A_0 will split into A_0^+ and A_0^- with A_0^+ being the trivial character of $S_4 \times \mathbb{Z}_2$. The full character table is shown in Appendix B.9.

Lemma 6.13. *Let R be the representation of G on the i -th cohomology group of B . Then the representation of $G \times \mathbb{Z}_2$ on this cohomology group is given by:*

$$[H^i(B)]_G = R \quad \Rightarrow \quad [H^i(B)]_{G \times \mathbb{Z}_2} = \begin{cases} R^+ & \text{if } i \text{ is even} \\ R^- & \text{if } i \text{ is odd} \end{cases} \quad (6.7)$$

Proof. From Arnold we know that products of the formulas $\omega_{jk} = \frac{1}{2\pi i} \frac{dz_j - dz_k}{z_j - z - k}$ form an additive basis for the cohomology groups of $E = S^{2n-3} - \Delta$.

We look at the \mathbb{Z}_2 action given by complex conjugation. Let $x = e^{i\theta}$. We have $dx = ie^{i\theta} d\theta$, $\bar{x} = e^{-i\theta}$ and $d\bar{x} = -ie^{-i\theta} d\theta$. Now we can see that

$$\begin{aligned}\frac{dx}{x} &= \frac{ie^{i\theta} d\theta}{e^{i\theta}} = i d\theta, \text{ and} \\ \frac{d\bar{x}}{\bar{x}} &= \frac{-ie^{-i\theta} d\theta}{e^{-i\theta}} = -i d\theta.\end{aligned}$$

If we now set $x = z_j - z_k$ we see that $\bar{\omega}_{jk} = -\omega_{jk}$. Since the basis of $H^p(E)$ consists of all products of the form

$$\omega_{k_1 l_1} \omega_{k_2 l_2} \cdots \omega_{k_p l_p}, \text{ where } k_s < l_s, l_1 < l_2 < \cdots < l_p,$$

we can see that if p is odd then the \mathbb{Z}_2 action is non-trivial whereas on the even degree cohomology groups \mathbb{Z}_2 acts trivially. This means that

$$[H^i(E)]_G = R \quad \Rightarrow \quad [H^i(E)]_{G \times \mathbb{Z}_2} = \begin{cases} R^+ & \text{if } i \text{ is even} \\ R^- & \text{if } i \text{ is odd} \end{cases} \quad (6.8)$$

The action of \mathbb{Z}_2 commutes with the maps p^* (since p^* is G -equivariant), but is “anti-equivariant” on the maps α , in that it reverses the direction of integration. If we now look at the Gysin sequence we see that for each n ,

$$0 \rightarrow H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{\alpha} H^{n-1}(B) \rightarrow 0,$$

since the Euler class is zero. From this we can see that $[H^n(E)] = [H^n(B)] + a_1[H^{n-1}(B)]$, where a_1 is the alternating character of \mathbb{Z}_2 . This proves the result. \square

In practical terms this means that the representation of $S_4 \times \mathbb{Z}_2$ on the cohomology groups of B change as shown in Table 6.9 below.

Cohomology group	Dimension	$[H^i(B)]_{S_4}$	$[H^i(B)]_{S_4 \times \mathbb{Z}_2}$
$H^0(B)$	1	A_0	A_0^+
$H^1(B)$	5	$E + T_2$	$E^- + T_2^-$
$H^2(B)$	6	$T_1 + T_2$	$T_1^+ + T_2^+$

(6.9)

There is more to be done to find the representations on the unstable spaces of the critical points - we must again find the isotropy subgroups of the critical points and

induce from the appropriate representation of that subgroup. Let κ be the generator for the \mathbb{Z}_2 part of $S_4 \times \mathbb{Z}_2$. Then the isotropy subgroups are as shown in Table 6.10 below.

Type	Index	Isotropy Subgroup
Square	0	$H_S = \langle \kappa(13), \kappa(12)(34), (1234) \rangle \cong D_4$
Isosceles	1	$H_I = \langle (23)\kappa \rangle \cong \mathbb{Z}_2$
Equilateral	2	$H_{Eq} = \langle \kappa(23), (123) \rangle \cong S_3$
Moulton	2	$H_M = \langle (14)(23), \kappa \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

(6.10)

We now look at the unstable spaces of the critical points. Since the Square points have index zero we induce from the trivial representation of H_S . We use the same basis for the unstable space of the Moulton configurations as shown earlier (Figure 6.11) and apply elements from the isotropy subgroup.

Figure 6.12 shows what happens to the basis as we apply the element $\kappa(14)(23) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. The resulting matrix representation is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

the determinant of which is minus one, the same as that of Figure 6.11 (which shows how $(14)(23)$ acts on the unstable space). Whereas if we apply κ only (Figure 6.13) we obtain a determinant of one. The character vector now looks like $\chi[1, (14)(23), \kappa, \kappa(14)(23)] = [1, -1, 1, -1]$ which corresponds to the character a_1^+ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (character table is in Appendix B.8).

We also find that we should induce from the alternating character of H_{Eq} . The tangent space to M at an equilateral critical point has an S_3 -character of $2E$. It is clear that the unstable space is 2-dimensional and that it is fixed under the action of $H_{Eq} \cong S_3$, the isotropy subgroup. This means that the unstable space must have an S_3 -character of E which gives us that the orientation character is A_1 and so this is the character we induce from.

The appropriate character to induce from in the isosceles case is the non-trivial character of $H_I \cong \mathbb{Z}_2$, see Appendix C.1.

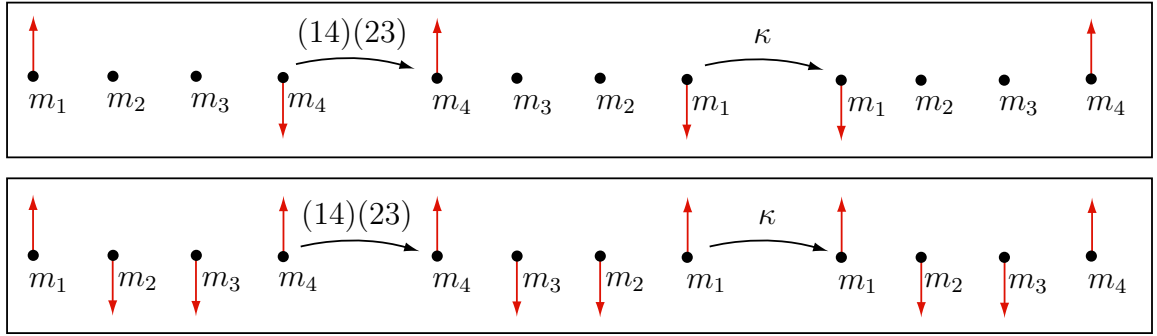


Figure 6.12: Applying the element $\kappa(14)(23)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the basis of the unstable space of the Moulton points — the final positions shown after rotation by π .

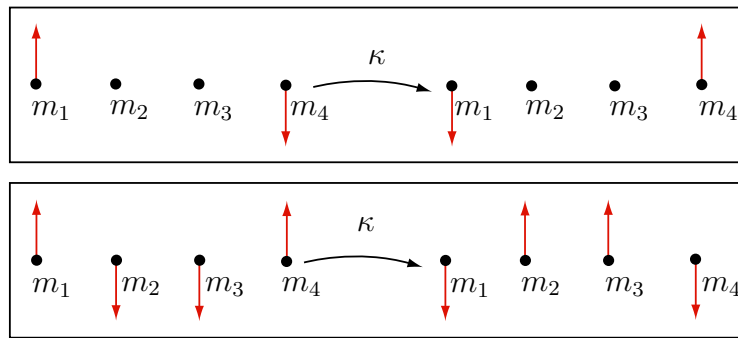


Figure 6.13: Applying the element κ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the basis of the unstable space of the Moulton points.

We then obtain the following results:

Type	Index	Character
Square	0	$\text{Ind}_{H_S}^{S_4 \times \mathbb{Z}_2}(a_1) = A_0^+ + E^+ + T_1^-$
Isosceles	1	$\text{Ind}_{H_U}^{S_4 \times \mathbb{Z}_2}(a_1) = A_0^- + A_1^+ + E^+ + E^-$ $+ 2T_1^+ + T_1^- + T_2^+ + 2T_2^-$
Equilateral	2	$\text{Ind}_{H_{E_4}}^{S_4 \times \mathbb{Z}_2}(a_1) = A_0^- + A_1^+ + T_1^+ + T_2^-$
Moulton	2	$\text{Ind}_{H_M}^{S_4 \times \mathbb{Z}_2}(a_1^+) = 2T_1^+ + 2T_2^+$

(6.11)

These inductions were computed using GAP [13]. Character tables for all groups used here are included in Appendix B.

