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ON DAVIS-JANUSZKIEWICZ HOMOTOPY TYPES II; COMPLETION AND GLOBALISATION

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Abstract. For any finite simplicial complex K, Davis and Januszkiewicz have defined a family of homotopy equivalent CW-complexes whose integral cohomology rings are isomorphic to the Stanley-Reisner algebra of K. Subsequently, Buchstaber and Panov gave an alternative construction, which they showed to be homotopy equivalent to the original examples. It is therefore natural to investigate the extent to which the homotopy type of a space X is determined by such a cohomology ring. Having analysed this problem rationally in Part I, we here consider it prime by prime, and utilise Lannes' T functor and Bousfield-Kan type obstruction theory to study the p-completion of X. We find the situation to be more subtle than for rationalisation, and confirm the uniqueness of the completion whenever K is a join of skeleta of simplices. We apply our results to the global problem by appealing to Sullivan's arithmetic square, and deduce integral uniqueness whenever the Stanley-Reisner algebra is a complete intersection.

1. Introduction

For any finite simplicial complex K, the Stanley-Reisner algebra $\mathbb{Z}[K]$ is an important combinatorial invariant [St], and may be graded by assigning dimension 2 to each of its generators. The corresponding R-algebra $R[K]$ is defined over any commutative ring R as $R \otimes \mathbb{Z}[K]$, and a topological space X *realises* $R[K]$ whenever $H^*(X; R)$ is isomorphic to $R[K]$ as graded R-algebras. In their pioneering work on toric topology [DJ], Davis and Januszkiewicz construct a family of realisations of $\mathbb{Z}[K]$ for every K, and show each of them to be homotopy equivalent to a certain universal example. We refer to a generic representative of this homotopy type as a *Davis-Januszkiewicz space* $DJ(K)$.

Questions of uniqueness then arise, and suggest that we investigate the relationship between the geometric properties of K and the number of homotopy types which realise $\mathbb{Z}[K]$. As usual, the problem is best approached by dealing separately with its rational and p-adic versions, and applying Sullivan's arithmetic square to recover global information. We consider the rational situation in [NR], and study the case in which $\mathbb{Q}[K]$ is a complete

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intersection. The rational homotopy type of any realisation of $\mathbb{Q}[K]$ is then unique, up to weak equivalence of nilpotent spaces.

Our purpose here is to address the p-adic version of the problem. We obtain uniqueness statements for a somewhat larger class of complexes K, from which we deduce global uniqueness for complete intersections by appealing to the arithmetic square. So far, we have little evidence to suggest that these results may be extended to more general K, either p-adically or globally; nor, however, do we have any example of an exotic Davis-Januszkiewicz space that realises $\mathbb{Z}[K]$ yet fails to be homotopy equivalent to $DJ(K)$. On the other hand, a simple cohomological argument proves that the suspension of any such space *is* equivalent to $\Sigma DJ(K)$, for every K.

We refer readers to [NR] for detailed background and prerequisites, and summarise the crucial points in Section 2 below, where we also state our main results explicitly. Unlike the rational version, we have found no helpful formulation of the p-adic problem in terms of model category theory. Nevertheless, we work in several algebraic and geometric categories which admit model structures, and appeal regularly to the language of homotopy colimits. We insist, for example, that our spaces lie in the model category TOP of k-spaces and continuous maps [V]. Sometimes we deal with based CW-complexes $(X, *)$, which lie in the pointed category TOP^+; then $*$ is a distinguished 0–cell and its inclusion into X is a cofibration.

So far as notation is concerned, we let p denote a fixed but arbitrary prime throughout our work. We usually abbreviate the cyclic group $\mathbb{Z}/p$ to C in the interests of notational simplicity, and write $\mathbb{Z}_p^\wedge$ for the p-adic integers. We denote the field of p elements by $\mathbb{F}_p$ and the topological group of unimodular complex numbers by T, in order to distinguish them from the underlying group $\mathbb{Z}/p$ and the underlying circle S^1 respectively. Over any commutative ring R with identity, we interpret the polynomial algebra generated by a set V of graded independent variables as the symmetric algebra $S_R(V)$ on V. Finally, for every pair of objects x_0 and x_1 of an arbitrary category C, we write $\mathrm{C}(x_0, x_1)$ for the set of morphisms $x_0 \to x_1$.

For any space X we consider the p-adic completion $X_p^\wedge$ described by Bousfield and Kan [BK], which agrees with Sullivan's original p-completion in all our cases. Given a p-complete space X_K realising $\mathbb{Z}_p^\wedge[K]$, our underlying strategy is to develop methods of comparing X_K with a canonical representative $hc(K)$ for $DJ(K)$, and each of sections 3 to 8 takes steps towards this goal. We organise the programme as follows.

In Section 3, we begin by assuming that X_K realises $\mathbb{F}_p[K]$, and apply Lannes' T-functor to compute the mod p cohomology ring of certain components of the mapping space $map\,(BC^\sigma, X_K)$ for appropriate elementary abelian p-groups C^σ. We extend our computations to p-adic cohomology in Section 4, by taking advantage of the fact that the mod p ring is zero in odd dimensions. In Section 5 we develop the corresponding results for certain components of $map\,(BT^\sigma, X_K)$, by interpreting $T_p^\wedge$ as a limit of cyclic subgroups $\mathbb{Z}/p^r$.

In order to bring these calculations to bear on the central problem, we provide a brief survey of higher limits of algebraic $\text{CAT}(K)$-diagrams in Section 6. These feature prominently in the obstruction theory associated to Bousfield and Kan's spectral sequence for homotopy limits, which we investigate in Section 7; in particular, we show that there are no obstructions to defining a homotopy equivalence $hc(K)^\wedge_p \to X_K$ when K is the skeleton of a simplex, and deduce p-adic uniqueness in such cases. We extend the results to iterated joins in Section 8, where we also confirm that homotopy classes of self-equivalences of X_K are classified by their action on cohomology. Finally, in Section 9, we employ the arithmetic square to combine our p-adic conclusions with the rational calculations of [NR], and prove global uniqueness for X_K when K is a complete intersection. The latter is equivalent to identifying K as the iterated join of a simplex with boundaries of simplices.

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2. Preliminaries and main results

We shall work with a canonical representative for $DJ(K)$, constructed as the homotopy colimit of a diagram of topological spaces. Our methods depend upon the cohomological and homotopy theoretic properties of such diagrams, so we recall the crucial definitions and notation in this section. Readers may consult [NR] for further motivation and details.

We consider an ordered set V of vertices $v_1, \ldots, v_m$. A *simplicial complex* K on $V = V(K)$ is given by a finite set of faces $\sigma \subseteq V$, which is closed with respect to the formation of subsets. The dimension $\dim(\sigma)$ of any face is given in terms of its cardinality by $|\sigma| - 1$, and $\dim K$ is the maximum dimension attained by its faces. We assume that the empty face $\varnothing$ belongs to K unless otherwise stated, and write $K^\times$ when we wish to emphasise that it has been omitted. The most basic example is the simplex $\Delta(V)$; it consists of all possible subsets of V, and therefore contains any K on V as a subcomplex. Its r–skeleton $\Delta^{(r)}(V)$ includes those faces $\sigma \subset V$ of dimension $\leq r$, for any $0 \leq r \leq m-1$. In particular, $\Delta^{(m-2)}(V)$ is the boundary $\partial(V) \subset \Delta(V)$, constructed by deleting the maximal face V. For any two disjoint simplicial complexes K and L, their *join* $K * L$ on $V(K) \cup V(L)$ has faces $\{\sigma \cup \tau : \sigma \in K, \tau \in L\}$; thus K and L are full subcomplexes.

Every face $\sigma \in K$ determines several subcomplexes. In particular, we need the simplex $\Delta(\sigma)$ on σ, and the *link* and *star* of σ, defined by

$$\ell_K(\sigma) := \{\tau \setminus \sigma : \sigma \subseteq \tau \in K\} \quad \text{and} \quad st_K(\sigma) := \{\tau : \sigma \cup \tau \in K\};$$

so $st_K(\sigma)$ coincides with $\varDelta(\sigma) * \ell_K(\sigma)$ as subcomplexes of K. Given any two complexes K and L, we note that

$$\ell_{K*L}(\sigma \cup \tau) \;=\; \ell_K(\sigma) * \ell_L(\tau) \quad \text{and} \quad st_{K*L}(\sigma \cup \tau) \;=\; st_K(\sigma) * st_L(\tau)$$

as subcomplexes of $K * L$, for every face $\sigma \in K$ and $\tau \in L$.

The vertices masquerade as algebraically independent variables of degree 2, and generate a graded polynomial algebra $S_R(V)$. For any subset $\omega \subseteq V$, we abbreviate the square-free monomial $\prod_\omega v_i$ to v_ω, which has degree $2|\omega|$. We then define the *Stanley-Reisner algebra* $R[K]$ as $S_R(V)/(v_\omega : \omega \notin K)$; so the inclusions of the full subcomplexes $K, L \subset K * L$ induce an isomorphism

$$R[K] \otimes R[L] \xrightarrow{\cong} R[K * L]. \tag{2.1}$$

Since $R[\varDelta(\sigma)] = S_R(\sigma)$ for every face σ, an example is provided by

$$S_R(\sigma) \otimes R[\ell_K(\sigma)] \xrightarrow{\cong} R[st_K(\sigma)]. \tag{2.2}$$

Any subcomplex $J \subseteq K$ induces a surjection $R[K] \to R[J]$, by annihilating the faces in $K \setminus J$; in particular, $K \subseteq \varDelta(V)$ induces the canonical projection $S_R(V) \to R[K]$.

Every K determines a finite category $\mathrm{CAT}(K)$, whose objects are the faces σ and morphisms the inclusions $i_{\sigma,\tau} \colon \sigma \subseteq \tau$. The empty face is initial, so the classifying space $B\mathrm{CAT}(K)$ is contractible, whereas $B\mathrm{CAT}(K^\times)$ is homeomorphic to the geometric realisation $|K|$. The maximal faces μ are characterised by the fact that they admit only identity morphisms. By construction, there is an isomorphism $\mathrm{CAT}(K) \times \mathrm{CAT}(L) \cong \mathrm{CAT}(K * L)$ for any K and L, which we use to identify $\mathrm{CAT}(\varDelta(\sigma)) \times \mathrm{CAT}(\ell_K(\sigma))$ with $\mathrm{CAT}(st_K(\sigma))$ as necessary.

An A-*diagram* in an arbitrary category R consists of a covariant functor $D \colon \mathrm{A} \to \mathrm{R}$ for some small category A.

Definitions 2.3. For any based CW-complex $(X, *)$, the $\mathrm{CAT}(K)$-diagram

$$X^K \colon \mathrm{CAT}(K) \longrightarrow \mathrm{TOP}$$

assigns the cartesian product X^σ to each face σ, where $X^K(\varnothing) = *$; its value on $i_{\sigma,\tau}$ is the cofibration $X^\sigma \to X^\tau$, where the extra coordinates are set to $*$. The constant functor cst_{X^V} assigns $X^V \cong X^m$ to each face, and the identity map id_{X^V} to each inclusion. The natural transformation $i_K \colon X^K \to \mathrm{cst}_{X^V}$ is induced by the inclusions $i_\sigma \colon X^\sigma \to X^V$, for every face σ.

We are interested in two particular values of X, which stem from the inclusion of the cyclic subgroup C in the circle T. To any subset $\sigma \subseteq V$, there corresponds the inclusion of the elementary abelian p-subgroup C^σ in the torus T^σ, whose classifying map represents the projection of a product of infinite dimensional lens spaces onto the corresponding product of complex projective spaces. We shall denote this map by $t_\sigma \colon BC^\sigma \to BT^\sigma$, and assume that it is a homomorphism of abelian topological groups, written multiplicatively. It may be interpreted as a natural transformation

$$t\colon BC^K \longrightarrow BT^K. \tag{2.4}$$

The colimit $c(K) := \operatorname{colim}_{\mathrm{CAT}(K)} BT^K$ lies in the category TOP_+ of pointed topological spaces, and is a subcomplex of BT^V via i_K.

Following [BP], we view $c(K)$ as a distinguished representative for $DJ(K)$. In [NR, §3], this property is expressed in terms of isomorphisms

$$H^*(c(K);R) \xrightarrow{\cong} \lim S_R(K) \xrightarrow{\cong} R[K], \tag{2.5}$$

where $S_R(K)$ is the $\mathrm{CAT}^{op}(K)$-diagram of graded commutative R-algebras whose value on σ is $S_R(\sigma)$, and on $\tau \supseteq \sigma$ is the canonical projection $p_{\tau,\sigma}$.

In order to investigate the homotopy theoretical properties of $c(K)$, it is natural to consider the *homotopy colimit* $hc(K) := \operatorname{hocolim}_{\mathrm{CAT}(K)} BT^K$. Following Hollender and Vogt's elaboration [HV] of the original definition of Bousfield and Kan [BK], we describe $hc(K)$ as the bar construction $B(*, \mathrm{CAT}(K), BT^K)$. As such, it lies in TOP^+. The fact that BT^K is cofibrant in the category of $\mathrm{CAT}(K)$-diagrams [NR, Lemma 2.7] ensures that the natural projection $hc(K) \to c(K)$ is a homotopy equivalence, and that either space may be used as a model for $DJ(K)$.