Using the information from the above tables we form the G -Morse and Poincaré polynomials. We obtain $\mathfrak{M}_t^G(-V) = A_0^+ + E^+ + T_1^- + t(A_0^- + A_1^+ + E^+ + E^- + 2T_1^+ + T_1^- + T_2^+ + 2T_2^-) + t^2(A_0^- + A_1^+ + 3T_1^+ + 2T_2^+ + T_2^-)$, and $P_t^G(B) = A_0^+ + t(E^- + T_2^-) + t^2(T_1^+ + T_2^+)$

respectively. We now set $t = -1$ and refer to Theorem 5.10:

$$\begin{aligned}
\mathfrak{M}_{t=-1}^G(-V) &= A_0^+ + E^+ + T_1^- - (A_0^- + A_1^+ + E^+ + E^- + 2T_1^+ + T_1^- + T_2^+ + 2T_2^-) \\
&\quad + (A_0^- + A_1^+ + 3T_1^+ + 2T_2^+ + T_2^-) \\
&= A_0^+ - E^- + T_1^+ + T_2^+ - T_2^- \\
P_{t=-1}^G(B) &= A_0^+ - (E^- + T_2^-) + (T_1^+ + T_2^+) \\
&= A_0^+ - E^- + T_1^+ + T_2^+ - T_2^-.
\end{aligned}$$

This shows that $P_{t=-1}^G(B) = \mathfrak{M}_{t=-1}^G(-V)$.

6.10 Critical Points of the 4-Body Problem

We have previously referred to the index 1 critical point configuration as ‘unknown’, but this is not strictly accurate. I have (as have numerous others before me) computed the stable configurations numerically from the equations of motion and the ‘unknown’ points have three masses at the vertices of an isosceles triangle with a fourth mass at the centre of mass, see Figure 6.14(a). This is in accordance with the earlier discussion. All four configurations are shown in Figure 6.14 for completeness.

6.11 The 5-Body Problem

We follow the steps taken for the 4-body problem and look at what information we can get from the G -Morse inequalities when all bodies have mass one.

There are three types of points that are well-known, they are:

- 24 Pentagonal classes (P), which have index 0 [26],
- 30 Square classes (S) of four masses with the fifth located at the origin, also with index 0, [29] and
- 60 Moulton classes (M) that have index 3 [23].

These classes are shown in Figure 6.15. The Poincaré polynomial of the space $B = \mathbb{C}P^3 - \tilde{\Delta}$ is $P_t(B) = (1 + 2t)(1 + 3t)(1 + 4t) = 1 + 9t + 26t^2 + 24t^3$. Using the Morse

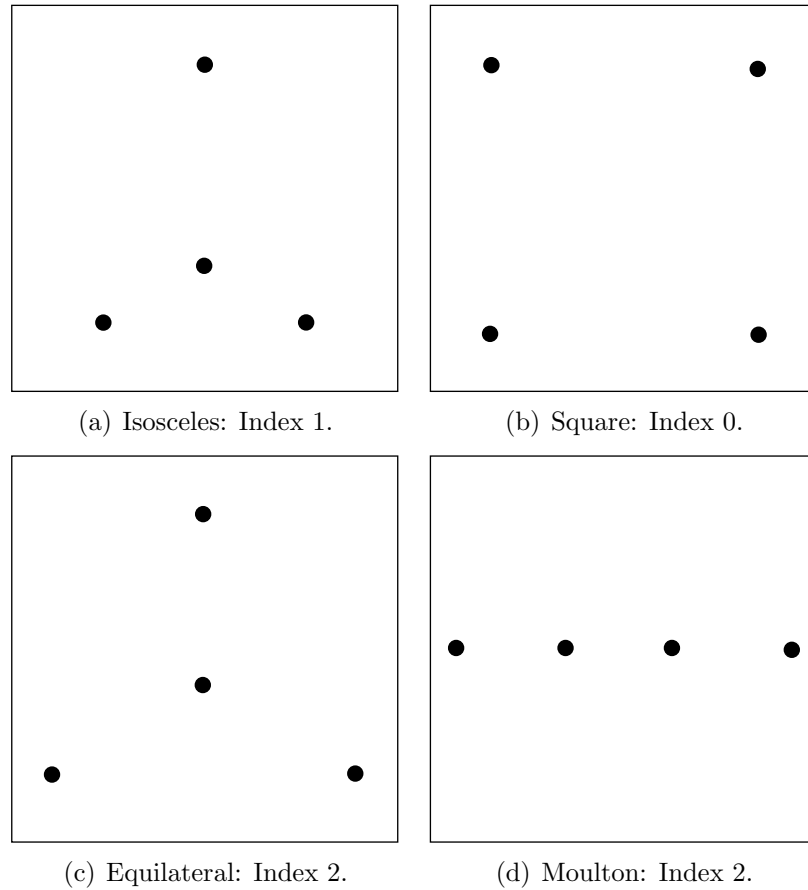


Figure 6.14: Relative equilibria configurations of four equal masses.

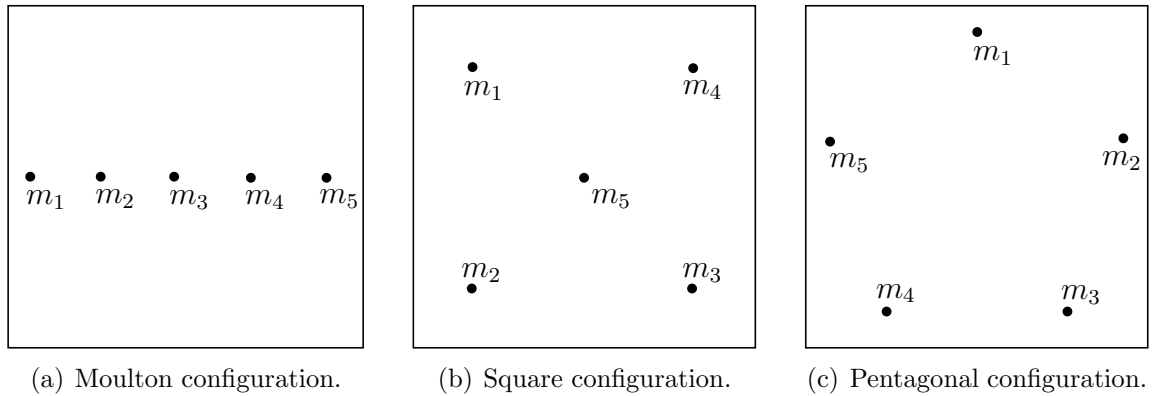
inequalities, assuming that the aforementioned classes are the only critical points of those indexes, it is easy to show that the number of classes of index 1 and 2 must be equal and there must be at least 62. We note that since the number of classes of index 1 and 2 is at least 62, these classes must be fixed by just the identity in S_5 (since the number of classes must divide the order of S_5) and that there must be 120 of each index. Armed with this information we can use the G -Morse inequalities to check these assumptions.

The isotropy subgroup of each configuration is shown in the following table.

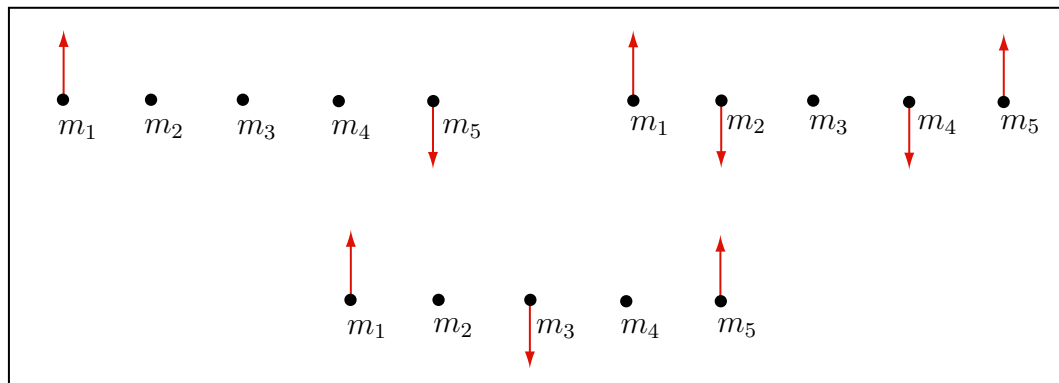
Type	Index	Isotropy Subgroup
Pentagonal	0	$H_P = \langle (12345) \rangle \cong \mathbb{Z}_5$
Square	0	$H_S = \langle (1234) \rangle \cong \mathbb{Z}_4$
Moulton	3	$H_M = \langle (12)(34) \rangle \cong \mathbb{Z}_2$

(6.12)

The S_5 characters for the zero-index configurations are clearly induced from the

Figure 6.15: Known relative equilibria for $n = 5$.

trivial characters of the isotopy subgroups. Following similar discussion to that of Section 6.9, we find that we should also induce from the trivial character of H_M for the Moulton points. A basis for this unstable space is shown in Figure 6.16 and full discussion can be found in Appendix C.2.

Figure 6.16: Basis for the unstable space of the Moulton points when $n = 5$.

Let A_0 and A_1 and be the trivial and alternating characters of S_5 respectively. Let G_i, H_i ($i = 1, 2$) and J be the four, five and six dimensional characters respectively. See Appendix B.3 for the full character tables.

The following table summarises the characters of S_5 on the critical points with predictions for the unknown configurations shown in [cerulean](#) - for these we have

calculated the character of the regular representation $\text{Ind}_1^{S_5}(1)$.

Type	Character
Pentagonal	$\text{Ind}_{H_P}^{S_5}(a_0) = A_0 + A_1 + H_1 + H_2 + 2J$
Square	$\text{Ind}_{H_S}^{S_5}(a_0) = A_0 + G_1 + G_2 + H_1 + 2H_2 + J$
Unknown Index 1	$\text{Ind}_{H_{U_1}}^{S_5}(1) = A_0 + A_1 + 4G_1 + 4G_2 + 5H_1 + 5H_2 + 6J$
Unknown Index 2	$\text{Ind}_{H_{U_2}}^{S_5}(1) = A_0 + A_1 + 4G_1 + 4G_2 + 5H_1 + 5H_2 + 6J$
Moulton	$\text{Ind}_{H_M}^{S_5}(a_0) = A_0 + A_1 + 2G_1 + 2G_2 + 3H_1 + 3H_2 + 2J$

(6.13)

These inductions were computed using GAP [13]. We know that the Poincaré polynomial of $E = S^7 - \Delta$ is given by

$$P_t(E) = (1+t)(1+2t)(1+3t)(1+4t) = 1 + 10t + 35t^2 + 50t^3 + 24t^4.$$

This gives Betti numbers of $b_0 = 1, b_1 = 10, b_2 = 35, b_3 = 50$ and $b_4 = 24$. Using the program from Appendix A, we see that the characters on the cohomology of E are as follows.

Cohomology group	Dimension	Character
$H^0(E)$	1	A_0
$H^1(E)$	10	$A_0 + G_1 + H_1$
$H^2(E)$	35	$2G_1 + 2H_1 + H_2 + 2J$
$H^3(E)$	50	$2G_1 + G_2 + 2H_1 + 2H_2 + 3J$
$H^4(E)$	24	$G_1 + G_2 + H_1 + H_2 + J$

(6.14)

Using the discussion from Section 6.7.1 we can check these calculations. We show that $[H^*(E)] = 2\text{Ind}_{\langle s \rangle}^{S_5}(1)$, where $s = (12)$:

$$\begin{aligned} \text{Ind}_{\langle s \rangle}^{S_5}(1) &= A_0 + 3G_1 + G_2 + 3H_1 + 2H_2 + 3J \\ [H^*(E)] &= A_0 + (A_0 + G_1 + H_1) + (2G_1 + 2H_1 + H_2 + 2J) \\ &\quad + (2G_1 + G_2 + 2H_1 + 2H_2 + 3J) + (G_1 + G_2 + H_1 + H_2 + J) \\ &= 2(A_0 + 3G_1 + G_2 + 3H_1 + 2H_2 + 3J) = 2\text{Ind}_{\langle s \rangle}^{S_5}(1). \end{aligned}$$

Using the information from the table (blue) and exactness of the Gysin sequence

we obtain the following sequence.

$$\begin{array}{ccccccc}
0 & \rightarrow & \overset{G_1+H_1}{[H^1(B)]} & \rightarrow & \overset{A_0+G_1+H_1}{[H^1(E)]} & \rightarrow & \overset{A_0}{[H^0(B)]} \\
& & \overset{G_1+H_1+H_2+J}{[H^2(B)]} & \rightarrow & \overset{2G_1+2H_1+H_2+2J}{[H^2(E)]} & \rightarrow & \overset{G_1+H_1}{[H^1(B)]} \\
& & \overset{G_1+G_2+H_1+H_2+J}{[H^3(B)]} & \rightarrow & \overset{2G_1+G_2+2H_1+2H_2+3J}{[H^3(E)]} & \rightarrow & \overset{G_1+H_1+H_2+2J}{[H^2(B)]} \\
& & \overset{0}{[H^4(B)]} & \rightarrow & \overset{G_1+G_2+H_1+H_2+J}{[H^4(E)]} & \rightarrow & \overset{G_1+G_2+H_1+H_2+J}{[H^3(B)]} \rightarrow 0
\end{array}$$

The results from this are summarised in the table below.

Cohomology	Dimension	Character
$H^0(B)$	1	A_0
$H^1(B)$	9	$G_1 + H_1$
$H^2(B)$	20	$G_1 + H_1 + H_2 + 2J$
$H^3(B)$	24	$G_1 + G_2 + H_1 + H_2 + J$

(6.15)

Using the information from Tables (6.13) and (6.15) we construct the G -Poincaré and Morse polynomials. We have:

$$P_t^G(B) = A_0 + t(G_1 + H_1) + t^2(G_1 + H_1 + H_2 + 2J) + t^3(G_1 + G_2 + H_1 + H_2 + J),$$

and

$$\begin{aligned}
\mathfrak{M}_t^G(-V) &= 2A_0 + A_1 + G_1 + G_2 + 2H_1 + 3H_2 + 3J + t[U_1] + t^2[U_2] \\
&\quad + t^3(A_0 + A_1 + 2G_1 + 2G_2 + 3H_1 + 3H_2 + 2J),
\end{aligned}$$

where $[U_i]$ is the character of G on the index i unknown configurations. We can say from this that $[U_1] \geq G_1 + H_1$ and $[U_2] \geq G_1 + H_1 + H_2 + J$. We set $t = -1$ and refer to Theorem 5.10 we can say that $P_{t=-1}^G(B) = \mathfrak{M}_{t=-1}^G(-V)$. We have

$$\begin{aligned}
P_{t=-1}^G(B) &= A_0 - (G_1 + H_1) + (G_1 + H_1 + H_2 + 2J) - (G_1 + G_2 + H_1 + H_2 + J) \\
&= A_0 - G_1 - G_2 - H_1 + J,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{M}_{t=-1}^G(-V) &= 2A_0 + A_1 + G_1 + G_2 + 2H_1 + 3H_2 + 3J - [U_1] + [U_2] \\
&\quad - (A_0 + A_1 + 2G_1 + 2G_2 + 3H_1 + 3H_2 + 2J) \\
&= A_0 - G_1 - G_2 - H_1 + J - [U_1] + [U_2],
\end{aligned}$$

which gives $[U_1] = [U_2]$. This tells us that the character of S_5 on the unstable spaces of these points is identical.

We now look at the G -Morse inequality:

$$\sum_{\text{ind}(\mathcal{O})=1} [TW_{\mathcal{O}}^-] - \sum_{\text{ind}(\mathcal{O})=0} [TW_{\mathcal{O}}^-] \geq [H^1(B)] - [H^0(B)]$$

to obtain:

$$\begin{aligned} -A_0 + G_1 + H_1 &\leq [U_1] - (2A_0 + A_1 + G_1 + G_2 + 2H_1 + 3H_2 + 3J) \\ [U_1] &\geq A_0 + A_1 + 2G_1 + G_2 + 3H_1 + 3H_2 + 3J. \end{aligned}$$

To summarise we have found the following:

1. the dimensions of $[U_1]$ and $[U_2]$ are 120,
2. $[U_1] \geq G_1 + H_1$ and $[U_2] \geq G_1 + H_1 + H_2 + J$,
3. $[U_1] = [U_2]$, and
4. $[U_1] \geq A_0 + A_1 + 2G_1 + G_2 + 3H_1 + 3H_2 + 3J$.

All of this information fits with our predictions and so we can say with some degree of certainty that we have found all the critical points of the system and that those points have the characters of S_5 shown in Table 6.13.