A deeper analysis of $hc(K)$ involves certain secondary structures associated to $\mathrm{CAT}(K)$, such as the *undercategory* $\sigma \downarrow \mathrm{CAT}(K)$. For any face σ of K, this is obtained by restricting attention to those objects τ for which $\sigma \subseteq \tau$. The *overcategory* $\mathrm{CAT}(K) \downarrow \sigma$ is defined by analogy, and $(\sigma \downarrow \mathrm{CAT}(K))^{op}$ is isomorphic to $\mathrm{CAT}^{op}(K) \downarrow \sigma$. Setting $P(\rho) := \rho \setminus \sigma$ defines a functor

$$P\colon \sigma \downarrow \mathrm{CAT}(K) \longrightarrow \mathrm{CAT}(\ell_K(\sigma)), \tag{2.6}$$

which is also an isomorphism. Writing the restriction of X^K to $\sigma \downarrow \mathrm{CAT}(K)$ as $X^{\sigma \downarrow K}$ and taking appropriate colimits then yields homeomorphisms

$$\operatorname{colim} X^{\sigma\downarrow K} \equiv X^\sigma \times \operatorname{colim} X^{\ell_K(\sigma)} \equiv \operatorname{colim} X^{st_K(\sigma)}, \tag{2.7}$$

where the former is induced by P. In case $X = BT$, we apply $H^*(-;R)$ and appeal to (2.5) to obtain isomorphisms

$$\lim S_R(\sigma \downarrow \mathrm{CAT}(K)) \cong S_R(\sigma) \otimes R[\ell_K(\sigma)] \cong R[st_K(\sigma)] \tag{2.8}$$

of graded commutative algebras.

To organise our proofs, it is convenient to formalise the following, in which $X = X_K$ denotes a space that realises $\mathbb{Z}[K]$ for any choice of K.

Definitions 2.9. The space X *reflects* K whenever a specific isomorphism $\theta_X\colon H^*(X;\mathbb{Z}) \to \mathbb{Z}[K]$ is given; then θ_X is the *reflector*. If the reflector is given with coefficients R, then X reflects K *over* R.

For any subcomplex $J \subseteq K$, a map $e\colon W_J \to X_K$ *reflects the pair* (K, J) whenever the square

$$\begin{array}{ccc} H^*(X_K;\mathbb{Z}) & \xrightarrow{e^*} & H^*(W_J;\mathbb{Z}) \\ \theta_X\big\downarrow & & \big\downarrow\theta_W \\ \mathbb{Z}[K] & \xrightarrow{p_{K,J}} & \mathbb{Z}[J] \end{array} \tag{2.10}$$

commutes, where $p_{K,J}$ denotes the canonical surjection.

For example, the classifying space BT^V reflects $\Delta(V)$ by means of the standard isomorphism $H^*(BT^V;\mathbb{Z}) \to S_{\mathbb{Z}}(V)$. More generally, the colimit $c(K)$ reflects K for any complex K, and the inclusion $c(K) \subset BT^V$ reflects the pair $(\Delta(V); K)$. This may be extended to any other X_K by interpreting BT^V as an Eilenberg-Mac Lane space $H(\mathbb{Z}^V, 2)$. Then θ_X determines a unique homotopy class of maps $q_X\colon X_K \to BT^V$, which represents the m-tuple of generators in $H^2(X_K;\mathbb{Z})$ and reflects $(\Delta(V); K)$ as before. Composing q_X with projection onto BT^σ creates a map $q_\sigma\colon X_K \to BT^\sigma$ for every face σ; its image in integral cohomology realises the polynomial subalgebra $\mathbb{Z}[\sigma] < \mathbb{Z}[K]$, via θ_X.

Canonical reflectors $\theta_{hc(K)}\colon H^*(hc(K)) \to \mathbb{Z}[K]$ arise by taking limits over $\mathrm{CAT}^{op}(K)$, and have good functorial properties. So we may often interchange $H^*(c(K))$ and $\mathbb{Z}[K]$ without further comment.

Given any e reflecting $(K; J)$, there is a homotopy commutative square

$$\begin{array}{ccc} W_J & \xrightarrow{e} & X_K \\ q_W\big\downarrow & & \big\downarrow q_X \\ BT^{V(J)} & \xrightarrow{j_{J,K}} & BT^{V(K)} \end{array} , \tag{2.11}$$

where $j_{J,K}$ denotes coordinatewise inclusion. This reflects the subpair

$$(K, J) \ \subseteq \ (\Delta(V(K)), \Delta(V(J))),$$

in the sense that it induces a commutative cube in cohomology when combined with the four reflectors.

For any noetherian local graded commutative algebra A over $\mathbb{Q}$, a choice of homogeneous generators establishes a presentation

$$0 \longrightarrow I_A \longrightarrow S \longrightarrow A \longrightarrow 0,$$

where S is a finitely generated graded polynomial algebra and $I_A \subset S$ is a graded ideal. Then A is a *complete intersection* whenever I_A is generated by a regular sequence of homogeneous elements; this definition is independent of the choice of generators by [BH, Theorem 2.3.3]. We note in [NR, Section 5] that the Stanley-Reisner algebra $\mathbb{Q}[K]$ is a complete intersection precisely when K is an iterated join of the form $\Delta(U_0) * \partial(U_1) * ... * \partial(U_t)$, for any partition $\{U_0, U_1, ..., U_t\}$ of V. In this case, an isomorphism

$$S_{\mathbb{Q}}(U_0) \otimes \mathbb{Q}[\partial(U_1)] \otimes \cdots \otimes \mathbb{Q}[\partial(U_t)] \ \cong \ \mathbb{Q}[K] \tag{2.12}$$

is given by (2.1) above.

Theorem 9.5. *Let X be a nilpotent CW-complex; if $\mathbb{Q}[K]$ is a complete intersection, then there is an isomorphism $\theta\colon H^*(X;\mathbb{Z}) \to \mathbb{Z}[K]$ if and only if there is a homotopy equivalence $f\colon hc(K) \to X$ such that $f^* = \theta$.*

We prove Theorem 9.5 by using the arithmetic square to combine the rational results of [NR] with the following p-adic statement, which refers to a larger class of complexes.

Theorem 8.8. *Let X be a p-complete CW-complex; if K is an iterated join $\Delta^{(r_1)}(U_1) * \cdots * \Delta^{(r_t)}(U_t)$ of skeleta of simplices, then there is an isomorphism $\theta\colon H^*(X;\mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f\colon hc(K)_p^\wedge \to X$ such that $f^* = \theta$.*

After suspension, however, the situation becomes much simpler.

Theorem 9.6. *Let X be a CW-complex that realises $\mathbb{Z}[K]$ for any K; then there is a homotopy equivalence $e\colon \Sigma X \to \Sigma hc(K)$.*

Our final result in the completed context is a homotopy classification of self-equivalences of $hc(K)$. We do not expect a global version to hold, even under the assumptions of Theorem 9.5.

Theorem 8.9. *Given K as in Theorem* 8.8*, any pair of self-equivalences $f, g\colon hc(K)_p^\wedge \to hc(K)_p^\wedge$ are homotopic if and only if $H^*(f;\mathbb{Z}_p^\wedge) = H^*(g;\mathbb{Z}_p^\wedge)$.*

For these K, it is also of interest to apply [GT, Theorem 10.2(3)], and deduce that the t-fold suspension $\Sigma^t \mathcal{Z}_K$ of the *moment-angle complex* [DJ], [BP] is homotopy equivalent to a wedge of spheres.

3. Stanley-Reisner algebras and the T-functor

We begin by focusing our attention on the mod p Stanley-Reisner algebra $\mathbb{F}_p[K]$, where K is a specific simplicial complex with generic face σ. It is convenient to denote $H^*(-;\mathbb{F}_p)$ by $H^*(-)$ throughout this section, and to abbreviate $H^*(BE)$ to HE for an arbitrary elementary abelian p-group E. Often, E will be of the form C^σ for some face σ of K.

For any space X reflecting K over $\mathbb{F}_p$, our aim is to construct maps $BE \to X$ with prescribed cohomological properties. We proceed by studying the mapping space $map(BE, X)$, and take advantage of the fact that its mod p cohomology ring may be computed by Lannes' functor T_E. We refer to Lannes [L] and Schwartz [Sc] for the basic properties of T_E, which we summarise below. We follow their notation by writing $\mathcal{U}$ for the category of unstable modules over the mod p Steenrod algebra $\mathcal{A}_p$, and $\mathcal{K}$ for the subcategory of unstable algebras over $\mathcal{A}_p$. In order to avoid technical difficulties, we shall only consider algebras that are finitely generated; thus $\mathcal{K}(A, A')$ is a finite set for any algebras A and A'.

By definition, T_E is left adjoint to the functor $HE \otimes -\colon \mathcal{U} \to \mathcal{U}$. The latter is compatible with the forgetful functor $\mathcal{K} \to \mathcal{U}$, and $T_E\colon \mathcal{K} \to \mathcal{K}$ is left adjoint to its restriction. By construction, T_{C^n} coincides with the n-fold iterate $T_C \cdots T_C$ for any n, and T_E preserves tensor products in $\mathcal{U}$ and $\mathcal{K}$.

For any space Y, evaluation defines a map $BE \times map\,(BE, Y) \to Y$. Passing to cohomology and left adjointing yields a $\mathcal{K}$-morphism

$$a_Y \colon T_E H^*(Y) \longrightarrow H^*(map\,(BE, Y)), \tag{3.1}$$

which is an isomorphism whenever Y is p-complete and $T_E H^*(Y)$ has finite type and vanishes in degree 1. So T_E computes the mod p cohomology of the mapping space under these conditions [L, Theorem 3.2.1].

The splitting of $map\,(BE, Y)$ into the disjoint union $\coprod map\,(BE, Y)_f$ of its connected components is mirrored by the algebraic properties of T_E, as follows. For each unstable algebra A and $\mathcal{K}$-morphism $\phi\colon A \to HE$, its adjoint restricts to a map $T_E^0 A \to \mathbb{F}_p$ on the zero-dimensional part. We write $\mathbb{F}_p(\phi)$ when it is necessary to emphasise the induced $T_E^0 A$-module structure on $\mathbb{F}_p$. The component of $T_E A$ associated to ϕ is defined by

$$T_E(A, \phi) \;:=\; T_E A \otimes_{T_E^0 A} \mathbb{F}_p(\phi), \tag{3.2}$$

and Lannes' linearisation principle [Sc, Theorem 3.8.6] leads to splittings

$$T_E A \;\cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} T_E(A, \phi) \qquad \text{and} \qquad T_E^0 A \;\cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} \mathbb{F}_p(\phi).$$

In particular, we may identify $T_E^0 A$ with the *p-Boolean algebra* $\mathbb{F}_p^{\mathcal{K}(A,HE)}$.

In the case that $A = H^*(Y)$ and $\phi = f^*$ for some $f\colon BE \to Y$, the morphism a_Y of (3.1) restricts to a $\mathcal{K}$-morphism

$$a_f\colon T_E(H^*(Y), f^*) \longrightarrow H^*(map\,(BE, Y)_f). \tag{3.3}$$

Since the algebraic components of $T_E H^*(Y)$ correspond bijectively to the connected components of $map\,(BE, Y))$, it follows that a_Y splits accordingly.

The following key example is based on the proof of [Sc, Proposition 9.8], in which $map\,(BE, BT^\omega)$ is identified with $E(\omega) \times BT^\omega$, where $E(\omega)$ denotes the discrete group of homomorphisms $E \to E^\omega$ for any subset $\omega \subseteq V$.

Example 3.4. As $\mathcal{A}_p$-algebras, $HT^\omega := H^*(BT^\omega)$ is canonically isomorphic to $S_{\mathbb{F}_p}(\omega)$, and multiplication on BT^ω induces a coproduct d. The $\mathcal{K}$-morphisms $\phi\colon HT^\omega \to HE$ correspond bijectively to group homomorphisms $E \to T^\omega$, and hence to $E(\omega)$. The coaugmentation $c_\phi\colon HT^\omega \to T_E(HT^\omega, \phi)$ is an isomorphism in $\mathcal{K}$, whose inverse r_ϕ is the adjoint of $(\phi \otimes 1) \circ d$.

For each $f\colon BE \to BT^\omega$, the composition

$$HT^\omega \xrightarrow{c_f} T_E(HT^\omega, f^*) \xrightarrow{a_f} H^*(map\,(BE, BT^\omega)_f) \tag{3.5}$$

computes the cohomology of the mapping space, and is induced by evaluation $map\,(BE, BT^\omega)_f \to BT^\omega$ at the base point. Its inverse is induced by the adjoint of the action $s_f\colon BE \times BT^\omega \to BT^\omega$, defined by $(b, u) \mapsto uf(b)$.

For any $\mathcal{K}$-morphism $\pi\colon A_1 \to A_2$, the induced homomorphism

$$T_E^0(\pi)\colon \bigoplus_{\phi \in \mathcal{K}(A_1, HE)} \mathbb{F}_p(\phi) \;\cong\; T_E^0 A_1 \;\longrightarrow\; T_E^0 A_2 \;\cong\; \bigoplus_{\psi \in \mathcal{K}(A_2, HE)} \mathbb{F}_p(\psi)$$

is determined by composition $\circ\,\pi\colon \mathcal{K}(A_2, HE) \to \mathcal{K}(A_1, HE)$ with π. For any $\phi \in \mathcal{K}(A_1, HE)$, we denote the subset $(\,\circ\,\pi)^{-1}(\phi) \subset \mathcal{K}(A_2, HE)$ by $\mathrm{ext}(\phi, \pi)$; its elements ψ satisfy $\psi \circ \pi = \phi$. We may then identify $T_E^0(\pi)$ with the direct sum of diagonal maps of the form $\mathbb{F}_p(\phi) \to \bigoplus_{\psi \in \mathrm{ext}(\phi,\pi)} \mathbb{F}_p(\psi)$, and obtain an isomorphism

$$T_E^0 A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \;\cong\; \bigoplus_{\psi \in \mathrm{ext}(\phi,\pi)} \mathbb{F}_p(\psi). \tag{3.6}$$

For any unstable algebra A in $\mathcal{K}$ we write A–$\mathcal{U}$ for the category whose objects M are simultaneously A-modules and unstable $\mathcal{A}_p$-modules, and whose structure maps $A \otimes M \to M$ are $\mathcal{A}_p$-linear; here $\mathcal{A}_p$ acts on $A \otimes M$ via the Cartan formula. The morphisms in A–$\mathcal{U}$ are both A- and $\mathcal{A}_p$-linear. Since the T-functor commutes with tensor products, T_E extends to a functor A–$\mathcal{U} \to (T_E A)$–$\mathcal{U}$, which we also label T_E. For every morphism $\phi\colon A \to HE$, we may then define the relative component

$$T_E(M, \phi) \;:=\; T_E M \otimes_{T_E^0 A} \mathbb{F}_p(\phi)$$

by analogy with (3.2), and obtain a splitting $T_E M \cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} T_E(M, \phi)$. Furthermore, the coaugmentation $A \to T_E(A, \phi)$ makes $T_E(M, \phi)$ into an A-module, and $T_E(-, \phi)$ becomes an endofunctor on A–$\mathcal{U}$.

In particular, the morphism π of (3.6) turns A_2 into an object of A_1–$\mathcal{U}$, where there are isomorphisms

$$\begin{aligned} T_E(A_2, \phi) \;&\cong\; T_E A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \\ &\cong\; T_E A_2 \otimes_{T_E^0 A_2} T_E^0 A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \\ &\cong\; T_E A_2 \otimes_{T_E^0 A_2} \big(\textstyle\bigoplus_{\psi \in \mathrm{ext}(\phi,\pi)} \mathbb{F}_p(\psi)\big) \\ &\cong\; \textstyle\bigoplus_{\psi \in \mathrm{ext}(\phi,\pi)} T_E(A_2, \psi). \end{aligned}$$

One additional property of the T-functor is that T_E commutes with finite limits; in other words, for any finite category C and diagram $F\colon \mathrm{C} \to \mathcal{K}$, there exists a $\mathcal{K}$-isomorphism $T_E(\lim F) \cong \lim T_E F$. In this situation, we are interested in computing the components of $T_E(\lim F)$. To avoid technical difficulties we assume that $F(c)$ is finitely generated as an algebra for every object c of C, and similarly for $A := \lim F$.

The limit is defined by an exact sequence of the form

$$0 \longrightarrow A \longrightarrow \prod_c F(c) \xrightarrow{\;\delta\;} \prod_{c_0 \to c_1} F(c_1),$$

and the natural projections $\pi_c\colon A \to F(c)$ turn δ into an A–$\mathcal{U}$-morphism. For every $\phi\colon A \to HE$ the functor $T_E(-, \phi)$ is exact on A–$\mathcal{U}$, because it is a retract of T_E. We therefore obtain an exact sequence

$$0 \longrightarrow T_E(A,\phi) \longrightarrow \prod_c T_E(F(c);\phi) \longrightarrow \prod_{c_0\to c_1} T_E(F(c_1),\phi),$$

which identifies $T_E(\lim_{\mathrm{C}} F,\phi)$ with $\lim_{\mathrm{C}} T_E(F,\phi)$.

If $\operatorname{ext}(\phi,\pi_c)$ is empty, then $T_E(F(c),\phi)=0$. So we consider the full subcategory $\mathrm{C}\langle F,\phi\rangle$ of C, whose objects are defined by insisting that $\operatorname{ext}(\phi,\pi_c)$ be non-empty. For any morphism $c'\to c$ with c in $\mathrm{C}\langle F,\phi\rangle$, it follows that c' also lies in $\mathrm{C}\langle F,\phi\rangle$; so there exists an isomorphism

$$T_E(\lim_{\mathrm{C}}F,\phi) \;\cong\; \lim_{\mathrm{C}\langle F,\phi\rangle}T_E(F,\phi). \tag{3.7}$$

We apply these considerations to the diagram $S_{\mathbb{F}_p}(K)\colon \mathrm{CAT}^{op}(K)\to\mathcal{K}$ of (2.5), whose objects are interpreted as $\mathcal{A}_p$-algebras via Example 3.4. In particular, $\lim_{\mathrm{CAT}^{op}(K)} S_{\mathbb{F}_p}(K)$ is isomorphic to $\mathbb{F}_p[K]$, and the $\mathcal{K}$-morphism $\pi_\sigma\colon \mathbb{F}_p[K]\to S_{\mathbb{F}_p}(\sigma)$ is the standard projection, for every $\sigma\in K$. We focus on the $\mathcal{K}$-morphisms $\phi_\sigma\colon \mathbb{F}_p[K]\to HC^\sigma$, defined in terms of (2.4) by $t_\sigma^*\circ\pi_\sigma$.

Proposition 3.8. *For any face σ of K, there is an isomorphism*

$$r_\sigma\colon T_{C^\sigma}(\mathbb{F}_p[K],\phi_\sigma)\longrightarrow \mathbb{F}_p[st_K(\sigma)]$$

as objects of $\mathcal{K}$.

Proof. The set $\operatorname{ext}(\phi_\sigma,\pi_\tau)$ contains the single element $\phi_{\tau,\sigma}\colon S_{\mathbb{F}_p}(\tau)\to HC^\sigma$, defined by $t_\sigma^*\circ p_{\tau,\sigma}$, if and only if $\sigma\subseteq\tau$. Otherwise, $\operatorname{ext}(\phi_\sigma,\pi_\tau)$ is empty; so for any particular σ, we may identify $\mathrm{CAT}(K)\langle S_{\mathbb{F}_p}(K),\phi_\sigma\rangle$ with $\sigma\downarrow\mathrm{CAT}(K)$.

For $\sigma\subseteq\tau$, Example 3.4 provides isomorphisms

$$r_\sigma(\tau)\colon T_{C^\sigma}\big(S_{\mathbb{F}_p}(\tau),\phi_{\tau,\sigma}\big)\longrightarrow S_{\mathbb{F}_p}(\tau)$$

that are functorial in τ, and so define an isomorphism

$$r_\sigma(K)\colon T_{C^\sigma}\big(S_{\mathbb{F}_p}(K),\phi_\sigma\big)\longrightarrow S_{\mathbb{F}_p}(\sigma\downarrow\mathrm{CAT}(K))$$

of $\mathrm{CAT}^{op}(K)\downarrow\sigma$-diagrams in $\mathcal{K}$. Then take limits, using (2.8) and (3.7). $\square$

Corollary 3.9. *For every maximal face μ of K, there is an isomorphism $r_\mu\colon T_{C^\mu}(\mathbb{F}_p[K],\phi_\mu)\to S_{\mathbb{F}_p}(\mu)$ as objects of $\mathcal{K}$.*

Proof. Since μ is maximal, its star is the simplex $\Delta(\mu)$. $\square$

We now introduce the crucial map $f_\sigma\colon BC^\sigma\to X$, and prove the main results of this section. It is convenient to write the the space $map\,(BC^\sigma,X)_{f_\sigma}$ as $m^\sigma(X)$, and evaluation at the identity of BC^σ as $ev\colon m^\sigma(X)\to X$.

Theorem 3.10. *If X is p-complete and reflects K by means of an isomorphism $\theta\colon H^*(X)\to\mathbb{F}_p[K]$, then:*

1. *there exists a map $f_\sigma\colon BC^\sigma\to X$ such that $f_\sigma^*=\phi_\sigma\circ\theta$, and it is uniquely determined up to homotopy;*
2. *the space $m^\sigma(X)$ is p-complete, and reflects $st_K(\sigma)$ over $\mathbb{F}_p$;*
3. *the map $ev\colon m^\sigma(X)\to X$ reflects $(K;st_K(\sigma))$ over $\mathbb{F}_p$;*
4. *for maximal μ, there is a homotopy equivalence $m^\mu(X)\simeq(BT^\mu)^\wedge_p$.*

Proof. (1) Every $\mathcal{K}$-morphism $H^*(X) \to HC^\sigma$ may be realised uniquely up to homotopy by [L, Théorème 3.1.1], because X is p-complete and $H^*(X)$ has finite type.

(2) By [L, Proposition 3.4.4], $m^\sigma(X)$ inherits completeness from X. Also, $T_{C^\sigma}(H^*(X), \phi_\sigma)$ is of finite type and vanishes in degree 1, by Proposition 3.8. So $a_\sigma := a_{f_\sigma} \colon T_{C^\sigma}(H^*(X), \phi_\sigma) \to H^*(m^\sigma(X))$ of (3.3) is an isomorphism, and the required reflector is $\theta_{m^\sigma(X)} := r_\sigma \circ T_{C^\sigma}(\theta) \circ a_\sigma^{-1}$.

(3) By construction, $ev^* \colon H^*(X) \to H^*(m^\sigma(X))$ is the composition of f_σ^* with the coaugmentation c_σ for T_{C^σ}. So we may combine (2) with the functorial properties of T to obtain a commutative ladder

$$
\begin{array}{ccccc}
H^*(X) & \xrightarrow{c_\sigma} & T_{C^\sigma}(H^*(X), \phi_\sigma) & \xrightarrow{a_\sigma} & H^*(m^\sigma(X)) \\
{\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle T_{C^\sigma}(\theta)} & & \big\downarrow{\scriptstyle\theta_{m^\sigma(X)}} \\
\mathbb{F}_p[K] & \xrightarrow{c_\sigma} & T_{C^\sigma}(\mathbb{F}_p[K], \phi_\sigma) & \xrightarrow{r_\sigma} & \mathbb{F}_p[st_K(\sigma)]
\end{array}
\quad , \tag{3.11}
$$

whose outer rectangle is a commutative square of the form (2.10); hence ev is the required reflector.

(4) It suffices to combine (2), Corollary 3.9, and the fact that the homotopy type of $(BT^\mu)^\wedge_p$ is determined by its mod p cohomology. $\square$

Remarks 3.12. (1) The mod p cohomology of X of Theorem 3.10 is concentrated in even dimensions, so $H^{ev}(X; \mathbb{Z}^\wedge_p)$ is torsion free and $H^{od}(X; \mathbb{Z}^\wedge_p)$ is zero. Since Theorem 3.10(3) implies that evaluation induces an epimorphism in mod p cohomology, the same must hold over $\mathbb{Z}^\wedge_p$.

(2) We may replace ϕ_σ in Proposition 3.8 with the homomorphism induced by the constant map $\mathrm{cst} \colon BC^\sigma \to X$, which we write as $\kappa_\sigma \colon \mathbb{F}_p[K] \to HC^\sigma$. Then $\mathrm{C}\langle F, \kappa_\sigma \rangle = \mathrm{C}$, and we obtain an isomorphism $T_{C^\sigma}(\mathbb{F}_p[K], \kappa_\sigma) \to \mathbb{F}_p[K]$. It follows that $ev \colon map\,(BC^\sigma, X)_{\mathrm{cst}} \to X$ induces an isomorphism in mod p cohomology, by adapting the proofs of Theorem 3.10(2) and (3); it is therefore an equivalence, because the source and target are p-complete.

4. Mapping spaces and p-adic cohomology: I

In this section we extend our calculations to p-adic cohomology for the mapping space $m^\sigma(X)$ of Theorem 3.10(2). We insist that X is p-complete and 1-connected, and reflects K over $\mathbb{Z}^\wedge_p$ by means of an isomorphism $\theta \colon H^*(X; \mathbb{Z}^\wedge_p) \to \mathbb{Z}^\wedge_p[K]$. The reduction of θ mod p is therefore a reflector over $\mathbb{F}_p$, and Theorem 3.10 applies.

For our first result, we consider the evaluation map of Theorem 3.10(3), and abbreviate $V(st_K(\sigma))$ to $V_K(\sigma)$ throughout.

Lemma 4.1. *For any face $\sigma \in K$, the map $q_{m^\sigma(X)}$ may be chosen so as to make the diagram*

$$\begin{array}{ccc} m^\sigma(X) & \xrightarrow{ev} & X \\ {\scriptstyle q_{m^\sigma(X)}}\Big\downarrow & & \Big\downarrow{\scriptstyle q_X} \\ (BT^{V_K(\sigma)})^\wedge_p & \xrightarrow{j_\sigma} & (BT^V)^\wedge_p \end{array} \tag{4.2}$$

homotopy commutative, where j_σ denotes coordinatewise inclusion.

Proof. Define $q_{m^\sigma(X)}$ by composing $q_X \circ ev$ with projection onto $(BT^{V_K(\sigma)})^\wedge_p$. The diagram induced by (4.2) in mod p cohomology is then commutative, by Theorem 3.10(3), and it remains to prove that $ev^* q_X^*(v_i)$ is also zero in $H^2(m^\sigma(X); \mathbb{Z}^\wedge_p)$ for any vertex $v_i \notin V_K(\sigma)$.