6.11.1 Looking at $S_5 \times \mathbb{Z}_2$

We now let $G = S_5 \times \mathbb{Z}_2$ in the hope that we can find more about the configurations of the index 1 and index 2 points. The representations of S_5 split according to whether the action of \mathbb{Z}_2 is orientation reversing or not. Following the discussion of Section 6.9.2 we use the same notation (superscripts of plus and minus) to give the representations of $S_2 \times \mathbb{Z}_2$. The character table of $S_5 \times \mathbb{Z}_2$ can be found in Appendix B.10.

Referring back to Lemma 6.13 we see that the characters of $S_5 \times \mathbb{Z}_2$ on the cohomology groups of $B = \mathbb{C}P^3 - \tilde{\Delta}$ are as in the following table.

Cohomology	Dimension	$[H^i(B)]_{S_5}$	$[H^i(B)]_{S_5 \times \mathbb{Z}_2}$
$H^0(B)$	1	A_0	A_0^+
$H^1(B)$	9	$G_1 + H_1$	$G_1^- + H_1^-$
$H^2(B)$	20	$G_1 + H_1 + H_2 + 2J$	$G_1^+ + H_1^+ + H_2^+ + 2J^+$
$H^3(B)$	24	$G_1 + G_2 + H_1 + H_2 + J$	$G_1^- + G_2^- + H_1^- + H_2^- + J^-$

(6.16)

To find the characters on the unstable spaces we need to work out which characters of the isotropy subgroups we should induce from. We follow the procedure from previous sections and find that we should induce from the following characters.

Type	Index	Subgroup	Character
Pentagonal	0	$H_P = \langle (12345), \kappa(12)(35) \rangle \cong \mathbb{Z}_5 \times \mathbb{Z}_2$	a_0
Square	0	$H_S = \langle (1234), \kappa(13), \kappa(12)(34) \rangle \cong D_4$	a_0
Moulton	3	$H_M = \langle (15)(24), \kappa \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	a_1^-

The character of $S_5 \times \mathbb{Z}_2$ on the index zero configurations is easily seen to be induced from the trivial character of the appropriate isotropy subgroup. Full details for the calculations involving the Moulton points can be found in Appendix C.2. We suspect that the isotropy subgroups of the unknown points are both \mathbb{Z}_2 , however we do not know which character of this to induce from. The two possibilities are:

- (i) $\text{Ind}_H^G(a_0) = A_0^+ + A_1^+ + 2G_1^+ + 2G_1^- + 2G_2^+ + 2G_2^- + 3H_1^+ + 2H_1^- + 3H_2^+ + 2H_2^- + 2J^+ + 4J^-$, and
- (ii) $\text{Ind}_H^G(a_1) = A_0^- + A_1^- + 2G_1^+ + 2G_1^- + 2G_2^+ + 2G_2^- + 2H_1^+ + 3H_1^- + 2H_2^+ + 3H_2^- + 4J^+ + 2J^-$.

The characters for the index 0 and index 3 points are shown in the following table.

Type	Index	Character
Pentagonal	0	$A_0^+ + A_1^+ + H_1^+ + H_2^+ + 2J^-$
Square	0	$A_0^+ + G_1^+ + G_2^- + H_1^+ + H_2^+ + H_2^- + J^-$
Moulton	3	$A_0^- + A_1^- + 2G_1^- + 2G_2^- + 3H_1^- + 3H_2^- + 2J^-$

These inductions were computed using GAP [13]. We now turn to the G -Morse inequalities; the G -Poincaré and G -Morse polynomials are

$$P_t^G(B) = A_0^+ + t(G_1^- + H_1^-) + t^2(G_1^+ + H_1^+ + H_2^+ + 2J^+) + t^3(G_1^- + G_2^- + H_1^- + H_2^- + J^-)$$

and

$$\begin{aligned} \mathfrak{M}_t^G(-V) &= 2A_0^+ + A_1^+ + G_1^+ + G_2^- + 2H_1^+ + 2H_2^+ + H_2^- + 3J^- + t[U_1] + t^2[U_2] \\ &\quad + t^3(A_0^- + A_1^- + 2G_1^- + 2G_2^- + 3H_1^- + 3H_2^- + 2J^-), \end{aligned}$$

respectively, with $[U_i]$ being the character of $S_5 \times \mathbb{Z}_2$ on the unstable space of the index i orbit. We now look at the G -Morse inequalities at each stage.

1. When $j = 1$:

$$\sum_{\text{ind}(\mathcal{O})=1} [TW_{\mathcal{O}}^-] - \sum_{\text{ind}(\mathcal{O})=0} [TW_{\mathcal{O}}^-] \geq [H_1(B)] - [H_0(B)].$$

This gives the following:

$$\begin{aligned} G_1^- + H_1^- - A_0^+ &\leq [U_1] - (2A_0^+ + A_1^+ + G_1^+ + G_2^- + 2H_1^+ + 2H_2^+ + H_2^- + 3J^-) \\ [U_1] &\geq A_0^+ + A_1^+ + G_1^+ + G_1^- + G_2^+ + G_2^- + 2H_1^+ + H_1^- + 2H_2^+ \\ &\quad + H_2^- + 3J^- \end{aligned}$$

which indicates that $[U_1] = \text{Ind}_{\mathbb{Z}_2}^{S_5 \times \mathbb{Z}_2}(a_0)$.

2. When $j = 2$:

$$\sum_{\text{ind}(\mathcal{O})=2} [TW_{\mathcal{O}}^-] - \sum_{\text{ind}(\mathcal{O})=1} [TW_{\mathcal{O}}^-] + \sum_{\text{ind}(\mathcal{O})=0} [TW_{\mathcal{O}}^-] \geq [H_2(B)] - [H_1(B)] + [H_0(B)]$$

gives the following:

$$\begin{aligned} [U_2] - [U_1] + (2A_0^+ + A_1^+ + G_1^+ + G_2^- + 2H_1^+ + 2H_2^+ + H_2^- + 3J^-) &\geq \\ A_0^+ - (G_1^- + H_1^-) + (G_1^+ + H_1^+ + H_2^+ + 2J^+) &. \end{aligned}$$

If we take $[U_1] = \text{Ind}_{\mathbb{Z}_2}^{S_5 \times \mathbb{Z}_2}(a_0)$, then this becomes

$$[U_2] \geq 2G_1^+ + G_1^- + 2G_2^+ + G_2^- + 2H_1^+ + H_1^- + 2H_2^+ + H_2^- + 4J^+ + J^-,$$

which indicates that $[U_2] = \text{Ind}_{\mathbb{Z}_2}^{S_5 \times \mathbb{Z}_2}(a_1)$.

3. We use Equation 5.6 with $t = -1$ to see if these assumptions are consistent and therefore reasonable. We are looking for

$$\mathfrak{M}_t^G(f) - P_t^G(M) = 0.$$

Firstly we look at the G -Morse polynomial

$$\begin{aligned} \mathfrak{M}_{t=-1}^G(-V) &= 2A_0^+ + A_1^+ + G_1^+ + G_2^- + 2H_1^+ + 2H_2^+ + H_2^- + 3J^- + -[U_1] + [U_2] \\ &\quad - (A_0^- + A_1^- + 2G_1^- + 2G_2^- + 3H_1^- + 3H_2^- + 2J^-), \end{aligned}$$

which, when we include our assumptions becomes

$$\mathfrak{M}_{t=-1}^G(-V) = A_0^+ + G_1^+ - 2G_1^- - G_2^- + H_1^+ - 2H_1^- + H_2^+ - H_2^- + 2J^+ - J^-.$$

Next we look at the G -Poincaré polynomial:

$$\begin{aligned} P_{t=-1}^G(B) &= A_0^+ - (G_1^- + H_1^-) + (G_1^+ + H_1^+ + H_2^+ + 2J^+) \\ &\quad - (G_1^- + G_2^- + H_1^- + H_2^- + J^-) \\ &= A_0^+ - G_1^+ + 2G_1^- + G_2^- - H_1^+ + 2H_1^- - H_2^+ + H_2^- - 2J^+ + J^-, \end{aligned}$$

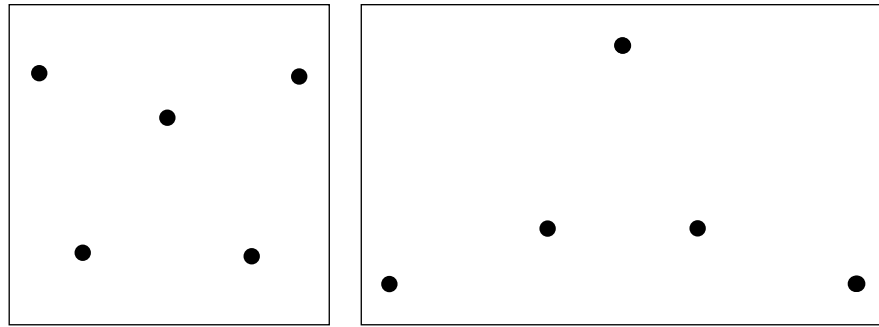
which is indeed equal to the G -Morse polynomial above. From this we can conclude that our assumptions are correct. It also implies that we were correct in assuming that $[U_1]_{S_5} = [U_2]_{S_5}$ in Section 6.11.