Let $q_i \colon X \to BT^\wedge_p$ represent v_i. The Serre spectral sequence of the induced spherical fibration

$$T^\wedge_p \longrightarrow F_i \longrightarrow X \tag{4.3}$$

shows that $H^*(F_i; \mathbb{F}_p)$ has finite type, and F_i is p-complete because q_i is a map of 1-connected p-complete spaces. Also, $q_i \circ f_\sigma \colon BC^\sigma \to BT^\wedge_p$ is null-homotopic for any $v_i \notin V_K(\sigma)$, so f_σ lifts to a map $f_i \colon BC^\sigma \to F_i$. By [Th], the functor $map\,(BC^\sigma, -)$ converts $F_i \to X \to BT^\wedge_p$ into a fibration

$$map\,(BC^\sigma, F_i)_{f_i} \xrightarrow{g_i} m^\sigma(X) \xrightarrow{k_i} map\,(BC^\sigma, BT^\wedge_p)_{\mathrm{cst}},$$

whose base is homotopy equivalent to $BT^\wedge_p$ by Example 3.4. We abbreviate the fibre to $m^\sigma(F_i)$, and note that it is path connected and p-complete, because k_i is a map of 1-connected p-complete spaces; and $H^*(m^\sigma(F_i); \mathbb{F}_p)$ is of finite type, because the same holds for $H^*(m^\sigma(X); \mathbb{F}_p)$. We may then deduce from [L, Proposition 3.4.4] that T_{C^σ} computes $H^*(m^\sigma(F_i); \mathbb{F}_p)$. But the Gysin sequence of (4.3) confirms that $H^*(F_i; \mathbb{Z}^\wedge_p)$ is torsion-free, so the same is also true for $H^*(m^\sigma(F_i); \mathbb{Z}^\wedge_p)$ by [KW, Theorem A].

Now $k_i^* v_i \in H^2(m^\sigma(X); R)$ is the Euler class $e(k_i)$ of the principal $T^\wedge_p$-fibration k_i for any coefficients R, and equals $ev^* q_X^*(v_i)$ by construction. So $e(k_i)$ vanishes mod p for every $v_i \notin V(st_K(\sigma))$, and we need only prove that the same is true p-adically. The mod p Gysin sequence shows that $g_i^* \colon H^*(m^\sigma(X); \mathbb{F}_p) \to H^*(m^\sigma(F_i); \mathbb{F}_p)$ is a monomorphism; and there are isomorphisms $H^*(Y; \mathbb{F}_p) \cong H^*(Y; \mathbb{Z}^\wedge_p) \otimes \mathbb{F}_p$ for $Y = m^\sigma(X)$ and $m^\sigma(F_i)$, because $H^*(Y; \mathbb{Z}^\wedge_p)$ is torsion-free in both cases. Then Nakayama's lemma confirms that g_i^* is also a p-adic monomorphism, and $e(k_i) = 0$ as sought. $\square$

Corollary 4.4. *For any maximal face $\mu \in K$, the map $q_{m^\mu(X)}$ is a homotopy equivalence and determines a homotopy commutative square*

$$\begin{array}{ccc} m^\mu(X) & \xrightarrow{ev} & X \\ {\scriptstyle q_{m^\mu(X)}}\Big\downarrow & & \Big\downarrow{\scriptstyle q_X} \\ (BT^\mu)^\wedge_p & \xrightarrow{j_\sigma} & (BT^V)^\wedge_p \end{array}.$$

Proof. The statement follows from Lemma 4.1 by setting $\sigma = \mu$, and identifying $q_{m^\mu(X)}$ as a homotopy equivalence by Theorem 3.10(4). □

Definition 4.5. For any maximal face μ, the map $h_\mu \colon BT^\mu \to X$ is the composition of $ev \circ q^{-1}_{m^\mu(X)}$ with completion. For any other face σ, the map $h_\sigma \colon BT^\sigma \to X$ is $h_\mu \circ i_{\sigma,\mu}$ for some choice of $\mu \supseteq \sigma$.

Remarks 4.6. (1) In Section 5, we shall show that the homotopy class of h_σ is independent of the choice of μ, and therefore that the diagram

(4.7)
$$\begin{tikzcd} BT^\sigma \arrow[rr, "i_{\sigma,\tau}"] \arrow[dr, "h_\sigma"'] & & BT^\tau \arrow[dl, "h_\tau"] \\ & X & \end{tikzcd}$$

is homotopy commutative for every $\sigma \subset \tau$. In fact h_μ reflects the pair $(K, \Delta(\mu))$ over $\mathbb{Z}_p^\wedge$ by construction, so the diagram induced by (4.7) is certainly commutative in p-adic cohomology.

(2) In [DL], Dehon and Lannes study maps $BT^r \to Y$ for spaces such that $H^*(Y;\mathbb{Z})$ is torsion free and concentrated in even degrees. They deduce that $H^*(-;\mathbb{Z})$ classifies such maps up to homotopy. Their proof establishes a similar result for the p-adic cohomology of p-complete spaces, and may also be adapted to show that the homotopy class of h_σ is independent of μ.

(3) For any face σ, Theorem 3.10(1) provides a homotopy

(4.8)
$$h_\sigma \circ t_\sigma \simeq f_\sigma \colon BC^\sigma \longrightarrow X.$$

Theorem 4.9. *If X is p-complete and reflects K over $\mathbb{Z}_p^\wedge$, then:*

(1) *the space $m^\sigma(X)$ reflects $st_K(\sigma)$ over $\mathbb{Z}_p^\wedge$;*
(2) *the map $ev \colon m^\sigma(X) \to X$ reflects the pair $(K, st_K(\sigma))$ over $\mathbb{Z}_p^\wedge$.*

Proof. (1) For any pair of faces $\sigma \subseteq \tau$, write $i_{\sigma,\tau} \circ t_\sigma$ as $t_{\sigma,\tau} \colon BC^\sigma \to BT^\tau$. Then (4.8) confirms that $h_\tau \circ t_{\sigma,\tau} \colon BC^\sigma \to X$ is homotopic to f_σ. Passing to mapping spaces and employing Example 3.4 defines a composition

$$l_{\sigma,\tau} \colon BT^\tau \xrightarrow{e_{\sigma,\tau}} map\,(BC^\sigma, BT^\tau)_{t_{\sigma,\tau}} \xrightarrow{h_\tau \circ} m^\sigma(X),$$

where $e_{\sigma,\tau}(u)$ acts by $b \mapsto ut_{\sigma,\tau}(b)$, and $ev \circ l_{\sigma,\tau} \simeq h_\tau$. Since $H^*(ev;\mathbb{Z}_p^\wedge)$ is epic by Remark 3.12(1), all choices of h_τ induce the same homomorphism $H^*(l_{\sigma,\tau};\mathbb{Z}_p^\wedge)$. So for any $\sigma \subseteq \tau \subseteq \rho$ we obtain a commutative triangle

(4.10)
$$\begin{tikzcd} & H^*(m^\sigma(X);\mathbb{Z}_p^\wedge) \arrow[dl] \arrow[dr] & \\ H^*(BT^\rho;\mathbb{Z}_p^\wedge) \arrow[rr] & & H^*(BT^\tau;\mathbb{Z}_p^\wedge)\,, \end{tikzcd}$$

thus defining a $(\textsc{cat}^{op}(K)\downarrow\sigma)$-diagram $HT^{\sigma\downarrow K}$ of graded commutative $\mathbb{Z}_p^\wedge$-algebras, and an induced homomorphism

(4.11)
$$\theta' \colon H^*(m^\sigma(X);\mathbb{Z}_p^\wedge) \longrightarrow \lim HT^{\sigma\downarrow K}.$$

By (2.8), the target is isomorphic to $\mathbb{Z}_p^\wedge[st_K(\sigma)]$.

Now consider the homomorphism $\theta''\colon H^*(m^\sigma(X);\mathbb{F}_p)\to\mathbb{F}_p[st_K(\sigma)]$, given by reducing θ' mod p; it may be identified with the isomorphism $\theta_{m^\sigma(X)}$ of (3.11) by noting the commutativity of the square

$$\begin{array}{ccc} T_{C^\sigma}(H^*(X;\mathbb{F}_p),\phi_\sigma) & \xrightarrow{a_\sigma} & H^*(m^\sigma(X);\mathbb{F}_p)\\ {\scriptstyle T_{C^\sigma}(\pi_\tau\circ\theta)}\big\downarrow & & \big\downarrow{\scriptstyle l^*_{\sigma,\tau}}\\ T_{C^\sigma}(S_{\mathbb{F}_p}(\tau),\phi_{\tau,\sigma}) & \xrightarrow{r_\sigma(\tau)} & S_{\mathbb{F}_p}(\tau)\end{array}$$

for any $\tau\supseteq\sigma$, and taking limits over $\text{CAT}^{op}(K)\downarrow\sigma$. The exact sequence $\mathbb{Z}_p^\wedge\xrightarrow{p}\mathbb{Z}_p^\wedge\to\mathbb{F}_p$ yields a commutative ladder of short exact sequences

$$\begin{array}{ccccc} H^*(m^\sigma(X);\mathbb{Z}_p^\wedge) & \xrightarrow{p} & H^*(m^\sigma(X);\mathbb{Z}_p^\wedge) & \longrightarrow & H^*(m^\sigma(X);\mathbb{F}_p)\\ {\scriptstyle\theta'}\big\downarrow & & {\scriptstyle\theta'}\big\downarrow & & \big\downarrow{\scriptstyle\theta''}\\ \mathbb{Z}_p^\wedge[st_K(\sigma)] & \xrightarrow{p} & \mathbb{Z}_p^\wedge[st_K(\sigma)] & \longrightarrow & \mathbb{F}_p[st_K(\sigma)]\end{array}, \tag{4.12}$$

because $H^*(m^\sigma(X);\mathbb{Z}_p^\wedge)$ is free. But θ'' is an isomorphism, so θ' is also an isomorphism, and is therefore the required reflector.

(2) The homomorphism $ev^*\colon H^*(X;\mathbb{Z}_p^\wedge)\to H^*(m^\sigma(X);\mathbb{Z}_p^\wedge)$ induces morphisms of triangles (4.10) over the inclusion functor $\sigma\downarrow\text{CAT}(K)\to\text{CAT}(K)$. The required square commutes, by Lemma 4.1. $\square$

Corollary 4.13. *If $st_K(\sigma)$ coincides with K, then $ev\colon m^\sigma(X)\to X$ is a homotopy equivalence and the reflectors satisfy $\theta'\circ ev^*=\theta_X$ over $\mathbb{Z}_p^\wedge$.*

Proof. Setting $\sigma\downarrow\text{CAT}(K)=\text{CAT}(K)$ in the proof of Theorem 4.9(2) shows that ev^* is an isomorphism of p-adic cohomology, as required. $\square$

Remark 4.14. In the special case $X=c(K)$ of Theorem 4.9, a more precise description of $m^\sigma(X)$ is possible by appeal to Example 3.4. The action $s_{t_{\sigma,\tau}}\colon BC^\sigma\times BT^\tau\to BT^\tau$ is given by $(b,u)\mapsto ut_{\sigma,\tau}(b)$ for every $\tau\supseteq\sigma$, and extends to a morphism $BC^\sigma\times BT^{\sigma\downarrow K}\to BT^K$ of diagrams over the inclusion $\sigma\downarrow\text{CAT}(K)\to\text{CAT}(K)$. Taking colimits and adjointing then creates a map

$$c(st_K(\sigma))\longrightarrow m^\sigma(c(K)),$$

which induces the isomorphism θ'' of (4.12) in mod p cohomology. So its p-completion is a homotopy equivalence.

5. Mapping spaces and p-adic cohomology: II

In this section we prove that the homotopy class of the map h_σ of Definition 4.5 is uniquely determined, and compute the p-adic cohomology of the space $map\,(BT^\sigma,X)_{h_\sigma}$. We retain the prevailing notation, whereby X is p-complete, simply connected, and reflects K over $\mathbb{Z}_p^\wedge$ by means of an isomorphism θ.

We focus on the action s_{t_σ} of BC^σ on BT^σ, given by $(b,u) \mapsto ut_\sigma(b)$ in Example 3.4 and denoted here by s_σ. There is a principal BC^σ-fibration

$$BC^\sigma \xrightarrow{j_\sigma} BC^\sigma \times BT^\sigma \xrightarrow{s_\sigma} BT^\sigma, \tag{5.1}$$

where j_σ includes the kernel of s_σ and is given by $b \mapsto (b^{-1}, t_\sigma(b))$. Any map $g\colon BT^\sigma \to X$ may be factorised through evaluation at the identity as

$$BT^\sigma \xrightarrow{g'} \mathit{map}\,(BC^\sigma, X)_{g\circ t_\sigma} \xrightarrow{ev} X, \tag{5.2}$$

where g' is the adjoint of $g \circ s_\sigma$, and satisfies $g'(u)(c) = g(ut_\sigma(c))$ for all $u \in BT^\sigma$ and $c \in BC^\sigma$.

Lemma 5.3. *The map*

$$\mathit{map}\,(BT^\sigma, X)_g \longrightarrow \mathit{map}\,(BT^\sigma, \mathit{map}\,(BC^\sigma, X)_{g\circ t_\sigma})_{g'}$$

induced by s_σ is a homotopy equivalence, whose inverse is induced by evaluation at the identity of BC^σ.