6.12 Critical Points of the 5-Body Problem

As with four masses we have been referring to the index 1 and 2 points as ‘unknown’, but thanks to the numerical work of Rick Moeckel [20] we are able to give pictorial representations of these unknown configurations, see Figures 6.17(a) and 6.17(b). All five configurations are shown in Figure 6.17 for completeness.

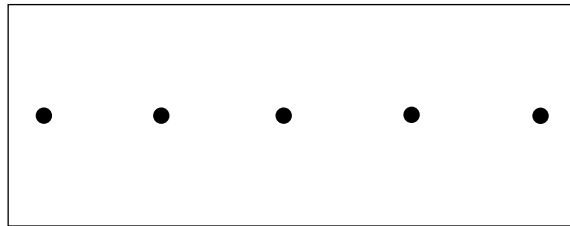
6.13 Another Look at the Total Cohomology

In the paper *Coxeter group actions on the complement of hyperplanes and special involutions* G. Felder and A.P. Veselov give two results directly related to our situation. They are

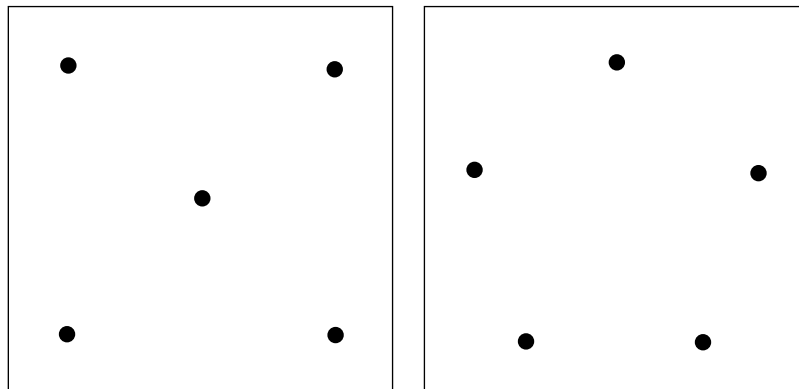


(a) Unknown 1: Index 1.

(b) Unknown 2: Index 2.



(c) Moulton: Index 3.



(d) Square: Index 0.

(e) Pentagonal: Index 0.

Figure 6.17: Relative equilibria configurations of five equal masses.

1. The representation of S_n acting on the complement M_n to the complexified reflection hyperplanes is equal to the induced representation of S_n from the trivial representation of the subgroup \mathbb{Z}_2 generated by the simple reflection $s = (12)$. That is

$$H^*(M_n) = 2\text{Ind}_{\langle s \rangle}^{S_n}(1).$$

2. The twisted action of S_n on M_n induces the regular representation (of S_n) on

the total cohomology $H^*(M_n)$. That is

$$H_\epsilon^*(M_n) = \rho.$$

Here ‘twisted action’ means we combine the usual action of the reflections with complex conjugation, i.e., $H_\epsilon^*(M_n) = \bigoplus_{k=0}^n \epsilon^k \otimes H^k(M_n)$.

Result 1 was previously mentioned, see Proposition 6.12. We now show that Result 2 is a direct consequence of Result 1.

We recall that $M_n \cong E = S^{2n-3} - \Delta$ and that since the Gysin sequence has zero Euler class it splits thus:

$$0 \rightarrow H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{\alpha} H^{n-1}(B) \rightarrow 0.$$

Here the \mathbb{Z}_2 action is complex conjugation and it is clear that since p^* is a pullback, \mathbb{Z}_2 acts trivially here, but this is not so on α . On α , which is integration around our fibre, we have the complex conjugation acting non-trivially — it reverses the direction of integration and so we pick up the alternating representation of \mathbb{Z}_2 . We now have that

$$[H^i(E)] = [H^i(B)] + \epsilon[H^{i-1}(B)],$$

where ϵ is the alternating character of \mathbb{Z}_2 . The S_n -Poincaré polynomial of E is therefore given by

$$P_{t,\epsilon}^{S_n}(E) = P_t^{S_n}(B) + \epsilon t P_t^{S_n}(B) = (1 + \epsilon t) P_t^{S_n}(B),$$

where S_n has the twisted action. We now set $t = 1$ to obtain

$$P_{1,\epsilon}^{S_n}(E) = (1 + \epsilon) P_1^{S_n}(B). \tag{6.17}$$

Since $(1 + \epsilon) = 0$ for $\sigma \in S_n$ odd and $P_{1,\epsilon}^{S_n}(E) = H_\epsilon^*(M_n)$, we can conclude that $H_\epsilon^*(M_n)(\sigma) = 0$ for odd σ .

It is easy to see that for $\sigma \in S_n$ even $\epsilon = 1$. This gives $(1 + \epsilon) = 2$ and it is clear that $H_\epsilon^*(M_n)(\sigma) = H^*(M_n)(\sigma)$. This means that when σ is an even element of S_n , $P_{1,\epsilon}^{S_n}(E)(\sigma) = H_\epsilon^*(M_n)(\sigma) = 2 \text{Ind}_{\langle \sigma \rangle}^{S_n}(1)$.

Now, suppose (V, ρ) is induced by (W, θ) and let χ_ρ and χ_θ be the corresponding characters of G and of H , then we can evaluate χ_ρ in the following way

$$\chi_\rho(u) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g^{-1}ug \in H}} \chi_\theta(g^{-1}ug). \quad (6.18)$$

We may now use Equation (6.18) to evaluate $\text{Ind}_{\langle s \rangle}^{S_n}(1)$ at even σ , when σ is not the identity in S_n , thus

$$\text{Ind}_{\langle s \rangle}^{S_n}(1)(\sigma) = \frac{1}{|\langle s \rangle|} \sum_{\substack{\tau \in S_n \\ \tau^{-1}\sigma\tau \in \langle s \rangle}} \chi_{\langle s \rangle}(\tau^{-1}\sigma\tau). \quad (6.19)$$

As σ is even there is no $\tau \in S_n$ such that $\tau^{-1}\sigma\tau \in \langle s \rangle$. This means that $\chi_{\langle s \rangle}(\sigma) = 0$ when σ is even.

Finally when $\sigma = e$, the identity element in S_n , we have that

$$2\text{Ind}_{\langle s \rangle}^{S_n}(1)(e) = 2\frac{|S_n|}{|\langle s \rangle|} = |S_n|.$$

From this it is apparent that $H_\epsilon^*(M_n) = \rho$, where ρ is the regular representation of S_n . Furthermore for the usual action of S_n ,

$$P_{1,\epsilon}^{S_n}(E) = 2P_1^{S_n}(B) = 2\text{Ind}_{\langle s \rangle}^{S_n}(1)$$

and so

$$P_1^{S_n}(B) = \text{Ind}_{\langle s \rangle}^{S_n}(1).$$

6.14 Conclusion

Our observations here lead us to two relative equilibria classes of the five-body problem both with some symmetry. The calculations allow us to say that either we have found all the relative equilibria solutions of the five-body problem or that if there are other classes relative equilibria that they must occur in pairs with differing parity indexes and they must have the same symmetries and therefore isotropy subgroups.

The n -body problem still remains largely unsolved, however the tools outlined here may shed some light on solutions of the equal mass case. The G -Morse inequalities allow us to describe this system in a new way; perhaps this will allow us to find more solutions of the n -body problem.