Proof. In the context of (5.1), $g \circ s_\sigma \circ j_\sigma$ is null-homotopic; it follows from Remark 3.12(2) that $ev\colon \mathit{map}\,(BC^\sigma, X)_{g\circ s_\sigma \circ j_\sigma} \to X$ is a homotopy equivalence. We may then apply Zabrodsky's Lemma [Z, 3.1], and deduce that $\circ\, s_\sigma\colon \mathit{map}\,(BT^\sigma, X) \to \mathit{map}\,(BC^\sigma \times BT^\sigma, X)_L$ is also an equivalence, where L denotes the set of homotopy classes whose composition with s_σ is trivial. Restricting to the component of g and adjointing yields the required equivalence; by construction, evaluation at the basepoint is a right inverse. $\square$

By analogy with (5.1), BC^σ also acts on $\mathit{map}\,(BC^\sigma, X)_{g\circ t_\sigma}$ according to the formula $(b,f) \mapsto fb$, where $fb(c) := f(bc)$.

Lemma 5.4. *The map $g'\colon BT^\sigma \to \mathit{map}\,(BC^\sigma, X)_{g\circ t_\sigma}$ is BC^σ-equivariant.*

Proof. For $(b,c) \in BC^\sigma \times BT^\sigma$, we have $g'(ut_\sigma(b))(c) = g(ut_\sigma(b)t_\sigma(c))$ by (5.2), and $g'b(c) = g(ut_\sigma(bc))$; these agree, as t_σ is a homomorphism. $\square$

We now insist that g induces the homomorphism $H^*(h_\sigma; \mathbb{Z}_p^\wedge)$, and write $g = g_\sigma$ to emphasise this choice. Thus $g_\sigma \circ t_\sigma \simeq f_\sigma$ by Theorem 3.10(1), and $\mathit{map}\,(BC^\sigma, X)_{g\circ t_\sigma}$ becomes $m^\sigma(X)$. Applying p-adic cohomology to (5.2) shows that $H^*(g'_\sigma; \mathbb{Z}_p^\wedge) = H^*(h'_\sigma; \mathbb{Z}_p^\wedge)$ by Remark 3.12(1), so we may study our uniqueness problem in terms of maps $BT^\sigma \to m^\sigma(X)$, or equivalently, by restricting attention to subcomplexes of the form $st_K(\sigma) \subseteq K$.

We may apply Lemma 4.1 to construct a map $q_\sigma\colon m^\sigma(X) \to (BT^\sigma)_p^\wedge$ such that $q_\sigma \circ g'_\sigma$ is completion, and consider the homotopy fibration

$$X_\ell \xrightarrow{i} m^\sigma(X) \xrightarrow{q_\sigma} (BT^\sigma)_p^\wedge. \tag{5.5}$$

Combining the Serre spectral sequence for i with Theorem 4.9 shows that i reflects $(st_K(\sigma), \ell_K(\sigma))$ over $\mathbb{Z}_p^\wedge$. We take θ' of (4.11) as the reflector for $m^\sigma(X)$, and its quotient by the appropriate ideals as the reflector for X_ℓ.

As a consequence of Lemma 5.4, we may apply the Borel construction to g'_σ, and obtain a homotopy pull-back diagram of principal BC^σ-fibrations

$$\begin{array}{ccc} BT^\sigma & \xrightarrow{g'_\sigma} & m^\sigma(X) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \overline{p}} \\ BT^\sigma & \xrightarrow{\overline{g}'_\sigma} & \overline{m}^\sigma(X) \end{array} , \tag{5.6}$$

where $\overline{m}^\sigma(X){:}=EBC^\sigma \times_{BC^\sigma} m^\sigma(X)$ and p denotes the pth power homomorphism. So $\overline{m}^\sigma(X)$ is simply connected, and applying $\pi_2(-)$ to (5.6) yields the commutative square

$$\begin{array}{ccc} \mathbb{Z}^\sigma & \longrightarrow & (\mathbb{Z}^\sigma)^\wedge_p \oplus (\mathbb{Z}^{V(\ell_K(\sigma))})^\wedge_p \\ {\scriptstyle p_*}\big\downarrow & & \big\downarrow{\scriptstyle p_*\oplus 1} \\ \mathbb{Z}^\sigma & \longrightarrow & (\mathbb{Z}^\sigma)^\wedge_p \oplus (\mathbb{Z}^{V(\ell_K(\sigma))})^\wedge_p \end{array} , \tag{5.7}$$

where the horizontal homomorphisms include the completed direct summand $(\mathbb{Z}^\sigma)^\wedge_p$, and p_* is multiplication by p.

We now identify $\pi_2(\overline{m}^\sigma(X))$ with $H^2(\overline{m}^\sigma(X);\mathbb{Z}^\wedge_p)$ by the universal coefficient theorem, and deduce the existence of a map $\overline{q}_\sigma\colon \overline{m}^\sigma(X) \to (BT^\sigma)^\wedge_p$ such that $\overline{q}_\sigma \circ \overline{g}'_\sigma$ is completion. The homotopy fibre of $\overline{q}_\sigma$ is X_ℓ because $\overline{q}_\sigma \circ \overline{p}$ and $p \circ q_\sigma$ are homotopic, and the Serre spectral sequence of the fibration $X_\ell \to \overline{m}^\sigma(X) \to (BT^\sigma)^\wedge_p$ shows that $H^*(\overline{m}^\sigma(X);\mathbb{Z}^\wedge_p)$ is of finite type, torsion free, and generated by 2–dimensional classes.

Lemma 5.8. *The space $\overline{m}^\sigma(X)$ reflects $st_K(\sigma)$ over $\mathbb{Z}^\wedge_p$; the reflector $\overline{\theta}'$ is determined by the commutative square*

$$\begin{array}{ccc} H^*(\overline{m}^\sigma(X);\mathbb{Z}^\wedge_p) & \xrightarrow{\overline{\theta}'} & S_{\mathbb{Z}^\wedge_p}(\sigma) \otimes \mathbb{Z}^\wedge_p[\ell_K(\sigma)] \\ {\scriptstyle \overline{p}^*}\big\downarrow & & \big\downarrow{\scriptstyle p\otimes 1} \\ H^*(m^\sigma(X);\mathbb{Z}^\wedge_p) & \xrightarrow{\theta'} & S_{\mathbb{Z}^\wedge_p}(\sigma) \otimes \mathbb{Z}^\wedge_p[\ell_K(\sigma)] \end{array} ,$$

where p acts on the polynomial generators of $S_{\mathbb{Z}^\wedge_p}(\sigma)$ as multiplication by p.

Proof. The Serre spectral sequence for the p-adic cohomology of the fibration $\overline{p}$ of (5.6) collapses at the E_2-page, and shows that $\overline{p}^*$ is a monomorphism. It is given in dimension 2 by the dual of (5.7), and the result follows. □

At this point it is helpful to work for a while with an arbitrary p-complete space Y that reflects $st_K(\sigma)$ over $\mathbb{Z}^\wedge_p$. We write the reflector as θ_Y, on the understanding that it agrees with θ' whenever Y is $m^\sigma(X)$; in these circumstances, we also identify $h_\sigma\colon BT^\sigma \to Y$ with $h'_\sigma\colon BT^\sigma \to m^\sigma(X)$. Of course h_σ induces the homomorphism $\pi_\sigma \circ \theta_Y$ in p-adic cohomology, where π_σ denotes the projection $S_{\mathbb{Z}^\wedge_p}(\sigma) \otimes \mathbb{Z}^\wedge_p[\ell_K(\sigma)] \to S_{\mathbb{Z}^\wedge_p}(\sigma)$. We write g_σ for any other map inducing the same homomorphism, and identify g_σ with g'_σ in case $Y = m^\sigma(X)$.

To analyse g_σ further, we consider its restriction $g_{\sigma/r}\colon BC_r^\sigma \to Y$, where $C_r < T$ denotes the cyclic subgroup of order p^r for any $r > 0$. We factorise t_σ into natural inclusions as

$$BC^\sigma \xrightarrow{\iota_{\sigma/r}} BC_r^\sigma \xrightarrow{t_{\sigma/r}} BT^\sigma\,,$$

so that $g_{\sigma/r} = g_\sigma \circ t_{\sigma/r}$. The induced homomorphism $H^*(h_{\sigma/r};\mathbb{Z}_p^\wedge)$ is given by $t_{\sigma/r}^* \circ \pi_\sigma \circ \theta_Y$, and Theorem 3.10(1) implies that $g'_{\sigma/1} \simeq h'_{\sigma/1}$ because both are homotopic to f_σ. We abbreviate $map\,(BC_r^\sigma, Y)_{h_{\sigma/r}}$ to $m_r^\sigma(Y)$ henceforth.

Proposition 5.9. *If Y is p-complete and reflects $st_K(\sigma)$ by means of an isomorphism $\theta_Y\colon H^*(Y;\mathbb{Z}_p^\wedge) \to S_{\mathbb{Z}_p^\wedge}(\sigma) \otimes \mathbb{Z}_p^\wedge[\ell_K(\sigma)]$, then:*

1. *there is a unique homotopy class of maps $h_{\sigma/r}\colon BC_r^\sigma \to Y$ such that $(h_{\sigma/r})^* = t_{\sigma/r}^* \circ \pi_\sigma \circ \theta_Y$;*
2. *the map $\circ\,\iota_{\sigma/r}\colon m_r^\sigma(Y) \to m^\sigma(Y)$ is a homotopy equivalence.*

We prove Proposition 5.9 by induction on r, following application to our main result. It is convenient to denote $map\,(BT^\sigma, X)_{h_\sigma}$ by $M^\sigma(X)$, and to rewrite the induced map $\circ\,t_\sigma\colon M^\sigma(X) \to m^\sigma(X)$ as u_σ.

Theorem 5.10. *If X is p-complete and reflects K over $\mathbb{Z}_p^\wedge$, then:*

1. *there is a unique homotopy class of maps $h_\sigma\colon BT^\sigma \to X$ that reflect $(K, \Delta(\sigma))$ over $\mathbb{Z}_p^\wedge$;*
2. *the map u_σ is a homotopy equivalence, and $ev\colon M^\sigma(X) \to X$ reflects $(K, st_K(\sigma))$ over $\mathbb{Z}_p^\wedge$;*
3. *there is a map $j\colon X_\ell \to M^\sigma(X)$ that reflects $(st_K(\sigma), \ell_K(\sigma))$ over $\mathbb{Z}_p^\wedge$;*
4. *there is a homotopy equivalence $M^\sigma(X) \simeq (BT^\sigma)_p^\wedge \times X_\ell$.*

Proof. (1) Write $Y = m^\sigma(X)$, and let $l(\sigma)$ be a set of representative maps $BT^\sigma \to Y$ for the homotopy classes that induce $\pi_\sigma \circ \theta'$ in p-adic cohomology. Since Y is p-complete and the natural map $\mathrm{hocolim}_r\, BC_r^\sigma \to BT^\sigma$ is a mod-p equivalence, the maps

$$map\,(BT^\sigma, Y)_{l(\sigma)} \longrightarrow map\,(\mathrm{hocolim}_r\, BC_r^\sigma, Y)_{l(\sigma)} \longrightarrow \mathrm{holim}_r\, m_r^\sigma(Y)$$

are both equivalences. Furthermore, evaluation at the identity induces compatible equivalences $m_r^\sigma(Y) \to Y$ for all $r \geq 1$, by Proposition 5.9. Thus $l(\sigma)$ contains only the homotopy class of h'_σ, and (5.2) implies the result.

(2) The proof of (1) shows that $ev \circ u_\sigma\colon M^\sigma(Y) \to Y$ is also evaluation at the identity, and an equivalence. So Corollary 4.13 confirms that u_σ is an equivalence. Then apply Lemma 5.3 and Corollary 4.13 to replace Y by X in the source and target respectively; the required reflectors are θ_X for X and $\theta' \circ (u_\sigma^{-1})^*$ for $M^\sigma(X)$.

(3) By (2), the fibration (5.5) may be rewritten as

$$X_\ell \xrightarrow{j} M^\sigma(X) \xrightarrow{r_\sigma} BT^\sigma, \tag{5.11}$$

where $i \simeq u_\sigma \circ j$ and $r_\sigma = q_\sigma \circ u_\sigma$. The reflectors are those of (2) and (5.5).

(4) An action $BT^\sigma \times M^\sigma(X) \to M^\sigma(X)$ is given by $(u,f) \mapsto fu$, where $fu(v) = f(uv)$ for any $u, v \in BT^\sigma$ and $f \in M^\sigma(X)$. Restricting to $BT^\sigma \times X_\ell$ along $1 \times j$ induces an isomorphism of p-adic cohomology, and is therefore an equivalence. $\square$

We now return to the proof of Proposition 5.9; the base cases $r = 1$ are immediate, so we assume that $r \geq 2$ throughout.

Proof. (1) Pulling the pth power map on BT^σ back along $t_{\sigma/r-1}$ and combining the result with (5.6) and (5.7) yields a homotopy commutative ladder

$$
\begin{array}{ccccccc}
BC_r^\sigma & \xrightarrow{t_{\sigma/r}} & BT^\sigma & \xrightarrow{g'_\sigma} & m^\sigma(Y) & \xrightarrow{q_\sigma} & BT^\sigma \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} & & \downarrow{\scriptstyle p} \\
BC_{r-1}^\sigma & \xrightarrow{t_{\sigma/r-1}} & BT^\sigma & \xrightarrow{\overline{g}'_\sigma} & \overline{m}^\sigma(Y) & \xrightarrow{\overline{q}_\sigma} & BT^\sigma
\end{array} \quad . \tag{5.12}
$$

Each vertical map has homotopy fibre BC^σ and induces a monomorphism in p-adic cohomology, so Lemma 5.8 confirms that $H^*(\overline{g}'_{\sigma/r-1}; \mathbb{Z}_p^\wedge)$ agrees with $H^*(\overline{h}'_{\sigma/r-1}; \mathbb{Z}_p^\wedge)$ as homomorphisms $H^*(\overline{m}^\sigma(Y); \mathbb{Z}_p^\wedge) \to H^*(BC_{r-1}^\sigma; \mathbb{Z}_p^\wedge)$. But $\overline{m}^\sigma(Y)$ also reflects $st_K(\sigma)$; so the inductive hypotheses show that $\overline{g}'_{\sigma/r-1}$ is homotopic to $\overline{h}'_{\sigma/r-1}$, and that evaluation

$$
map\,(BC_{r-1}^\sigma, \overline{m}^\sigma(Y))_{\overline{h}'_{\sigma/r-1}} \xrightarrow{\circ\iota_{\sigma/r-1}} map\,(BC^\sigma, \overline{m}^\sigma(Y))_{\overline{h}'_{\sigma/1}} \xrightarrow{ev} \overline{m}^\sigma(Y) \tag{5.13}
$$

at the identity is a homotopy equivalence.

Let l_Y denote the set of homotopy classes of lifts $BC_r^\sigma \to m^\sigma(Y)$ of the map $\overline{p} \circ g'_{\sigma/r} \colon BC_r^\sigma \to \overline{m}^\sigma(Y)$. Applying the functor $map\,(BC_r^\sigma, -)$ to (5.12) creates a homotopy pullback square

$$
\begin{array}{ccc}
map\,(BC_r^\sigma, m^\sigma(Y))_{l_Y} & \xrightarrow{q_\sigma\circ} & map\,(BC_r^\sigma, BT^\sigma)_{l_B} \\
{\scriptstyle \overline{p}\circ}\downarrow & & \downarrow{\scriptstyle p\circ} \\
map\,(BC_r^\sigma, \overline{m}^\sigma(Y))_{\overline{p}\circ g'_{\sigma/r}} & \xrightarrow{\overline{q}_\sigma\circ} & map\,(BC_r^\sigma, BT^\sigma)_{p\circ t_{\sigma/r}}
\end{array} \quad , \tag{5.14}
$$

where l_B denotes a set of representatives for homotopy classes of lifts of $p \circ t_{\sigma/r}$. The vertical maps are principal $map\,(BC_r^\sigma, BC^\sigma)$-fibrations by [Th], and $\overline{q}_\sigma \circ$ may be identified with the corresponding map

$$
map\,(BC_{r-1}^\sigma, \overline{m}^\sigma(Y)_{\overline{h}'_{\sigma/r-1}} \longrightarrow map\,(BC_{r-1}^\sigma, BT^\sigma)_{t_{\sigma/r-1}\circ p}
$$

by Zabrodsky's lemma. It follows from the inductive hypothesis that the lower fibration of (5.14) is equivalent to $\overline{q}_\sigma \colon \overline{m}^\sigma(Y) \to BT^\sigma$, and therefore that the common homotopy fibre of the horizontal maps is Y_ℓ, which is 1-connected. Thus $q_\sigma \circ$ induces a bijection $l_Y \leftrightarrow l_B$ of components. But we are given that $H^*(g'_{\sigma/r}; \mathbb{Z}_p^\wedge) = H^*(h'_{\sigma/r}; \mathbb{Z}_p^\wedge)$, from which we deduce that $q_\sigma \circ g'_{\sigma/r} \simeq q_\sigma \circ h'_{\sigma/r}$; so $g'_{\sigma/r}$ and $h'_{\sigma/r}$ define the same element of l_Y, and are therefore homotopic. Applying $ev \colon m^\sigma(Y) \to Y$ completes the proof.

(2) The pullback square (5.14) combines with the inductive hypotheses to show that $ev\colon map\,(BC^\sigma, m^\sigma(Y))_{h'_{\sigma/r}} \to m^\sigma(Y)$ is an equivalence, and Corollary 4.13 allows us to replace $m^\sigma(Y)$ with Y, and $h'_{\sigma/r}$ with $h_{\sigma/r}$. Then $\circ\,\iota_{\sigma/r}$ is also an equivalence, by analogy with (5.13). □

6. Higher limits of diagrams

In this section we recall the basic properties of higher limits of diagrams $\Phi\colon \mathrm{C} \to \mathrm{AB}$ of abelian groups, where C is usually of the form $\mathrm{CAT}^{op}(K)$. These are crucial for later sections, and also featured in [NR, Section 3].

Following Oliver's elaboration [O] of [BK], we interpret $\lim^i_{\mathrm{C}} \Phi$ as the ith cohomology group of the cochain complex $(C^*(\mathrm{C}, \Phi), \delta)$, where

$$C^n(\mathrm{C};\Phi) \;=\; \prod_{c_0\to c_1\to\ldots\to c_n} \Phi(c_n) \tag{6.1}$$

for every $n \geq 0$, and the product is taken over all chains of morphisms in C. The differential $\delta\colon C^n(\mathrm{C};\Phi) \to C^{n+1}(\mathrm{C};\Phi)$ is the alternating sum $\sum_{k=0}^r (-1)^k \delta^k$, where δ^k is defined on $u \in C^n(\mathrm{C};\Phi)$ by

$$\delta^k(u)(c_0 \to \ldots \to c_{n+1}) := \begin{cases} u(c_0 \to \ldots \to \widehat{c_k} \to \ldots \to c_{n+1}) & \text{for } k \leq n \\ \Phi(c_n \to c_{n+1})u(c_1 \to \ldots \to c_n) & \text{for } k = n+1. \end{cases}$$

We may replace $C^*(\mathrm{C};\Phi)$ by its quotient $N^*(\mathrm{C},\Phi)$ of normalised cochains, obtained by restricting the products (6.1) to chains of non-identities. So

$$\lim{}^i_{\mathrm{C}}\Phi \;:=\; H^i(C^*(\mathrm{C};\Phi),\delta) \;\cong\; H^i(N^*(\mathrm{C};\Phi),\delta)$$

for $i \geq 0$, and the limits are themselves abelian groups. The same construction works for small diagrams in an arbitrary abelian category A, and the limits inherit any additional algebraic structure from the objects of A.

Given a constant diagram $\mathrm{cst}_M\colon \mathrm{C} \to \mathrm{AB}$, it follows that its higher limits are determined by the cohomology of the classifying space $B\mathrm{C}$, via an isomorphism $\lim^i_{\mathrm{C}} \mathrm{cst}_M \cong H^i(B\mathrm{C}; M)$. In particular, $\lim^i_{\mathrm{CAT}^{op}(K)} \mathrm{cst}_M$ vanishes for $i \geq 1$ because $B\mathrm{CAT}^{op}(K)$ is contractible, and $\lim^i_{\mathrm{CAT}^{op}(K^\times)} \mathrm{cst}_M$ is isomorphic to $H^i(K;M)$ because $B\mathrm{CAT}^{op}(K^\times)$ is homeomorphic to the realisation $|K|$. Both are isomorphic to M when $i = 0$.

For any face $\sigma \in K$, we refer to a diagram $\Phi\colon \mathrm{CAT}^{op}(K) \to \mathrm{AB}$ as *σ-atomic* when $\Phi(\tau) = 0$ for every $\tau \neq \sigma$. We consider the following generalisation of [NR, Lemma 3.9], in which it is convenient to set $\widetilde{H}^n(\varnothing; M) := 0$ for $n \geq 0$, and $\widetilde{H}^{-1}(\varnothing; M) := M$, for any abelian group M.

Proposition 6.2. *If Φ is σ-atomic, then there exists an isomorphism*

$$\lim{}^i_{\mathrm{CAT}^{op}(K)} \Phi \;\cong\; \widetilde{H}^{i-1}(\ell_K(\sigma); \Phi(\sigma))$$

for any $i \geq 0$.

Proof. Let $J = \ell_K(\sigma)$, and define $\Phi' \colon \text{CAT}^{op}(J) \to \text{AB}$ by $\Phi'(\tau \setminus \sigma) := \Phi(\tau)$ for every $\tau \supseteq \sigma$. Thus Φ' is $\varnothing$-atomic, and $\Phi' \circ P^{op} = \Phi|_{\text{CAT}^{op}(K)\downarrow\sigma}$ in terms of the functor P of (2.6). The isomorphisms

$$N^*(\text{CAT}^{op}(K), \Phi) \;\cong\; N^*(\text{CAT}^{op}(K)\!\downarrow\!\sigma, \Phi) \;\cong\; N^*(\text{CAT}^{op}(J), \Phi')$$

of normalised chain complexes determine an isomorphism

$$\lim{}^i_{\text{CAT}^{op}(K)} \Phi \;\cong\; \lim{}^i_{\text{CAT}^{op}(J)} \Phi'$$

for every $i \geq 0$.

If $J = \{\varnothing\}$, the claim is clear. Otherwise, write $\Phi'(\varnothing)$ as M and consider the short exact sequence

$$0 \longrightarrow \Phi' \longrightarrow \text{cst}_M \longrightarrow \varUpsilon \longrightarrow 0 \tag{6.3}$$

of $\text{CAT}^{op}(J)$-diagrams, where $\varUpsilon$ is trivial on $\varnothing$ and constant on the subcategory $\text{CAT}^{op}(J^\times)$. The long exact sequence associated to (6.3) combines with the isomorphism $N^*(\text{CAT}^{op}(J), \varUpsilon) \cong N^*(\text{CAT}^{op}(J^\times), \text{cst}_M)$ to ensure that

$$\lim{}^i_{\text{CAT}^{op}(J)} \Phi' \;\cong\; \lim{}^{i-1}_{\text{CAT}^{op}(J)} \varUpsilon \;\cong\; \widetilde{H}^{i-1}(J; M)$$

for $i \geq 2$, and reduces to the short exact sequence

$$0 \longrightarrow M \longrightarrow H^0(J; M) \longrightarrow \lim{}^1 \Phi' \longrightarrow 0$$

for $i = 1$. So the result holds for all i. □

We may associate a σ-atomic diagram Φ_σ to every Φ, by the formula

$$\Phi_\sigma(\tau) = \begin{cases} \Phi(\sigma) & \text{when } \tau = \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Such diagrams play a fundamental rôle in proving vanishing results for higher limits; in particular, they act as building blocks for the following notions of filtration, which depend on the poset structure of $\text{CAT}^{op}(K)$.

Given Φ as above, Φ_s, $\Phi_{\geq s} \colon \text{CAT}^{op}(K) \to \text{AB}$ are defined by

$$\Phi_s(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } |\sigma| = s \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Phi_{\geq s}(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } |\sigma| \geq s \\ 0 & \text{otherwise.} \end{cases}$$

Thus Φ_s decomposes as the product diagram $\prod_{|\sigma|=s} \Phi_\sigma$. Also, $\Phi_{\geq 0} = \Phi$ and $\Phi_{\geq \dim K+1} = \Phi_{\dim K+1}$ hold by definition, and $\Phi_{\geq \dim K+2}$ is trivial. Furthermore, there is a short exact sequence

$$0 \longrightarrow \Phi_s \longrightarrow \Phi_{\geq s} \longrightarrow \Phi_{\geq s+1} \longrightarrow 0 \tag{6.4}$$

for every $s > 0$, because there are no morphisms $\sigma \to \tau$ when $|\tau| < |\sigma|$.

Lemma 6.5. *If* $\lim{}^i_{\text{CAT}^{op}(K)} \Phi_\sigma = 0$ *for every face* $\sigma \in K$*, then we have that* $\lim{}^i_{\text{CAT}^{op}(K)} \Phi = 0$.

Proof. The decomposition of Φ_s into atomic factors ensures that it is zero, and the long exact sequence associated to (6.4) confirms that

$$\lim{}^{i}_{\mathrm{CAT}^{op}(K)} \Phi_{\geq s} \longrightarrow \lim{}^{i}_{\mathrm{CAT}^{op}(K)} \Phi_{\geq s+1}$$

is a monomorphism for every $0 \leq s \leq \dim K + 1$. So the composition

$$\lim{}^{i}_{\mathrm{CAT}^{op}(K)} \Phi \longrightarrow \lim{}^{i}_{\mathrm{CAT}^{op}(K)} \Phi_{\geq \dim K+2} \;=\; 0$$

is also a monomorphism, and the result follows. □

Lemma 6.6. *If $i \geq \dim K + 2$, then we have that $\lim^{i}_{\mathrm{CAT}^{op}(K)} \Phi = 0$.*

Proof. The cochain group $N^i(\mathrm{CAT}^{op}(K), \Psi)$ vanishes for $i \geq \dim K + 2$. □

Given any two diagrams $\Phi\colon \mathrm{CAT}^{op}(K) \to \mathrm{AB}$ and $\Psi\colon \mathrm{CAT}^{op}(L) \to \mathrm{AB}$, we introduce their *external product* $\Phi \times \Psi\colon \mathrm{CAT}^{op}(K * L) \to \mathrm{AB}$ by means of the formula $\Phi \times \Psi(\sigma \cup \tau) := \Phi(\sigma) \times \Psi(\tau)$.