Appendix A

Finding the Character of S_n on the Cohomology of $\mathbb{C}P^{n-2} - \tilde{\Delta}$

The following program in MATLAB that computes the character of S_n on the Cohomology groups of $\mathbb{C}P^{n-2} - \tilde{\Delta}$ using the method described in Section 6.4 with basis from Arnol'd [3]. This was developed in collaboration with David Szotten.

```
function character = findCharacter(N, P, varargin)
%findCharacter character of S_n acting on H^p
% character = findCharacter(n, p) calculates the character of S_n
% acting on the p th cohomology group
%
% character = findCharacter(n, p, 'display') pauses and displays each
% representation using spy()
%
% Example:
%     char = findCharacter(4, 2);
%     disp(char);

% by David Szotten and Gemma Lloyd
% University of Manchester
% 2008

if nargin < 2
    error('Missing input argument. Usage: findCharacter(N, P)')
end

parser = inputParser;
parser.addOptional('option', '', ...
    @(x)any(strcmpi(x,{'disp', 'display', ''})));
parser.addOptional('progress', '', ...
    @(x)any(strcmpi(x,{'progress', ''})));
```

```

parser.parse(varargin{:});

if isempty(parser.Results.option)
    disp = 0;
else
    disp = 1;
end

allForms = generateBasis(N, P, [], []);

basisHash = hashForms(N, P, allForms);

%% setup permutations
allPerms{3} = [
    1 2 3;
    2 1 3;
    2 3 1
];

allPerms{4} = [
    1 2 3 4;
    2 1 3 4;
    2 1 4 3;
    2 3 1 4;
    2 3 4 1
];

allPerms{5} = [
    1 2 3 4 5;
    2 1 3 4 5;
    2 1 4 3 5;
    2 3 1 4 5;
    2 1 4 5 3;
    2 3 4 1 5;
    2 3 4 5 1
];

perms = allPerms{N};

nPerms = size(perms,1);

character = zeros(nPerms,1);

if ~isempty(parser.Results.progress)
    progBar = progressBar(nPerms);
end

for currPermNr = 1:nPerms

    representation = zeros( size(allForms,1) );

    currPerm = perms(currPermNr,:);

    permutedForms = zeros( size(allForms));

```



```

% sort within pairs
for currForm = 1:size(allForms, 1)
    permutedForms(currForm,:) = currPerm(allForms(currForm,:));
    permutedForms(currForm,:) = ...
        sortWithinPairs( permutedForms(currForm,:) );

    [basisForms, minusOnes] = expandInBasis( permutedForms(currForm,:));

    % loop through all basis vectors in expansion
    for currFoundBasisN = 1:size(basisForms,1)
        % hash creates a unique index for each basis vector and is used to
        % find the index in the original list of basis elements
        expansionNHash = hashForms(N, P, basisForms(currFoundBasisN,:));
        matchingBasis = (expansionNHash == basisHash);

        % add an entry in the matrix corresponding to the basis element
        representation( currForm, matchingBasis) = ...
            representation( currForm, matchingBasis) + ...
            minusOnes(currFoundBasisN);
    end
end

character(currPermNr) = trace(representation);

% display debug information if needed
if ( disp == 1 )
    spy(representation);
    title(num2str(currPerm));
    pause
end

if ~isempty(parser.Results.progress)
    progBar(currPermNr);
end

end

character = character';
end

function basis = generateBasis(N, P, formSoFar, basisSoFar)
% generate the basis vectors recursively
% we build up individual basis elements in formSoFar and
% when they are the correct length, we add to basisSoFar

% when we have enough pairs, stop and add current to list
if ( length(formSoFar) == 2*P )
    basis = [basisSoFar; formSoFar];
    return
else

    % deal with the initial case
    % lastL is 1_{i-1}
    if (isempty(formSoFar))
        lastL = 0;
    else

```

```

    lastL = formSoFar(end);
end

% these are the possible values for k_i and l_i for a basis vector
% we need l_i > l_{i-1} and l_i > k_i
% lastL is l_{i-1}
for k = 1:N
    for l = ( max(lastL, k)+1 ):N
        basisSoFar = generateBasis(N, P, [formSoFar k l], basisSoFar);
    end
end
end

basis = basisSoFar;
end

function hash = hashForms(N, P, forms)
% hash key for each form, decimal equivalent of the form as a 2P digit
% N-ary number

hash = zeros( size(forms,1), 1);
for i = 1:2*P
    hash = hash + (forms(:, i) - 1) * N^i;
end
end

function formList = sortWithinPairs( formList)
% sorts each pair: [ 4 3 2 1] -> [ 3 4 1 2 ]
% each row is considered at a time

for row = 1:size( formList, 1)
    for currPair = 1:2:size( formList, 2)
        if( formList(row, currPair) > formList(row, currPair+1) )
            formList(row, [currPair currPair+1]) = ...
                formList(row, [currPair+1 currPair]);
        end
    end
end
end

function [foundForms, minusOnes] = expandInBasis(form)
foundForms = form;
% takes a general element and expands it as a sum of basis vectors.
% coefficients are stored in minusOnes

% we look for repeated l values. if any, we replace the form by
% two others using the relation kmlm = klm - klml and repeat until
% all are basis vectors.
row = 1;
while row <= size(foundForms, 1)
    currentForm = foundForms(row, :);
    matches = firstMatchingPair( currentForm );
    if isempty(matches)
        row = row + 1;
    end
end

```

```

else
    % we have a match
    % we move the matching pairs to the front, making sure
    % not to change the sign (by swapping the order of the matching pairs).
    totalDistanceToMove = matches(1) - 1 + matches(2) - 2;
    matchingPair = currentForm( [2*matches(1)-1 2*matches(1), ...
        2*matches(2)-1, 2*matches(2)] );
    % if we moved the pairs an odd number of steps, swap the matching pairs
    % to reverse sign
    if ( mod(totalDistanceToMove, 2) == 1)
        matchingPair = matchingPair( [ 3 4 1 2] );
    end
    currentForm( [2*matches(1)-1, 2*matches(1), ...
        2*matches(2)-1, 2*matches(2)] ) = [];

    % now we replace the matching pairs at the front by a linear
    % combination and tack on the rest of the element

    %note that for the second factor, the first two pairs are
    %swapped to make sign positive
    replacement = [
        [matchingPair( [1 3 3 4 ] ) currentForm];
        [matchingPair( [1 4 1 3 ] ) currentForm] ];

    replacement = sortWithinPairs(replacement);

        foundForms = replaceRow(foundForms, row, replacement);

        % row is left to keep checking from the first replacement
    end
end

%sort the pairs
[foundForms, minusOnes] = sortPairs(foundForms);

end

function matches = firstMatchingPair(form)
%make sure we are only looking at one form at a time
if( size(form,1) > 1)
    error('expandInBasis:firstMathcingPair: more than one form!');
end

%if nothing found:
matches = [];

%only looking at the 1 values
lVals = form(2:2:end);

for first = 1:length(lVals)
    for second = first + 1 : length(lVals)
        if ( lVals(first) == lVals(second) )
            matches = [first second];
            return
        end
    end
end

```

```

end

end

function newMatrix = replaceRow(matrix, row, newRows)
    newMatrix = [
        matrix(1:row-1, :);
        newRows;
        matrix(row+1:end,:) ];
end

function [formList, minusOnes] = sortPairs( formList )

    nForms = size( formList, 1);
    minusOnes = ones(nForms, 1);

    for row = 1:nForms

        %only need to compare l values

        kVals = formList(row, 1:2:end);
        lVals = formList(row, 2:2:end);

        % get ix, the ordering to sort the lValues
        [tmp, ix] = sort(lVals);

        % apply this to both k and l values
        formList(row, 1:2:end) = kVals(ix);
        formList(row, 2:2:end) = lVals(ix);

        % need to find the sign of the applied permutation
        minusOnes(row) = minusOnes(row) * permutationSign(ix);

    end

end

function sign = permutationSign(perm)
    prod = 1;
    for j = 1:length(perm)-1
        for i = j+1 : length(perm)
            prod = prod * (perm(i) - perm(j) );
        end
    end

    if( prod > 0)
        sign = 1;
    else
        sign = -1;
    end

end

```

Appendix B

Character Tables

B.1 Symmetric Group on 3 Elements, S_3

In this table elements are denoted as products of disjoint cycles.

For example $c_2 = (12)$ and $c_3 = (123)$.

S_3	e	c_2	c_3
A_0	1	1	1
A_1	1	-1	1
E	2	0	-1

B.2 Symmetric Group on 4 Elements, S_4

In this table elements are denoted as products of disjoint cycles.

For example $c_2 = (12)$, $c_4 = (1234)$ and $c_2c_2 = (12)(34)$.

S_4	e	c_2	c_3	c_4	c_2c_2
A_0	1	1	1	1	1
A_1	1	-1	1	-1	1
E	2	0	-1	0	2
T_1	3	-1	0	1	-1
T_2	3	1	0	-1	-1

B.3 Symmetric Group on 5 Elements, S_5

In this table elements are denoted as products of disjoint cycles.

For example $c_2 = (12)$, $c_5 = (12345)$ and $c_2c_3 = (12)(345)$.