Lemma 6.7. *For any $i \geq 1$, there is an isomorphism*

$$\lim{}^{i}_{\mathrm{CAT}^{op}(K*L)} \Phi \times \Psi \;\cong\; \lim{}^{i}_{\mathrm{CAT}^{op}(K)} \Phi \times \lim{}^{i}_{\mathrm{CAT}^{op}(L)} \Psi.$$

Proof. This follows directly from the isomorphism

$$C^*(\mathrm{CAT}^{op}(K * L); \Phi \times \Psi) \;\cong\; C^*(\mathrm{CAT}^{op}(K); \Phi) \times C^*(\mathrm{CAT}^{op}(L); \Psi)$$

of cochain complexes. □

7. p-adic homotopy uniqueness for $\Delta^{(r)}(V)$

In this section we show that the homotopy type of the p-adic completion $hc(K)^{\wedge}_p$ is determined by its p-adic cohomology, for any r-skeleton $\Delta^{(r)}(V)$ of the $(m-1)$-simplex. We restrict attention to 1-connected spaces, so that completion is well-behaved. We also maintain the notation of previous sections, but make the additional assumption that any space denoted by X or X_J is p-complete, for every simplicial complex J.

Our aim is to prove that $K = \Delta^{(r)}(V)$ satisfies the following.

Conditions 7.1.

(C1) *There is an isomorphism $\theta\colon H^*(X; \mathbb{Z}^{\wedge}_p) \to \mathbb{Z}^{\wedge}_p[K]$ if and only if there is a homotopy equivalence $f\colon hc(K)^{\wedge}_p \to X$ such that $f^* = \theta$.*

(C2) *Any two self-equivalences $f, g\colon hc(K)^{\wedge}_p \to hc(K)^{\wedge}_p$ are homotopic if and only if $H^*(f; \mathbb{Z}^{\wedge}_p) = H^*(g; \mathbb{Z}^{\wedge}_p)$.*

To confirm (C1), it suffices to assume the existence of θ and construct a mod p equivalence $f\colon hc(K) \to X$ such that $f^* = \theta$. Our strategy is to apply the results of Sections 4 and 5.

By Corollary 4.4, there are maps $h_\sigma\colon BT^\sigma \to X$ for all faces $\sigma \in K$, which induce the restrictions of θ in p-adic cohomology. By Theorem 5.10(1), the h_σ are compatible up to homotopy with the inclusions $BT^\sigma \to BT^\tau$ for all $\sigma \subseteq \tau$. They therefore define a map $(hc(K))^{(1)} \to X$ on the 1-skeleton of the homotopy colimit $hc(K)$. The Bousfield-Kan spectral sequence for homotopy inverse limits, together with work of Wojtkowiak [W] clarifying the situation for low dimensions, provides an obstruction theory for extending this map to $hc(K)$. The obstruction groups are given by the higher limits

$$\lim{}^{i+1}_{\mathrm{CAT}^{op}(K)} \pi_i(M^\sigma(X));$$

if these groups vanish for all $i \geq 1$, then the desired extension exists. So for any $i \geq 1$, it is convenient to define the diagram

$$\Pi_i \colon \mathrm{CAT}^{op}(K) \longrightarrow \mathrm{AB}$$

by assigning $\pi_i(M^\sigma(X))$ to each face σ, and the homomorphism induced by $M^\tau(X) \to M^\sigma(X)$ to each morphism $\tau \supseteq \sigma$.

The obstruction theory also applies to the question of uniqueness. There is a restriction map

$$R : [hc(K), X] \longrightarrow \lim{}^0_{\mathrm{CAT}^{op}(K)}[BT^K, X]$$

of homotopy classes, whose target may be identified with $[hc(K)^{(1)}, X]$. If, for some element $h_1 := (h_\sigma)_{\sigma \in K}$ of the latter, the limits $\lim^i_{\mathrm{CAT}^{op}(K)} \Pi_i$ vanish for all $i \geq 1$, then $R^{-1}(h_1)$ contains at most one element; in other words, there is at most one extension $hc(K) \to X$ of h_1, up to homotopy.

So we are interested in simplicial complexes K that satisfy the following.

Condition 7.2. (VHL) *For any p-complete space X equipped with an isomorphism $\theta\colon H^*(X; \mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge[K]$, the groups*

$$\lim{}^i_{\mathrm{CAT}^{op}(K)} \Pi_i \quad \text{and} \quad \lim{}^{i+1}_{\mathrm{CAT}^{op}(K)} \Pi_i$$

vanish for all $i \geq 1$.

Proposition 7.3. *If K satisfies* (VHL)*, then it satisfies* (C1) *and* (C2)*.*

Proof. If K satisfies (VHL), the above considerations show that there exists a map $f\colon hc(K) \to X$ which induces θ. It therefore determines the homotopy equivalence $f\colon hc(K)_p^\wedge \to X$ required for (C1).

Suppose that $f, g\colon hc(K)_p^\wedge \to hc(K)_p^\wedge$ are self-equivalences that satisfy $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$. Without loss of generality, we may assume that f is the identity. Then for every face $\sigma \in K$, the restriction $g|_{BT^\sigma}$ reflects the pair $(K, \Delta(\sigma))$ over $\mathbb{Z}_p^\wedge$, and so by Proposition 5.10(1) is homotopic to h_σ; in particular, g lies in $R^{-1}((h_\sigma)_{\sigma\in K})$. All obstruction groups vanish by assumption, so 1 and g are homotopic, as required for (C2). $\square$

We now turn to the distinguishing properties of the skeleta $\Delta^{(r)}(V)$. For any space X that reflects K over $\mathbb{Z}_p^\wedge$, we denote the homotopy fibre of the map $q_X\colon X \to BT^V$ of Lemma 4.1 by F_X.

Lemma 7.4. *For $K = \Delta^{(r-1)}(V)$ and any face σ, we have that*

1. *the space F_X is $2r$-connected;*
2. *$\ell_K(\sigma)$ is empty if σ is maximal, and is $\Delta^{(r-|\sigma|-1)}(V \setminus \sigma)$ otherwise.*

Proof. (1) The missing faces of K have cardinality at least $r+1$, so q_X induces an isomorphism of p-adic cohomology in dimensions $\leq 2r+1$. The same is therefore true in homology. Both X and BT^V are simply connected and p-complete, so $\pi_j(F_X)=0$ for $1\leq j\leq 2r$ by Whitehead's Theorem.

(2) The faces ρ of $\ell_K(\sigma)$ are restricted only by the requirements that they are subsets of $V\setminus\sigma$, and that $|\sigma|+|\rho|\leq r$. □

Lemma 7.4(2) combines with (2.11) and Theorem 5.10(3),(4) to create a homotopy pullback square

$$\begin{array}{ccc} X_{\ell_K(\sigma)} & \longrightarrow & M^\sigma(X) \\ {\scriptstyle q_{\ell_K(\sigma)}}\big\downarrow & & \big\downarrow{\scriptstyle q_{M^\sigma(X)}} \\ (BT^{V\setminus\sigma})^\wedge_p & \longrightarrow & (BT^V)^\wedge_p \end{array} \tag{7.5}$$

for every non-maximal face $\sigma\in K$; when σ is maximal, there is an equivalence $M^\sigma(X)\simeq(BT^\sigma)^\wedge_p$.

Theorem 7.6. *Suppose that $K=\Delta^{(r)}(V)$, and X reflects K over $\mathbb{Z}^\wedge_p$: then we have that $\lim^j_{\mathrm{CAT}^{op}(K)}\mathit{\Pi}_i=0$ for all $j\geq i$.*

Proof. If $i=1$, then $\mathit{\Pi}_1(\sigma)=0$ for every face σ by Theorem 5.10(2); so $\lim^j_{\mathrm{CAT}^{op}(K)}\mathit{\Pi}_1=0$ for $j\geq 1$.

If $i=2$, then $\mathit{\Pi}_2(\sigma)=(\mathbb{Z}^\sigma)^\wedge_p$ when σ is maximal, and $(\mathbb{Z}^V)^\wedge_p$ otherwise, by (7.5). So there exists a short exact sequence

$$0\longrightarrow\mathit{\Pi}_2\longrightarrow\mathrm{cst}_{(\mathbb{Z}^V)^\wedge_p}\longrightarrow\mathit{\Psi}\longrightarrow 0,$$

where $\mathit{\Psi}$ is non-zero only on maximal faces, and decomposes as a product of atomic diagrams. Then $\lim^j_{\mathrm{CAT}^{op}(K)}\mathit{\Pi}_2\cong\lim^{j-1}_{\mathrm{CAT}^{op}(K)}\mathit{\Psi}$, and Proposition 6.2 confirms that $\lim^j_{\mathrm{CAT}^{op}(K)}\mathit{\Pi}_2=0$ for $j\geq 2$.

If $i\geq 3$, abbreviate the atomic diagrams $(\mathit{\Pi}_i)_\sigma$ to $\mathit{\Pi}_\sigma$, and the groups $\lim^j_{\mathrm{CAT}^{op}(K)}\mathit{\Pi}_\sigma$ to l^j_σ. It follows from Lemma 7.4(1) and (7.5) that $\mathit{\Pi}_\sigma=0$ for all σ satisfying $|\sigma|\leq r-i/2$, because the fibre of $q_{\ell_K(\sigma)}$ is $2(r-|\sigma|)$-connected. So $l^j_\sigma=0$ for $j\geq i$. The remaining faces satisfy $i>2(r-|\sigma|)$, and therefore $i>(r-|\sigma|)$. But Proposition 6.2 and Lemma 7.4(2) identify l^j_σ with $\widetilde{H}^{j-1}(\Delta^{(r-|\sigma|-1)}(V\setminus\sigma);\mathit{\Phi}(\sigma))$. Hence $l^j_\sigma=0$ for $j\geq i$ in these cases also, and the result follows by applying Lemma 6.5. □

Theorem 7.6 shows that K satisfies (VHL), so the following is an immediate consequence of Proposition 7.3.

Corollary 7.7. *Every complex $\Delta^{(r)}(V)$ satisfies conditions* (C1) *and* (C2).

For low dimensional K, our main results hold without restriction.

Theorem 7.8.

(1) *For* $\dim(K) \leq 1$*, there is an isomorphism* $\theta\colon H^*(X;\mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge[K]$ *if and only if there is a homotopy equivalence* $f\colon hc(K)_p^\wedge \to X$ *such that* $f^* = \theta$*;*

(2) *for* $dim(K) = 0$*, then two self equivalences* $f, g\colon hc(K)_p^\wedge \to hc(K)_p^\wedge$ *are homotopic if and only if* $H^*(f;\mathbb{Z}_p^\wedge) = H^*(g;\mathbb{Z}_p^\wedge)$*.*

Proof. Since all the relevant mapping spaces are simply connected, we need only check that $\lim^j \Pi_i(\sigma)$ vanishes when $j \geq 3$ for (1) and when $j \geq 2$ for (2). These both hold for dimensional reasons. □

8. Joins of complexes and higher limits

We have shown in Section 7 that skeleta of simplices satisfy condition (VHL), and the aim of Theorem 8.7 is to extend the class of complexes with this property. Preparing for the proof occupies most of this section, throughout which we insist that $X = X_{K*L}$ reflects $K * L$ over $\mathbb{Z}_p^\wedge$. We maintain our convention that any space denoted by X or X_J is p-complete, for every simplicial complex J.

Lemma 8.1. *For any* i_K *reflecting* $(K*L, K)$ *and* i_L *reflecting* $(K*L, L)$ *over* $\mathbb{Z}_p^\wedge$*, the homotopy pullback of the diagram*

$$X_K \xrightarrow{i_K} X \xleftarrow{i_L} X_L$$

is p*-complete and contractible.*

Proof. The pullback is p-complete by [BK, Section II.5], and the Eilenberg-Moore spectral sequence computes its mod p cohomology. The E_2-term is given by

$$\mathrm{Tor}^j_{\mathbb{F}_p[K]\otimes\mathbb{F}_p[L]}(\mathbb{F}_p[K], \mathbb{F}_p[L]) = \begin{cases} 0 & \text{for } j \neq 0 \\ \mathbb{F}_p & \text{for } j = 0 \end{cases},$$

so the spectral sequence collapses and the pullback is contractible. □

The homotopy fibre of i_L is therefore the loop space ΩX_K, and its inclusion into X_L is null-homotopic.

Lemma 8.2. *For any* i_L *reflecting* $(K*L, L)$ *over* $\mathbb{Z}_p^\wedge$*, and any face* $\tau \in L$*, the square*

$$\begin{array}{ccc} M^\tau(X_L) & \xrightarrow{ev} & X_L \\ {\scriptstyle M^\tau(i_L)}\downarrow & & \downarrow{\scriptstyle i_L} \\ M^\tau(X) & \xrightarrow{ev} & X \end{array}$$

is a homotopy pullback, and functorial in τ*.*

Proof. First apply the functor $map(BT^\tau, -)$ to i_L, and use evaluation at the base point to obtain a commutative square whose upper left-hand entry is $M^\tau(X_L)$. By Theorems 4.14 and 5.10(2), the upper evaluation reflects $(L, st_L(\tau))$ over $\mathbb{Z}_p^\wedge$; and there is a canonical map $d\colon M^\tau(X_L) \to P$ to the pullback P.