S_5	e	c_2	c_2c_2	c_3	c_2c_3	c_4	c_5
A_0	1	1	1	1	1	1	1
A_1	1	-1	1	1	-1	-1	1
G_1	4	2	0	1	-1	0	-1
G_2	4	-2	0	1	1	0	-1
H_1	5	1	1	-1	1	-1	0
H_2	5	-1	1	-1	-1	1	0
J	6	0	-2	0	0	0	1

B.4 Cyclic Group of Order 2, \mathbb{Z}_2

In this table κ is the generator of \mathbb{Z}_2 .

\mathbb{Z}_2	1	κ
A_0	1	1
A_1	1	-1

B.5 Cyclic Group of Order 3, \mathbb{Z}_3

Since we are looking at \mathbb{Z}_3 as a subgroup of S_m , with $(m \geq 3)$ we use the rotations of S_m as our representatives of the conjugacy classes. So we look at $\mathbb{Z}_3 = \langle (123) \rangle$.

\mathbb{Z}_3	1	(123)	(132)
A_0	1	1	1
E	2	-1	-1

B.6 Cyclic Group of Order 4, \mathbb{Z}_4

Since we are looking at \mathbb{Z}_4 as a subgroup of S_m with ($m \geq 4$) we use a rotation of S_m as the generator for \mathbb{Z}_4 . We will use $\mathbb{Z}_4 = \langle (1234) \rangle$.

\mathbb{Z}_4	0	(1234)	(13)(24)	(1432)
A_0	1	1	1	1
A_1	1	-1	1	-1
E	2	0	-2	0

B.7 Cyclic Group of Order 5, \mathbb{Z}_5

Since we are looking at \mathbb{Z}_5 as a subgroup of S_m with ($m \geq 5$) we use a rotation of S_m as the generator for \mathbb{Z}_5 . We will use $\mathbb{Z}_5 = \langle (12345) \rangle$.

\mathbb{Z}_5	1	(12345)	(12345) ²	(12345) ³	(12345) ⁴
A_0	1	1	1	1	1
E_1	2	γ	η	η	γ
E_2	2	η	γ	γ	η

where

- $\gamma = 2 \cos(\frac{2\pi}{5}) = -\frac{1}{2}(1 - \sqrt{5})$ =golden ratio
- $\eta = 2 \cos(\frac{4\pi}{5}) = -\frac{1}{2}(1 + \sqrt{5})$

B.8 $\mathbb{Z}_2 \times \mathbb{Z}_2$

We will be considering $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a subgroup of $S_m \times \mathbb{Z}_2$ with ($m \geq 4$) and so we have generators $(c_2c_2, 1)$ and $(1, \kappa)$, where $c_2 = (12)$ for example.

$\mathbb{Z}_2 \times \mathbb{Z}_2$	(1, 1)	(c_2c_2 , 1)	(1, κ)	(c_2c_2 , κ)
A_0^+	1	1	1	1
A_0^-	1	1	-1	-1
A_1^+	1	-1	1	-1
A_1^-	1	-1	-1	1

B.9 $S_4 \times \mathbb{Z}_2$

Here conjugacy classes are denoted (a, b) where a is a cycle type in S_5 and κ is the generator of \mathbb{Z}_2 .

	e	$(c_2, 1)$	$(c_3, 1)$	$(c_4, 1)$	$(c_2c_2, 1)$	$(1, \kappa)$	(c_2, κ)	(c_3, κ)	(c_4, κ)	(c_2c_2, κ)
A_0^+	1	1	1	1	1	1	1	1	1	1
A_0^-	1	1	1	1	1	-1	-1	-1	-1	-1
A_1^+	1	-1	1	-1	1	1	-1	1	-1	1
A_1^-	1	-1	1	-1	1	-1	1	-1	1	-1
E^+	2	0	-1	0	2	2	0	-1	0	2
E^-	2	0	-1	0	2	-2	0	1	0	-2
T_1^+	3	-1	0	1	-1	3	-1	0	1	-1
T_1^-	3	-1	0	1	-1	-3	1	0	-1	1
T_2^+	3	1	0	-1	-1	3	1	0	-1	-1
T_2^-	3	1	0	-1	-1	-3	-1	0	1	1

Here e denotes the identity $(1, 1)$.

B.10 $S_5 \times \mathbb{Z}_2$

The character table for $S_5 \times \mathbb{Z}_2$ spans two pages. This page holds conjugacy classes that look like $(a, 1)$, where a is the cycle type of S_5 and 1 is the identity element of \mathbb{Z}_2 .

$S_5 \times \mathbb{Z}_2$	$(e, 1)$	$(c_2, 1)$	$(c_2c_2, 1)$	$(c_3, 1)$	$(c_2c_3, 1)$	$(c_4, 1)$	$(c_5, 1)$
A_0^+	1	1	1	1	1	1	1
A_0^-	1	1	1	1	1	1	1
A_1^+	1	-1	1	1	-1	-1	1
A_1^-	1	-1	1	1	-1	-1	1
G_1^+	4	2	0	1	-1	0	-1
G_1^-	4	2	0	1	-1	0	-1
G_2^+	4	-2	0	1	1	0	-1
G_2^-	4	-2	0	1	1	0	-1
H_1^+	4	1	1	-1	1	-1	0
H_1^-	4	1	1	-1	1	-1	0
H_2^+	5	-1	1	-1	-1	1	0
H_2^-	5	-1	1	-1	-1	1	0
J^+	6	0	-2	0	0	0	1
J^-	6	0	-2	0	0	0	1

This page holds the conjugacy classes that look like (a, κ) , where a gives the cycle type of S_5 and κ is the generator of \mathbb{Z}_2 .

$S_5 \times \mathbb{Z}_2$	(e, κ)	(c_2, κ)	(c_2c_2, κ)	(c_3, κ)	(c_2c_3, κ)	(c_4, κ)	(c_5, κ)
A_0^+	1	1	1	1	1	1	1
A_0^-	-1	-1	-1	-1	-1	-1	-1
A_1^+	1	-1	1	1	-1	-1	1
A_1^-	-1	1	-1	-1	1	1	-1
G_1^+	4	2	0	1	-1	0	-1
G_1^-	-4	-2	0	-1	1	0	1
G_2^+	4	-2	0	1	1	0	-1
G_2^-	-4	2	0	-1	-1	0	1
H_1^+	4	1	1	-1	1	-1	0
H_1^-	-4	-1	-1	1	-1	1	0
H_2^+	5	-1	1	-1	-1	1	0
H_2^-	-5	1	-1	1	1	-1	0
J^+	6	0	-2	0	0	0	1
J^-	-6	0	2	0	0	0	-1

B.11 Dihedral Group of Order Eight, D_4

We consider D_4 as a subgroup of $S_m \times \mathbb{Z}_2$ where $m \geq 4$. We consider D_4 as the symmetry group of the square. There are three generators for D_4 : ρ is rotation by $\frac{\pi}{2}$, τ is reflection in a line joining two vertices and τ' is reflection in a line joining the midpoint of two edges. Keeping to generators of $S_m \times \mathbb{Z}_2$ means that we are looking at $D_4 \cong \langle (12)(34)\kappa, (13)\kappa, (1234) \rangle$. Here $\rho = (1234)$, $\tau = (13)\kappa$ and $\tau' = (12)(34)\kappa$.

D_4	1	(1234)	(1234) ²	(13) κ	(12)(34) κ
A_0	1	1	1	1	1
A_1	1	1	1	-1	-1
B_1	1	-1	1	1	-1
B_2	1	-1	1	-1	1
E	2	0	-2	0	0

Appendix C

Finding the Character on the Unstable Manifold

This appendix contains the explicit calculations regarding which character of the isotropy subgroup to induce from. The notation here is as in Chapter 6 - we use lowercase letters to denote characters of isotropy subgroups to avoid confusion with similarly named characters of the group acting on the space as a whole.

Note C.1. As in previous discussions (see Section 6.9.1), we will be looking at the determinant of the following matrix representations since we are interested in the orientation representation.

C.1 $S_4 \times \mathbb{Z}_2$: Isosceles Points

Here we look at the Isosceles configuration when the group acting on the space is $S_4 \times \mathbb{Z}_2$. The index of these points is one and so we are looking for a single perturbation that lies in the unstable space, such a perturbation is shown in Figure C.1 (the perturbations on masses m_2 and m_3 are at an angle of $\arctan(\frac{1}{5})$ to the horizontal and on m_1 the perturbation is horizontal).

The isotropy subgroup of $S_4 \times \mathbb{Z}_2$ of this configuration is $H_I = \langle \kappa(23) \rangle$, so we look at the action of this group on the basis. It is clear that the orientation is reversed and so we are looking to induce from the non-trivial character of \mathbb{Z}_2 , see Figure C.2.

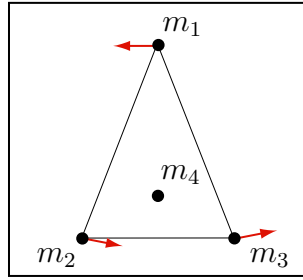


Figure C.1: Basis for the unstable space of the Isosceles points when $n = 4$.

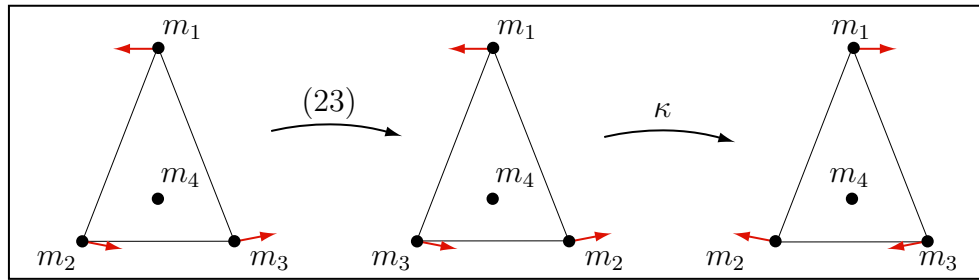


Figure C.2: How $(23)\kappa$ acts on the unstable space of the Isosceles points..

C.2 S_5 : Moulton Points

Here we look at the Moulton points when the group acting on the whole space is S_5 .

A basis for the unstable space is given in Figure 6.16 and is shown again below. The

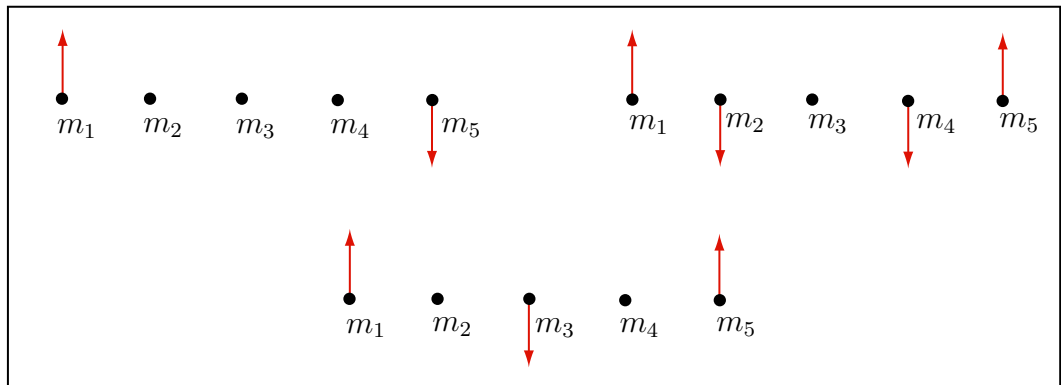
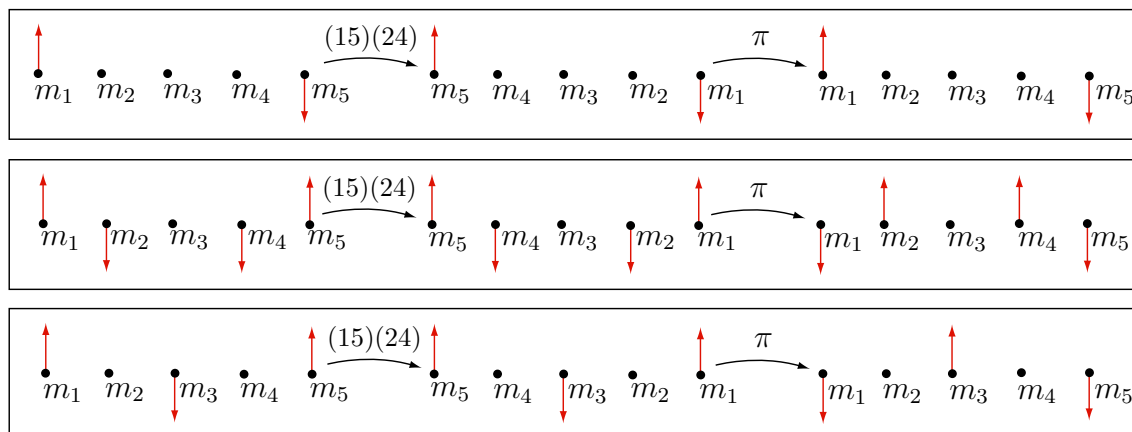


Figure C.3: Basis for the unstable space of the Moulton points when $n = 5$.

isotropy subgroup of S_5 of this configuration is $H_M = \langle (15)(24) \rangle \cong \mathbb{Z}_2$. The following figure (Figure C.4) shows what happens when we apply $(15)(24)$ to the basis. The

Figure C.4: How $(15)(24)$ acts on the unstable space of the Moulton points.

matrix representation of this is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which has determinant one. Therefore the character vector of the representation we are looking for is $\chi[1, c_2 c_2] = [1, 1]$ — this means that we induce from the trivial representation of \mathbb{Z}_2 .

C.3 $S_5 \times \mathbb{Z}_2$: Moulton Points

We look at the Moulton points when the group acting on the whole space is $S_5 \times \mathbb{Z}_2$. A basis for the unstable space is the same as for when the group acting is S_5 (Figure C.3). The isotropy subgroup is $H_M = \langle (15)(24), \kappa \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\kappa \in \mathbb{Z}_2$ is complex conjugation. We can use the result from above but still must look at the elements κ and $(15)(24)\kappa$ - Figures C.5 and C.6 do just that.

The determinants of the matrix representations are

$$\det(M_\kappa) = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \text{and} \quad \det(M_{\kappa(15)(24)}) = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1,$$

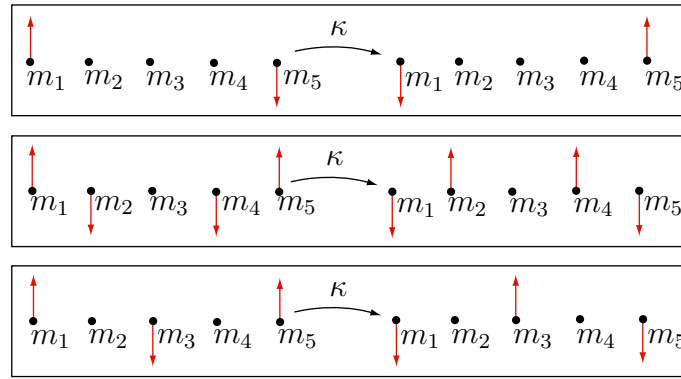


Figure C.5: How $\kappa \in H_M$ acts on the unstable space of the Moulton points.

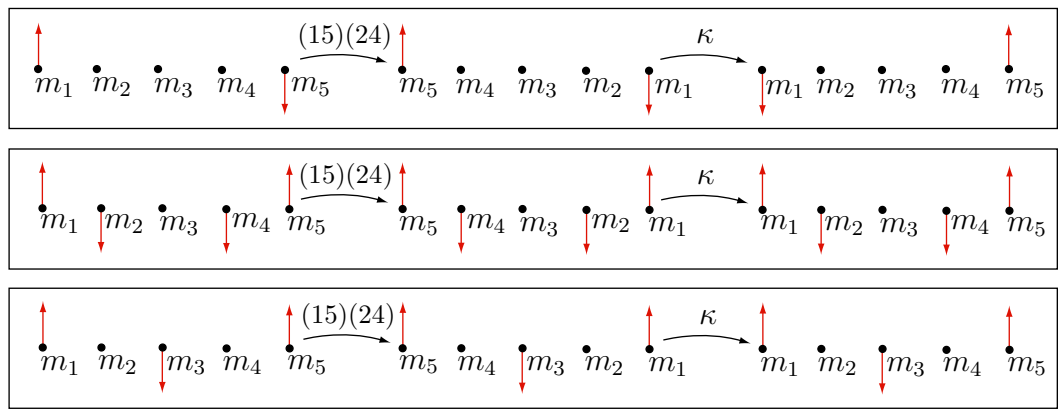


Figure C.6: How $\kappa(15)(24) \in H_M$ acts on the unstable space of the Moulton points — the final positions shown after rotation by π .

respectively. This means that we are looking for a character taking values

$$\chi[1, (15)(24), \kappa, \kappa(15)(24)] = [1, -1, -1, 1].$$

It follows that we should induce from the character a_1^- of $\mathbb{Z}_2 \times \mathbb{Z}_2$, see Appendix B.8.

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