The Eilenberg-Moore spectral sequence computes $H^*(P;\mathbb{F}_p)$. The isomorphisms

$$\mathrm{Tor}^j_{\mathbb{F}_p[K]\otimes\mathbb{F}_p[L]}(\mathbb{F}_p[K]\otimes\mathbb{F}_p[st_L(\tau)],\mathbb{F}_p[L]) \;\cong\; \mathrm{Tor}^j_{\mathbb{F}_p}(\mathbb{F}_p[st_L(\tau)],\mathbb{F}_p)$$

show that the higher Tor groups vanish, and that the spectral sequence collapses at the E_2-page. It follows that d is a homotopy equivalence, as required. Functoriality of the construction is immediate. □

Lemma 8.3. *Suppose that L satisfies condition* (VHL)*; then there exists a map $j_L\colon hc(L)\to X$ reflecting $(K*L,L)$ over $\mathbb{Z}_p^\wedge$, and it is unique up to homotopy.*

Proof. Consider a maximal face μ of the subcomplex $K\subseteq K*L$, so that $\ell_{K*L}(\mu)=L$. The fibre inclusion $X_L\to M^\mu(X)$ of Theorem 5.10(3) may be combined with the equivalence $hc(L)\to X_L$ that arises because L satisfies (VHL). Then composition with evaluation at the basepoint defines j_L as $hc(L)\to M^\mu(X)\to X$, which reflects $(K*L,K)$ over $\mathbb{Z}_p^\wedge$ by Theorem 5.10.

In order to prove homotopy uniqueness, we utilise the obstruction theory of Section 7. In this case the obstruction groups are $\lim^i_{\mathrm{CAT}^{op}(L)}\pi_i(M^\tau(X))$ for $i\geq 1$, and it remains to show that they vanish.

First apply Lemmas 8.1 and 8.2 to the maps j_K and j_L. The homotopy exact sequences of the principal fibrations $\Omega hc(K)\to M^\tau(hc(L))\to M^\tau(X)$ decompose into short exact sequences

$$0\longrightarrow\pi_i(M^\tau(hc(L)))\longrightarrow\pi_i(M^\tau(X))\longrightarrow\mathrm{cst}_{\pi_i(hc(K))}\longrightarrow 0 \tag{8.4}$$

of $\mathrm{CAT}^{op}(L)$-diagrams. But L satisfies (VHL) by assumption, and the higher limits of any constant diagram are zero on $\mathrm{CAT}^{op}(L)$. So (8.4) implies that $\lim^i_{\mathrm{CAT}^{op}(L)}\pi_i(M^\tau(X))=0$ for $i\geq 1$, as required. □

Lemma 8.5. *Suppose that K and L satisfy* (VHL)*; then for any $\sigma\in K$ and $\tau\in L$ there exist maps $l_K(\sigma,\tau)$ reflecting $(st_K(\sigma)*st_L(\tau),st_K(\sigma))$ and $l_L(\tau,\sigma)$ reflecting $(st_K(\sigma)*st_L(\tau),st_L(\tau))$ over $\mathbb{Z}_p^\wedge$, such that the diagram*

$$M^\sigma(hc(K))\xrightarrow{l_K(\sigma,\tau)}M^{\sigma\cup\tau}(X)\xleftarrow{l_L(\tau,\sigma)}M^\tau(hc(L))$$

is homotopy functorial in σ and τ, and has contractible homotopy pullback.

Proof. In order to define $l_K(\sigma,\tau)$ and $l_L(\tau,\sigma)$, observe that the restrictions of $h_{\sigma\cup\tau}\colon BT^{\sigma\cup\tau}\to X$ to BT^σ and BT^τ are homotopic to h_σ and h_τ respectively, by Theorem 5.10(1). So the adjoint properties of mapping spaces provide homeomorphisms

$$M^\tau(M^\sigma(X))\;\equiv\;M^{\sigma\cup\tau}(X)\;\equiv\;M^\sigma(M^\tau(X)). \tag{8.6}$$

By Theorem 5.10(2) and Lemma 8.3, there exists $j_L(\sigma)\colon hc(L)\to M^\sigma(X)$ reflecting $(st_K(\sigma)*L,L)$ over $\mathbb{Z}_p^\wedge$. Then $l_L(\tau,\sigma)$ is induced by applying $map\,(BT^\tau,-)$ to $j_L(\sigma)$, and reflects $(st_K(\sigma)*st_L(\tau),st_L(\tau))$ over $\mathbb{Z}_p^\wedge$ by Theorem 5.10(2). The construction is functorial in τ, and homotopy functorial in σ because Lemma 8.3 defines $j_L(\sigma)$ uniquely up to homotopy. Interchanging $\sigma \in K$ and $\tau \in L$ defines $l_K(\sigma,\tau)$, which reflects $(st_K(\sigma)*st_L(\tau), st_K(\sigma))$ over $\mathbb{Z}_p^\wedge$; the construction is functorial in σ and homotopy functorial in τ.

The homotopy pullback of the diagram is contractible by Lemma 8.1. $\square$

Proposition 8.7. *The class of simplicial complexes satisfying* (VHL) *is closed under the formation of finite joins.*

Proof. Let K and L satisfy (VHL). By Lemma 8.5, there are isomorphisms

$$\pi_i(M^\sigma(hc(K))) \times \pi_i(M^\tau(hc(L))) \;\cong\; \pi_i(M^{\tau\cup\sigma}(X))$$

that are functorial on $\mathrm{CAT}^{op}(K) \times \mathrm{CAT}^{op}(L)$, and so on $\mathrm{CAT}^{op}(K*L)$. The result follows from Lemma 6.7, by induction on the number of factors. $\square$

We believe that other combinatorial operations on simplicial complexes may also respect condition (VHL).

Theorem 8.8. *Let X be a p-complete CW-complex; if K is an iterated join $\Delta^{(r_1)}(U_1) * \cdots * \Delta^{(r_t)}(U_t)$ of skeleta of simplices, then there is an isomorphism $\theta\colon H^*(X;\mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f\colon hc(K)_p^\wedge \to X$ such that $f^* = \theta$.*

Proof. Any such K satisfies condition (VHL) by Theorem 7.6 and Proposition 8.7, and therefore satisfies (C1) by Proposition 7.3. $\square$

Theorem 8.9. *Given K as in Theorem 8.8, any pair of self-equivalences $f, g\colon hc(K)_p^\wedge \to hc(K)_p^\wedge$ are homotopic if and only if $H^*(f;\mathbb{Z}_p^\wedge) = H^*(g;\mathbb{Z}_p^\wedge)$.*

Proof. As before, K satisfies (C2) by Proposition 7.3. $\square$

9. The integral homotopy type

Our final aim is to prove Theorems 9.5 and 9.6. For any space Y, we denote the product of all its p-adic completions by $Y^\wedge := \prod_p Y_p^\wedge$, and its rationalisation by Y_0; its finite adele type $Y_{\mathbb{A}_f}$ is both the rationalisation of $Y^\wedge$ and the formal completion of Y_0. When Y is nilpotent, these spaces fit into Sullivan's arithmetic pullback square

$$\begin{array}{ccc} Y & \xrightarrow{\wedge} & Y^\wedge \\ \big\downarrow & & \big\downarrow \\ Y_0 & \xrightarrow{\wedge} & Y_{\mathbb{A}_f} \end{array} \quad . \tag{9.1}$$

For any partition $\mathcal{U} := \{U_0, \ldots, U_t\}$ of V, we write $K(\mathcal{U})$ for the iterated join $\Delta(U_0) * \partial(U_1) * \cdots * \partial(U_t)$. So there is an isomorphism

$$\mathbb{Q}[K] \;\cong\; \mathbb{Q}[\Delta(U_0)] \otimes \mathbb{Q}[\partial(U_1)] \otimes \cdots \otimes \mathbb{Q}[\partial(U_t)],$$

which implies the following observation of [NR, Section 5].

Proposition 9.2. *The simplicial complexes $K(\mathcal{U})$ are precisely those for which $\mathbb{Q}[K]$ is a complete intersection.*

For the rest of this section we insist that X is a 1-connected space realising the complete intersection $\mathbb{Q}[K(\mathcal{U})]$, and maintain our convention of denoting the homotopy fibre of $q_X\colon X \to BT^V$ by F_X. In particular, the cofibre of any $q_{hc(\partial(U))}$ is the Thom complex $Th(U)$ of the universal product of line bundles over BT^U, so $q_{hc(\partial(U))}$ is equivalent to the projection of the corresponding sphere bundle, and has homotopy fibre the $(2|U|-1)$-dimensional sphere $S(U)$; hence $F_{hc(K(\mathcal{U}))}$ is given by $S(\mathcal{U}) := S(U_1) \times \cdots \times S(U_t)$.

Lemma 9.3. *The p-adic completion, rationalisation, and finite adele type of the fibration $F_X \to X \to BT^V$ are all themselves fibrations; moreover, F_X is 1-connected and rationally equivalent to $S(\mathcal{U})$.*

Proof. The first statement follows from [BK, Section II.5] because BT^V is 1-connected; the 1-connectedness of F_X arises directly from the definitions. Part I of [NR, Proposition 5.11] confirms the existence of a homotopy equivalence $h\colon hc(K(\mathcal{U}))_0 \to X_0$, for which $q_X \circ h \simeq q_{hc(K(\mathcal{U}))}$ as rationalised maps. So h lifts to a rational equivalence $S(\mathcal{U}) \to F_X$ of homotopy fibres. □

Now let $q^\sharp\colon X^\sharp \to BT^V$ denote the homotopy pullback of the fibration $q_X^\wedge$ along the completion map $BT^V \to (BT^V)^\wedge$; then the inclusion $F_X^\wedge \to X^\sharp$ of the fibre is homotopic to the principal T^V-fibration classified by $q^\sharp$.

Lemma 9.4. *The homotopy fibre $F^\sharp$ of the induced map $X \to X^\sharp$ is a product $\prod_{i=1}^t H(\mathbb{Z}^\wedge/\mathbb{Z}, 2|U_i|-2)$ of Eilenberg-Mac Lane spaces.*

Proof. By construction, there is a homotopy pullback diagram

$$\begin{array}{ccc} F_X & \xrightarrow{\wedge} & F_X^\wedge \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & X^\sharp \end{array},$$

in which the homotopy fibre of the horizontal maps agrees with that of $(F_X)_0 \to (F_X)_{\mathbb{A}_f}$. Since the fibre of $S(\mathcal{U})_0 \to S(\mathcal{U})_{\mathbb{A}_f}$ has the required form, the result follows from (9.1) and Lemma 9.3. □

Theorem 9.5. *Let X be a nilpotent CW-complex; if $\mathbb{Q}[K]$ is a complete intersection, then there is an isomorphism $\theta\colon H^*(X;\mathbb{Z}) \to \mathbb{Z}[K]$ if and only if there is a homotopy equivalence $f\colon hc(K) \to X$ such that $f^* = \theta$.*

Proof. Using Proposition 9.2, write K as $K(\mathcal{U})$ for some partition $\mathcal{U}$ of V. Then θ extends to an isomorphism $H^*(X;\mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge[K(\mathcal{U})]$ for each prime p, which is induced by an equivalence $hc(K(\mathcal{U}))_p^\wedge \to X_p^\wedge$ via Theorem 8.8. So there exists a map $f'\colon hc(K(\mathcal{U})) \to X^\wedge$, such that $q_X^\wedge \circ f'$ factors through the completion $BT^V \to (BT^V)^\wedge$ using arguments of Sections 7 and 8. In other words, the square

$$\begin{array}{ccc} hc(K(\mathcal{U})) & \xrightarrow{q_{hc(K(\mathcal{U}))}} & BT^V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \wedge} \\ X^\wedge & \xrightarrow{q_X^\wedge} & (BT^V)^\wedge \end{array},$$

is homotopy commutative, and f' factors through $f''\colon hc(K(\mathcal{U})) \to X^\sharp$.

In order to lift f'' to X, note that the obstructions lie in the groups $H^{i+1}(hc(K(\mathcal{U})); \pi_i(F^\sharp))$. But $\pi_i(F^\sharp) = 0$ in odd dimensions by Lemma 9.4; so all obstructions vanish, and a lift $f\colon hc(K(\mathcal{U})) \to X$ exists. Then $f^* = \theta$ by construction, and f is an equivalence by Whitehead's theorem, because the source and target are 1-connected. □

Theorem 9.6. *Let X be a CW-complex that realises $\mathbb{Z}[K]$ for any K; then there is a homotopy equivalence $e\colon \Sigma X \to \Sigma hc(K)$.*

Proof. The Stanley-Reisner algebra $\mathbb{Z}[K]$ decomposes additively as a sum $\oplus_{\sigma\in K} A_\sigma$ of graded subgroups, generated by monomials whose support is a particular σ. Every A_σ is realised in $H^*(X;\mathbb{Z})$ by a map $q'_\sigma\colon X \to \wedge_\sigma BT$, defined by composing q_σ with projection onto the iterated smash product. Using the cogroup structure of ΣX, we sum the suspensions $\Sigma q'_\sigma$ over all faces σ, to obtain a map

$$h_X\colon \Sigma X \longrightarrow \bigvee_{\sigma\in K} \Sigma(\wedge_\sigma BT).$$

This induces a cohomology isomorphism of 1-connected CW-complexes, and is therefore a homotopy equivalence. Identical reasoning applies to $hc(K)$, so it suffices to define h as $h^{-1}_{c(K)} \circ h_X$. □

We may use the splitting of [BBCG, Theorem 1.18] as an elegant alternative to $h_{c(K)}$; in either event, Proposition 9.6 identifies X with the wedge $\bigvee_{\sigma\in K} Th(\sigma)$ of Thom complexes, after one suspension.

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